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INVERSE ELASTIC SCATTERING FOR A RANDOM SOURCE*

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Abstract. Consider the inverse random source scattering problem for the two-dimensional time-harmonic elastic wave equation with a linear load. The source is modeled as a microlocally isotropic generalized Gaussian random function whose covariance operator is a classical pseudodifferential operator. The goal is to recover the principal symbol of the covariance operator from the displacement measured in a domain away from the source. For such a distributional source, we show that the direct problem has a unique solution by introducing an equivalent Lippmann–Schwinger integral equation. For the inverse problem, we demonstrate that, with probability one, the principal symbol of the covariance operator can be uniquely determined by the amplitude of the displacement averaged over the frequency band, generated by a single realization of the random source. The analysis employs the Born approximation, asymptotic expansions of the Green tensor, and microlocal analysis of the Fourier integral operators.

Key words. inverse source problem, elastic wave equation, Lippmann–Schwinger integral equation, Gaussian random function, uniqueness

AMS subject classifications. 78A46, 65C30

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1. Introduction.

The inverse source scattering problems are to recover the unknown sources from the radiated wave field which is generated by the unknown sources. These problems are motivated by significant applications in diverse scientific areas such as medical imaging [3, 24, 36] and antenna design and synthesis [21]. Driven by these applications, the inverse source scattering problems have been extensively studied by many researchers in both mathematical and engineering communities. Consequently, a great deal of mathematical and numerical results are available, especially for deterministic sources [1, 7, 14, 21, 23]. It is known that the inverse source problem, in general, does not have a unique solution at a single frequency due to the existence of nonradiating sources [9, 18, 22, 25]. There are two approaches to overcome the nonuniqueness issue: one is to seek the minimum energy solution [34], which represents the pseudoinverse solution for the inverse source problem; the other is the use of multifrequency data to achieve uniqueness and gain increasing stability [13, 15, 16, 20, 31].

In many situations, the source, hence the wave field, may not be deterministic but is rather modeled by random processes [8]. Due to the extra challenge of randomness and uncertainties, little is known for the inverse random source scattering problems. In [10, 11, 12, 17, 28, 29], the random source was assumed to be driven by an additive white noise. Mathematical modeling and numerical computation were proposed for a class of inverse source problems for acoustic and elastic waves. The method requires

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one to know the expectation of the scattering data, which needs to be measured corresponding to a fairly large number of realizations of the source.

Recently, a different model was proposed in [19, 33] to describe random functions. The random function is considered to be a generalized Gaussian random function whose covariance is represented by a classical pseudodifferential operator. The authors studied an inverse problem for the two-dimensional random Schrödinger equation where the potential function was random. It was shown that the principal symbol of the covariance operator can be uniquely determined by the backscattered far field [19] or backscattered field [33], generated from a single realization of the random potential by using either plane waves [19] or a point source [33] as the incident field. A related work can be found in [26], where the authors considered an inverse scattering problem in a half-space with an impedance boundary condition where the impedance function was random. In [30], the inverse random source scattering problems were considered for the time-harmonic acoustic and elastic waves in a homogeneous and isotropic medium. The source was assumed to be a microlocally isotropic generalized Gaussian random function. It was shown that the amplitude of the scattering field averaged over the frequency band, obtained from a single realization of the random source, determines uniquely the principal symbol of the covariance operator. In this paper, we study an inverse random source scattering problem for the two-dimensional elastic wave equation with a linear load inside a homogeneous and isotropic medium. This paper significantly extends our previous work on the inverse random source problem for elastic waves. The techniques also differ greatly because a more complicated model equation is considered.

The wave propagation is governed by the stochastic elastic wave equation

$$\mu\Delta\boldsymbol{u} + (\lambda+\mu)\nabla\nabla\cdot\boldsymbol{u} + \omega^2\boldsymbol{u} - \boldsymbol{M}\boldsymbol{u} = \boldsymbol{f} \quad \text{in } \mathbb{R}^2, \tag{1.1}$$

where $\boldsymbol{u} \in \mathbb{C}^2$ is the complex-valued displacement vector, $\omega > 0$ is the angular frequency, λ and μ are the Lamé constants satisfying $\mu > 0, \lambda + 2\mu > 0$, which implies that the second order partial differential operator $\Delta^* := \mu\Delta + (\lambda+\mu)\nabla\nabla\cdot$ is strongly elliptic [35], and $\boldsymbol{M} \in \mathbb{R}^{2\times 2}$ is a deterministic real-valued symmetric matrix with a compact support contained in $D \subset \mathbb{R}^2$ and represents the matrix of a linear load inside a known homogeneous and isotropic elastic solid [6]. The randomness of (1.1) comes from the external source $\boldsymbol{f} = (f_1, f_2)^\top$. Throughout, we make the following assumption.

Assumption 1.1. *The domain D is bounded, simply connected, and Lipschitz. The source $\boldsymbol{f} = (f_1, f_2)^\top$ is compactly supported in D and $f_j, j = 1, 2$ are microlocally isotropic Gaussian random fields of the same order $m \in [2, \frac{5}{2})$ in D. Each covariance operator C_{f_j} is a classical pseudodifferential operator having the same principal symbol $\phi(x)|\xi|^{-m}$ with $\phi \in C_0^\infty(D), \phi \geq 0$. Moreover, the source $\boldsymbol{f}$ is assumed to be bounded almost surely with $\mathbb{E}(f_j) = 0$ and $\mathbb{E}(f_1 f_2) = 0$.*

Since (1.1) is imposed in the whole space $\mathbb{R}^2$, an appropriate radiation condition is needed to complete the problem formulation. By the Helmholtz decomposition, the displacement $\boldsymbol{u}$ can be decomposed into the compressional part $\boldsymbol{u}_{\rm p}$ and the shear part $\boldsymbol{u}_{\rm s}$ away from the source:

$$\boldsymbol{u} = -\frac{1}{\kappa_{\rm p}^2}\nabla\nabla\cdot\boldsymbol{u} + \frac{1}{\kappa_{\rm s}^2}\mathbf{curl}\,\mathrm{curl}\boldsymbol{u} := \boldsymbol{u}_{\rm p} + \boldsymbol{u}_{\rm s} \quad \text{in } \mathbb{R}^2 \setminus \overline{D}.$$

For a scalar function u and a vector function $\boldsymbol{u} = (u_1, u_2)^\top$, the vector and scalar curl operators are defined by

$$\mathbf{curl} u = (\partial_{x_2} u, -\partial_{x_1} u)^\top, \quad \mathrm{curl} \boldsymbol{u} = \partial_{x_1} u_2 - \partial_{x_2} u_1.$$

The Kupradze–Sommerfeld radiation condition requires that $\boldsymbol{u}_{\rm p}$ and $\boldsymbol{u}_{\rm s}$ satisfy the Sommerfeld radiation condition:

$$\lim_{r\to\infty} r^{\frac{1}{2}} \left(\partial_r \boldsymbol{u}_{\rm p} - {\rm i}\kappa_{\rm p} \boldsymbol{u}_{\rm p}\right) = 0, \quad \lim_{r\to\infty} r^{\frac{1}{2}} \left(\partial_r \boldsymbol{u}_{\rm s} - {\rm i}\kappa_{\rm s} \boldsymbol{u}_{\rm s}\right) = 0, \quad r = |x|, \tag{1.2}$$

where $\kappa_{\rm p}$ and $\kappa_{\rm s}$ are known as the compressional wavenumber and the shear wavenumber, respectively, and are defined by

$$\kappa_{\rm p} = \frac{\omega}{(\lambda + 2\mu)^{1/2}} = c_{\rm p}\omega, \qquad \kappa_{\rm s} = \frac{\omega}{\mu^{1/2}} = c_{\rm s}\omega.$$

Here

$$c_{\rm p} = (\lambda + 2\mu)^{-1/2}, \quad c_{\rm s} = \mu^{-1/2}.$$

Note that $c_{\rm p}$ and $c_{\rm s}$ are independent of ω and $c_{\rm p} < c_{\rm s}$.

Given $\omega, \lambda, \mu, \boldsymbol{M}$, and $\boldsymbol{f}$, *the direct scattering problem* is to determine $\boldsymbol{u}$ which satisfies (1.1)–(1.2). For $m \in [2, 5/2)$, the random source is a rough field and belongs to the Sobolev space with a negative smoothness index almost surely. A careful study is needed to show the well-posedness of the direct scattering problem for such a distributional source. Using Green's theorem and the Kupradze–Sommerfeld radiation condition, we show that the direct scattering problem is equivalent to the Lippmann–Schwinger equation. By the Fredholm alternative along with the unique continuation principle, we prove that the Lippmann–Schwinger equation has a unique solution which, almost surely, belongs to the Sobolev space with a positive smoothness index $\varepsilon \in (0, p/2)$ for some $p \geq 2$. Thus the well-posedness is established for the direct scattering problem.

Given ω, λ, μ, and $\boldsymbol{M}$, *the inverse scattering problem* is to determine $\phi(x)$, the microcorrelation strength of the source, from the displacement measured in a bounded domain $U \subset \mathbb{R}^2 \setminus \overline{D}$, which stands for the measurement domain and is required to satisfy the following assumption.

ASSUMPTION 1.2. *The measurement domain U is bounded, simply connected, Lipschitz, and convex and has a positive distance to D.*

In addition, the following assumption is imposed on $\boldsymbol{M}$.

ASSUMPTION 1.3. *The matrix $\boldsymbol{M} = (\boldsymbol{M}_{ij})_{2\times 2}$ is a deterministic and real-valued symmetric matrix with $\boldsymbol{M}_{ij} \in C_0^1(\overline{D})$ for $i, j = 1, 2$.*

The following result concerns the uniqueness of the inverse scattering problem and is the main result of this paper.

THEOREM 1.4. *Let $\boldsymbol{f}, U$, and $\boldsymbol{M}$ satisfy Assumptions* 1.1, 1.2, *and* 1.3, *respectively. Then for all $x \in U$, it holds almost surely that*

$$\lim_{Q\to\infty} \frac{1}{Q-1} \int_1^Q \omega^{m+1} |\boldsymbol{u}(x,\omega)|^2 d\omega = a \int_{\mathbb{R}^2} \frac{1}{|x-y|} \phi(y) dy, \tag{1.3}$$

where $a = \frac{1}{32\pi}\left(c_{\rm s}^{3-m} + c_{\rm p}^{3-m}\right)$ is a constant. Moreover, the function ϕ can be uniquely determined from the integral equation (1.3) *for all $x \in U$.*

For any finite Q, the scattering data given in the left-hand side of (1.3) is random in the sense that it depends on realization of the source, while (1.3) shows that in the limit $Q \to \infty$, the scattering data becomes statistically stable, i.e., it is independent of realization of the source. Hence, Theorem 1.4 shows that the amplitude of the displacement averaged over the frequency band, measured from a single realization of the random source, can uniquely determine the microcorrelation strength function ϕ. The proof of Theorem 1.4 combines the Born approximation, asymptotic expansions of the Green tensor, and microlocal analysis of integral operators.

For clarity, we briefly explain the steps of the proof for the main result. As mentioned above, the direct scattering problem is equivalent to the Lippmann–Schwinger equation, which has a unique solution under Assumption 1.3. Considering the Born series of the Lippmann–Schwinger equation $\sum_{n=0}^{\infty} \boldsymbol{u}_n(x,\omega)$ (see (4.1) for the definition of $\boldsymbol{u}_n(x,\omega)$), we may show that the Born series converges to the solution of the direct scattering problem when the angular frequency ω is large enough. Therefore

$$\boldsymbol{u}(x,\omega) = \boldsymbol{u}_0(x,\omega) + \boldsymbol{u}_1(x,\omega) + \boldsymbol{b}(x,\omega), \quad \boldsymbol{b}(x,\omega) := \sum_{n=2}^{\infty} \boldsymbol{u}_n(x,\omega). \tag{1.4}$$

For the leading term $\boldsymbol{u}_0(x,\omega)$, it is the solution of the random source problem in a homogeneous medium for the time-harmonic elastic wave without the linear load, which was considered in [30]. If the random source $\boldsymbol{f}$ satisfies the Assumption 1.1, it was shown in [30] that

$$\lim_{Q\to\infty} \frac{1}{Q-1} \int_1^Q \omega^{m+1} |\boldsymbol{u}_0(x,\omega)|^2 d\omega = a \int_{\mathbb{R}^2} \frac{1}{|x-y|} \phi(y) dy, \quad x \in U, \tag{1.5}$$

where a is some positive constant. In this work, it is required to consider the two extra terms $\boldsymbol{u}_1(x,\omega)$ and $\boldsymbol{b}(x,\omega)$, which are nontrivial. For the term $\boldsymbol{u}_1(x,\omega)$, we show that

$$\lim_{Q\to\infty} \frac{1}{Q-1} \int_1^Q \omega^{m+1} |\boldsymbol{u}_1(x,\omega)|^2 d\omega = 0, \quad x \in U. \tag{1.6}$$

It is quite technical and takes half of the main body text of the paper to (1.6). The major ingredients are the asymptotic expansions of the Green tensor and the microlocal analysis of integral operator. For the remainder $\boldsymbol{b}(x,\omega)$, we may show that

$$\lim_{Q\to\infty} \frac{1}{Q-1} \int_1^Q \omega^{m+1} |\boldsymbol{b}(x,\omega)|^2 d\omega = 0, \quad x \in U. \tag{1.7}$$

The main result (1.3) can be obtained by combining (1.4)–(1.7) and using the Cauchy–Schwarz inequality.

