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Inverse moving source problems in electrodynamics

Guanghui Hu[1], **Yavar Kian**[2], **Peijun Li**[3] **and Yue Zhao**[4,5]

[1] Beijing Computational Science Research Center, Beijing 100193, People's Republic of China

[2] Aix Marseille Univ, Université de Toulon, CNRS, CPT, Marseille, France

[3] Department of Mathematics, Purdue University, West Lafayette, Indiana 47907, United States of America

[4] School of Mathematics and Statistics, Central China Normal University, Wuhan 430079, People's Republic of China

E-mail: hu@csrc.ac.cn, yavar.kian@univ-amu.fr, lipeijun@math.purdue.edu and zhaoyueccnu@163.com

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Abstract

This paper is concerned with the uniqueness of two inverse moving source problems in electrodynamics with partial boundary data. We show that (1) if the temporal source function is compactly supported, then the spatial source profile function or the orbit function can be uniquely determined by the tangential trace of the electric field measured on part of a sphere; (2) if the temporal function is given by a Dirac distribution, then the impulsive time point and the source location can be uniquely determined at four receivers on a sphere.

Keywords: inverse moving source problems, Maxwell's equations, uniqueness

(Some figures may appear in colour only in the online journal)

1. Introduction

Consider the time-dependent Maxwell equations in a homogeneous medium:

$$\mu\partial_t \boldsymbol{H}(\boldsymbol{x},t) + \nabla\times\boldsymbol{E}(\boldsymbol{x},t) = 0, \quad \varepsilon\partial_t\boldsymbol{E}(\boldsymbol{x},t) - \nabla\times\boldsymbol{H}(\boldsymbol{x},t) = -\sigma\boldsymbol{E} + \tilde{F}(\boldsymbol{x},t), \quad \boldsymbol{x}\in\mathbb{R}^3,\ t>0, \tag{1.1}$$

where $\boldsymbol{E}$ and $\boldsymbol{H}$ are the electric and magnetic fields, respectively, the source function $\tilde{F}$ is known as the electric current density, ε and μ are the dielectric permittivity and the magnetic permeability, respectively, and σ is the electric conductivity and is assumed to be zero. Since the medium is homogeneous, we assume, without loss of generality, that $\varepsilon = \mu = 1$.

[5] Author to whom any correspondence should be addressed.

Eliminating the magnetic field $\boldsymbol{H}$ from (1.1), we obtain the Maxwell system for the electric field $\boldsymbol{E}$:

$$\partial_t^2 \boldsymbol{E}(\boldsymbol{x},t) + \nabla \times (\nabla \times \boldsymbol{E}(\boldsymbol{x},t)) = \partial_t \tilde{F}(\boldsymbol{x},t) =: \boldsymbol{F}(\boldsymbol{x},t), \quad \boldsymbol{x} \in \mathbb{R}^3,\ t > 0, \tag{1.2}$$

which is supplemented by the homogeneous initial conditions

$$\boldsymbol{E}(\boldsymbol{x},0) = \partial_t \boldsymbol{E}(\boldsymbol{x},0) = 0, \quad \boldsymbol{x} \in \mathbb{R}^3. \tag{1.3}$$

The electrodynamic field is assumed to be excited by a moving point source radiating over a finite time period. Specifically, the source function $\boldsymbol{F}$ is assumed to be given in the following form:

$$\boldsymbol{F}(\boldsymbol{x},t) = \boldsymbol{J}(\boldsymbol{x} - \boldsymbol{a}(t))\, g(t),$$

where $\boldsymbol{J} : \mathbb{R}^3 \to \mathbb{R}^3$ is the source profile function, $g : \mathbb{R}_+ \to \mathbb{R}$ the temporal function, and $\boldsymbol{a} : \mathbb{R}_+ \to \mathbb{R}^3$ is the orbit function of the moving source. Hence the source term is assumed to be a product of the spatially moving source function $\boldsymbol{J}(\boldsymbol{x} - \boldsymbol{a}(t))$ and the temporal function $g(t)$. Physically, the spatially moving source function can be thought as an approximation of a pulsed signal which is transmitted by a moving antenna, whereas the temporal function is usually used to model the evolution of source magnitude in time. Throughout, we make the following assumptions:

(1) The profile function $\boldsymbol{J}(\boldsymbol{x})$ is compactly supported in $B_{\hat{R}} := \{\boldsymbol{x} : |\boldsymbol{x}| < \hat{R}\}$ for some $\hat{R} > 0$;
(2) the source radiates only over a finite time period $[0, T_0]$ for some $T_0 > 0$, i.e. $g(t) = 0$ for $t \geqslant T_0$ and $t \leqslant 0$;
(3) the source moves in a bounded domain, i.e. $|\boldsymbol{a}(t)| < R_1$ for all $t \in \mathbb{R}_+$ and some $R_1 > 0$.

These assumptions imply that the source term $\boldsymbol{F}$ is supported in $B_R \times (0, T_0)$ for $R > \hat{R} + R_1$. Unless otherwise stated, we take $T := T_0 + \hat{R} + R_1 + R$ and set $\Gamma_R := \{\boldsymbol{x} \in \mathbb{R}^3 : |\boldsymbol{x}| = R\}$. Denote by $\boldsymbol{\nu}$ the unit normal vector on Γ_R and let $\Gamma \subset \Gamma_R$ be an open subset with a positive Lebesgue measure.

In this work, we study the inverse moving source problems of determining the profile function $\boldsymbol{J}(\boldsymbol{x})$ and the orbit function $\boldsymbol{a}(t)$ from boundary measurements of the tangential trace of the electric field over a finite time interval, $\boldsymbol{E}(\boldsymbol{x},t) \times \boldsymbol{\nu}|_{\Gamma \times [0,T]}$. Specifically, we consider the following two inverse problems.

(i) IP1. Assume that $\boldsymbol{a}(t)$ is known, the inverse problem is to determine $\boldsymbol{J}$ from the measurement $\boldsymbol{E}(\boldsymbol{x},t) \times \boldsymbol{\nu}, \boldsymbol{x} \in \Gamma, t \in (0,T)$.
(ii) IP2. Assume that $\boldsymbol{J}$ is a known vector function, the inverse problem is to determine $\boldsymbol{a}(t), t \in (0, T_0)$ from the measurement $\boldsymbol{E}(\boldsymbol{x},t) \times \boldsymbol{\nu}, \boldsymbol{x} \in \Gamma, t \in (0,T)$.

The IP1 is a linear inverse source problem, whereas the IP2 is a nonlinear inverse source problem. The inverse source problems arise from many scientific and industrial areas such as antenna design and synthesis, biomedical imaging, and photo-acoustic tomography [2]. The time-dependent inverse source problems have attracted considerable attention [3, 11, 14, 15, 19, 20, 25]. However, the inverse moving source problems are rarely studied for the wave propagation. We refer to [8] on the inverse moving source problems by using the time-reversal method and to [21, 22] for the inverse problems of moving obstacles. Numerical methods can be found in [16, 18, 24] to identify the orbit of a moving acoustic point source. To the best of our knowledge, the uniqueness result is not available for the inverse moving source problem, which is the focuse of this paper.

Recently, a Fourier method was proposed for solving inverse source problems for the time-dependent Lamé system [4] and the Maxwell system [9], where the source term is assumed to be the product of a spatial function and a temporal function. These work were motivated by the studies on the uniqueness and increasing stability in recovering compactly supported source terms with multiple frequency data [5–7, 12, 13, 26]. It is known that there is no uniqueness for the inverse source problems with a single frequency data due to the existence of non-radiating sources [1, 23]. In [4, 9], the idea was to use the Fourier transform and combine with Huygens' principle to reduce the time-dependent inverse problem into an inverse problem in the Fourier domain with multi-frequency data. The idea was further extended in [10] to handle the time-dependent source problems in elastodynamics where the uniqueness and stability were studied.

In this paper, we use partial boundary measurements of dynamical Dirichlet data over a finite time interval to recover either the source profile function or the orbit function. In sections 3 and 4.2, we show that the ideas of [4, 9] and [10] can be used to recover the source profile function as well as the moving trajectory which lies on a flat surface. For general moving orbit functions, we apply the moment theory to deduce the uniqueness under *a priori* assumptions on the path of the moving source, see section 4.1. When the compactly supported temporal function shrinks to a Dirac distribution, we show in section 5 that the data measured at four discrete receivers on a sphere is sufficient to uniquely determine the impulsive time point and to the source location. This work is a nontrivial extension of the Fourier approach from recovering the spatial sources to recovering the orbit functions. The latter is nonlinear and more difficult to handle.

The rest of the paper is organized as follows. In section 2, we present some preliminary results concerning the regularity and well-posedness of the direct problem. Sections 3 and 4 are devoted to the uniqueness of IP1 and IP2, respectively. In section 5, we show the uniqueness to recover a Dirac distribution of the source function by using a finite number of receivers.

2. The direct problem

In addition to those assumptions given in the previous section, we give some additional conditions on the source functions:

$$\boldsymbol{J} \in H^2(\mathbb{R}^3), \quad \operatorname{div} \boldsymbol{J} = 0 \ \text{ in } \ \mathbb{R}^3, \quad g \in C^1(\mathbb{R}_+), \quad \boldsymbol{a} \in C^1(\mathbb{R}_+).$$

It follows from [1] that any source function can be decomposed into a sum of radiating and non-radiating parts. The non-radiating part cannot be determined and gives rise to the non-uniqueness issue. By the divergence-free condition of $\boldsymbol{J}$, we eliminate non-radiating sources in order to ensure the uniqueness of the inverse problem. Since the source term $\boldsymbol{J}$ has a compact support in $B_R \times (0, T)$, we may show the following result by Huygens' principle.

Lemma 2.1. *It holds that* $\boldsymbol{E}(\boldsymbol{x}, t) = 0$ *for all* $\boldsymbol{x} \in B_R, t > T$.

The proof of lemma 2.1 is similar to that of lemma 2.1 in [9]. It states that the electric field $\boldsymbol{E}$ over B_R must vanish after time T. This property of the electric field plays an important role in the mathematical justification of the Fourier approach.

Noting $\nabla \cdot \boldsymbol{J} = 0$, taking the divergence on both sides of (1.2), and using the initial conditions (1.3), we have

$$\partial_t^2 (\nabla \cdot \boldsymbol{E}(\boldsymbol{x}, t)) = 0, \quad \boldsymbol{x} \in \mathbb{R}^3, \ t > 0$$

and

$$\nabla \cdot \boldsymbol{E}(\boldsymbol{x},0) = \partial_t(\nabla \cdot \boldsymbol{E}(\boldsymbol{x},0)) = 0.$$

Therefore, $\nabla \cdot \boldsymbol{E}(\boldsymbol{x},t) = 0$ for all $\boldsymbol{x} \in \mathbb{R}^3$ and $t > 0$. In view of the identify $\nabla \times \nabla \times = -\Delta + \nabla\nabla$, we obtain from (1.2) and (1.3) that

$$\begin{cases} \partial_t^2 \boldsymbol{E}(\boldsymbol{x},t) - \Delta \boldsymbol{E}(\boldsymbol{x},t) = \boldsymbol{J}(\boldsymbol{x}-\boldsymbol{a}(t))g(t), & \boldsymbol{x} \in \mathbb{R}^3,\ t>0, \\ \boldsymbol{E}(\boldsymbol{x},0) = \partial_t \boldsymbol{E}(\boldsymbol{x},0) = 0, & \boldsymbol{x} \in \mathbb{R}^3. \end{cases} \tag{2.1}$$

We briefly introduce some notation on functional spaces with the time variable. Given the Banach space X with norm $||\cdot||_X$, the space $C([0,T];X)$ consists of all continuous functions $f : [0,T] \to X$ with the norm

$$||f||_{C([0,T];X)} := \max_{t\in[0,T]} ||f(t,\cdot)||_X.$$

The Sobolev space $W^{m,p}(0,T;X)$, where both m and p are positive integers such that $1 \leqslant m < \infty, 1 \leqslant p < \infty$, comprises all functions $f \in L^2(0,T;X)$ such that $\partial_t^k f, k = 0,1,2,\cdots,m$ exist in the weak sense and belong to $L^p(0,T;X)$. The norm of $W^{m,p}(0,T;X)$ is given by

$$||f||_{W^{m,p}(0,T;X)} := \left(\int_0^T \sum_{k=0}^m ||\partial_t^k f(t,\cdot)||_X^p \right)^{1/p}.$$

Denote $H^m = W^{m,2}$.