The paper is organized as follows. In section 2, we introduce some necessary notation including Sobolev spaces, generalized Gaussian random functions, and some properties of the Hankel function of the first kind. Section 3 addresses the direct scattering problem; sections 4 and 5 study the inverse scattering problem. In section 3, the well-posedness of the direct scattering problem is established for a distributional source. Using the Riesz–Fredholm theory and the Sobolev embedding theorem, we show that the direct scattering problem is equivalent to a uniquely solvable Lippmann–Schwinger equation. Section 4 presents the Born approximation of the solution to the Lippmann–Schwinger integral equation. Section 5 examines the second term in the Born approximation via the microlocal analysis. The paper is concluded with some general remarks in section 6.

2. Preliminaries. In this section, we introduce some necessary notation and properties of Sobolev spaces, generalized Gaussian random functions, and the Hankel functions.

2.1. Sobolev spaces. Let $C_0^\infty(\mathbb{R}^2)$ be the set of smooth functions with compact support and $\mathcal{D}'(\mathbb{R}^2)$ be the set of generalized (distributional) functions. Given $1 < p < \infty, s \in \mathbb{R}$, define the Sobolev space

$$H^{s,p}(\mathbb{R}^2) = \{h = (I - \Delta)^{-\frac{s}{2}} g : g \in L^p(\mathbb{R}^2)\},$$

which has the norm

$$\|h\|_{H^{s,p}(\mathbb{R}^2)} = \|(I - \Delta)^{\frac{s}{2}} h\|_{L^p(\mathbb{R}^2)}.$$

With the definition of Sobolev spaces in the whole space, the Sobolev space $H^{s,p}(V)$ for any Lipschitz domain $V \subset \mathbb{R}^2$ can be defined as the restriction to V of the elements in $H^{s,p}(\mathbb{R}^2)$ with the norm

$$\|h\|_{H^{s,p}(V)} = \inf\{\|g\|_{H^{s,p}(\mathbb{R}^2)} : g|_V = h\}.$$

By [27], for $s \in \mathbb{R}$ and $1 < p < \infty$, $H_0^{s,p}(V)$ can be defined as the space of all distributions $h \in H^{s,p}(\mathbb{R}^2)$ satisfying $\text{supp} h \subset \overline{V}$ with the norm

$$\|h\|_{H_0^{s,p}(V)} = \|h\|_{H^{s,p}(\mathbb{R}^2)}.$$

It is known that $C_0^\infty(V)$ is dense in $H_0^{s,p}(V)$ for any $1 < p < \infty, s \in \mathbb{R}$; $C_0^\infty(V)$ is dense in $H^{s,p}(V)$ for any $1 < p < \infty, s \leq 0$; and $C^\infty(\overline{V})$ is dense in $H^{s,p}(V)$ for any $1 < p < \infty, s \in \mathbb{R}$. In addition, by [27, Propositions 2.4 and 2.9], for any $s \in \mathbb{R}$ and $p, q \in (1, \infty)$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$H_0^{-s,q}(V) = (H^{s,p}(V))' \quad \text{and} \quad H^{-s,q}(V) = (H_0^{s,p}(V))',$$

where the prime denotes the dual space.

Alternatively, the Sobolev spaces can be defined as follows [2]: For an integer $m \geq 1$ and $1 < p < \infty$, the integer order Sobolev space $H^{m,p}(V)$ can be defined by

$$H^{m,p}(V) := \{f \in L^p(V), \quad \frac{\partial^\beta f}{\partial x^\beta} \in L^p(V), \quad |\beta| \leq m\},$$

which is equipped with the norm

$$\|f\|_{H^{m,p}(V)} := \left[\sum_{|\beta| \leq m} \int_V \left|\frac{\partial^\beta f}{\partial x^\beta}\right|^p dx\right]^{\frac{1}{p}}.$$

Here $\beta = (\beta_1, \beta_2)$ is a multiple index and $|\beta| = \beta_1 + \beta_2$, $\frac{\partial^\beta f}{\partial x^\beta} = \frac{\partial^{|\beta|} f}{\partial x_1^{\beta_1} \partial x_2^{\beta_2}}$. With the definition of the integer order Sobolev space, the fractional order Sobolev space is defined by the complex interpolation between L^p and the integer order Sobolev space. Specifically, if $s > 0$ and m is the smallest integer greater than s, the space $H^{s,p}(V)$ is defined by

$$H^{s,p}(V) := [L^p(V), H^{m,p}(V)]_{s/m}. \tag{2.1}$$

The following two lemmas will be used in the subsequent analysis. The proofs of Lemmas 2.1 and 2.2 can be found in [33, Lemma 2] and [37, Proposition 1], respectively.

LEMMA 2.1. *Assume that* $\epsilon > 0$, $1 < r < \infty$, $\frac{1}{r} + \frac{1}{r'} = 1$, $g \in H^{\epsilon,2r}_{\rm loc}(\mathbb{R}^2)$, $h \in H_0^{-\epsilon,r'}(\mathbb{R}^2)$. *Then* $gh \in H_0^{-\epsilon,\tilde r}(\mathbb{R}^2)$ *and satisfies*

$$\|gh\|_{H_0^{-\epsilon,\tilde r}(\mathbb{R}^2)} \lesssim \|g\|_{H^{\epsilon,2r}(\mathbb{R}^2)}||h||_{H_0^{-\epsilon,r'}(\mathbb{R}^2)},$$

where $\tilde r = \frac{2r}{2r-1}$.

LEMMA 2.2. *Assume that* $s > 0$, $1 < \tilde p < \infty$, *and* $\frac{1}{\tilde p} = \frac{1}{q_1} + \frac{1}{q_2} = \frac{1}{r_1} + \frac{1}{r_2}, q_1, r_1 \in (1, \infty], q_2, r_2 \in (1, \infty)$. *Then the following estimate holds:*

$$\|gh\|_{H^{s,\tilde p}(\mathbb{R}^2)} \lesssim \|g\|_{L^{q_1}(\mathbb{R}^2)}||h||_{H^{s,q_2}(\mathbb{R}^2)} + \|h\|_{L^{r_1}(\mathbb{R}^2)}\|g\|_{H^{s,r_2}(\mathbb{R}^2)}.$$

Throughout the paper, $a \lesssim b$ stands for $a \le Cb$, where C is a positive constant and its specific value is not required but should be clear from the context.

2.2. Generalized Gaussian random functions. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. The function h is said to be a generalized Gaussian random function if $h : \Omega \to \mathcal{D}'(\mathbb{R}^2)$ is a mapping such that, for each $\hat\omega \in \Omega$, the realization $h(\hat\omega)$ is a real-valued linear functional on $C_0^\infty(\mathbb{R}^2)$ and the function

$$\hat\omega \in \Omega \to \langle h(\hat\omega), \psi\rangle \in \mathbb{R}$$

is a Gaussian random variable for all $\psi \in C_0^\infty(\mathbb{R}^2)$. The distribution of h is determined by its expectation $\mathbb{E}h$ and the covariance Cov$h$ defined as

$$\mathbb{E}h : \psi \in C_0^\infty(\mathbb{R}^d) \longmapsto \mathbb{E}\langle h, \psi\rangle \in \mathbb{R},$$
$$\mathrm{Cov}h : (\psi_1, \psi_2) \in C_0^\infty(\mathbb{R}^d)^2 \longmapsto \mathrm{Cov}(\langle h, \psi_1\rangle, \langle h, \psi_2\rangle) \in \mathbb{R},$$

where $\mathbb{E}\langle h, \psi\rangle$ denotes the expectation of $\langle h, \psi\rangle$ and

$$\mathrm{Cov}(\langle h, \psi_1\rangle, \langle h, \psi_2\rangle) = \mathbb{E}((\langle h, \psi_1\rangle - \mathbb{E}\langle h, \psi_1\rangle)(\langle h, \psi_2\rangle - \mathbb{E}\langle h, \psi_2\rangle))$$

denotes the covariance of $\langle h, \psi_1\rangle$ and $\langle h, \psi_2\rangle$. The covariance operator $\mathrm{Cov}_h : C_0^\infty(\mathbb{R}^2) \to \mathcal{D}'(\mathbb{R}^2)$ is defined by

$$\langle \mathrm{Cov}_h\psi_1, \psi_2\rangle = \mathrm{Cov}(\langle h, \psi_1\rangle, \langle h, \psi_2\rangle) = \mathbb{E}(\langle h - \mathbb{E}h, \psi_1\rangle\langle h - \mathbb{E}h, \psi_2\rangle). \tag{2.2}$$

Since the covariance operator Cov_h is continuous, the Schwartz kernel theorem shows that there exists a unique $C_h \in \mathcal{D}'(\mathbb{R}^2 \times \mathbb{R}^2)$, usually called the covariance function, such that

$$\langle C_h, \psi_1 \otimes \psi_2\rangle = \langle \mathrm{Cov}_h\psi_1, \psi_2\rangle \quad \forall \psi_1, \psi_2 \in C_0^\infty(\mathbb{R}^2). \tag{2.3}$$

By (2.2) and (2.3), it is easy to see that

$$C_h(x, y) = \mathbb{E}((h(x) - \mathbb{E}h(x))(h(y) - \mathbb{E}h(y))).$$

In this paper, we are interested in the generalized, microlocally isotropic Gaussian random function, which is defined as follows (cf. [33, Definition 1]).

DEFINITION 2.3. *A generalized Gaussian random function* h *on* $\mathbb{R}^2$ *is called microlocally isotropic of order* m *in* D *if the realizations of* h *are almost surely supported in the domain* D *and its covariance operator* Cov_h *is a classical pseudodifferential operator having the principal symbol* $\phi(x)|\xi|^{-m}$, *where* $\phi \in C_0^\infty(\mathbb{R}^2)$ *satisfies* $\mathrm{supp}\phi \subset D$ *and* $\phi(x) \ge 0$ *for all* $x \in \mathbb{R}^2$.

In particular, we pay attention to the case $m \in [2, 5/2)$, which corresponds to rough fields. The following results will also be used in the subsequent analysis. The proofs of Lemmas 2.4 and 2.5 can be found in [33, Theorem 2 and Proposition 1].

LEMMA 2.4. *Let h be a generalized, microlocally isotropic Gaussian random function of order m in D. If $m = 2$, then $h \in H^{-\varepsilon,p}(D)$ almost surely for all $\varepsilon > 0, 1 < p < \infty$. If $m \in (2, 5/2)$, then $h \in C^{\alpha}(D)$ almost surely for all $\alpha \in (0, \frac{m-2}{2})$.*

LEMMA 2.5. *Let h be a microlocally isotropic Gaussian random field of order $m \in [2, 5/2)$. Then the Schwartz kernel of the covariance operator Cov_h has the form*

$$C_h(x,y) = \begin{cases} c_0(x,y)\log|x-y| + r_1(x,y) & \text{for} \quad m = 2, \\ c_0(x,y)|x-y|^{m-2} + r_1(x,y) & \text{for} \quad m \in (2, 5/2), \end{cases}$$

where $c_0 \in C_0^\infty(D \times D)$ and $r_1 \in C_0^\alpha(D \times D)$ for any $\alpha < 1$.

2.3. Properties of the Hankel function. In this subsection, we present some asymptotic expansions of the Hankel function of the first kind for small and large arguments. Let $H_n^{(1)}$ be the Hankel function of the first kind with order n. Recall the definition

$$H_n^{(1)}(t) = J_n(t) + \mathrm{i}Y_n(t),$$

where J_n and Y_n are the Bessel functions of the first and second kind with order n, respectively. They admit the following expansions:

$$J_n(t) = \sum_{p=0}^{\infty} \frac{(-1)^p}{p!(n+p)!}\left(\frac{t}{2}\right)^{n+2p}, \tag{2.4}$$

$$Y_n(t) = \frac{2}{\pi}\left\{\ln\frac{t}{2} + \gamma\right\} J_n(t) - \frac{1}{\pi}\sum_{p=0}^{n-1}\frac{(n-1-p)!}{p!}\left(\frac{2}{t}\right)^{n-2p}$$
$$- \frac{1}{\pi}\sum_{p=0}^{\infty}\frac{(-1)^p}{p!(n+p)!}\left(\frac{t}{2}\right)^{n+2p}\{\psi(p+n) + \psi(p)\}, \tag{2.5}$$

where $\gamma := \lim_{p\to\infty}\{\sum_{j=1}^{p} j^{-1} - \ln p\}$ denotes the Euler constant, $\psi(0) = 0$, $\psi(p) = \sum_{j=1}^{p} j^{-1}$, and the finite sum in (2.5) is set to be zero for $n = 0$.

Using the expansions (2.4) and (2.5), we may verify as $t \to 0$ that

$$H_0^{(1)}(t) = \frac{2\mathrm{i}}{\pi}\ln\frac{t}{2} + b_0 + O\Big(t^2\ln\frac{t}{2}\Big), \tag{2.6}$$

$$H_1^{(1)}(t) = -\frac{2\mathrm{i}}{\pi}\frac{1}{t} + \frac{\mathrm{i}}{\pi}t\ln\frac{t}{2} + b_1 t + O\Big(t^3\ln\frac{t}{2}\Big), \tag{2.7}$$

$$H_2^{(1)}(t) = -\frac{4\mathrm{i}}{\pi}\frac{1}{t^2} - \frac{\mathrm{i}}{\pi} + \frac{\mathrm{i}}{4\pi}t^2\ln\frac{t}{2} + b_2 t^2 + O\Big(t^4\ln\frac{t}{2}\Big), \tag{2.8}$$

$$H_3^{(1)}(t) = -\frac{16\mathrm{i}}{\pi}\frac{1}{t^3} - \frac{2\mathrm{i}}{\pi}\frac{1}{t} - \frac{\mathrm{i}}{4\pi}t + \frac{\mathrm{i}}{24\pi}t^3\ln\frac{t}{2} + b_3 t^3 + O\Big(t^5\ln\frac{t}{2}\Big), \tag{2.9}$$

where $b_0 = 1 + \frac{2\mathrm{i}}{\pi}\gamma$, $b_1 = \frac{1}{2} + \frac{\mathrm{i}}{\pi}\gamma - \frac{\mathrm{i}}{2\pi}$, $b_2 = \frac{\gamma\mathrm{i}}{4\pi} - \frac{3\mathrm{i}}{16\pi} + \frac{1}{8}$, $b_3 = \frac{\gamma\mathrm{i}}{24\pi} + \frac{1}{48} - \frac{11\mathrm{i}}{288\pi}$. Denote

$$\Gamma_n(z,\omega) = \kappa_{\mathrm{s}}^n H_n^{(1)}(\kappa_{\mathrm{s}}|z|) - \kappa_{\mathrm{p}}^n H_n^{(1)}(\kappa_{\mathrm{p}}|z|).$$

Noting (2.6)–(2.9), we have from a direct calculation that the following asymptotic expansions hold as $|z| \to 0$:

$$\Gamma_1(z,\omega) = \frac{\mathrm{i}}{\pi}|z|\left(\kappa_{\mathrm{s}}^2 \ln\frac{\kappa_{\mathrm{s}}|z|}{2} - \kappa_{\mathrm{p}}^2 \ln\frac{\kappa_{\mathrm{p}}|z|}{2}\right) + b_1(\kappa_{\mathrm{s}}^2 - \kappa_{\mathrm{p}}^2)|z| + O\Big(|z|^3 \ln\frac{|z|}{2}\Big), \tag{2.10}$$

$$\Gamma_2(z,\omega) = \frac{\mathrm{i}}{4\pi}\left(\kappa_{\mathrm{s}}^4 \ln\frac{\kappa_{\mathrm{s}}|z|}{2} - \kappa_{\mathrm{p}}^4 \ln\frac{\kappa_{\mathrm{p}}|z|}{2}\right)|z|^2 - \frac{\mathrm{i}}{\pi}(\kappa_{\mathrm{s}}^2 - \kappa_{\mathrm{p}}^2) + O(|z|^2), \tag{2.11}$$

$$\Gamma_3(z,\omega) = \frac{2\mathrm{i}}{\pi}(\kappa_{\mathrm{p}}^2 - \kappa_{\mathrm{s}}^2)\frac{1}{|z|} + \frac{\mathrm{i}}{4\pi}(\kappa_{\mathrm{p}}^4 - \kappa_{\mathrm{s}}^4)|z| + O\Big(|z|^3 \ln\frac{|z|}{2}\Big). \tag{2.12}$$

For a large argument, i.e., as $|z| \to \infty$, it follows from [5, equations (9.2.7)–(9.2.10)] and [32, equation (5.11.4)] that the Hankel function of the first kind $H_n^{(1)}$ has the asymptotics

$$\begin{aligned} H_n^{(1)}(z) =& \sqrt{\frac{1}{z}} e^{\mathrm{i}(z-(\frac{n}{2}+\frac{1}{4})\pi)} \\ & \times \Big(\sum_{j=0}^{N} a_j^{(n)} z^{-j} + O(|z|^{-N-1})\Big), \quad |\arg z| \le \pi - \delta, \end{aligned} \tag{2.13}$$

where δ is a small positive number and the coefficients $a_j^{(n)} = (-2\mathrm{i})^j \sqrt{\frac{2}{\pi}}(n,j)$ with

$$(n,0) = 1, \quad (n,j) = \frac{(4n^2-1)(4n^2-3^2)\cdots(4n^2-(2j-1)^2)}{2^{2j} j!}.$$

Using the first N terms in the asymptotic of $H_n^{(1)}(\kappa|z|)$, we define

$$H_{n,N}^{(1)}(\kappa|z|) = \sqrt{\frac{1}{\kappa|z|}} e^{\mathrm{i}(\kappa|z|-(\frac{n}{2}+\frac{1}{4})\pi)} \sum_{j=0}^{N} a_j^{(n)} \left(\frac{1}{\kappa|z|}\right)^j. \tag{2.14}$$

Denoting $\Gamma_{n,N}(\kappa|z|) = H_n^{(1)}(\kappa|z|) - H_{n,N}^{(1)}(\kappa|z|)$, it is easy to show from (2.13) that

$$\big|\Gamma_{n,N}(\kappa|z|)\big| \le c\Big(\frac{1}{\kappa|z|}\Big)^{N+\frac{3}{2}}. \tag{2.15}$$

3. The direct scattering problem. This section aims to establish the well-posedness of the direct scattering problem for a distributional source. Based on Green's theorem and the Kupradze–Sommerfeld radiation, the direct problem is equivalently formulated as a Lippmann–Schwinger equation, which is shown to have a unique solution by using the Riesz–Fredholm theory and the Sobolev embedding theorem.

By Lemma 2.4, we have that $\boldsymbol{f} \in H^{-\varepsilon,p}(D)^2$ almost surely for all $\varepsilon > 0$, $1 < p < \infty$ if $m = 2$; and $\boldsymbol{f} \in C^{0,\alpha}(D)^2$ almost surely for all $\alpha \in (0, \frac{m-2}{2})$ if $m \in (2, 5/2)$. Therefore, it suffices to show that the scattering problem (1.1)–(1.2) has a unique solution for such a deterministic source $\boldsymbol{f} \in H^{-\varepsilon,p}(D)^2$.

Introduce the Green tensor $\boldsymbol{G}(x,y,\omega) \in \mathbb{C}^{2\times 2}$ for the Navier equation

$$\boldsymbol{G}(x,y,\omega) = \frac{1}{\mu}\Phi(x,y,\kappa_{\mathrm{s}})\boldsymbol{I} + \frac{1}{\omega^2}\nabla_x\nabla_x^\top(\Phi(x,y,\kappa_{\mathrm{s}}) - \Phi(x,y,\kappa_{\mathrm{p}})), \tag{3.1}$$

where $\boldsymbol{I}$ is the 2×2 identity matrix, $\Phi(x, y, \kappa) = \frac{\mathrm{i}}{4} H_0^{(1)}(\kappa |x-y|)$ is the fundamental solution for the two-dimensional Helmholtz equation, and $\nabla_x \nabla_x^\top$ is defined by

$$\nabla_x \nabla_x^\top \varphi = \begin{bmatrix} \partial^2_{x_1x_1}\varphi & \partial^2_{x_1x_2}\varphi \\ \partial^2_{x_2x_1}\varphi & \partial^2_{x_2x_2}\varphi \end{bmatrix}$$

for some scalar function φ defined in $\mathbb{R}^2$. It is easy to note that the Green tensor $\boldsymbol{G}(x, y, \omega)$ is symmetric with respect to the variables x and y.

In order to obtain the well-posedness of the scattering problem (1.1)–(1.2), we first derive a Lippmann–Schwinger equation which is equivalent to the direct scattering problem, and then we show that the Lippmann–Schwinger equation has a unique solution.

THEOREM 3.1. *For some* $p \geq 2$, $\frac{1}{p} + \frac{1}{p'} = 1$, $0 < \varepsilon < \frac{2}{p}$, $\boldsymbol{f} \in H_0^{-\varepsilon, p'}(D)^2$, *if* $\boldsymbol{M}$ *satisfies Assumption* 1.3, *then the scattering problem* (1.1)–(1.2) *is equivalent to the Lippmann–Schwinger equation*

$$\boldsymbol{u}(x) + \int_D \boldsymbol{G}(x, y, \omega)\boldsymbol{M}(y)\boldsymbol{u}(y)dy = -\int_D \boldsymbol{G}(x, y, \omega)\boldsymbol{f}(y)dy, \quad x \in \mathbb{R}^2. \tag{3.2}$$

Proof. Let $\boldsymbol{u} \in H^{\varepsilon, p}_{\mathrm{loc}}(\mathbb{R}^2)^2$ be a solution to (3.2), and then we have

$$\boldsymbol{u}(x) = -\int_D \boldsymbol{G}(x, y, \omega)\boldsymbol{M}(y)\boldsymbol{u}(y)dy - \int_D \boldsymbol{G}(x, y, \omega)\boldsymbol{f}(y)dy, \quad x \in \mathbb{R}^2.$$

Since the Green tensor $\boldsymbol{G}(x, y, \omega)$ and its derivatives satisfy the Kupradze–Sommerfeld radiation condition, we conclude that $\boldsymbol{u}$ also satisfies the Kupradze–Sommerfeld radiation condition. By (3.1), the Green tensor $\boldsymbol{G}(x, y, \omega)$ satisfies

$$\mu\Delta\boldsymbol{G}(x, y, \omega) + (\lambda + \mu)\nabla\nabla \cdot \boldsymbol{G}(x, y, \omega) + \omega^2\boldsymbol{G}(x, y, \omega) = -\delta(x - y)\boldsymbol{I}. \tag{3.3}$$

Letting $y = 0$ and taking the Fourier transform with respect to x on both sides of (3.3) yields

$$\left[(4\pi^2\mu|\xi|^2 - \omega^2)\boldsymbol{I} + 4\pi^2(\lambda + \mu)\xi \cdot \xi^\top\right] \widehat{\boldsymbol{G}}(\xi) = \boldsymbol{I}, \quad \xi \in \mathbb{R}^2. \tag{3.4}$$

Note that the integral in (3.2) is a convolution since $\boldsymbol{G}(x, y, \omega)$ is a function of $x - y$. Taking the Fourier transform on both sides of (3.2) leads to

$$\hat{\boldsymbol{u}}(\xi) = -\widehat{\boldsymbol{G}}(\xi)(\hat{\boldsymbol{f}}(\xi) + \widehat{\boldsymbol{M}\boldsymbol{u}}(\xi)). \tag{3.5}$$

Multiplying $(4\pi^2\mu|\xi|^2 - \omega^2)\boldsymbol{I} + 4\pi^2(\lambda + \mu)\xi \cdot \xi^\top$ on both sides of (3.5) and using (3.4) gives

$$\left[(4\pi^2\mu|\xi|^2 - \omega^2)\boldsymbol{I} + 4\pi^2(\lambda + \mu)\xi \cdot \xi^\top\right] \hat{\boldsymbol{u}}(\xi) + \widehat{\boldsymbol{M}\boldsymbol{u}}(\xi) = -\hat{\boldsymbol{f}}(\xi).$$

Taking the inverse Fourier transform yields

$$\mu\Delta\boldsymbol{u} + (\lambda + \mu)\nabla\nabla \cdot \boldsymbol{u} + \omega^2\boldsymbol{u} - \boldsymbol{M}\boldsymbol{u} = \boldsymbol{f} \quad \text{in } \mathbb{R}^2.$$

Hence, $\boldsymbol{u}$ is the solution of the direct scattering problem (1.1)–(1.2).