Now we state the regularity of the solution for the initial value problem (2.1). The proof follows similar arguments to the proof of lemma 2.3 in [9] by taking $p = 2$.

Lemma 2.2. *The initial value problem (2.1) admits a unique solution*

$$\boldsymbol{E} \in C(0,T;H^3(\mathbb{R}^3))^3 \cap H^\tau(0,T;H^{2-\tau+1}(\mathbb{R}^3))^3, \quad \tau = 1,2,$$

which satisfies

$$\|\boldsymbol{E}\|_{C([0,T];H^3(\mathbb{R}^3))^3} + \|\boldsymbol{E}\|_{H^\tau(0,T;H^{2-\tau+1}(\mathbb{R}^3))^3} \leqslant C\|g\|_{L^2(0,T)}\|\boldsymbol{J}\|_{H^2(\mathbb{R}^3)^3},$$

where C is a positive constant depending on R.

Applying the Sobolev embedding theorem, it follows from lemma 2.2 that

$$\boldsymbol{E} \in C([0,T];H^2(\mathbb{R}^3))^3 \cap C^1([0,T];H^1(\mathbb{R}^3))^3.$$

Denote by $\mathbb{I}$ the 3-by-3 identity matrix and by H the Heaviside step function. Recall the Green tensor $\mathbb{G}(\boldsymbol{x},t)$ to the Maxwell system (see e.g. [9])

$$\mathbb{G}(\boldsymbol{x},t) = \frac{1}{4\pi|\boldsymbol{x}|}\delta'(|\boldsymbol{x}|-t)\mathbb{I} - \nabla\nabla^\top\Big(\frac{1}{4\pi|\boldsymbol{x}|}H(|\boldsymbol{x}|-t)\Big),$$

which satisfies

$$\partial_t^2 \mathbb{G}(\boldsymbol{x},t) + \nabla\times(\nabla\times\mathbb{G}(\boldsymbol{x},t)) = -\delta(t)\delta(\boldsymbol{x})\,\mathbb{I}$$

with the homogeneous initial conditions

$$\mathbb{G}(\boldsymbol{x},0) = \partial_t \mathbb{G}(\boldsymbol{x},0) = 0, \quad |\boldsymbol{x}| \neq 0.$$

Taking the Fourier transform of $\mathbb{G}(\boldsymbol{x}, t)$ with respect to the time variable yields

$$\hat{\mathbb{G}}(\boldsymbol{x}, \kappa) = \Big(g(\boldsymbol{x}, \kappa)\mathbb{I} + \frac{1}{\kappa^2}\nabla\nabla^\top g(\boldsymbol{x}, \kappa)\Big), \tag{2.2}$$

which is known as the Green tensor to the reduced time-harmonic Maxwell system with the wavenumber κ. Here g is the fundamental solution of the three-dimensional Helmholtz equation and is given by

$$g(\boldsymbol{x}, \kappa) = \frac{1}{4\pi}\frac{\mathrm{e}^{\mathrm{i}\kappa|\boldsymbol{x}|}}{|\boldsymbol{x}|}.$$

It is clear to verify that $\hat{\mathbb{G}}(\boldsymbol{x}, \kappa)$ satisfies

$$\nabla \times (\nabla \times \hat{\mathbb{G}}) - \kappa^2\hat{\mathbb{G}} = \delta(\boldsymbol{x})\mathbb{I}, \quad \boldsymbol{x} \in \mathbb{R}^3,\ |\boldsymbol{x}| \neq 0.$$

3. Determination of the source profile function

In this section we consider IP1. Below we state the uniqueness result. The idea of the proof is to adopt the Fourier approach of [9] to the case of a moving point source.

Theorem 3.1. *Suppose that the orbit function* $\boldsymbol{a}$ *is given and that* $\int_0^{T_0} g(t)\mathrm{d}t \neq 0$. *Then the source profile function* $\boldsymbol{J}(\boldsymbol{x})$ *can be uniquely determined by the partial data set* $\{\boldsymbol{E}(\boldsymbol{x}, t) \times \boldsymbol{\nu} : \boldsymbol{x} \in \Gamma, t \in (0, T)\}$.

Proof. Assume that there are two functions $\boldsymbol{J}_1$ and $\boldsymbol{J}_2$ which satisfy

$$\begin{cases} \partial_t^2\boldsymbol{E}_1(\boldsymbol{x}, t) + \nabla \times (\nabla \times \boldsymbol{E}_1(\boldsymbol{x}, t)) = \boldsymbol{J}_1(\boldsymbol{x} - \boldsymbol{a}(t))\, g(t), & \boldsymbol{x} \in \mathbb{R}^3,\ t > 0, \\ \boldsymbol{E}_1(\boldsymbol{x}, 0) = \partial_t\boldsymbol{E}_1(\boldsymbol{x}, 0) = 0, & \boldsymbol{x} \in \mathbb{R}^3, \end{cases}$$

and

$$\begin{cases} \partial_t^2\boldsymbol{E}_2(\boldsymbol{x}, t) + \nabla \times (\nabla \times \boldsymbol{E}_2(\boldsymbol{x}, t)) = \boldsymbol{J}_2(\boldsymbol{x} - \boldsymbol{a}(t))\, g(t), & \boldsymbol{x} \in \mathbb{R}^3,\ t > 0, \\ \boldsymbol{E}_2(\boldsymbol{x}, 0) = \partial_t\boldsymbol{E}_2(\boldsymbol{x}, 0) = 0, & \boldsymbol{x} \in \mathbb{R}^3. \end{cases}$$

It suffices to show $\boldsymbol{J}_1(\boldsymbol{x}) = \boldsymbol{J}_2(\boldsymbol{x})$ in B_R if $\boldsymbol{E}_1(\boldsymbol{x}, t) \times \boldsymbol{\nu} = \boldsymbol{E}_2(\boldsymbol{x}, t) \times \boldsymbol{\nu}$ for all $x \in \Gamma, t \in (0, T)$.

Let $\boldsymbol{E} = \boldsymbol{E}_1 - \boldsymbol{E}_2$ and

$$\boldsymbol{f}(\boldsymbol{x}, t) = \boldsymbol{J}_1(\boldsymbol{x} - \boldsymbol{a}(t))\, g(t) - \boldsymbol{J}_2(\boldsymbol{x} - \boldsymbol{a}(t))\, g(t).$$

Then we have

$$\begin{cases} \partial_t^2\boldsymbol{E}(\boldsymbol{x}, t) + \nabla \times (\nabla \times \boldsymbol{E}(\boldsymbol{x}, t)) = \boldsymbol{f}(\boldsymbol{x}, t), & \boldsymbol{x} \in \mathbb{R}^3,\ t > 0, \\ \boldsymbol{E}(\boldsymbol{x}, 0) = \partial_t\boldsymbol{E}(\boldsymbol{x}, 0) = 0, & \boldsymbol{x} \in \mathbb{R}^3, \\ \boldsymbol{E}(\boldsymbol{x}, t) \times \boldsymbol{\nu} = 0, & \boldsymbol{x} \in \Gamma,\ t > 0. \end{cases}$$

Denote by $\hat{\boldsymbol{E}}(\boldsymbol{x}, \kappa)$ the Fourier transform of $\boldsymbol{E}(\boldsymbol{x}, t)$ with respect to the time t, i.e.

$$\hat{\boldsymbol{E}}(\boldsymbol{x}, \kappa) = \int_{\mathbb{R}} \boldsymbol{E}(\boldsymbol{x}, t)\mathrm{e}^{-\mathrm{i}\kappa t}\mathrm{d}t, \quad \boldsymbol{x} \in B_R,\ \kappa \in \mathbb{R}^+. \tag{3.1}$$

By lemma 2.1, the improper integral on the right-hand side of (3.1) makes sense and it holds that

$$\hat{\boldsymbol{E}}(\boldsymbol{x},\kappa)=\int_0^T \boldsymbol{E}(\boldsymbol{x},t)\mathrm{e}^{-\mathrm{i}\kappa t}\mathrm{d}t,\quad \boldsymbol{x}\in B_R,\ \kappa>0.$$

Hence

$$\hat{\boldsymbol{E}}(\boldsymbol{x},\kappa)\times\boldsymbol{\nu}=0,\quad \forall\boldsymbol{x}\in\Gamma,\ \kappa\in\mathbb{R}^+.$$

Taking the Fourier transform of (1.2) with respect to the time t, we obtain

$$\nabla\times(\nabla\times\hat{\boldsymbol{E}})-\kappa^2\hat{\boldsymbol{E}}=\int_0^T \boldsymbol{f}(\boldsymbol{x},t)\mathrm{e}^{-\mathrm{i}\kappa t}\mathrm{d}t,\quad \boldsymbol{x}\in\mathbb{R}^3. \tag{3.2}$$

Since $\mathrm{supp}(\boldsymbol{J})\subset B_{\hat{R}}$ and $|\boldsymbol{a}(t)|<R_1$, it is clear to note that $\hat{\boldsymbol{E}}$ is analytic with respect to $\boldsymbol{x}$ in a neighbourhood of $\Gamma_R\supseteq\Gamma$ and $\hat{\boldsymbol{E}}$ satisfies the Silver–Müller radiation condition:

$$\lim_{r\to\infty}((\nabla\times\hat{\boldsymbol{E}})\times\boldsymbol{x}-\mathrm{i}\kappa r\hat{\boldsymbol{E}})=0,\quad r=|\boldsymbol{x}|,$$

for any fixed frequency $\kappa>0$. In fact, the radiation condition of $\hat{\boldsymbol{E}}$ can be straightforwardly derived from the expression of $\boldsymbol{E}$ in terms of the Green tensor $\mathbb{G}(\boldsymbol{x},t)$ together with the radiation condition of $\hat{\mathbb{G}}(\boldsymbol{x};\kappa)$. The details may be found in [9]. Hence, we have $\hat{\boldsymbol{E}}(\boldsymbol{x},\kappa)\times\boldsymbol{\nu}=0$ on the whole boundary Γ_R. It follows from (2.2) that

$$\hat{\boldsymbol{E}}(\boldsymbol{x},\kappa)=\int_{\mathbb{R}^3}\hat{\mathbb{G}}(\boldsymbol{x}-\boldsymbol{y},\kappa)\int_0^T \boldsymbol{f}(\boldsymbol{y},t)\mathrm{e}^{-\mathrm{i}\kappa t}\mathrm{d}t\,\mathrm{d}\boldsymbol{y}.$$

Let $\hat{\boldsymbol{E}}\times\boldsymbol{\nu}$ and $\hat{\boldsymbol{H}}\times\boldsymbol{\nu}$ be the tangential trace of the electric and the magnetic fields in the frequency domain, respectively. In the Fourier domain, there exists a capacity operator $T: H^{-1/2}(\mathrm{div},\Gamma_R)\to H^{-1/2}(\mathrm{div},\Gamma_R)$ such that the following transparent boundary condition can be imposed on Γ_R (see e.g. [17]):

$$\hat{\boldsymbol{H}}\times\boldsymbol{\nu}=T(\hat{\boldsymbol{E}}\times\boldsymbol{\nu})\quad \text{on }\Gamma_R. \tag{3.3}$$

This implies that $\hat{\boldsymbol{H}}\times\boldsymbol{\nu}$ is uniquely determined by $\hat{\boldsymbol{E}}\times\boldsymbol{\nu}$ on Γ_R, provided $\hat{\boldsymbol{H}}$ and $\hat{\boldsymbol{E}}$ are radiating solutions. The transparent boundary condition (3.3) can be equivalently written as

$$(\nabla\times\hat{\boldsymbol{E}})\times\boldsymbol{\nu}=\mathrm{i}\kappa T(\hat{\boldsymbol{E}}\times\boldsymbol{\nu})\quad \text{on }\Gamma_R. \tag{3.4}$$