Conversely, if $\boldsymbol{u}$ is a solution of the direct scattering problem (1.1)–(1.2), we show that $\boldsymbol{u}$ satisfies the Lippmann–Schwinger equation (3.2). It follows from (1.1) that

$$\mu\Delta\boldsymbol{u} + (\lambda + \mu)\nabla\nabla \cdot \boldsymbol{u} + \omega^2\boldsymbol{u} = \boldsymbol{M}\boldsymbol{u} + \boldsymbol{f} \quad \text{in } \mathbb{R}^2,$$

where $\boldsymbol{u} \in H^{\varepsilon,p}_{\rm loc}(\mathbb{R}^2)^2$ and $\boldsymbol{M}_{ij} \in C^1_0(\overline{D})$. Noting that $\boldsymbol{f} \in H_0^{-\varepsilon,p'}(\mathbb{R}^2)^2$, we have that $\boldsymbol{M}\boldsymbol{u} + \boldsymbol{f} \in H_0^{-\varepsilon,p'}(\mathbb{R}^2)^2$. An application of Lemma 4.1 in [30] shows that for some fixed $x \in \mathbb{R}^2$, $\boldsymbol{G}(x,\cdot,\omega) \in [L^2_{\rm loc}(\mathbb{R}^2) \cap H^{1,\hat{p}}_{\rm loc}(\mathbb{R}^2)]^{2\times 2}$ for $\hat{p} \in (1,2)$. Since $0 < \varepsilon < \frac{2}{p}$, a simple calculation gives that $\frac{1}{p} - \frac{\varepsilon}{2} > 0$. Let $\widetilde{\delta} = \frac{1}{p} - \frac{\varepsilon}{2}$ and define $\widetilde{p} := \frac{2}{1+\widetilde{\delta}} < 2$, and then $\frac{1}{\widetilde{p}} - \frac{1}{2} < \frac{1}{p} - \frac{\varepsilon}{2}$. It follows from the Sobolev embedding theorem that $H^{1,\widetilde{p}}_{\rm loc}(\mathbb{R}^2)$ is embedded into $H^{\varepsilon,p}_{\rm loc}(\mathbb{R}^2)$, which implies that $\boldsymbol{G}(x,\cdot,\omega) \in [H^{\varepsilon,p}_{\rm loc}(\mathbb{R}^2)]^{2\times 2}$. Choose a large enough ball B_r such that $D \subset B_r$, and then we have in the sense of distributions that

$$\int_{B_r} \boldsymbol{G}(x,y,\omega)\left[\mu\Delta\boldsymbol{u}(y) + (\lambda+\mu)\nabla\nabla\cdot\boldsymbol{u}(y) + \omega^2\boldsymbol{u}(y)\right]dy$$
$$= \int_{B_r} \boldsymbol{G}(x,y,\omega)[\boldsymbol{M}(y)\boldsymbol{u}(y) + \boldsymbol{f}(y)]dy.$$

Denote by T the operator that maps $\boldsymbol{u}$ to the left-hand side of the above equation. For $\boldsymbol{\psi} \in C^\infty(\mathbb{R}^2)^2$, by similar arguments as those in the proof of Lemma 4.3 in [30], we obtain

$$T\boldsymbol{\psi}(x) = -\boldsymbol{\psi}(x) + \int_{\partial B_r} [\boldsymbol{G}(x,y,\omega)P\boldsymbol{\psi}(y) - P\boldsymbol{G}(x,y,\omega)\boldsymbol{\psi}(y)]ds(y),$$

where $P\boldsymbol{\psi} := \mu\frac{\partial\boldsymbol{\psi}}{\partial\boldsymbol{\nu}} + (\lambda+\mu)(\nabla\cdot\boldsymbol{\psi})\boldsymbol{\nu}$ and $\boldsymbol{\nu}$ is the unit normal vector on the boundary ∂B_r.

Approximating $\boldsymbol{u}$ with smooth functions, we get

$$-\boldsymbol{u}(x) + \int_{\partial B_r} [\boldsymbol{G}(x,y,\omega)P\boldsymbol{u}(y) - P\boldsymbol{G}(x,y,\omega)\boldsymbol{u}(y)]ds(y)$$
$$= \int_{B_r} \boldsymbol{G}(x,y,\omega)[\boldsymbol{M}(y)\boldsymbol{u}(y) + \boldsymbol{f}(y)]dy.$$

Using the radiation condition yields

$$\lim_{r\to\infty} \int_{\partial B_r} [\boldsymbol{G}(x,y,\omega)P\boldsymbol{u}(y) - P\boldsymbol{G}(x,y,\omega)\boldsymbol{u}(y)]ds(y) = 0.$$

Therefore,

$$\boldsymbol{u}(x) + \int_D \boldsymbol{G}(x,y,\omega)\boldsymbol{M}(y)\boldsymbol{u}(y)dy = -\int_D \boldsymbol{G}(x,y,\omega)\boldsymbol{f}(y)dy, \quad x \in \mathbb{R}^2,$$

which shows that $\boldsymbol{u}$ satisfies the Lippmann–Schwinger equation (3.2) and completes the proof. ∎

The Lippmann–Schwinger equation (3.2) can be written in the operator form

$$(I + K_\omega)\boldsymbol{u} = -H_\omega\boldsymbol{f}, \tag{3.6}$$

where the operators H_ω and K_ω are defined by

$$(H_\omega\boldsymbol{f})(x) = \int_D \boldsymbol{G}(x,y,\omega)\boldsymbol{f}(y)dy, \quad x \in D, \tag{3.7}$$

$$(K_\omega\boldsymbol{u})(x) = \int_D \boldsymbol{G}(x,y,\omega)\boldsymbol{M}(y)\boldsymbol{u}(y)dy, \quad x \in D. \tag{3.8}$$

Lemma 3.2. *Assume that* $p \geq 2$, $\frac{1}{p} + \frac{1}{p'} = 1$, $0 < \varepsilon < \frac{2}{p}$, *and* $\boldsymbol{M}$ *satisfies Assumption* 1.3. *Then the operators* $H_\omega : H_0^{-s}(D)^2 \to H^s(D)^2$ *and* $K_\omega : H^{\varepsilon,p}(D)^2 \to H^{\varepsilon,p}(D)^2$ *are bounded for* $s \in (0,1)$. *Moreover,* $K_\omega : H^{\varepsilon,p}(D)^2 \to H^{\varepsilon,p}(D)^2$ *is compact.*

Proof. We study the asymptotic expansion of the Green tensor $\boldsymbol{G}(x,y,\omega)$ when $|x-y| \to 0$. Recall the Green tensor,

$$\boldsymbol{G}(x,y,\omega) = \frac{1}{\mu}\Phi(x,y,\kappa_{\rm s})\boldsymbol{I} + \frac{1}{\omega^2}\nabla_x\nabla_x^\top(\Phi(x,y,\kappa_{\rm s}) - \Phi(x,y,\kappa_{\rm p})),$$

and the recurrence relation for the Hankel function of the first kind [32, (5.6.3)],

$$\frac{d}{dt}[t^{-n}H_n^{(1)}(t)] = -t^{-n}H_{n+1}^{(1)}(t).$$

A direct calculation shows for $i,j = 1,2$ that

$$\begin{aligned}&\partial^2_{x_ix_j}[\Phi(x,y,\kappa_{\rm s}) - \Phi(x,y,\kappa_{\rm p})]\\ &= -\frac{\rm i}{4}\frac{1}{|x-y|}\Gamma_1(x-y,\omega)\delta_{ij} + \frac{\rm i}{4}\frac{(x_i-y_i)(x_j-y_j)}{|x-y|^2}\Gamma_1(x-y,\omega),\end{aligned} \tag{3.9}$$

where δ_{ij} is the Kronecker delta function. Substituting (2.10)–(2.11) into (3.9) gives

$$\begin{aligned}&\partial^2_{x_ix_j}[\Phi(x,y,\kappa_{\rm s}) - \Phi(x,y,\kappa_{\rm p})]\\ &= \frac{1}{4\pi}\left(\kappa_{\rm s}^2\ln\frac{\kappa_{\rm s}|x-y|}{2} - \kappa_{\rm p}^2\ln\frac{\kappa_{\rm p}|x-y|}{2}\right)\delta_{ij} + O(1).\end{aligned} \tag{3.10}$$

Comparing (3.10) with (2.6), we conclude that the singularity of $\nabla_x\nabla_x^\top(\Phi(x,y,\kappa_{\rm s}) - \Phi(x,y,\kappa_{\rm p}))$ is not exceeding the singularity of $\Phi(x,y,\kappa_{\rm s})\boldsymbol{I}$ when $|x-y| \to 0$. It follows from Lemma 2.1 that $H_\omega : H_0^{-s}(D)^2 \to H^s(D)^2$ is bounded for $s \in (0,1)$.

For $\boldsymbol{u} \in H^{\varepsilon,p}(D)^2$ and $\boldsymbol{M}_{ij} \in C_0^1(D) \subset H_0^{-\varepsilon,p_1'}(D)$, by Lemma 2.1, we obtain that $\boldsymbol{M}_{ij}\boldsymbol{u}$ is a well-defined element of $H_0^{-\varepsilon,p'}(D)^2$ and

$$\|\boldsymbol{M}_{ij}\boldsymbol{u}\|_{H_0^{-\varepsilon,p'}(D)^2} \lesssim \|\boldsymbol{M}_{ij}\|_{H_0^{-\varepsilon,p_1'}(D)}\|\boldsymbol{u}\|_{H^{\varepsilon,p}(D)^2}. \tag{3.11}$$

For some fixed $\varepsilon \in (0,\frac{2}{p})$, we define $\tilde{\delta} = \frac{1}{p} - \frac{\varepsilon}{2} \in (0,1)$ and $s = 1-\tilde{\delta} \in (0,1)$. It is clear to note that $\frac{1}{2} - \frac{s}{2} < \frac{1}{p} - \frac{\varepsilon}{2}$. The Sobolev embedding theorem implies that $H^s(D)$ is embedded compactly into $H^{\varepsilon,p}(D)$ and $H_0^{-\varepsilon,p'}(D)$ is embedded compactly into $H_0^{-s}(D)$. Noting that $K_\omega\boldsymbol{u} = H_\omega(\boldsymbol{M}\boldsymbol{u})$ and $\boldsymbol{M}\boldsymbol{u} \in H_0^{-\varepsilon,p'}(D)^2$, which is embedded compactly into $H_0^{-s}(D)^2$, and that $H_\omega : H_0^{-s}(D)^2 \to H^s(D)^2$ is bounded, we claim from (3.11) that $K_\omega : H^{\varepsilon,p}(D)^2 \to H^{\varepsilon,p}(D)^2$ is bounded and compact. ∎

Now we present the existence of a unique solution of the direct scattering problem (1.1)–(1.2).

Theorem 3.3. *Let* $\boldsymbol{f} \in H_0^{-\varepsilon,p'}(D)^2$ *with* $0 < \varepsilon < \frac{2}{p}$ *and* $\boldsymbol{M}$ *satisfy Assumption* 1.3. *Then the Lippmann–Schwinger equation* (3.6) *has a unique solution* $\boldsymbol{u} \in H_{\rm loc}^{\varepsilon,p}(\mathbb{R}^2)^2$, *which implies that the scattering problem* (1.1)–(1.2) *has a unique solution* $\boldsymbol{u} \in H_{\rm loc}^{\varepsilon,p}(\mathbb{R}^2)^2$ *which satisfies the stability estimate*

$$\|\boldsymbol{u}\|_{H_{\rm loc}^{\varepsilon,p}(\mathbb{R}^2)^2} \lesssim \|\boldsymbol{f}\|_{H_0^{-\varepsilon,p'}(\mathbb{R}^2)^2}.$$

Proof. For the Lippmann–Schwinger equation $(I+K_\omega)\boldsymbol{u} = -H_\omega \boldsymbol{f}$, by Lemma 3.2, we obtain that $H_\omega \boldsymbol{f} \in H^{\varepsilon,p}(D)^2$ for $\boldsymbol{f} \in H_0^{-\varepsilon,p'}(D)^2$ and $I+K_\omega : H^{\varepsilon,p}(D)^2 \to H^{\varepsilon,p}(D)^2$ is a Fredholm operator. Thus, by the Fredholm alternative, it suffices to show that $(I+K_\omega)\boldsymbol{u} = 0$ has only the trivial solution $\boldsymbol{u} = 0$.

For $(I+K_\omega)\boldsymbol{u} = 0$, we have

$$\boldsymbol{u}(x) = -\int_D \boldsymbol{G}(x,y,\omega)\boldsymbol{M}(y)\boldsymbol{u}(y)dy, \quad x \in \mathbb{R}^2,$$

which implies that $\boldsymbol{u}$ is smooth in $\mathbb{R}^2 \setminus \overline{D}$ and

$$\hat{\boldsymbol{u}}(\xi) = -\widehat{\boldsymbol{G}}(\xi)\widehat{\boldsymbol{M}\boldsymbol{u}}(\xi). \tag{3.12}$$

Multiplying $(4\pi^2\mu|\xi|^2 - \omega^2)\boldsymbol{I} + 4\pi^2(\lambda+\mu)\xi\cdot\xi^\top$ on both sides of (3.12) and using (3.4) gives

$$\left[4\pi^2\mu|\xi|^2 + 4\pi^2(\lambda+\mu)\xi\cdot\xi^\top - \omega^2\right]\hat{\boldsymbol{u}}(\xi) = -\widehat{\boldsymbol{M}\boldsymbol{u}}(\xi).$$

Taking the inverse Fourier transform of the above equation yields

$$\mu\Delta\boldsymbol{u} + (\lambda+\mu)\nabla\nabla^\top\cdot\boldsymbol{u} + \omega^2\boldsymbol{u} = \boldsymbol{M}\boldsymbol{u} \quad \text{in } \mathbb{R}^2. \tag{3.13}$$

By the Helmholtz decomposition, there exists two scalar potential functions ψ_1 and ψ_2 such that

$$\boldsymbol{u} = \nabla\psi_1 + \mathbf{curl}\psi_2 = (\partial_{x_1}\psi_1, \partial_{x_2}\psi_1)^\top + (\partial_{x_2}\psi_2, -\partial_{x_1}\psi_2)^\top. \tag{3.14}$$

Substituting (3.14) into (3.13) gives that

$$\nabla[(\lambda+2\mu)\Delta\psi_1 + \omega^2\psi_1] + \mathbf{curl}[\mu\Delta\psi_2 + \omega^2\psi_2] = \boldsymbol{M}\nabla\psi_1 + \boldsymbol{M}\mathbf{curl}\psi_2 \quad \text{in } \mathbb{R}^2,$$

which implies that

$$\begin{aligned}(\lambda+2\mu)\Delta(\nabla\psi_1) + \omega^2(\nabla\psi_1) &= \boldsymbol{M}\nabla\psi_1,\\ \mu\Delta(\mathbf{curl}\psi_2) + \omega^2(\mathbf{curl}\psi_2) &= \boldsymbol{M}\mathbf{curl}\psi_2.\end{aligned}$$