Next we introduce the functions $\hat{\boldsymbol{E}}^{\mathrm{inc}}$ and $\hat{\boldsymbol{H}}^{\mathrm{inc}}$ by

$$\hat{\boldsymbol{E}}^{\mathrm{inc}}(x)=\boldsymbol{p}\mathrm{e}^{-\mathrm{i}\kappa\boldsymbol{x}\cdot\boldsymbol{d}}\quad\text{and}\quad \hat{\boldsymbol{H}}^{\mathrm{inc}}(x)=\boldsymbol{q}\mathrm{e}^{-\mathrm{i}\kappa\boldsymbol{x}\cdot\boldsymbol{d}}, \tag{3.5}$$

where $\boldsymbol{d}\in\mathbb{S}^2$ is a unit vector and $\boldsymbol{p},\boldsymbol{q}$ are two unit polarization vectors satisfying $\boldsymbol{p}\cdot\boldsymbol{d}=0$, $\boldsymbol{q}=\boldsymbol{p}\times\boldsymbol{d}$. It is easy to verify that $\hat{\boldsymbol{E}}^{\mathrm{inc}}$ and $\hat{\boldsymbol{H}}^{\mathrm{inc}}$ satisfy the homogeneous time-harmonic Maxwell equations in $\mathbb{R}^3$:

$$\nabla\times(\nabla\times\hat{\boldsymbol{E}}^{\mathrm{inc}})-\kappa^2\hat{\boldsymbol{E}}^{\mathrm{inc}}=0 \tag{3.6}$$

and

$$\nabla \times (\nabla \times \hat{\boldsymbol{H}}^{\text{inc}}) - \kappa^2 \hat{\boldsymbol{H}}^{\text{inc}} = 0. \tag{3.7}$$

Let $\boldsymbol{\xi} = \kappa \boldsymbol{d}$ with $|\boldsymbol{\xi}| = \kappa \in (0, \infty)$. We have from (3.5) that $\hat{\boldsymbol{E}}^{\text{inc}} = \boldsymbol{p}\mathrm{e}^{-\mathrm{i}\boldsymbol{\xi}\cdot\boldsymbol{x}}$ and $\hat{\boldsymbol{H}}^{\text{inc}} = \boldsymbol{q}\mathrm{e}^{-\mathrm{i}\boldsymbol{\xi}\cdot\boldsymbol{x}}$. Multiplying both sides of (3.2) by $\hat{\boldsymbol{E}}^{\text{inc}}$ and using the integration by parts over B_R and (3.6), we have from $\hat{\boldsymbol{E}}(\boldsymbol{x}, \kappa) \times \boldsymbol{\nu} = 0$ on Γ_R and the transparent boundary condition (3.4) that

$$\begin{aligned}
&\int_{B_R} \int_0^T \boldsymbol{f}(\boldsymbol{x}, t)\mathrm{e}^{-\mathrm{i}\kappa t} \cdot \hat{\boldsymbol{E}}^{\text{inc}} \mathrm{d}t\, \mathrm{d}\boldsymbol{x} \\
&= \int_{B_R} (\nabla \times (\nabla \times \hat{\boldsymbol{E}}) - \kappa^2 \hat{\boldsymbol{E}}) \cdot \hat{\boldsymbol{E}}^{\text{inc}} \mathrm{d}\boldsymbol{x} \\
&= \int_{\Gamma_R} \nu \times (\nabla \times \hat{\boldsymbol{E}}) \cdot \hat{\boldsymbol{E}}^{\text{inc}} - \nu \times (\nabla \times \hat{\boldsymbol{E}}^{\text{inc}}) \cdot \hat{\boldsymbol{E}} \mathrm{d}s \\
&= -\int_{\Gamma_R} \left(\mathrm{i}\kappa T(\hat{\boldsymbol{E}} \times \boldsymbol{\nu}) \cdot \hat{\boldsymbol{E}}^{\text{inc}} + (\hat{\boldsymbol{E}} \times \boldsymbol{\nu}) \cdot (\nabla \times \hat{\boldsymbol{E}}^{\text{inc}}) \right) \mathrm{d}s \\
&= 0.
\end{aligned} \tag{3.8}$$

Hence from (3.8) we obtain

$$\int_{B_R} \int_0^T \boldsymbol{p}\mathrm{e}^{-\mathrm{i}\boldsymbol{\xi}\cdot\boldsymbol{x}} \cdot g(t)\boldsymbol{J}_1(\boldsymbol{x} - \boldsymbol{a}(t))\mathrm{e}^{-\mathrm{i}\kappa t} \mathrm{d}t\mathrm{d}\boldsymbol{x} = \int_{B_R} \int_0^T \boldsymbol{p}\mathrm{e}^{-\mathrm{i}\boldsymbol{\xi}\cdot\boldsymbol{x}} \cdot g(t)\boldsymbol{J}_2(\boldsymbol{x} - \boldsymbol{a}(t))\mathrm{e}^{-\mathrm{i}\kappa t} \mathrm{d}t\mathrm{d}\boldsymbol{x}.$$

By Fubini's theorem, it is easy to obtain

$$\boldsymbol{p} \cdot \hat{\boldsymbol{J}}_1(\kappa \boldsymbol{d}) \int_0^T g(t)\mathrm{e}^{-\mathrm{i}\kappa \boldsymbol{d}\cdot\boldsymbol{a}(t)}\mathrm{e}^{-\mathrm{i}\kappa t}\mathrm{d}t = \boldsymbol{p} \cdot \hat{\boldsymbol{J}}_2(\kappa \boldsymbol{d}) \int_0^T g(t)\mathrm{e}^{-\mathrm{i}\kappa \boldsymbol{d}\cdot\boldsymbol{a}(t)}\mathrm{e}^{-\mathrm{i}\kappa t}\mathrm{d}t. \tag{3.9}$$

Taking the limit $\kappa \to 0^+$ yields

$$\lim_{\kappa \to 0} \int_0^T g(t)\mathrm{e}^{-\mathrm{i}\kappa \boldsymbol{d}\cdot\boldsymbol{a}(t)}\mathrm{e}^{-\mathrm{i}\kappa t}\mathrm{d}t = \int_0^T g(t)\mathrm{d}t > 0.$$

Hence, there exist a small positive constant δ such that for all $\kappa \in (0, \delta)$,

$$\int_0^T g(t)\mathrm{e}^{-\mathrm{i}\kappa \boldsymbol{d}\cdot\boldsymbol{a}(t)}\mathrm{e}^{-\mathrm{i}\kappa t}\mathrm{d}t \neq 0,$$

which together with (3.9) implies that

$$\boldsymbol{p} \cdot \hat{\boldsymbol{J}}_1(\kappa \boldsymbol{d}) = \boldsymbol{p} \cdot \hat{\boldsymbol{J}}_2(\kappa \boldsymbol{d}).$$

Similarly, we may deduce from (3.7) and the integration by parts that

$$\boldsymbol{q} \cdot \hat{\boldsymbol{J}}_1(\kappa \boldsymbol{d}) = \boldsymbol{q} \cdot \hat{\boldsymbol{J}}_2(\kappa \boldsymbol{d}) \quad \text{for all } \boldsymbol{d} \in \mathbb{S}^2,\ \kappa \in (0, \delta).$$

On the other hand, since $\boldsymbol{J}_i,\ i = 1, 2$ is compactly supported in $B_{\hat{R}}$ and $\nabla_{\boldsymbol{x}} \cdot \boldsymbol{J}_i = 0$ in $B_{\hat{R}}$, we have

$$\begin{aligned}\int_{\mathbb{R}^3} \boldsymbol{d}\mathrm{e}^{-\mathrm{i}\kappa\boldsymbol{x}\cdot\boldsymbol{d}}\cdot\boldsymbol{J}_i(\boldsymbol{x})\mathrm{d}\boldsymbol{x} &= -\frac{1}{\mathrm{i}\kappa}\int_{B_{\hat{R}}} \nabla\mathrm{e}^{-\mathrm{i}\kappa\boldsymbol{x}\cdot\boldsymbol{d}}\cdot\boldsymbol{J}_i(\boldsymbol{x})\mathrm{d}\boldsymbol{x} \\ &= \frac{1}{\mathrm{i}\kappa}\int_{B_{\hat{R}}} \mathrm{e}^{-\mathrm{i}\kappa\boldsymbol{x}\cdot\boldsymbol{d}}\nabla\cdot\boldsymbol{J}_i(\boldsymbol{x})\mathrm{d}\boldsymbol{x} = 0.\end{aligned}$$

This implies that $\boldsymbol{d}\cdot\hat{\boldsymbol{J}}_i(\kappa\boldsymbol{d}) = 0$. Since $\boldsymbol{p},\boldsymbol{q},\boldsymbol{d}$ are orthonormal vectors, they form an orthonormal basis in $\mathbb{R}^3$. It follows from the previous identities that

$$\begin{aligned}\hat{\boldsymbol{J}}_1(\kappa\boldsymbol{d}) &= \boldsymbol{p}\cdot\hat{\boldsymbol{J}}_1(\kappa\boldsymbol{d})\boldsymbol{p} + \boldsymbol{q}\cdot\hat{\boldsymbol{J}}_1(\kappa\boldsymbol{d})\boldsymbol{q} + \boldsymbol{d}\cdot\hat{\boldsymbol{J}}_1(\kappa\boldsymbol{d})\boldsymbol{d} \\ &= \boldsymbol{p}\cdot\hat{\boldsymbol{J}}_2(\kappa\boldsymbol{d})\boldsymbol{p} + \boldsymbol{q}\cdot\hat{\boldsymbol{J}}_2(\kappa\boldsymbol{d})\boldsymbol{q} + \boldsymbol{d}\cdot\hat{\boldsymbol{J}}_2(\kappa\boldsymbol{d})\boldsymbol{d} \\ &= \hat{\boldsymbol{J}}_2(\kappa\boldsymbol{d})\end{aligned}$$

for all $\boldsymbol{d}\in\mathbb{S}^2$ and $\kappa\in(0,\delta)$. Noting that $\hat{\boldsymbol{J}}_i$, $i=1,2$, are analytical functions in $\mathbb{R}^3$, we obtain $\hat{\boldsymbol{J}}_1(\boldsymbol{\xi}) = \hat{\boldsymbol{J}}_2(\boldsymbol{\xi})$ for all $\boldsymbol{\xi}\in\mathbb{R}^3$, which completes the proof by taking the inverse Fourier transform. □

4. Determination of moving orbit function

In this section, we assume that the source profile function $\boldsymbol{J}$ is given. To prove the uniqueness for IP2, we consider two cases:

Case (i): the orbit $\{\boldsymbol{a}(t) : t\in[0,T_0]\}\subset B_{R_1}\cap\mathbb{R}^3$ is a curve lying in three dimensions;

Case (ii): $\{\boldsymbol{a}(t) : t\in[0,T_0]\}\subset B_{R_1}\cap\Pi$, where Π is a plane in three dimensions.

The second case means that the path of the moving source lies on a bounded flat surface in three dimensions. Cases (i) and (ii) will be discussed separately in the subsequent two subsections.

4.1. Uniqueness to IP2 in case (i)

Before stating the uniqueness result, we need an auxillary lemma.

Lemma 4.1. *Let $f_1, f_2, g\in C^1[0,L]$ be functions such that*

$$f_1' > 0, f_2' > 0, g > 0 \text{ on } (0,L); \quad f_1(0) = f_2(0).$$

In addition, suppose that

$$\int_0^L f_1^n(s)g(s)\mathrm{d}s = \int_0^L f_2^n(s)g(s)\mathrm{d}s \tag{4.1}$$

for all integers $n = 0,1,2\cdots$. Then it holds that $f_1 = f_2$ on $[0,L]$.