Letting $\boldsymbol{u}_{\rm p} = \nabla\psi_1$ and $\boldsymbol{u}_{\rm s} = \mathbf{curl}\psi_2$, we obtain that

$$\begin{cases}\Delta\boldsymbol{u}_{\rm p} + \kappa_{\rm p}^2\boldsymbol{u}_{\rm p} = \frac{1}{\lambda+2\mu}\boldsymbol{M}\boldsymbol{u}_{\rm p} \quad \text{in } \mathbb{R}^2,\\ \lim\limits_{r\to\infty} r^{\frac12}\left(\partial_r\boldsymbol{u}_{\rm p} - \mathrm{i}\kappa_{\rm p}\boldsymbol{u}_{\rm p}\right) = 0\end{cases} \tag{3.15}$$

and

$$\begin{cases}\Delta\boldsymbol{u}_{\rm s} + \kappa_{\rm s}^2\boldsymbol{u}_{\rm s} = \frac{1}{\mu}\boldsymbol{M}\boldsymbol{u}_{\rm s} \quad \text{in } \mathbb{R}^2,\\ \lim\limits_{r\to\infty} r^{\frac12}\left(\partial_r\boldsymbol{u}_{\rm s} - \mathrm{i}\kappa_{\rm s}\boldsymbol{u}_{\rm s}\right) = 0.\end{cases} \tag{3.16}$$

Since $\mathrm{supp}\boldsymbol{M}_{ij} \subset D$, it follows from (3.15)–(3.16) that $\boldsymbol{u}_{\rm p}$ and $\boldsymbol{u}_{\rm s}$ satisfy the homogeneous Helmholtz equation in $\mathbb{R}^2 \setminus \overline{D}$ and the Sommerfeld radiation condition. Hence, $\boldsymbol{u}_{\rm p}$ and $\boldsymbol{u}_{\rm s}$ admit the following asymptotic expansions:

$$\boldsymbol{u}_{\rm p}(x) = \frac{e^{\mathrm{i}\kappa_{\rm p}|x|}}{4\pi|x|^{\frac12}}\boldsymbol{u}_{{\rm p},\infty}(\hat{x}) + o(|x|^{\frac12}), \quad \boldsymbol{u}_{\rm s}(x) = \frac{e^{\mathrm{i}\kappa_{\rm s}|x|}}{4\pi|x|^{\frac12}}\boldsymbol{u}_{{\rm s},\infty}(\hat{x}) + o(|x|^{\frac12}). \tag{3.17}$$

Noting that $\boldsymbol{u}_{\rm p}$ satisfies the Sommerfeld radiation condition, when $r \to \infty$, we have

$$\int_{\partial B_r} |\partial_r \boldsymbol{u}_{\rm p} - {\rm i}\kappa_{\rm p}\boldsymbol{u}_{\rm p}|^2\, ds = \int_{\partial B_r} (|\partial_r \boldsymbol{u}_{\rm p}|^2 + \kappa_{\rm p}^2 |\boldsymbol{u}_{\rm p}|^2) ds + 2\kappa_{\rm p}{\rm Im}\int_{\partial B_r} \boldsymbol{u}_{\rm p}\partial_\nu \overline{\boldsymbol{u}}_{\rm p} ds \to 0.$$

Combining the second Green theorem and (3.15)–(3.16), we get

$$\begin{aligned}&\int_{\partial B_r} \boldsymbol{u}_{\rm p}\partial_\nu \overline{\boldsymbol{u}}_{\rm p} ds = \int_{B_r} |\nabla \boldsymbol{u}_{\rm p}|^2 dx - \kappa_{\rm p}^2 \int_{B_r} |\boldsymbol{u}_{\rm p}|^2 dx \\ &+ \frac{1}{\lambda+2\mu}\int_{B_r} \left(\boldsymbol{M}_{11}|u_{{\rm p},1}|^2 + M_{22}|u_{{\rm p},2}|^2 + M_{12}u_{{\rm p},1}\overline{u_{{\rm p},2}} + M_{21}\overline{u_{{\rm p},1}}u_{{\rm p},2}\right) dx,\end{aligned}$$

where $u_{{\rm p},1}$ and $u_{{\rm p},2}$ are the components of $\boldsymbol{u}_{\rm p}$. Since M is real-valued and symmetric, taking the imaginary part of the above equation leads to ${\rm Im}\int_{\partial B_r} \boldsymbol{u}_{\rm p}\partial_\nu \overline{\boldsymbol{u}}_{\rm p} ds = 0$, which yields $\lim_{r\to\infty}\int_{\partial B_r} |\boldsymbol{u}_{\rm p}|^2 dx = 0$. Using (3.17), we obtain $\int_{\partial B_1} |\boldsymbol{u}_{{\rm p},\infty}|^2 ds = 0$, which implies $\boldsymbol{u}_{{\rm p},\infty} = 0$, so $\boldsymbol{u}_{\rm p}(x) = 0$ in $\mathbb{R}^2 \setminus \overline{D}$. Similarly, we can obtain $\boldsymbol{u}_{\rm s} = 0$ in $\mathbb{R}^2 \setminus \overline{D}$. Thus, we have $\boldsymbol{u} = 0$ in $\mathbb{R}^2 \setminus \overline{D}$. Since $\boldsymbol{M}_{ij} \in C_0^1(\overline{D})$, it follows from the unique continuation (e.g., [4]) that $\boldsymbol{u} = 0$ in $\mathbb{R}^2$, which shows that $I + K_\omega$ is injective and completes the proof. ∎

4. Born approximation. As shown in the previous section, the direct scattering problem is equivalent to the Lippmann–Schwinger equation

$$\boldsymbol{u}(x) + \int_D \boldsymbol{G}(x,y,\omega)\boldsymbol{M}(y)\boldsymbol{u}(y)dy = -\int_D \boldsymbol{G}(x,y,\omega)\boldsymbol{f}(y)dy, \quad x \in \mathbb{R}^2.$$

Consider the Born sequence of the Lippmann–Schwinger equation

$$\boldsymbol{u}_n(x) := (-K_\omega \boldsymbol{u}_{n-1})(x), \quad n = 1, 2, \ldots, \tag{4.1}$$

where the initial guess is given by

$$\boldsymbol{u}_0(x) := (-H_\omega \boldsymbol{f})(x),$$

which is called the Born approximation to the solution of the Lippmann–Schwinger equation. Here, K_ω and H_ω are operators given by (3.7) and (3.8), respectively.

We aim to show that for sufficiently large ω and $x \in U$, the Born series $\sum_{n=0}^\infty \boldsymbol{u}_n(x)$ converges to the solution $\boldsymbol{u}(x)$ and the higher order terms decay in an appropriate way.

Lemma 4.1. *For any $1 \le p \le 2 \le r \le \infty$, $s \in (0,1)$, and $\omega \ge 1$, the following estimates hold:*

$$\begin{aligned}\|H_\omega\|_{H_0^{-s,p}(D)^2 \to H^{s,r}(D)^2} &\lesssim \omega^{-1+2[s+(\frac{1}{p}-\frac{1}{r})]},\\ \|K_\omega\|_{H^{s,2p}(D)^2 \to H^{s,2p}(D)^2} &\lesssim \omega^{-1+2[s+(1-\frac{1}{p})]},\\ \|K_\omega\|_{H^{s,2p}(D)^2 \to L^\infty(U)^2} &\lesssim \omega^{1+2s-\frac{1}{p}},\end{aligned}$$

where the constant $c = c(\hat\omega)$ in the inequalities is finite almost surely.

The proof of Lemma 4.1 can be found in [33, Lemma 5]. By Lemma 4.1, we have for large enough ω that

$$(I + K_\omega)\sum_{n=0}^{N} \boldsymbol{u}_n = \boldsymbol{u}_0 + (-1)^N K_\omega^{N+1}\boldsymbol{u}_0 \to \boldsymbol{u}_0 \quad \text{as } N \to \infty. \tag{4.2}$$

Since $(I+K_\omega)^{-1}\boldsymbol{u}_0=\boldsymbol{u}$, taking the inverse of the operator $I+K_\omega$ in (4.2) leads to

$$\boldsymbol{u}(x,\omega)=\boldsymbol{u}_0(x,\omega)+\boldsymbol{u}_1(x,\omega)+\boldsymbol{b}(x,\omega), \tag{4.3}$$

where $\boldsymbol{b}(x,\omega):=\sum_{n=2}^\infty \boldsymbol{u}_n(x,\omega)$. With the convergence of the Born approximation (4.3), we can analyze each term in the Born approximation. For the leading term $\boldsymbol{u}_0$, we have the following result [30, Theorem 4.6].

THEOREM 4.2. *Let $\boldsymbol{f}$ satisfy Assumption* 1.1. *For all $x\in U$, it holds almost surely that*

$$\lim_{Q\to\infty}\frac{1}{Q-1}\int_1^Q \omega^{m+1}|\boldsymbol{u}_0(x,\omega)|^2 d\omega = a\int_{\mathbb{R}^2}\frac{1}{|x-y|}\phi(y)dy,$$

where a is a constant given in Theorem 1.4.

Now we analyze the term $\boldsymbol{b}(x,\omega)$. For $n\ge 2$, by Lemma 4.1, we get

$$\begin{aligned}
&\|\boldsymbol{u}_n(x,\omega)\|_{L^\infty(U)^2} = \|K_\omega^n\boldsymbol{u}_0\|_{L^\infty(U)^2}\\
&\le \|K_\omega\|_{H^{\varepsilon,p}(D)^2\to L^\infty(U)^2}\|K_\omega\|^{n-1}_{H^{\varepsilon,p}(D)^2\to H^{\varepsilon,p}(D)^2}\\
&\quad\times \|H_\omega\|_{H_0^{-\varepsilon,p'}(D)^2\to H^{\varepsilon,p}(D)^2}\|\boldsymbol{f}\|_{H_0^{-\varepsilon,p'}(D)^2}\\
&\lesssim \omega^{1+2\varepsilon-\frac{2}{p}}\omega^{(n-1)[-1+2(\varepsilon+1-\frac{2}{p})]}\omega^{-1+2[\varepsilon+\frac{1}{p'}-\frac{1}{p}]}\\
&\lesssim \omega^{4\varepsilon+2-\frac{6}{p}}\omega^{(n-1)[-1+2(\varepsilon+\frac{1}{p'}-\frac{1}{p})]},
\end{aligned}$$

which gives

$$\sum_{n=2}^\infty \|\boldsymbol{u}_n\|_{L^\infty(U)^2}\lesssim \omega^{4\varepsilon+2-\frac{6}{p}}\frac{\omega^{-1+2(\varepsilon+1-\frac{2}{p})}}{1-\omega^{-1+2(\varepsilon+1-\frac{2}{p})}}\lesssim \omega^{6\varepsilon+3-\frac{10}{p}}.$$

Since $0<\varepsilon<\frac{2}{p}$ and $p>2$, we can choose suitable ε,p such that $\varepsilon'=6\varepsilon+5-\frac{10}{p}$ is small enough and

$$\sum_{n=2}^\infty ||\boldsymbol{u}_n||_{L^\infty(U)^2}\lesssim \omega^{-2+\varepsilon'}. \tag{4.4}$$

Hence, when $Q\to\infty$,

$$\frac{1}{Q-1}\int_1^Q \omega^{m+1}|\boldsymbol{b}(x,\omega)|^2 d\omega \lesssim \frac{1}{Q-1}\int_1^Q \omega^\alpha d\omega = \frac{1}{\alpha+1}\frac{Q^{\alpha+1}-1}{Q-1}\to 0, \tag{4.5}$$

where $\alpha=m+2\varepsilon'-3$. Noting that $m\in[2,5/2)$, we have $\alpha\in(-1,0)$, which is used in (4.5).

5. The analysis of $\boldsymbol{u}_1(x,\omega)$.

In this section, we consider the term $\boldsymbol{u}_1(x,\omega)$ in the Born series (4.1), which is given by

$$\boldsymbol{u}_1(x,\omega)=\int_D\int_D \boldsymbol{G}(x,y,\omega)\boldsymbol{M}(y)\boldsymbol{G}(y,z,\omega)\boldsymbol{f}(z)dydz,\quad x\in U. \tag{5.1}$$

It turns out the term $\boldsymbol{u}_1(x,\omega)$ is very difficult to analyze. Fortunately, after tedious calculations, we find out that the contribution of $\boldsymbol{u}_1$ can be ignored. We present the main result of this section.

THEOREM 5.1. *Let $\boldsymbol{f}$, U, and $\boldsymbol{M}$ satisfy Assumptions* 1.1, 1.2, *and* 1.3, *respectively. Then for $x \in U$, it holds almost surely that*

$$\lim_{Q\to\infty} \frac{1}{Q-1}\int_1^Q \omega^{m+1}|\boldsymbol{u}_1(x,\omega)|^2 d\omega = 0. \tag{5.2}$$

The proof of Theorem 5.1 requires the asymptotic expansions of Green's tensor and microlocal analysis of integral operators. Since the whole proof is lengthy, we split it into several parts, and which consist of Lemmas 5.2, 5.3, and 5.6.

Recalling the Green tensor in (3.1), a direct computation shows

$$\begin{aligned}\boldsymbol{G}(x,y,\omega) = &\left(\frac{\mathrm{i}}{4\mu}H_0^{(1)}(\kappa_{\mathrm{s}}|x-y|) - \frac{\mathrm{i}}{4\omega^2}\frac{1}{|x-y|}\Gamma_1(x-y,\omega)\right)\boldsymbol{I}\\ &+ \frac{\mathrm{i}}{4\omega^2}\frac{1}{|x-y|^2}\Gamma_2(x-y,\omega)(x-y)\cdot(x-y)^\top,\end{aligned} \tag{5.3}$$

where $x-y = (x_1-y_1, x_2-y_2)^\top$ and Γ_1, Γ_2 are given in (2.10), (2.11). Noting the definition of $H_{n,N}^{(1)}$ in (2.14), we define the notation $\Theta_n(z,\omega) := \kappa_{\mathrm{s}}^n H_{n,0}^{(1)}(\kappa_{\mathrm{s}}|z|) - \kappa_{\mathrm{p}}^n H_{n,0}^{(1)}(\kappa_{\mathrm{p}}|z|)$,

$$\begin{aligned}\boldsymbol{G}_0(x,y,\omega) = &\left(\frac{\mathrm{i}}{4\mu}H_{0,0}^{(1)}(\kappa_{\mathrm{s}}|x-y|) - \frac{\mathrm{i}}{4\omega^2}\frac{1}{|x-y|}\Theta_1(x-y,\omega)\right)\boldsymbol{I}\\ &+ \frac{\mathrm{i}}{4\omega^2}\frac{1}{|x-y|^2}\Theta_2(x-y,\omega)(x-y)\cdot(x-y)^\top,\end{aligned} \tag{5.4}$$

and

$$\boldsymbol{u}_{1,l}(x,\omega) := \int_D\int_D \boldsymbol{G}_0(x,y,\omega)\boldsymbol{M}(y)\boldsymbol{G}(y,z,\omega)\boldsymbol{f}(z)dydz, \quad x\in U. \tag{5.5}$$

Now we estimate the order of the difference $\boldsymbol{u}_1 - \boldsymbol{u}_{1,l}$ with respect to the angular frequency ω.

LEMMA 5.2. *For $\boldsymbol{u}_1(x,\omega)$ and $\boldsymbol{u}_{1,l}(x,\omega)$ given by* (5.1) *and* (5.5), *respectively, we have*

$$|\boldsymbol{u}_1(x,\omega) - \boldsymbol{u}_{1,l}(x,\omega)| \lesssim \omega^{-\frac{5}{2}+\varepsilon_l}, \quad x\in U, \tag{5.6}$$

where ε_l is a sufficient small positive number.

Proof. A simple calculation yields

$$\begin{aligned}&|\boldsymbol{u}_1(x,\omega) - \boldsymbol{u}_{1,l}(x,\omega)|\\ &= \left|\int_D \left(\boldsymbol{G}(x,y,\omega) - \boldsymbol{G}_0(x,y,\omega)\right)\boldsymbol{M}(y)\int_D \boldsymbol{G}(y,z,\omega)\boldsymbol{f}(z)dzdy\right|\\ &\lesssim \|\boldsymbol{G}(x,y,\omega) - \boldsymbol{G}_0(x,y,\omega)\|_{L^{p'}(D)^{2\times 2}}\|H_\omega \boldsymbol{f}\|_{L^p(D)^2}.\end{aligned}$$

Since $x\in U$, $y\in D$, there exists $c_1, c_2 > 0$ such that $c_1 < |x-y| < c_2$. By (2.15), we have

$$\|\Gamma_{n,0}(\kappa|x-\cdot|)\|_{L^{p'}(D)} \lesssim \kappa^{-\frac{3}{2}}. \tag{5.7}$$

A direct computation shows that $\nabla\Gamma_{n,0}(\kappa|x-\cdot|) \lesssim \kappa^{-\frac{1}{2}}$. Hence

$$\|\nabla\Gamma_{n,0}(\kappa|x-\cdot|)\|_{L^{p'}(D)} \lesssim \kappa^{-\frac{1}{2}}. \tag{5.8}$$

By (5.7) and (5.8), we get

$$\|\Gamma_{n,0}(\kappa|x-\cdot|)\|_{H^{\varepsilon,p'}(D)} \lesssim \kappa^{-\frac{3}{2}+\varepsilon}.$$

Therefore,

$$\|\boldsymbol{G}(x,\cdot,\omega)-\boldsymbol{G}_0(x,\cdot,\omega)\|_{H^{\varepsilon,p'}(D)^{2\times 2}} \lesssim \omega^{-\frac{3}{2}+\varepsilon}. \tag{5.9}$$

It follows from Lemma 4.1 that we obtain

$$\|H_\omega \boldsymbol{f}\|_{H^{\varepsilon,p}(D)^2} \le \|H_\omega\|_{H_0^{-\varepsilon,p'}(D)^2\to H^{\varepsilon,p}(D)^2}\|\boldsymbol{f}\|_{H_0^{-\varepsilon,p'}(D)^2} \lesssim \omega^{-1+2(\varepsilon+1-\frac{2}{p})}, \tag{5.10}$$

where we use the fact that $\|\boldsymbol{f}\|_{H_0^{-\varepsilon,p'}(D)^2}$ is bounded almost surely. Denoting $\varepsilon_l = 3\varepsilon+2(1-\frac{2}{p})$ which can be sufficient small for suitably chosen ε and p due to $p\ge 2$ and $0<\varepsilon<\frac{2}{p}$, we conclude the result from (5.9) and (5.10). ◻

In order to analyze the term $\boldsymbol{u}_{1,l}$, we replace the Green tensor $\boldsymbol{G}(y,z,\omega)$ in $\boldsymbol{u}_{1,l}$ by $\boldsymbol{G}_0(y,z,\omega)$ and define

$$\boldsymbol{u}_{1,r}(x,\omega) = \int_D\int_D \boldsymbol{G}_0(x,y,\omega)\boldsymbol{M}(y)\boldsymbol{G}_0(y,z,\omega)\boldsymbol{f}(z)dydz, \quad x\in U. \tag{5.11}$$

LEMMA 5.3. *For $\boldsymbol{u}_{1,l}(x,\omega)$ and $\boldsymbol{u}_{1,r}(x,\omega)$ given by* (5.5) *and* (5.11)*, respectively, the estimate*

$$|\boldsymbol{u}_{1,l}(x,\omega)-\boldsymbol{u}_{1,r}(x,\omega)| \lesssim \omega^{-2+\varepsilon}, \quad x\in U, \tag{5.12}$$

holds for any $\varepsilon\in(0,\frac{1}{5})$.

Proof. By (5.5) and (5.11), a direct calculation shows that

$$\begin{aligned}\boldsymbol{u}_{1,l}(x,\omega)-\boldsymbol{u}_{1,r}(x,\omega) &= \int_D\int_D \boldsymbol{G}_0(x,y,\omega)\boldsymbol{M}(y)\left(\boldsymbol{G}(y,z,\omega)-\boldsymbol{G}_0(y,z,\omega)\right)\boldsymbol{f}(z)dydz\\ &= \Bigg(\sum_{j,k,l=1}^{2} I_{jkl}^{(1)}, \sum_{j,k,l=1}^{2} I_{jkl}^{(2)}\Bigg)^\top,\end{aligned}$$

where

$$I_{jkl}^{(i)} := \int_D\int_D \boldsymbol{G}_{0,ij}(x,y,\omega)\boldsymbol{M}_{jk}(y)\left(\boldsymbol{G}_{kl}(y,z,\omega)-\boldsymbol{G}_{0,kl}(y,z,\omega)\right)f_l(z)dydz$$

for $i,j,k,l=1,2$. Here, $\boldsymbol{G}_{ij}$ and $\boldsymbol{G}_{0,ij}$ represent the elements of the matrix $\boldsymbol{G}$ and $\boldsymbol{G}_0$, respectively.

Now we only focus on the analysis of the term $I_{111}^{(1)}$ and show the details; the other terms can be analyzed in a similar way. In the dual sense, we have

$$\begin{aligned}I_{111}^{(1)} = \langle &\boldsymbol{G}_{11}(y,z,\omega)-\boldsymbol{G}_{0,11}(y,z,\omega),\\ &\boldsymbol{G}_{0,11}(x,y,\omega)\boldsymbol{M}_{11}(y)f_1(z)\rangle_{(H^{\varepsilon,\tilde{p}}(D\times D),H_0^{-\varepsilon,\tilde{p}'}(D\times D))}.\end{aligned} \tag{5.13}$$

By (5.3) and (5.4), we can split $\boldsymbol{G}_{11}(y,z,\omega)-\boldsymbol{G}_{0,11}(y,z,\omega)$ into three terms:

$$\boldsymbol{G}_{11}(y,z,\omega)-\boldsymbol{G}_{0,11}(y,z,\omega) = g_0(y-z,\omega)+g_1(y-z,\omega)+g_2(y-z,\omega),$$

where

$$g_0(y-z,\omega)=\frac{\mathrm{i}}{4\mu}\Gamma_{0,0}(\kappa_{\mathrm{s}}|y-z|),$$
$$g_1(y-z,\omega)=-\frac{\mathrm{i}}{4\omega^2}\frac{1}{|y-z|}[\kappa_{\mathrm{s}}\Gamma_{1,0}(\kappa_{\mathrm{s}}|y-z|)-\kappa_{\mathrm{p}}\Gamma_{1,0}(\kappa_{\mathrm{p}}|y-z|)],$$
$$g_2(y-z,\omega)=\frac{\mathrm{i}}{4\omega^2}\frac{(y_1-z_1)^2}{|y-z|^2}[\kappa_{\mathrm{s}}^2\Gamma_{2,0}(\kappa_{\mathrm{s}}|y-z|)-\kappa_{\mathrm{p}}^2\Gamma_{2,0}(\kappa_{\mathrm{p}}|y-z|)].$$

Note $y,z\in D$ and D is a bounded domain. Next is to estimate the term $\|\boldsymbol{G}_{11}(y,z,\omega)-\boldsymbol{G}_{0,11}(y,z,\omega)\|_{H^{\varepsilon,\tilde{p}}(D\times D)}$, which requires estimating $\|g_j(z,\omega)\|_{H^{\varepsilon,\tilde{p}}(B)}, j=0,1,2$ for some bounded domain containing the origin.

We analyze the three terms one by one. For large $\kappa_{\mathrm{s}}|z|$, it is easy to note from (2.15) that

$$|g_0(z,\omega)|\lesssim(\kappa_{\mathrm{s}}|z|)^{-\frac{3}{2}}. \tag{5.14}$$

For small $\kappa_{\mathrm{s}}|z|$, using (2.6) and (2.14) gives that

$$|g_0(z,\omega)|\lesssim(\kappa_{\mathrm{s}}|z|)^{-\frac{1}{2}}=(\kappa_{\mathrm{s}}|z|)^{-\frac{3}{2}}(\kappa_{\mathrm{s}}|z|)\lesssim(\kappa_{\mathrm{s}}|z|)^{-\frac{3}{2}}. \tag{5.15}$$

By (5.14) and (5.15), we obtain that

$$\|g_0(z,\omega)\|^{\tilde{p}}_{L^{\tilde{p}}(B)}\lesssim\int_B\omega^{-\frac{3}{2}\tilde{p}}|z|^{-\frac{3}{2}\tilde{p}}dz\lesssim\omega^{-\frac{3}{2}\tilde{p}}\int_0^R r^{1-\frac{3}{2}\tilde{p}}dr\lesssim\omega^{-\frac{3}{2}\tilde{p}} \tag{5.16}$$

holds for $\tilde{p}<\frac{4}{3}$, where $R=\max\{|z|,z\in B\}$. Since

$$\nabla g_0(z,\omega)=\frac{\mathrm{i}}{4\mu}\nabla\left(H_0^{(1)}(\kappa_{\mathrm{s}}|z|)-a_0^{(0)}\sqrt{\frac{1}{\kappa_{\mathrm{s}}|z|}}e^{\mathrm{i}(\kappa_{\mathrm{s}}|z|-\frac{\pi}{4})}\right)$$
$$=\frac{\mathrm{i}}{4\mu}\frac{z}{|z|}\left(-\kappa_{\mathrm{s}}H_1^{(1)}(\kappa_{\mathrm{s}}|z|)+\frac{1}{2}a_0^{(0)}\kappa_{\mathrm{s}}^{-\frac{1}{2}}|z|^{-\frac{3}{2}}e^{\mathrm{i}(\kappa_{\mathrm{s}}|z|-\frac{\pi}{4})}-\mathrm{i}a_0^{(0)}\kappa_{\mathrm{s}}^{\frac{1}{2}}|z|^{-\frac{1}{2}}e^{\mathrm{i}(\kappa_{\mathrm{s}}|z|-\frac{\pi}{4})}\right),$$

we have for large $\kappa_{\mathrm{s}}|z|$ that

$$|\nabla g_0(z,\omega)|\lesssim\kappa_{\mathrm{s}}^{\frac{1}{2}}|z|^{-\frac{1}{2}}+\kappa_{\mathrm{s}}^{-\frac{1}{2}}|z|^{-\frac{3}{2}}+\kappa_{\mathrm{s}}^{\frac{1}{2}}|z|^{-\frac{1}{2}}\lesssim\kappa_{\mathrm{s}}^{\frac{1}{2}}|z|^{-\frac{1}{2}}. \tag{5.17}$$