Proof. Without loss of generality we assume that $f_1(0) = f_2(0) = 0$. Otherwise, we may consider the functions $s\to f_j(s) - f_j(0)$ in place of f_j. To prove lemma 4.1, we first show $f_1(L) = f_2(L)$ and then apply the moment theory to get $f_1\equiv f_2$.

Assume without loss of generality that $f_1(L) > f_2(L)$. Write $f_1(L) = c$ and $\sup_{x\in(0,L)} g(x) = M$. Since $f_1'(s) > 0$ and $f_1(0) = 0$, we have $c > 0$. Therefore, there exists sufficiently small positive numbers $\epsilon > 0$ and $\delta_1, \delta_2 > 0$ such that

$$\begin{aligned} f_1(s) \geqslant c - \delta_1,\ f_2(s) \leqslant c - 2\delta_1,\ g(s) \geqslant \delta_2 \quad &\text{for all} \quad s \in [L-2\epsilon, L-\epsilon], \\ f_1(s) > f_2(s) \quad &\text{for all} \quad s \in [L-2\epsilon, L]. \end{aligned}$$

Using the above relations, we deduce from (4.1) that

$$\begin{aligned} 0 &= \int_0^L f_1^n(s)g(s) - f_1^n(s)g(s)\mathrm{d}s \\ &= \int_{L-\epsilon}^L f_1^n(s)g(s) - f_2^n(s)g(s)\mathrm{d}s + \int_{L-2\epsilon}^{L-\epsilon} f_1^n(s)g(s) - f_2^n(s)g(s)\mathrm{d}s \\ &\quad + \int_0^{L-2\epsilon} f_1^n(s)g(s) - f_2^n(s)g(s)\mathrm{d}s \\ &\geqslant \int_{L-2\epsilon}^{L-\epsilon} f_1^n(s)g(s) - f_2^n(s)g(s)\mathrm{d}s - \int_0^{L-2\epsilon} f_2^n(s)g(s)\mathrm{d}s \\ &\geqslant \epsilon\delta_2\Big[(c-\delta_1)^n - (c-2\delta_1)^n\Big] - (L-2\epsilon)M(c-2\delta_1)^n \\ &\geqslant (c-\delta_1)^n\Big[\epsilon\delta_2 - (\epsilon\delta_2 + (L-2\epsilon)M)\Big(\frac{c-2\delta_1}{c-\delta_1}\Big)^n\Big], \end{aligned}$$

which means that

$$(c-\delta_1)^n\Big[\epsilon\delta_2 - (\epsilon\delta_2 + (L-2\epsilon)M)\Big(\frac{c-2\delta_1}{c-\delta_1}\Big)^n\Big] \leqslant 0$$

for all integers $n = 0, 1, 2\cdots$. However, since $\frac{c-2\delta_1}{c-\delta_1} < 1$, there exists a sufficiently large integer $N > 0$ such that

$$\epsilon\delta_2 - (\epsilon\delta_2 + (L-2\epsilon)M)\Big(\frac{c-2\delta_1}{c-\delta_1}\Big)^N > 0.$$

Then we obtain

$$(c-\delta_1)^N\Big[\epsilon\delta_2 - (\epsilon\delta_2 + (L-2\epsilon)M)\Big(\frac{c-2\delta_1}{c-\delta_1}\Big)^N\Big] > 0,$$

which is a contradiction. Therefore, we obtain $f_1(L) = f_2(L)$.

Denote $c = f_1(0) = f_2(0)$ and $d = f_1(L) = f_2(L)$. Since f_j is monotonically increasing, the relation $\tau = f_j(s)$ implies that $s = f_j^{-1}(\tau)$ for all $s \in [0, L]$ and $\tau \in [c, d]$. Using the change of variables, we get

$$\int_0^L f_j^n(s)g(s)\mathrm{d}s = \int_c^d \tau^n g\circ(f_j^{-1}(\tau))(f_j^{-1})'(\tau)\mathrm{d}\tau, \quad j = 1, 2.$$

Hence, it follows from (4.1) that

$$\int_c^d \tau^n \mathrm{d}\mu = \int_c^d \tau^n \mathrm{d}\nu, \tag{4.2}$$

where μ and ν are two Lebesgue measures such that

$$\mathrm{d}\mu = g \circ (f_1^{-1}(\tau))(f_1^{-1})'(\tau)\mathrm{d}\tau,$$
$$\mathrm{d}\nu = g \circ (f_2^{-1}(\tau))(f_2^{-1})'(\tau)\mathrm{d}\tau.$$

By the Stone–Weierstrass theorem, it is easy to note from (4.2) that $\mathrm{d}\mu = \mathrm{d}\nu$, which means

$$g \circ (f_1^{-1}(\tau))(f_1^{-1})'(\tau) = g \circ (f_2^{-1}(\tau))(f_2^{-1})'(\tau) \quad \text{for all } \tau \in [c,d]. \tag{4.3}$$

Introduce two functions:

$$F_1(\tau) = \int_0^{f_1^{-1}(\tau)} g(s)\mathrm{d}s, \quad F_2(\tau) = \int_0^{f_2^{-1}(\tau)} g(s)\mathrm{d}s.$$

Hence, from (4.3) we deduce $F_1'(\tau) = F_2'(\tau)$ for $\tau \in [c,d]$. Moreover, since $f_1^{-1}(c) = f_2^{-1}(c) = 0$, we have $F_1(c) = F_2(c) = 0$ and then $F_1(\tau) = F_2(\tau)$ for $\tau \in [c,d]$, i.e.

$$\int_0^{f_1^{-1}(\tau)} g(s)\mathrm{d}s = \int_0^{f_2^{-1}(\tau)} g(s)\mathrm{d}s. \tag{4.4}$$

From (4.4), it is easy to know that $f_1^{-1}(\tau) = f_2^{-1}(\tau)$ for all $\tau \in [c,d]$. Otherwise, suppose $f_1^{-1}(\tau_0) \neq f_2^{-1}(\tau_0)$ at some point $\tau_0 \in [c,d]$. Since $g(s) > 0$ for all $s \in (0,L)$, we obtain that

$$\int_0^{f_1^{-1}(\tau_0)} g(s)\mathrm{d}s \neq \int_0^{f_2^{-1}(\tau_0)} g(s)\mathrm{d}s,$$

which is a contradiction. Consequently, we obtain $f_1^{-1} = f_2^{-1}$ and thus $f_1(s) = f_2(s)$ for all $s \in [0,L]$. The proof is complete. □

Our uniqueness result for the determination of $\boldsymbol{a}$ is stated as follows.

Theorem 4.2. *Assume that $g(t) > 0$ for $t \in (0,T_0)$ and that $\boldsymbol{a}(0) = \boldsymbol{O} \in \mathbb{R}^3$ is located at the origin and that each component $a_j, j = 1,2,3$ of $\boldsymbol{a}$ satisfies $|a_i'(t)| < 1$ for $t \in [0,T_0]$. Then the function $\boldsymbol{a}(t), t \in [0,T_0]$ can be uniquely determined by the data set $\{\boldsymbol{E}(\boldsymbol{x},t) \times \boldsymbol{\nu} : \boldsymbol{x} \in \Gamma, t \in (0,T)\}$.*

Proof. Assume that there are two orbit functions $\boldsymbol{a}$ and $\boldsymbol{b}$ such that

$$\begin{cases} \partial_t^2 \boldsymbol{E}_1(\boldsymbol{x},t) + \nabla \times (\nabla \times \boldsymbol{E}_1(\boldsymbol{x},t)) = \boldsymbol{J}(x - \boldsymbol{a}(t))g(t), & \boldsymbol{x} \in \mathbb{R}^3,\ t > 0, \\ \boldsymbol{E}_1(\boldsymbol{x},0) = \partial_t \boldsymbol{E}_1(\boldsymbol{x},0) = 0, & \boldsymbol{x} \in \mathbb{R}^3, \end{cases}$$

and

$$\begin{cases} \partial_t^2 \boldsymbol{E}_2(\boldsymbol{x},t) + \nabla \times (\nabla \times \boldsymbol{E}_2(\boldsymbol{x},t)) = \boldsymbol{J}(x - \boldsymbol{b}(t))g(t), & \boldsymbol{x} \in \mathbb{R}^3,\ t > 0, \\ \boldsymbol{E}_2(\boldsymbol{x},0) = \partial_t \boldsymbol{E}_2(\boldsymbol{x},0) = 0, & \boldsymbol{x} \in \mathbb{R}^3. \end{cases}$$

Here we assume that $\boldsymbol{b}(0) = \boldsymbol{O}$ and $|b_j'(t)| < 1$ for $t \in [0, t_0]$ and $j = 1, 2, 3$. We need to show $\boldsymbol{a}(t) = \boldsymbol{b}(t)$ in $(0,T_0)$ if $\boldsymbol{E}_1(\boldsymbol{x}, t) \times \boldsymbol{\nu}(\boldsymbol{x}) = \boldsymbol{E}_2(\boldsymbol{x}, t) \times \boldsymbol{\nu}$ for $x \in \Gamma, t \in (0, T)$.

For each unit vector $\boldsymbol{d}$, we can choose two unit polarization vectors $\boldsymbol{p}, \boldsymbol{q}$ such that $\boldsymbol{p} \cdot \boldsymbol{d} = 0, \boldsymbol{q} = \boldsymbol{p} \times \boldsymbol{d}$. Letting $\boldsymbol{E} = \boldsymbol{E}_1 - \boldsymbol{E}_2$ and following similar arguments as those of theorem 3.1, we obtain

$$\boldsymbol{p} \cdot \hat{\boldsymbol{J}}(\kappa \boldsymbol{d}) \int_0^T g(t) \mathrm{e}^{-\mathrm{i}\kappa \boldsymbol{d} \cdot \boldsymbol{a}(t)} \mathrm{e}^{-\mathrm{i}\kappa t} \mathrm{d}t = \boldsymbol{p} \cdot \hat{\boldsymbol{J}}(\kappa \boldsymbol{d}) \int_0^T g(t) \mathrm{e}^{-\mathrm{i}\kappa \boldsymbol{d} \cdot \boldsymbol{b}(t)} \mathrm{e}^{-\mathrm{i}\kappa t} \mathrm{d}t, \quad (4.5)$$

$$\boldsymbol{q} \cdot \hat{\boldsymbol{J}}(\kappa \boldsymbol{d}) \int_0^T g(t) \mathrm{e}^{-\mathrm{i}\kappa \boldsymbol{d} \cdot \boldsymbol{a}(t)} \mathrm{e}^{-\mathrm{i}\kappa t} \mathrm{d}t = \boldsymbol{q} \cdot \hat{\boldsymbol{J}}(\kappa \boldsymbol{d}) \int_0^T g(t) \mathrm{e}^{-\mathrm{i}\kappa \boldsymbol{d} \cdot \boldsymbol{b}(t)} \mathrm{e}^{-\mathrm{i}\kappa t} \mathrm{d}t, \quad (4.6)$$

and

$$\boldsymbol{d} \cdot \hat{\boldsymbol{J}}(\kappa \boldsymbol{d}) = 0,$$

which means

$$\hat{\boldsymbol{J}}(\kappa \boldsymbol{d}) = \boldsymbol{p} \cdot \hat{\boldsymbol{J}}(\kappa \boldsymbol{d}) \boldsymbol{p} + \boldsymbol{q} \cdot \hat{\boldsymbol{J}}(\kappa \boldsymbol{d}) \boldsymbol{q}.$$

Therefore, since $\boldsymbol{J} \neq 0$, for each unit vector $\boldsymbol{d}$ there exists a sequence $\{\kappa_j\}_{j=1}^{+\infty}$ such that $\lim_{j \to 0} \kappa_j = 0$ and for each κ_j, either $\boldsymbol{p} \cdot \hat{\boldsymbol{J}}(\kappa_j \boldsymbol{d}) \neq 0$ or $\boldsymbol{q} \cdot \hat{\boldsymbol{J}}(\kappa_j \boldsymbol{d}) \neq 0$. Hence from (4.5)–(4.6) we have

$$\int_0^T \mathrm{e}^{-\mathrm{i}\kappa_j \boldsymbol{d} \cdot \boldsymbol{a}(t)} \mathrm{e}^{-\mathrm{i}\kappa_j t} g(t) \mathrm{d}t = \int_0^T \mathrm{e}^{-\mathrm{i}\kappa_j \boldsymbol{d} \cdot \boldsymbol{b}(t)} \mathrm{e}^{-\mathrm{i}\kappa_j t} g(t) \mathrm{d}t, \quad j = 1, 2, \cdots. \quad (4.7)$$

Expanding $\mathrm{e}^{-\mathrm{i}\kappa_j \boldsymbol{d} \cdot \boldsymbol{a}(t)} \mathrm{e}^{-\mathrm{i}\kappa_j t}$ and $\mathrm{e}^{-\mathrm{i}\kappa_j \boldsymbol{d} \cdot \boldsymbol{a}(t)} \mathrm{e}^{-\mathrm{i}\kappa_j t}$ into power series with respect to κ_j, we write (4.7) as

$$\sum_{n=0}^{\infty} \frac{\alpha_n}{n!} \kappa_j^n = \sum_{n=0}^{\infty} \frac{\beta_n}{n!} \kappa_j^n, \quad (4.8)$$

where

$$\alpha_n := \int_0^T (\boldsymbol{d} \cdot \boldsymbol{a}(t) + t)^n g(t) \mathrm{d}t, \quad \beta_n := \int_0^T (\boldsymbol{d} \cdot \boldsymbol{b}(t) + t)^n g(t) \mathrm{d}t, \quad n = 1, 2 \cdots.$$

In view of the fact that $\mathrm{supp}(g) \subset [0, T_0]$, we get

$$\alpha_n = \int_0^{T_0} (\boldsymbol{d} \cdot \boldsymbol{a}(t) + t)^n g(t) \mathrm{d}t, \quad \beta_n = \int_0^{T_0} (\boldsymbol{d} \cdot \boldsymbol{b}(t) + t)^n g(t) \mathrm{d}t, \quad n = 1, 2 \cdots.$$

Since (4.8) holds for all κ_j and $\lim_{j \to \infty} \kappa_j = 0$, it is easy to conclude that $\alpha_n = \beta_n$ for $n = 0, 1, 2 \cdots$. Choosing $\boldsymbol{d} = (1, 0, 0)$, we have

$$(a_1(t) + t)' = 1 + a_1'(t) > 0, \quad (b_1(t) + t)' = 1 + b_1'(t) > 0, \quad a_1(0) = b_1(0).$$

It follows from $\alpha_n = \beta_n$ and lemma 4.1 that $a_1(t) = b_1(t)$ for $t \in [0, T_0]$. Similarly letting $\boldsymbol{d} = (0, 1, 0)$ and $\boldsymbol{d} = (0, 0, 1)$ we have $a_2(t) = b_2(t)$ and $a_3(t) = b_3(t)$ for $t \in [0, T_0]$, respectively, which proves that $\boldsymbol{a}(t) = \boldsymbol{b}(t)$ for $t \in [0, T_0]$. □

Remark 4.3. In theorem 4.2, it is stated that we can only recover the function $\boldsymbol{a}(t)$ over the finite time period $[0,T_0]$ because the moving source radiates in this time period, i.e. $\mathrm{supp}(g) = [0, T_0]$. The information of $\boldsymbol{a}(t)$ for $t > T_0$ cannot be retrieved. The monotonicity assumption $a_j' \geqslant 0$ for $j = 1, 2, 3$ can be replaced by the following condition: there exist three linearly independent unit directions $\boldsymbol{d}_j, j = 1, 2, 3$ such that

$$|\boldsymbol{d}_j \cdot \boldsymbol{a}'(t)| < 1, \quad t \in [0, T_0],\ j = 1, 2, 3.$$

Note that this condition can always be fulfilled if the source moves along a straight line with the speed less than one.

4.2. Uniqueness to IP2 in case (ii)

For simplicity of notation, let $\tilde{\boldsymbol{x}} = (x_1, x_2)$ for $\boldsymbol{x} = (x_1, x_2, x_3)$ and $\mathbb{R}^2 = \{\boldsymbol{x} \in \mathbb{R}^3 : x_3 = 0\}$. Let $\tilde{\boldsymbol{a}}(t) \in \mathbb{R}^2$ for all $t \in [0, T_0]$. In this subsection, we assume that

$$\boldsymbol{F}(\boldsymbol{x}, t) = \boldsymbol{J}(\tilde{\boldsymbol{x}} - \tilde{\boldsymbol{a}}(t))\, h(x_3)\, g(t), \quad \boldsymbol{x} \in \mathbb{R}^3,\ t \in \mathbb{R}_+,$$

where $\boldsymbol{J}(\tilde{\boldsymbol{x}}) = (J_1(\tilde{\boldsymbol{x}}), J_2(\tilde{\boldsymbol{x}}), 0) \in H^2(\mathbb{R}^2)^3$ depends only on $\tilde{\boldsymbol{x}}$ and $h \in H^2(\mathbb{R})$, $\mathrm{supp}(h) \subset (-\hat{R}, \hat{R})\sqrt{2}/2$. Moreover, we assume that h does not vanish identically and

$$\mathrm{supp}(\boldsymbol{J}) \subset \{\tilde{\boldsymbol{x}} \in \mathbb{R}^2 : |\tilde{\boldsymbol{x}}| < \hat{R}\sqrt{2}/2\}, \quad \partial_{x_1}\boldsymbol{J}_1(\tilde{\boldsymbol{x}}) + \partial_{x_2}\boldsymbol{J}_2(\tilde{\boldsymbol{x}}) = 0.$$

The temporal function g is defined the same as in the previous sections. The above assumptions imply that we still have $\mathrm{supp}(\boldsymbol{F}) \subset B_{\hat{R}} \times [0, T_0]$ and $\mathrm{div}\, \boldsymbol{F} = 0$ in $\mathbb{R}^3$. We consider the inhomogeneous Maxwell system

$$\begin{cases} \partial_t^2 \boldsymbol{E}(\boldsymbol{x}, t) + \nabla \times (\nabla \times \boldsymbol{E}(\boldsymbol{x}, t)) = \boldsymbol{J}(\tilde{\boldsymbol{x}} - \tilde{\boldsymbol{a}}(t))\, h(x_3)\, g(t), & \boldsymbol{x} \in \mathbb{R}^3,\ t > 0, \\ \boldsymbol{E}(\boldsymbol{x}, 0) = \partial_t \boldsymbol{E}(\boldsymbol{x}, 0) = 0, & \boldsymbol{x} \in \mathbb{R}^3. \end{cases} \tag{4.9}$$

Since the equation (4.9) is a special case of (1.2), the results of lemmas 2.1 and 2.2 also apply to this case.

For our inverse problem, it is assumed that $\boldsymbol{J} \in \mathcal{A}$ is a given source function, where the admissible set

$$\mathcal{A} = \{\boldsymbol{J} = (J_1, J_2, 0) : J_i(0) > J_i(\tilde{\boldsymbol{x}}) \ \text{ for } \ i = 1 \text{ or } i = 2 \text{ and all } \tilde{\boldsymbol{x}} \neq 0\}.$$

The x_3-dependent function h is also assumed to be given. We point out that these *a priori* information of $\boldsymbol{J}$ and h are physically reasonable, while $\boldsymbol{J}$ and h can be regarded as approximation of the Dirac functions (for example, Gaussian functions) with respect to $\tilde{\boldsymbol{x}}$ and x_3, respectively. Our aim is to recover the unknown orbit function $\tilde{\boldsymbol{a}}(t) \in C^1([0, T_0])^2$ which has a upper bound $|\tilde{\boldsymbol{a}}(t)| \leqslant R_1$ for some $R_1 > 0$ and for all $t \in [0, T_0]$. Let $R > \hat{R} + R_1$ and $T = T_0 + R + \hat{R} + R_1$.

Below we prove that the tangential trace of the dynamical magnetic field on $\Gamma_R \times (0, T)$ can be uniquely determined by that of the electric field. It will be used in the subsequent uniqueness proof with the data measured on the whole surface Γ_R.

Lemma 4.4. *Assume that the electric field* $\boldsymbol{E} \in C([0, T]; H^2(\mathbb{R}^3))^3 \cap C^1([0, T]; H^1(\mathbb{R}^3))^3$ *satisfies*

$$\begin{cases} \partial_t^2 \boldsymbol{E}(\boldsymbol{x},t) + \nabla\times(\nabla\times\boldsymbol{E}(\boldsymbol{x},t)) = 0, & |\boldsymbol{x}|>R,\ t\in(0,T),\\ \boldsymbol{E}(\boldsymbol{x},0)=\partial_t\boldsymbol{E}(\boldsymbol{x},0)=0, & \boldsymbol{x}\in\mathbb{R}^3. \end{cases}$$

If $\boldsymbol{E}\times\boldsymbol{\nu}=0$ *on* $\Gamma_R\times(0,T)$, *then* $(\nabla\times\boldsymbol{E})\times\boldsymbol{\nu}=0$ *on* $\Gamma_R\times(0,T)$.

Proof. Let us assume that $\boldsymbol{E}\times\boldsymbol{\nu}=0$ on $\Gamma_R\times(0,T)$ and consider $\boldsymbol{V}$ defined by

$$\boldsymbol{V}(\boldsymbol{x},t)=\int_0^t \boldsymbol{E}(\boldsymbol{x},s)\mathrm{d}s,\quad (\boldsymbol{x},t)\in\mathbb{R}^3\times(0,T).$$

In view of (4.4) and the fact that $\boldsymbol{E}(\boldsymbol{x},t)\times\boldsymbol{\nu}=0$ on $\Gamma_R\times(0,T)$, we find

$$\begin{cases} \partial_t^2 \boldsymbol{V}(\boldsymbol{x},t) + \nabla\times(\nabla\times\boldsymbol{V}(\boldsymbol{x},t)) = 0, & |\boldsymbol{x}|>R,\ t\in(0,T),\\ \boldsymbol{V}(\boldsymbol{x},0)=\partial_t\boldsymbol{V}(\boldsymbol{x},0)=0, & \boldsymbol{x}\in\mathbb{R}^3,\\ \partial_t\boldsymbol{V}(\boldsymbol{x},t)\times\boldsymbol{\nu}(x)=0, & (\boldsymbol{x},t)\in\Gamma_R\times(0,T). \end{cases}\tag{4.10}$$

We define the energy $\mathcal{E}$ associated to $\boldsymbol{V}$ on $\Omega:=\{x\in\mathbb{R}^3:\ |x|>R\}$

$$\mathcal{E}(t):=\int_\Omega(|\partial_t\boldsymbol{V}(\boldsymbol{x},t)|^2+|\nabla_x\times\boldsymbol{V}(\boldsymbol{x},t)|^2)\mathrm{d}\boldsymbol{x},\quad t\in[0,T].$$

Since $\boldsymbol{E}\in C([0,T];H^2(\mathbb{R}^3))^3\cap C^1([0,T];H^1(\mathbb{R}^3))^3$, we have

$$\boldsymbol{V}\in C([0,T];H^2(\mathbb{R}^3))^3\cap C^1([0,T];H^1(\mathbb{R}^3))^3\cap C^2([0,T];L^2(\mathbb{R}^3))^3.$$

It follows that $\mathcal{E}\in C^1([0,T])$. Moreover, we get

$$\mathcal{E}'(t)=2\int_\Omega[\partial_t^2\boldsymbol{V}(x,t)\cdot\partial_t\boldsymbol{V}(\boldsymbol{x},t)+(\nabla_x\times\boldsymbol{V}(\boldsymbol{x},t))\cdot(\nabla_{\boldsymbol{x}}\times\partial_t\boldsymbol{V}(\boldsymbol{x},t))]\,\mathrm{d}\boldsymbol{x}.$$