For small $\kappa_{\mathrm{s}}|z|$, we get

$$|\nabla g_0(z,\omega)|\lesssim\kappa_{\mathrm{s}}(\kappa_{\mathrm{s}}|z|)^{-1}+\kappa_{\mathrm{s}}^{-\frac{1}{2}}|z|^{-\frac{3}{2}}+\kappa_{\mathrm{s}}^{\frac{1}{2}}|z|^{-\frac{1}{2}}\lesssim\kappa_{\mathrm{s}}^{-\frac{1}{2}}|z|^{-\frac{3}{2}}. \tag{5.18}$$

By (5.17) and (5.18), we conclude for $\tilde{p}<\frac{4}{3}$ that

$$\|\nabla g_0(z,\omega)\|^{\tilde{p}}_{L^{\tilde{p}}(B)}\lesssim\int_B\omega^{\frac{1}{2}\tilde{p}}|z|^{-\frac{1}{2}\tilde{p}}dz+\int_B\omega^{-\frac{1}{2}\tilde{p}}|z|^{-\frac{3}{2}\tilde{p}}dz\lesssim\omega^{\frac{1}{2}\tilde{p}}. \tag{5.19}$$

Using (5.16) and (5.19), we have for $\tilde{p}<\frac{4}{3}$ that

$$||g_0(z,\omega)||^{\tilde{p}}_{H^{1,\tilde{p}}(B)}\lesssim\omega^{-\frac{3}{2}\tilde{p}}+\omega^{\frac{1}{2}\tilde{p}}\lesssim\omega^{\frac{1}{2}\tilde{p}}. \tag{5.20}$$

From (2.1), we have

$$H^{\varepsilon,\tilde{p}}(B):=[L^{\tilde{p}}(B),H^{1,\tilde{p}}(B)]_{\varepsilon}. \tag{5.21}$$

Using (5.16) and (5.20)–(5.21), we arrive at

$$\|g_0(z,\omega)\|_{H^{\varepsilon,\tilde p}(B)} \lesssim \|g_0(z,\omega)\|^{1-\varepsilon}_{L^{\tilde p}(B)} \|g_0(z,\omega)\|^{\varepsilon}_{H^{1,\tilde p}(B)} \lesssim \omega^{-\frac{3}{2}+2\varepsilon}. \tag{5.22}$$

Now we analyze the term $g_1(z,\omega)$ which is given by

$$g_1(z,\omega) = -\frac{\mathrm{i}}{4\omega^2}\frac{1}{|z|}[\kappa_{\mathrm{s}}\Gamma_{1,0}(\kappa_{\mathrm{s}}|z|) - \kappa_{\mathrm{p}}\Gamma_{1,0}(\kappa_{\mathrm{p}}|z|)].$$

For large $\omega|z|$, it follows from (2.15) that

$$\begin{aligned} |g_1(z,\omega)| &\lesssim \omega^{-2}|z|^{-1}[\omega(\omega|z|)^{-\frac{3}{2}}] \lesssim \omega^{-5/2}|z|^{-5/2} \\ &= \frac{(\omega|z|)^{-\frac{3}{2}}}{\omega|z|} \lesssim (\omega|z|)^{-\frac{3}{2}}. \end{aligned} \tag{5.23}$$

For small $\omega|z|$, by (2.5) and (2.8), we have

$$|g_1(z,\omega)| \lesssim (\omega|z|)^{-\frac{3}{2}}. \tag{5.24}$$

Combining (5.23) and (5.24) implies for $\tilde p < \frac{4}{3}$ that

$$\|g_1(z,\omega)\|^{\tilde p}_{L^{\tilde p}(B)} \lesssim \int_B \omega^{-\frac{3}{2}\tilde p}|z|^{-\frac{3}{2}\tilde p}dz \lesssim \omega^{-\frac{3}{2}\tilde p}\int_0^R r^{1-\frac{3}{2}\tilde p}dr \lesssim \omega^{-\frac{3}{2}\tilde p}.$$

For convenience, we split g_1 into two parts by $g_1(z,\omega) = g_{11}(z,\omega) + g_{12}(z,\omega)$ with

$$\begin{aligned} g_{11}(z,\omega) &= -\frac{\mathrm{i}}{4\omega^2}\frac{1}{|z|}\Gamma_1(z,\omega), \\ g_{12}(z,\omega) &= \frac{\mathrm{i}}{4\omega^2}\frac{1}{|z|}\Theta_1(z,\omega) = -\frac{\mathrm{i}}{4}a_0^{(1)}e^{-\frac{3}{4}\pi \mathrm{i}}(c_{\mathrm{p}}^{\frac{1}{2}}e^{\mathrm{i}\kappa_{\mathrm{p}}|z|} - c_{\mathrm{s}}^{\frac{1}{2}}e^{\mathrm{i}\kappa_{\mathrm{s}}|z|})\omega^{-\frac{3}{2}}|z|^{-\frac{3}{2}}. \end{aligned}$$

For large $\omega|z|$, by (2.7), we have

$$|g_{11}(x,\omega)| \lesssim \omega^{-\frac{3}{2}}|z|^{-\frac{3}{2}}. \tag{5.25}$$

For small $\omega|z|$, by (2.5), we have

$$|g_{11}(x,\omega)| \lesssim \left|\ln\frac{\omega|z|}{2}\right| \lesssim \omega^{-\frac{3}{2}}|z|^{-\frac{3}{2}}. \tag{5.26}$$

Combining (5.25) and (5.26) yields for $\tilde p < \frac{4}{3}$ that

$$||g_{11}(z,\omega)||^{\tilde p}_{L^{\tilde p}(B)} \lesssim \int_B \omega^{-\frac{3}{2}\tilde p}|z|^{-\frac{3}{2}\tilde p}dz \lesssim \omega^{-\frac{3}{2}\tilde p}\int_0^R r^{1-\frac{3}{2}\tilde p}dr \lesssim \omega^{-\frac{3}{2}\tilde p}. \tag{5.27}$$

For $\nabla g_{11}(z,\omega)$, we have

$$\nabla g_{11}(z,\omega) = \frac{\mathrm{i}}{4\omega^2}\frac{z}{|z|^2}\Gamma_2(z,\omega).$$

For large $\omega|z|$, (2.7) implies

$$|\nabla g_{11}(z,\omega)| \lesssim \omega^{-\frac{1}{2}}|z|^{-\frac{3}{2}}. \tag{5.28}$$

For small $\omega|z|$, (2.6) implies

$$|\nabla g_{11}(z,\omega)| \lesssim |z|^{-1} \lesssim \omega^{-\frac{1}{2}}|z|^{-\frac{3}{2}}. \tag{5.29}$$

Following (5.28) and (5.29), we get for $\tilde{p} < \frac{4}{3}$ that

$$\|\nabla g_{11}(z,\omega)\|^{\tilde{p}}_{L^{\tilde{p}}(B)} \lesssim \int_B \omega^{-\frac{1}{2}\tilde{p}}|z|^{-\frac{3}{2}\tilde{p}}dz \lesssim \omega^{-\frac{1}{2}\tilde{p}} \int_0^R r^{1-\frac{3}{2}\tilde{p}}dr \lesssim \omega^{-\frac{1}{2}\tilde{p}}. \tag{5.30}$$

Using (5.27) and (5.30), we have that

$$||g_{11}(z,\omega)||^{\tilde{p}}_{H^{1,\tilde{p}}(B)} \lesssim \omega^{-\frac{3}{2}\tilde{p}} + \omega^{-\frac{1}{2}\tilde{p}} \lesssim \omega^{-\frac{1}{2}\tilde{p}}, \tag{5.31}$$

which gives after combining (5.21) and (5.27) that

$$\|g_{11}(z,\omega)\|_{H^{\varepsilon,\tilde{p}}(B)} \lesssim \|g_{11}(z,\omega)\|^{1-\varepsilon}_{L^{\tilde{p}}(B)}\|g_{11}(z,\omega)\|^{\varepsilon}_{H^{1,\tilde{p}}(B)} \lesssim \omega^{-\frac{3}{2}+\varepsilon}. \tag{5.32}$$

Since

$$g_{12}(z,\omega) = -\frac{\mathrm{i}}{4}a_0^{(1)}e^{-\frac{3}{4}\pi\mathrm{i}}(c_{\mathrm{p}}^{\frac{1}{2}}e^{\mathrm{i}\kappa_{\mathrm{p}}|z|} - c_{\mathrm{s}}^{\frac{1}{2}}e^{\mathrm{i}\kappa_{\mathrm{s}}|z|})\omega^{-\frac{3}{2}}|z|^{-\frac{3}{2}},$$

it suffices to prove that $\omega^{-\frac{3}{2}}|z|^{-\frac{3}{2}} \in H^{\varepsilon,\tilde{p}}(B)$. By the Slobodeckij seminorm, we need to prove

$$|\omega^{-\frac{3}{2}}|z|^{-\frac{3}{2}}|^{\tilde{p}}_{\varepsilon,\tilde{p},B} = \omega^{-\frac{3}{2}\tilde{p}} \int_B\int_B \frac{||z|^{-\frac{3}{2}} - |z'|^{-\frac{3}{2}}|^{\tilde{p}}}{|z-z'|^{2+\varepsilon\tilde{p}}}dzdz' < \infty,$$

which requires showing the following two lemmas: one is the integrability criterion and the other is Young's inequality for convolutions [2, Theorem 2.24].

Lemma 5.4. *For the n-dimensional space, we have*

$$\int_{|x|\le 1} \frac{1}{|x|^\rho}dx < \infty \quad \text{if and only if} \quad \rho < n.$$

This lemma is fundamental and can be easily proved by using the polar coordinates.

Lemma 5.5. *Let $s_1, s_2, s_3 \ge 1$ and suppose that $\frac{1}{s_1} + \frac{1}{s_2} + \frac{1}{s_3} = 2$. It holds that*

$$\left|\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} h_1(x)h_2(x-y)h_3(y)dxdy\right| \le \|h_1\|_{s_1}\|h_2\|_{s_2}\|h_3\|_{s_3}$$

for any $h_1 \in L^{s_1}(\mathbb{R}^n)$, $h_2 \in L^{s_2}(\mathbb{R}^n)$, $h_3 \in L^{s_3}(\mathbb{R}^n)$.

Since

$$\begin{aligned} ||z|^{-\frac{3}{2}} - |z'|^{-\frac{3}{2}}| &= \left|\frac{(|z'|^{\frac{1}{2}} - |z|^{\frac{1}{2}})(|z'| + |z'|^{\frac{1}{2}}|z|^{\frac{1}{2}} + |z|)}{|z|^{\frac{3}{2}}|z'|^{\frac{3}{2}}}\right| \\ &\le \left|\frac{(|z'|-|z|)(|z'|^{\frac{1}{2}} + |z|^{\frac{1}{2}})^2}{|z|^{\frac{3}{2}}|z'|^{\frac{3}{2}}(|z'|^{\frac{1}{2}} + |z|^{\frac{1}{2}})}\right| \le \frac{|z'-z|(|z'|^{\frac{1}{2}} + |z|^{\frac{1}{2}})}{|z|^{\frac{3}{2}}|z'|^{\frac{3}{2}}}, \end{aligned}$$

we have

$$\int_B\int_B \frac{||z|^{-\frac32}-|z'|^{-\frac32}|^{\tilde p}}{|z-z'|^{2+\varepsilon\tilde p}}dzdz' \le \int_B\int_B \frac{(|z'|^{\frac12}+|z|^{\frac12})^{\tilde p}}{|z|^{\frac32\tilde p}|z'|^{\frac32\tilde p}|z-z'|^{2+\varepsilon\tilde p-\tilde p}}dzdz'$$
$$\lesssim \int_B\int_B \frac{1}{|z|^{\tilde p}|z'|^{\frac32\tilde p}|z-z'|^{2+\varepsilon\tilde p-\tilde p}}dzdz' + \int_B\int_B \frac{1}{|z|^{\frac32\tilde p}|z'|^{\tilde p}|z-z'|^{2+\varepsilon\tilde p-\tilde p}}dzdz'$$
$$:= I_1 + I_2.$$

We choose $\tilde p = \frac{10}{9}$, $\varepsilon = \frac15$, $s_1 = \frac{89}{50}$, $s_2 = \frac{59}{50}$, and $s_3 = \frac{5251}{3102}$, and then we have $\frac{1}{s_1}+\frac{1}{s_2}+\frac{1}{s_3} = 2$ and $\tilde p s_1 < 2$, $\frac32\tilde p s_2 < 2$, $(2+\varepsilon\tilde p-\tilde p)s_3 < 2$. A direct application of Lemmas 5.4 and 5.5 leads to

$$I_1 = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \chi_B(z)|z|^{-\tilde p}\chi_B(z')|z'|^{-\frac32\tilde p}\chi_{B_{2R}}(z-z')|z-z'|^{-(2+\varepsilon\tilde p-\tilde p)}dzdz'$$
$$\le \||z|^{-\tilde p}\|_{s_1}\||z|^{-\frac32\tilde p}\|_{s_2}\||z|^{-(2+\varepsilon\tilde p-\tilde p)}\|_{s_3} < \infty,$$

where B_{2R} is the ball with radius $2R$ and center at the origin, and χ_B is the characteristic function of the domain B which equals to 1 in B and vanishes outside of B. We can prove $I_2 < \infty$ by a similar argument. Therefore,

$$\|g_{12}(z,\omega)\|_{H^{\frac15,\frac{10}{9}}(B)} \lesssim \omega^{-\frac32}. \tag{5.33}$$