Integrating by parts in $\boldsymbol{x}\in\Omega$ and applying (4.10), we obtain

$$\begin{aligned}\mathcal{E}'(t)&=2\int_\Omega[\partial_t^2\boldsymbol{V}+\nabla_{\boldsymbol{x}}\times(\nabla_{\boldsymbol{x}}\times\boldsymbol{V})]\cdot\partial_t\boldsymbol{V}(\boldsymbol{x},t)\,\mathrm{d}\boldsymbol{x}\\ &\quad+2\int_{\Gamma_R}(\nabla_{\boldsymbol{x}}\times\boldsymbol{V})\cdot(\nu\times\partial_t\boldsymbol{V}(\boldsymbol{x},t))\mathrm{d}s\\ &=0.\end{aligned}$$

This proves that $\mathcal{E}$ is a constant function. Since

$$\mathcal{E}(0)=\int_\Omega(|\partial_t\boldsymbol{V}(\boldsymbol{x},0)|^2+|\nabla_x\times\boldsymbol{V}(\boldsymbol{x},0)|^2)\mathrm{d}\boldsymbol{x}=0,$$

we deduce $\mathcal{E}(t)=0$ for all $t\in[0,T]$. In particular, we have

$$\int_\Omega|\boldsymbol{E}(\boldsymbol{x},t)|^2\,\mathrm{d}\boldsymbol{x}=\int_\Omega|\partial_t\boldsymbol{V}(\boldsymbol{x},t)|^2\,\mathrm{d}\boldsymbol{x}\leqslant\mathcal{E}(t)=0,\quad t\in[0,T].$$

This proves that

$$\boldsymbol{E}(\boldsymbol{x},t)=0,\quad|\boldsymbol{x}|>R,\ t\in(0,T),$$

which implies that $(\nabla \times \boldsymbol{E}) \times \boldsymbol{\nu} = 0$ on $\Gamma_R \times (0, T)$ and completes the proof. □

In the following lemma, we present a uniqueness result for recovering $\tilde{\boldsymbol{a}}$ from the tangential trace of the electric field measured on Γ_R. Our arguments are inspired by a recent uniqueness result [10] to inverse source problems in elastodynamics. Compared to the uniqueness result of theorem 4.2, the slow moving assumption of the source is not required in the following theorem 4.5.

Theorem 4.5. *Assume that $g(t) > 0$ for $t \in (0, T_0)$, $\boldsymbol{J} \in \mathcal{A}$ and the non-vanishing function h are both known. Then the function $\tilde{\boldsymbol{a}}(t), t \in [0, T_0]$ can be uniquely determined by the data set $\{\boldsymbol{E}(\boldsymbol{x}, t) \times \boldsymbol{\nu} : \boldsymbol{x} \in \Gamma_R, t \in (0, T)\}$.*

Proof. Assume that there are two functions $\tilde{\boldsymbol{a}}$ and $\tilde{\boldsymbol{b}}$ such that

$$\begin{cases} \partial_t^2 \boldsymbol{E}_1(\boldsymbol{x}, t) + \nabla \times (\nabla \times \boldsymbol{E}_1(\boldsymbol{x}, t)) = \boldsymbol{J}(\tilde{\boldsymbol{x}} - \tilde{\boldsymbol{a}}(t)) h(x_3) g(t), & \boldsymbol{x} \in \mathbb{R}^3,\ t > 0, \\ \boldsymbol{E}_1(\boldsymbol{x}, 0) = \partial_t \boldsymbol{E}_1(\boldsymbol{x}, 0) = 0, & \boldsymbol{x} \in \mathbb{R}^3, \end{cases} \tag{4.11}$$

and

$$\begin{cases} \partial_t^2 \boldsymbol{E}_2(\boldsymbol{x}, t) + \nabla \times (\nabla \times \boldsymbol{E}_2(\boldsymbol{x}, t)) = \boldsymbol{J}(\tilde{\boldsymbol{x}} - \tilde{\boldsymbol{b}}(t)) h(x_3) g(t), & \boldsymbol{x} \in \mathbb{R}^3,\ t > 0, \\ \boldsymbol{E}_2(\boldsymbol{x}, 0) = \partial_t \boldsymbol{E}_2(\boldsymbol{x}, 0) = 0, & \boldsymbol{x} \in \mathbb{R}^3. \end{cases} \tag{4.12}$$

It suffices to show that $\tilde{\boldsymbol{a}}(t) = \tilde{\boldsymbol{b}}(t)$ in $(0, T_0)$ if $\boldsymbol{E}_1(\boldsymbol{x}, t) \times \boldsymbol{\nu} = \boldsymbol{E}_2(\boldsymbol{x}, t) \times \boldsymbol{\nu}$ for $x \in \Gamma_R, t \in (0, T)$. Denote $\boldsymbol{E} = \boldsymbol{E}_1 - \boldsymbol{E}_2$ and

$$\boldsymbol{f}(\tilde{\boldsymbol{x}}, t) = \boldsymbol{J}(\tilde{\boldsymbol{x}} - \tilde{\boldsymbol{a}}(t)) g(t) - \boldsymbol{J}(\tilde{\boldsymbol{x}} - \tilde{\boldsymbol{b}}(t)) g(t).$$

Subtracting (4.11) from (4.12) yields

$$\begin{cases} \partial_t^2 \boldsymbol{E}(\boldsymbol{x}, t) + \nabla \times (\nabla \times \boldsymbol{E}(\boldsymbol{x}, t)) = \boldsymbol{f}(\tilde{\boldsymbol{x}}, t) h(x_3) g(t), & \boldsymbol{x} \in \mathbb{R}^3,\ t > 0, \\ \boldsymbol{E}(\boldsymbol{x}, 0) = \partial_t \boldsymbol{E}(\boldsymbol{x}, 0) = 0, & \boldsymbol{x} \in \mathbb{R}^3. \end{cases} \tag{4.13}$$

Since h does not vanish identically, we can always find an interval $\Lambda = (a_-, a_+) \subset \mathbb{R}_+$ such that

$$\int_{-\hat{R}\sqrt{2}/2}^{\hat{R}\sqrt{2}/2} \mathrm{e}^{\lambda x_3} h(x_3) \mathrm{d}x_3 \neq 0, \quad \forall \lambda \in \Lambda. \tag{4.14}$$

Set $H := \{(x_1, x_2) : a_-^2 < x_2^2 - x_1^2 < a_+^2, x_1 > 0, x_2 > 0\}$, which is an open set in $\mathbb{R}^2$. We choose a test function $\boldsymbol{F}(\boldsymbol{x}, t)$ of the form

$$\boldsymbol{F}(\boldsymbol{x}, t) = \boldsymbol{p} \mathrm{e}^{-\mathrm{i}\kappa_1 t} \mathrm{e}^{-\mathrm{i}\kappa_2 \tilde{\boldsymbol{d}} \cdot \tilde{\boldsymbol{x}}} \mathrm{e}^{\sqrt{\kappa_2^2 - \kappa_1^2} x_3},$$

where $\tilde{\boldsymbol{d}} = (d_1, d_2)$ is a unit vector, $\tilde{\boldsymbol{p}} = (p_1, p_2)$ is a unit vector orthogonal to $\tilde{\boldsymbol{d}}$, $\boldsymbol{d} := (\tilde{\boldsymbol{d}}, 0) \in \mathbb{R}^3$, $\boldsymbol{p} := (\tilde{\boldsymbol{p}}, 0) \in \mathbb{R}^3$ and κ_1, κ_2 are positive constants such that $(\kappa_1, \kappa_2) \in H$. It is easy to verify that

$$\partial_t^2 \boldsymbol{F}(\boldsymbol{x}, t) + \nabla \times (\nabla \times \boldsymbol{F}(\boldsymbol{x}, t)) = 0. \tag{4.15}$$

Since $\boldsymbol{E}(\boldsymbol{x},t)\times\boldsymbol{\nu}=0$ on Γ_R, from lemma 4.4, we also have $(\nabla\times\boldsymbol{E}(\boldsymbol{x},t))\times\boldsymbol{\nu}=0$ on Γ_R. Consequently, multiplying both sides of the Maxwell system by $\boldsymbol{F}$ and using integration by parts over $[0,T]\times B_R$, we can obtain from (4.15) that

$$\begin{aligned}
&\int_0^T\int_{B_R}\boldsymbol{f}(\tilde{\boldsymbol{x}},t)h(x_3)\cdot\boldsymbol{F}(\boldsymbol{x},t)\mathrm{d}\boldsymbol{x}\mathrm{d}t\\
&=\int_0^T\int_{B_R}\Big(\partial_t^2\boldsymbol{E}(\boldsymbol{x},t)+\nabla\times(\nabla\times\boldsymbol{E}(\boldsymbol{x},t))\Big)\cdot\boldsymbol{F}(\boldsymbol{x},t)\mathrm{d}\boldsymbol{x}\mathrm{d}t\\
&=\int_0^T\int_{\Gamma_R}\boldsymbol{\nu}\times(\nabla\times\boldsymbol{E}(\boldsymbol{x},t))\cdot\boldsymbol{F}(\boldsymbol{x},t)-\boldsymbol{\nu}\times(\nabla\times\boldsymbol{F}(\boldsymbol{x},t))\cdot\boldsymbol{E}(\boldsymbol{x},t)\mathrm{d}s\mathrm{d}t\\
&=\int_0^T\int_{\Gamma_R}\boldsymbol{\nu}\times(\nabla\times\boldsymbol{E}(\boldsymbol{x},t))\cdot\boldsymbol{F}(\boldsymbol{x},t)-(\boldsymbol{E}(\boldsymbol{x},t)\times\boldsymbol{\nu})\cdot(\nabla\times\boldsymbol{F}(\boldsymbol{x},t))\mathrm{d}s\mathrm{d}t\\
&=0.
\end{aligned}$$

Note that in the last step we have used lemma 4.4. Recalling the definition of $\boldsymbol{F}$ and $\boldsymbol{f}$, we obtain from the previous identity that

$$\Big(\int_{-\hat{R}\sqrt{2}/2}^{\hat{R}\sqrt{2}/2}\mathrm{e}^{\sqrt{\kappa_2^2-\kappa_1^2}x_3}h(x_3)\mathrm{d}x_3\Big)\,\boldsymbol{p}\cdot\int_0^T\int_{B_{\hat{R}}}\boldsymbol{f}(\tilde{\boldsymbol{x}},t)\mathrm{e}^{-\mathrm{i}\kappa_1 t}\mathrm{e}^{-\mathrm{i}\kappa_2\tilde{\boldsymbol{d}}\cdot\tilde{\boldsymbol{x}}}\mathrm{d}\tilde{\boldsymbol{x}}\mathrm{d}t=0.$$

In view of (4.14) and the choice of κ_1,κ_2, we get

$$\boldsymbol{p}\cdot\int_0^T\int_{B_{\hat{R}}}\boldsymbol{f}(\tilde{\boldsymbol{x}},t)\mathrm{e}^{-\mathrm{i}\kappa_1 t}\mathrm{e}^{-\mathrm{i}\kappa_2\tilde{\boldsymbol{d}}\cdot\tilde{\boldsymbol{x}}}\mathrm{d}\tilde{\boldsymbol{x}}\mathrm{d}t=0.$$

For a vector $\boldsymbol{v}(\tilde{\boldsymbol{x}},t)\in\mathbb{R}^3$, denote by $\hat{\boldsymbol{v}}(\boldsymbol{\xi}),\boldsymbol{\xi}\in\mathbb{R}^3$ the Fourier transform of $\boldsymbol{v}$ with respect to the variable $(\tilde{\boldsymbol{x}},t)$, i.e.

$$\hat{\boldsymbol{v}}(\boldsymbol{\xi})=\int_{\mathbb{R}^3}\boldsymbol{v}(\tilde{\boldsymbol{x}},t)\mathrm{e}^{-\mathrm{i}\boldsymbol{\xi}\cdot(\tilde{\boldsymbol{x}},t)}\mathrm{d}\tilde{\boldsymbol{x}}\mathrm{d}t.$$

Consequently, it holds that

$$\boldsymbol{p}\cdot\hat{\boldsymbol{f}}(\kappa_2\tilde{\boldsymbol{d}},\kappa_1)=0$$

for all $\kappa_2>\kappa_1>0$ and $|\tilde{\boldsymbol{d}}|=1$.

On the other hand, since $\partial_{x_1}J_1+\partial_{x_2}J_2=0$, fixing $\tilde{\boldsymbol{f}}=(f_1,f_2)$, we have $\nabla_{\tilde{\boldsymbol{x}}}\cdot\tilde{\boldsymbol{f}}=0$. Hence,

$$\begin{aligned}
&\boldsymbol{d}\cdot\int_0^T\int_{B_{\hat{R}}}\boldsymbol{f}(\tilde{\boldsymbol{x}},t)\mathrm{e}^{-\mathrm{i}\kappa_1 t}\mathrm{e}^{-\mathrm{i}\kappa_2\tilde{\boldsymbol{d}}\cdot\tilde{\boldsymbol{x}}}\mathrm{d}\tilde{\boldsymbol{x}}\mathrm{d}t\\
&=-\frac{1}{\mathrm{i}\kappa_2}\int_0^T\int_{B_{\hat{R}}}\tilde{\boldsymbol{f}}(\tilde{\boldsymbol{x}},t)\cdot\nabla_{\tilde{\boldsymbol{x}}}\mathrm{e}^{-\mathrm{i}\kappa_2\tilde{\boldsymbol{d}}\cdot\tilde{\boldsymbol{x}}}\mathrm{d}\tilde{\boldsymbol{x}}\mathrm{d}t\\
&=\frac{1}{\mathrm{i}\kappa_2}\int_0^T\int_{B_{\hat{R}}}\nabla_{\tilde{\boldsymbol{x}}}\cdot\tilde{\boldsymbol{f}}(\tilde{\boldsymbol{x}},t)\mathrm{e}^{-\mathrm{i}\kappa_2\tilde{\boldsymbol{d}}\cdot\tilde{\boldsymbol{x}}}\mathrm{d}\tilde{\boldsymbol{x}}\mathrm{d}t\\
&=0,
\end{aligned}$$

which means $\boldsymbol{d}\cdot\hat{\boldsymbol{f}}(\kappa_2\tilde{\boldsymbol{d}},\kappa_1)=0$ for all $(\kappa_1,\kappa_2)\in H$ and $|\tilde{\boldsymbol{d}}|=1$. Since both $\tilde{\boldsymbol{d}}$ and $\tilde{\boldsymbol{p}}$ are orthonormal vectors in $\mathbb{R}^2$, they form an orthonormal basis in $\mathbb{R}^2$. Therefore we have

$$\hat{\boldsymbol{f}}(\kappa_2\tilde{\boldsymbol{d}},\kappa_1)=\boldsymbol{d}\cdot\hat{\boldsymbol{f}}(\kappa_2\tilde{\boldsymbol{d}},\kappa_1)\boldsymbol{d}+\boldsymbol{p}\cdot\hat{\boldsymbol{f}}(\kappa_2\tilde{\boldsymbol{d}},\kappa_1)\boldsymbol{p}=0$$

for all $(\kappa_1,\kappa_2)\in H$ and $|\tilde{\boldsymbol{d}}|=1$. Since $\hat{\boldsymbol{f}}$ is analytic in $\mathbb{R}^3$ and $\{(\kappa_1,\kappa_2\tilde{\boldsymbol{d}}):(\kappa_1,\kappa_2)\in H,\ |\tilde{\boldsymbol{d}}|=1\}$ is an open set in $\mathbb{R}^3$, we have $\hat{\boldsymbol{f}}(\boldsymbol{\xi})=0$ for all $\boldsymbol{\xi}\in\mathbb{R}^3$, which means $\boldsymbol{f}(\tilde{\boldsymbol{x}},t)\equiv 0$ and then

$$\boldsymbol{J}(\tilde{\boldsymbol{x}}-\tilde{\boldsymbol{a}}(t))g(t)=\boldsymbol{J}(\tilde{\boldsymbol{x}}-\tilde{\boldsymbol{b}}(t))g(t)$$

for all $\tilde{\boldsymbol{x}}\in\mathbb{R}^2$ and $t>0$. This particularly gives

$$\boldsymbol{J}(\tilde{\boldsymbol{x}}-\tilde{\boldsymbol{a}}(t))=\boldsymbol{J}(\tilde{\boldsymbol{x}}-\tilde{\boldsymbol{b}}(t))\quad\text{for all}\quad t\in(0,T_0),\ \ \tilde{\boldsymbol{x}}\in\mathbb{R}^2. \tag{4.16}$$

Assume that there exists one time point $t_0\in(0,T_0)$ such that $\tilde{\boldsymbol{a}}(t_0)\neq\tilde{\boldsymbol{b}}(t_0)$. By choosing $\tilde{\boldsymbol{x}}=\tilde{\boldsymbol{a}}(t_0)$ we deduce from (4.16) that

$$\boldsymbol{J}(0)=\boldsymbol{J}(\tilde{\boldsymbol{a}}(t_0)-\tilde{\boldsymbol{b}}(t_0)),$$

which is a contradiction to our assumption that $\boldsymbol{J}\in\mathcal{A}$. This finishes the proof of $\tilde{\boldsymbol{a}}(t)=\tilde{\boldsymbol{b}}(t)$ for $t\in[0,T_0]$. □

Remark 4.6. The proof of theorem 4.5 does not depend on the Fourier transform of the electromagnetic field in time, but it requires the data measured on the whole surface Γ_R. However, the Fourier approach presented in the proof of theorems 3.1 and 4.2 straightforwardly carries over to the proof of theorem 4.5 without any additional difficulties. Particulary, the result of theorem 4.5 remains valid with the partial data $\{\boldsymbol{E}(\boldsymbol{x},t)\times\boldsymbol{\nu}:\boldsymbol{x}\in\Gamma\subset\Gamma_R,t\in(0,T)\}$.

Remark 4.7. In the case of the scalar wave equation,

$$\begin{cases}\partial_t^2u(\boldsymbol{x},t)+\nabla\times(\nabla\times u(\boldsymbol{x},t))=J(\tilde{\boldsymbol{x}}-\tilde{\boldsymbol{a}}(t))\,h(x_3)\,g(t), & \boldsymbol{x}\in\mathbb{R}^3,\ t>0,\\ u(\boldsymbol{x},0)=\partial_tu(\boldsymbol{x},0)=0, & \boldsymbol{x}\in\mathbb{R}^3,\end{cases}$$

where $J:\mathbb{R}^2\to\mathbb{R}_+$ is a scalar function compactly supported on $\{(x_1,x_2)\in\mathbb{R}^2:x_1^2+x_2^2<\hat{R}^2\}$. Then, following the same arguments as in the proof of theorem 4.5, one can prove that $\tilde{\boldsymbol{a}}(t),t\in[0,T_0]$ can be uniquely determined by the data set $\{u(\boldsymbol{x},t):\boldsymbol{x}\in\Gamma\subset\Gamma_R,t\in(0,T)\}$.

5. Inverse moving source problem for a delta distribution

As seen in the previous sections, when the temporal function g is supported on $[0,T_0]$, it is possible to recover the moving orbit function $\boldsymbol{a}(t)$ for $t\in[0,T_0]$. In this section we consider the case where the temporal function shrinks to the Dirac distribution $g(t)=\delta(t-t_0)$ with some unknown time point $t_0>0$. Our aim is to determine t_0 and $\boldsymbol{a}(t_0)$ from the electric data at a finite number of measurement points.

Consider the following initial value problem of the time-dependent Maxwell equation

$$\begin{cases}\partial_t^2\boldsymbol{E}(\boldsymbol{x},t)+\nabla\times(\nabla\times\boldsymbol{E}(\boldsymbol{x},t))=-\boldsymbol{J}(x-\boldsymbol{a}(t))\delta(t-t_0), & \boldsymbol{x}\in\mathbb{R}^3,\ t>0,\\ \boldsymbol{E}(\boldsymbol{x},0)=\partial_t\boldsymbol{E}(\boldsymbol{x},0)=0, & \boldsymbol{x}\in\mathbb{R}^3.\end{cases} \tag{5.1}$$

Since $\nabla\cdot\boldsymbol{J}=0$, the electric field $\boldsymbol{E}(\boldsymbol{x})$ in this case can be expressed as

$$\begin{aligned}\boldsymbol{E}(\boldsymbol{x},t)&=\int_0^\infty\int_{\mathbb{R}^3}\mathbb{G}(\boldsymbol{x}-\boldsymbol{y},t-s)\boldsymbol{J}(\boldsymbol{y}-\boldsymbol{a}(s))\delta(s-t_0)\mathrm{d}\boldsymbol{y}\mathrm{d}s\\&=\int_0^\infty\int_{\mathbb{R}^3}\frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\delta(|\boldsymbol{x}-\boldsymbol{y}|-(t-s))\boldsymbol{J}(\boldsymbol{y}-\boldsymbol{a}(s))\delta(s-t_0)\mathrm{d}\boldsymbol{y}\mathrm{d}s\\&\quad-\int_0^\infty\int_{\mathbb{R}^3}\nabla_{\boldsymbol{x}}\nabla_{\boldsymbol{x}}^\top\Big(\frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}H(|\boldsymbol{x}-\boldsymbol{y}|+s-t)\Big)\boldsymbol{J}(\boldsymbol{y}-\boldsymbol{a}(s))\delta(s-t_0)\mathrm{d}\boldsymbol{y}\mathrm{d}s\\&=\int_0^\infty\int_{\mathbb{R}^3}\frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\delta(|\boldsymbol{x}-\boldsymbol{y}|-(t-s))\boldsymbol{J}(\boldsymbol{y}-\boldsymbol{a}(s))\delta(s-t_0)\mathrm{d}\boldsymbol{y}\mathrm{d}s\\&\quad-\int_0^\infty\int_{\mathbb{R}^3}\nabla_{\boldsymbol{y}}\nabla_{\boldsymbol{y}}^\top\Big(\frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}H(|\boldsymbol{x}-\boldsymbol{y}|+s-t)\Big)\boldsymbol{J}(\boldsymbol{y}-\boldsymbol{a}(s))\delta(s-t_0)\mathrm{d}\boldsymbol{y}\mathrm{d}s\\&=\int_{\mathbb{R}^3}\frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\delta(|\boldsymbol{x}-\boldsymbol{y}|-(t-t_0))\boldsymbol{J}(\boldsymbol{y}-\boldsymbol{a}(t_0))\mathrm{d}\boldsymbol{y}.\end{aligned}\tag{5.2}$$

Before stating the main theorem of this section, we describe the strategy for the choice of four measurement points (or receivers) on the sphere Γ_R. The geometry is shown in figure 1. First, we choose arbitrarily three different points $\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3\in\Gamma_R$. Denote by P the uniquely determined plane passing through $\boldsymbol{x}_1$, $\boldsymbol{x}_2$ and $\boldsymbol{x}_3$, and by L the line passing through the origin and perpendicular to P. Obviously the straight line L has two intersection points with Γ_R. Choose one of the intersection points with the longer distance to plane P as the fourth point $\boldsymbol{x}_4$. If the two intersection points have the same distance to P, we can choose either one of them as $\boldsymbol{x}_4$. By our choice of $\boldsymbol{x}_j, j=1,2,3,4$, they cannot lie on one side of any plane passing through the origin, if the plane P determined by $\boldsymbol{x}_j, j=1,2,3$ does not pass through the origin.

Theorem 5.1. *Let the measurement positions $\boldsymbol{x}_j\in\Gamma_R$, $j=1,\cdots,4$ be given as above and let $\boldsymbol{J}$ be specified as in the introduction part. We assume additionally that* $\mathrm{supp}(\boldsymbol{J})=B_{\hat{R}}$ *and there exists a small constant $\delta>0$ such that $|J_i(\boldsymbol{x})|>0$ for all $\hat{R}-\delta\leqslant|\boldsymbol{x}|\leqslant\hat{R}$ and $i=1,2,3$. Then both t_0 and $\boldsymbol{a}(t_0)$ can be uniquely determined by the data set $\{\boldsymbol{E}(\boldsymbol{x}_j,t):j=1,\cdots,4,\ t\in(0,T)\}$, where $T=t_0+\hat{R}+R_1+R$.*

Proof. Analogously to lemma 2.1, one can prove that $\boldsymbol{E}(\boldsymbol{x},t)=0$ for all $\boldsymbol{x}\in B_R$ and $t>T$. Taking the Fourier transform of $\boldsymbol{E}(\boldsymbol{x},t)$ in (3.1) with respect to t and making use of the representation of $\boldsymbol{E}$ in (5.2), we obtain

$$\begin{aligned}\hat{\boldsymbol{E}}(\boldsymbol{x},\kappa)&=\int_{\mathbb{R}^3}\frac{\mathrm{e}^{\mathrm{i}\kappa(t_0+|\boldsymbol{x}-\boldsymbol{y}|)}}{|\boldsymbol{x}-\boldsymbol{y}|}\boldsymbol{J}(\boldsymbol{y}-\boldsymbol{a}(t_0))\mathrm{d}\boldsymbol{y}\\&=\mathrm{e}^{\mathrm{i}\kappa t_0}\int_0^\infty\mathrm{e}^{\mathrm{i}\kappa\rho}\frac{1}{\rho}\int_{\Gamma_\rho(\boldsymbol{x})}\boldsymbol{J}(\boldsymbol{y}-\boldsymbol{a}(t_0))\mathrm{d}\boldsymbol{y}\mathrm{d}\rho,\end{aligned}\tag{5.3}$$

where $\Gamma_\rho(\boldsymbol{x}):=\{\boldsymbol{y}\in\mathbb{R}^3:|\boldsymbol{y}-\boldsymbol{x}|=\rho\}$. Assume that there are two orbit functions $\boldsymbol{a}$ and $\boldsymbol{b}$ and two time points t_0 and $\tilde{t}_0$ such that

$$\begin{cases}\partial_t^2\boldsymbol{E}_1(\boldsymbol{x},t)+\nabla\times(\nabla\times\boldsymbol{E}_1(\boldsymbol{x},t))=-\boldsymbol{J}(x-\boldsymbol{a}(t))\delta(t-t_0), & \boldsymbol{x}\in\mathbb{R}^3,\ t>0,\\ \boldsymbol{E}_1(\boldsymbol{x},0)=\partial_t\boldsymbol{E}_1(\boldsymbol{x},0)=0, & \boldsymbol{x}\in\mathbb{R}^3,\end{cases}$$

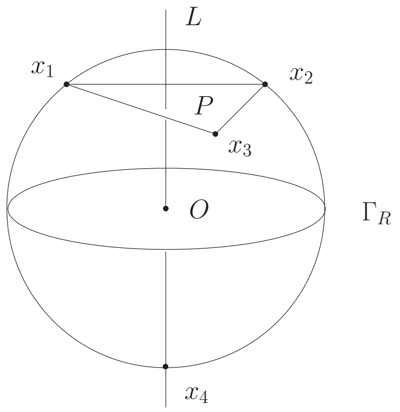

Figure 1. Geometry of the four measurement points.

and

$$\begin{cases}\partial_t^2 \boldsymbol{E}_2(\boldsymbol{x},t) + \nabla\times(\nabla\times\boldsymbol{E}_2(\boldsymbol{x},t)) = -\boldsymbol{J}(x-\boldsymbol{b}(t))\delta(t-\tilde{t}_0), & \boldsymbol{x}\in\mathbb{R}^3,\ t>0,\\ \boldsymbol{E}_2(\boldsymbol{x},0) = \partial_t\boldsymbol{E}_2(\boldsymbol{x},0) = 0, & \boldsymbol{x}\in\mathbb{R}^3.\end{cases}$$

We need to prove $t_0=\tilde{t}_0$ and $\boldsymbol{a}(t_0)=\boldsymbol{b}(\tilde{t}_0)$ under the condition $\boldsymbol{E}_1(\boldsymbol{x}_j,t)=\boldsymbol{E}_2(\boldsymbol{x}_j,t)$ for $t\in[0,T]$ and $j=1,2,3,4$. Below we denote by $\boldsymbol{x}\in\Gamma_R$ one of the measurement points $\boldsymbol{x}_j$ $(j=1,\cdots,4)$. Introduce the functions $\boldsymbol{F},\boldsymbol{F}_a,\boldsymbol{F}_b$: $\mathbb{R}_+\to\mathbb{R}$ as follows:

$$\begin{aligned}\boldsymbol{F}(\rho) &= \frac{1}{\rho}\int_{\Gamma_\rho(\boldsymbol{x})}\boldsymbol{J}(\boldsymbol{y})\mathrm{d}\boldsymbol{y},\\ \boldsymbol{F}_a(\rho) &= \frac{1}{\rho}\int_{\Gamma_\rho(\boldsymbol{x})}\boldsymbol{J}(\boldsymbol{y}-\boldsymbol{a}(t_0))\mathrm{d}\boldsymbol{y},\\ \boldsymbol{F}_b(\rho) &= \frac{1}{\rho}\int_{\Gamma_\rho(\boldsymbol{x})}\boldsymbol{J}(\boldsymbol{y}-\boldsymbol{b}(\tilde{t}_0))\mathrm{d}\boldsymbol{y}.\end{aligned}$$

Since $\mathrm{supp}(\boldsymbol{J})=B_{\hat{R}}$ and by our assumption, each component $J_j(\boldsymbol{x})$ $(j=1,2,3)$ is either positive or negative in a small neighborhood of $\Gamma_{\hat{R}}$, we can obtain that

$$\begin{aligned}&\inf\{\rho\in\mathrm{supp}(\boldsymbol{F})\}=|\boldsymbol{x}|-\hat{R}, &&\sup\{\rho\in\mathrm{supp}(\boldsymbol{F})\}=|\boldsymbol{x}|+\hat{R},\\ &\inf\{\rho\in\mathrm{supp}(\boldsymbol{F}_a)\}=|\boldsymbol{x}-\boldsymbol{a}(t_0)|-\hat{R}, &&\sup\{\rho\in\mathrm{supp}(\boldsymbol{F}_a)\}=|\boldsymbol{x}-\boldsymbol{a}(t_0)|+\hat{R},\\ &\inf\{\rho\in\mathrm{supp}(\boldsymbol{F}_b)\}=|\boldsymbol{x}-\boldsymbol{b}(\tilde{t}_0)|-\hat{R}, &&\sup\{\rho\in\mathrm{supp}(\boldsymbol{F}_b)\}=|\boldsymbol{x}-\boldsymbol{b}(\tilde{t}_0)|+\hat{R}.\end{aligned} \tag{5.4}$$

Since $\boldsymbol{E}_1(\boldsymbol{x},t)=\boldsymbol{E}_2(\boldsymbol{x},t)$, $t\in[0,T]$ for some point $\boldsymbol{x}\in\partial B_R$, from (5.3) we have

$$\mathrm{e}^{\mathrm{i}\kappa t_0}\hat{\boldsymbol{F}}_a(\kappa)=\mathrm{e}^{\mathrm{i}\kappa\tilde{t}_0}\hat{\boldsymbol{F}}_b(\kappa)$$

for all $\kappa>0$, which means

$$\hat{\boldsymbol{F}}_a(\kappa)=\mathrm{e}^{-\mathrm{i}\kappa(t_0-\tilde{t}_0)}\hat{\boldsymbol{F}}_b(\kappa). \tag{5.5}$$

Recalling the property of the Fourier transform,

$$\boldsymbol{F}_b(\rho \widehat{- (t_0} - \tilde{t}_0))(\kappa) = \mathrm{e}^{-\mathrm{i}\kappa(t_0-\tilde{t}_0)}\hat{\boldsymbol{F}}_b(\kappa),$$

we deduce from (5.5) that

$$\boldsymbol{F}_b(\rho - (t_0 - \tilde{t}_0)) = \boldsymbol{F}_a(\rho), \quad \rho \in \mathbb{R}^+.$$

Particularly,

$$\begin{aligned} \inf\{\mathrm{supp}(\boldsymbol{F}_b(\cdot - (t_0 - \tilde{t}_0)))\} &= \inf\{\mathrm{supp}(\boldsymbol{F}_a(\cdot))\}, \\ \sup\{\mathrm{supp}(\boldsymbol{F}_b(\cdot - (t_0 - \tilde{t}_0)))\} &= \sup\{\mathrm{supp}(\boldsymbol{F}_a(\cdot))\}. \end{aligned}$$

Therefore, we derive from (5.4) that

$$\begin{aligned} |\boldsymbol{x} - \boldsymbol{b}(\tilde{t}_0)| - \hat{R} + (t_0 - \tilde{t}_0) &= |\boldsymbol{x} - \boldsymbol{a}(t_0)| - \hat{R}, \\ |\boldsymbol{x} - \boldsymbol{b}(\tilde{t}_0)| + \hat{R} + (t_0 - \tilde{t}_0) &= |\boldsymbol{x} - \boldsymbol{a}(t_0)| + \hat{R}, \end{aligned}$$

which means

$$|\boldsymbol{x} - \boldsymbol{b}(\tilde{t}_0)| - |\boldsymbol{x} - \boldsymbol{a}(t_0)| = \tilde{t}_0 - t_0. \tag{5.6}$$

Physically, the right and left hand sides of the above identity represent the difference of the flight time between $\boldsymbol{x}$ and $\boldsymbol{a}(t_0)$, $\boldsymbol{b}(\tilde{t}_0)$. Note that the wave speed has been normalized to one for simplicity.

Finally, we prove that the identity (5.6) cannot hold simultaneously for our choice of measurement points $\boldsymbol{x}_j \in \Gamma_R$ $(j = 1, \cdots, 4)$. Obviously, the set $\{\boldsymbol{x} \in \mathbb{R}^3 : |\boldsymbol{x} - \boldsymbol{b}(\tilde{t}_0)| - |\boldsymbol{x} - \boldsymbol{a}(t_0)| = t_0 - \tilde{t}_0\}$ represents one sheet of a hyperboloid. This implies that $\boldsymbol{x}_j$ $(j = 1, 2, 3, 4)$ should be located on one half sphere of radius R excluding the corresponding equator, which is a contradiction to our choice of $\boldsymbol{x}_j$. Then we have $t_0 = \tilde{t}_0$ and (5.6) then becomes

$$|\boldsymbol{x} - \boldsymbol{b}(t_0)| - |\boldsymbol{x} - \boldsymbol{a}(t_0)| = 0.$$

This implies that $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4$ should be on the same plane. This is also a contradiction to our choice of $\boldsymbol{x}_i$, $i = 1, \cdots, 4$. Then we have $\boldsymbol{a}(t_0) = \boldsymbol{b}(t_0)$. □

Remark 5.2. If the source term on the right hand side of (5.1) takes the form

$$\boldsymbol{F}(\boldsymbol{x}, t) = -\boldsymbol{J}(x - \boldsymbol{a}(t)) \sum_{j=1}^{m} \delta(t - t_j),$$

with the impulsive time points

$$t_1 < t_2 < \cdots < t_m, \quad |t_{j+1} - t_j| > R.$$

One can prove that the set $\{(t_j, \boldsymbol{a}(t_j)) : j = 1, 2, \cdots, m\}$ can be uniquely determined by $\{\boldsymbol{E}(\boldsymbol{x}_j, t) : j = 1, \cdots, 4,\ t \in (0, T)\}$, where $T = t_m + \hat{R} + R_1 + R$. In fact, for $2 \leqslant j \leqslant m$, one can prove that $(t_j, \boldsymbol{a}(t_j))$ can be uniquely determined by $\{\boldsymbol{E}(\boldsymbol{x}_j, t) : j = 1, \cdots, 4,\ t \in (T_{j-1}, T_j)\}$, where $T_j = T_{j-1} + t_j$ and $T_1 := t_1 + \hat{R} + R_1 + R$.

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ORCID iDs

Guanghui Hu https://orcid.org/0000-0002-8485-9896
Yavar Kian https://orcid.org/0000-0002-5588-3600
Peijun Li https://orcid.org/0000-0001-5119-6435
Yue Zhao https://orcid.org/0000-0001-5939-8410

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