Next we analyze the term $g_2(z,\omega)$, which is given by

$$g_2(z,\omega) = \frac{\mathrm{i}}{4\omega^2}\frac{z_1^2}{|z|^2}[\kappa_{\mathrm{s}}^2\Gamma_{2,0}(\kappa_{\mathrm{s}}|z|) - \kappa_{\mathrm{p}}^2\Gamma_{2,0}(\kappa_{\mathrm{p}}|z|)].$$

For large $\omega|z|$, (2.15) shows that

$$|g_2(z,\omega)| \lesssim \frac{1}{\omega^2}\left(\kappa_{\mathrm{s}}^2(\kappa_{\mathrm{s}}|z|)^{-\frac32} + \kappa_{\mathrm{p}}^2(\kappa_{\mathrm{p}}|z|)^{-\frac32}\right) \lesssim (\omega|z|)^{-\frac32}. \tag{5.34}$$

For small $\omega|z|$, we have from (2.11) that

$$|g_2(z,\omega)| \lesssim (\kappa_{\mathrm{s}}|z|)^{-\frac12} + (\kappa_{\mathrm{p}}|z|)^{-\frac12} \lesssim (\omega|z|)^{-\frac12} \lesssim (\omega|z|)^{-\frac32}. \tag{5.35}$$

Thus, (5.34) and (5.35) implies for $\tilde p < \frac43$ that

$$\|g_2(z,\omega)\|_{L^{\tilde p}(B)}^{\tilde p} \lesssim \int_B \omega^{-\frac32\tilde p}|z|^{-\frac32\tilde p}dz \lesssim \omega^{-\frac32\tilde p}\int_0^R r^{1-\frac32\tilde p}dr \lesssim \omega^{-\frac32\tilde p}. \tag{5.36}$$

A direct computation shows that

$$\nabla g_2(z,\omega) = \frac{\mathrm{i}}{2\omega^2}z_1\boldsymbol{e}_1 a_0^{(2)}|z|^{-\frac52}e^{-\frac54\pi\mathrm{i}}\left[\kappa_{\mathrm{p}}^{\frac32}e^{\mathrm{i}\kappa_{\mathrm{p}}|z|} - \kappa_{\mathrm{s}}^{\frac32}e^{\mathrm{i}\kappa_{\mathrm{s}}|z|}\right]$$
$$+ \frac{\mathrm{i}}{2\omega^2}\frac{z_1}{|z|^2}\boldsymbol{e}_1\Gamma_2(z,\omega) - \frac{\mathrm{i}}{4\omega^2}\frac{z}{|z|}\frac{z_1^2}{|z|^2}\Gamma_3(z,\omega) + \frac{\mathrm{i}}{4\omega^2}a_0^{(2)}\frac{z_1^2}{|z|^2}\frac{z}{|z|}e^{-\frac54\pi\mathrm{i}} \times$$
$$\left[\left(\frac52\kappa_{\mathrm{s}}^{\frac32}|z|^{-\frac32} - \mathrm{i}\kappa_{\mathrm{s}}^{\frac52}|z|^{-\frac12}\right)e^{\mathrm{i}\kappa_{\mathrm{s}}|z|} - \left(\frac52\kappa_{\mathrm{p}}^{\frac32}|z|^{-\frac32} - \mathrm{i}\kappa_{\mathrm{p}}^{\frac52}|z|^{-\frac12}\right)e^{\mathrm{i}\kappa_{\mathrm{p}}|z|}\right].$$

For large $\omega|z|$, we know from (2.7) that

$$|\nabla g_2(z,\omega)| \lesssim \omega^{-\frac12}|z|^{-\frac32} + \omega^{\frac12}|z|^{-\frac12} \lesssim \omega^{\frac12}|z|^{-\frac12}. \tag{5.37}$$

For small $\omega|z|$, we obtain from (2.11) and (2.12) that

$$|\nabla g_2(z,\omega)| \lesssim \omega^{-\frac{1}{2}}|z|^{-\frac{3}{2}} + \omega^{\frac{1}{2}}|z|^{-\frac{1}{2}} + |z|^{-1} \lesssim \omega^{-\frac{1}{2}}|z|^{-\frac{3}{2}}. \tag{5.38}$$

By (5.37) and (5.38), we conclude for $\tilde{p} < \frac{4}{3}$ that

$$\|\nabla g_2(z,\omega)\|^{\tilde{p}}_{L^{\tilde{p}}(B)} \lesssim \int_B \omega^{\frac{1}{2}\tilde{p}}|z|^{-\frac{1}{2}\tilde{p}}dz + \int_B \omega^{-\frac{1}{2}\tilde{p}}|z|^{-\frac{3}{2}\tilde{p}}dz \lesssim \omega^{\frac{1}{2}\tilde{p}}. \tag{5.39}$$

Using (5.36) and (5.39), we get

$$\|g_2(z,\omega)\|^{\tilde{p}}_{H^{1,\tilde{p}}(B)} \lesssim \omega^{-\frac{3}{2}\tilde{p}} + \omega^{\frac{1}{2}\tilde{p}} \lesssim \omega^{\frac{1}{2}\tilde{p}}. \tag{5.40}$$

It follows from (5.21), (5.36), and (5.40) that

$$\|g_2(z,\omega)\|_{H^{\varepsilon,\tilde{p}}(B)} \lesssim \|g_2(z,\omega)\|^{1-\varepsilon}_{L^{\tilde{p}}(B)}\|g_2(z,\omega)\|^{\varepsilon}_{H^{1,\tilde{p}}(B)} \lesssim \omega^{-\frac{3}{2}+2\varepsilon}. \tag{5.41}$$

Noting that D is a bounded domain, and combining (5.22), (5.32), (5.33), and (5.41), we obtain for any $\varepsilon \in (0, \frac{1}{5}]$ and $\tilde{p} \in [1, \frac{10}{9}]$ that

$$\|\boldsymbol{G}_{11}(y,z,\omega) - \boldsymbol{G}_{0,11}(y,z,\omega)\|_{H^{\varepsilon,\tilde{p}}(D\times D)} \lesssim \omega^{-\frac{3}{2}+2\varepsilon}.$$

Since $\boldsymbol{G}_{0,11}(x,y,\omega)$ is smooth for $x \in U$ and $y \in D$, $\boldsymbol{M}_{11}(y) \in C^1_0(\overline{D})$, and $f_1(z) \in H^{-\varepsilon,\tilde{p}}(D)$ for any $\varepsilon > 0$ and $1 < \tilde{p} < \infty$, we have $\boldsymbol{G}_{0,11}(x,y,\omega)\boldsymbol{M}_{11}(y)f_1(z) \in H_0^{-\varepsilon,\tilde{p}}(D\times D)$. Moreover,

$$\begin{aligned}\boldsymbol{G}_{0,11}(x,y,\omega) = {}& \frac{\mathrm{i}}{4\mu}\frac{e^{-\frac{\pi}{4}\mathrm{i}}}{|x-y|^{\frac{1}{2}}}a_0^{(0)}c_{\mathrm{s}}^{-\frac{1}{2}}e^{\mathrm{i}\kappa_{\mathrm{s}}|x-y|}\omega^{-\frac{1}{2}} - \frac{\mathrm{i}}{4}\frac{e^{-\frac{3}{4}\pi\mathrm{i}}}{|x-y|^{\frac{3}{2}}}a_0^{(1)} \\ &\times\left(c_{\mathrm{s}}^{\frac{1}{2}}e^{\mathrm{i}\kappa_{\mathrm{s}}|x-y|} - c_{\mathrm{p}}^{\frac{1}{2}}e^{\mathrm{i}\kappa_{\mathrm{p}}|x-y|}\right)\omega^{-\frac{3}{2}} \\ &+ \frac{\mathrm{i}}{4}\frac{e^{-\frac{5}{4}\pi\mathrm{i}}(x_1-y_1)^2}{|x-y|^{5/2}}a_0^{(2)}\left(c_{\mathrm{s}}^{\frac{3}{2}}e^{\mathrm{i}\kappa_{\mathrm{s}}|x-y|} - c_{\mathrm{p}}^{\frac{3}{2}}e^{\mathrm{i}\kappa_{\mathrm{p}}|x-y|}\right)\omega^{-\frac{1}{2}}.\end{aligned}$$

Thus, we obtain for sufficient large ω that

$$\|\boldsymbol{G}_{0,11}(x,y,\omega)\boldsymbol{M}_{11}(y)f_1(z)\|_{H_0^{-\varepsilon,\tilde{p}}(D\times D)} \lesssim \omega^{-\frac{1}{2}}. \tag{5.42}$$

Substituting (5.41) and (5.42) into (5.13) yields that $|I^{(1)}_{111}| \lesssim \omega^{-2+\varepsilon}$ holds for any $\varepsilon \in (0, \frac{1}{5}]$.

Repeating a similar proof, we can obtain estimates for $I^{(1)}_{112}, \dots, I^{(2)}_{222}$ and get (5.12). The details are omitted for brevity. ∎

Noting (5.6), we have

$$|\boldsymbol{u}_1(x,\omega) - \boldsymbol{u}_{1,r}(x,\omega)| \lesssim \omega^{-2+\varepsilon},$$

which gives

$$\begin{aligned}\frac{1}{Q-1}\int_1^Q \omega^{m+1}|\boldsymbol{u}_1(x,\omega)|^2 d\omega \lesssim {}& \frac{2}{Q-1}\int_1^Q \omega^{m+1}|\boldsymbol{u}_{1,r}(x,\omega)|^2 d\omega \\ &+ \frac{2}{Q-1}\int_1^Q \omega^{m-3+2\varepsilon}d\omega.\end{aligned}$$

It is easy to verify that

$$\frac{2}{Q-1}\int_1^Q \omega^{m-3+2\varepsilon}d\omega \to 0$$

for $m \in [2, 5/2)$ and small enough ε. To prove (5.2), it is sufficient to prove the following result.

LEMMA 5.6. *For the item $\boldsymbol{u}_{1,r}(x,\omega)$ given by* (5.11), *we have*

$$\lim_{Q\to\infty}\frac{1}{Q-1}\int_1^Q \omega^{m+1}|\boldsymbol{u}_{1,r}(x,\omega)|^2 d\omega = 0, \quad x\in U. \tag{5.43}$$

Proof. It follows from a straightforward but tedious calculation that the vector $\boldsymbol{u}_{1,r}(x,\omega)$ can be decomposed into three parts according to the order of ω in the following form:

$$\boldsymbol{u}_{1,r}(x,\omega) = \boldsymbol{v}_1(x,\omega)\omega^{-1} + \boldsymbol{v}_2(x,\omega)\omega^{-2} + \boldsymbol{v}_3(x,\omega)\omega^{-3}, \tag{5.44}$$

where

$$\begin{aligned}
\boldsymbol{v}_1(x,\omega) = &-\frac{e^{-\frac{\pi}{2}\mathrm{i}}}{16\mu^2 c_{\mathrm{s}}} a_0^{(0)^2}\int_D\int_D e^{\mathrm{i}\kappa_{\mathrm{s}}(|x-y|+|y-z|)}\frac{\boldsymbol{M}(y)\boldsymbol{f}(z)}{|x-y|^{\frac12}|y-z|^{\frac12}}dydz - \frac{e^{-\frac{3}{2}\pi\mathrm{i}}}{16\mu}a_0^{(0)}a_0^{(2)}\\
&\times\Bigg[\int_D\int_D\Big(c_{\mathrm{s}}e^{\mathrm{i}\kappa_{\mathrm{s}}(|x-y|+|y-z|)} - c_{\mathrm{p}}^{\frac32}c_{\mathrm{s}}^{-\frac12}e^{\mathrm{i}(\kappa_{\mathrm{s}}|x-y|+\kappa_{\mathrm{p}}|y-z|)}\Big)\frac{\boldsymbol{M}(y)\boldsymbol{J}(y-z)\boldsymbol{f}(z)}{|x-y|^{\frac12}|y-z|^{\frac52}}dydz\\
&+\int_D\int_D\Big(c_{\mathrm{s}}e^{\mathrm{i}\kappa_{\mathrm{s}}(|x-y|+|y-z|)} - c_{\mathrm{p}}^{\frac32}c_{\mathrm{s}}^{-\frac12}e^{\mathrm{i}(\kappa_{\mathrm{s}}|y-z|+\kappa_{\mathrm{p}}|x-y|)}\Big)\frac{\boldsymbol{J}(x-y)\boldsymbol{M}(y)\boldsymbol{f}(z)}{|x-y|^{\frac52}|y-z|^{\frac12}}dydz\Bigg]\\
&+\frac{e^{-\frac52\pi\mathrm{i}}}{16}a_0^{(2)^2}\int_D\int_D\Big(-c_{\mathrm{s}}^3e^{\mathrm{i}\kappa_{\mathrm{s}}(|x-y|+|y-z|)} + c_{\mathrm{s}}^{\frac32}c_{\mathrm{p}}^{\frac32}e^{\mathrm{i}(\kappa_{\mathrm{s}}|x-y|+\kappa_{\mathrm{p}}|y-z|)}\\
&+c_{\mathrm{s}}^{\frac32}c_{\mathrm{p}}^{\frac32}e^{\mathrm{i}(\kappa_{\mathrm{p}}|x-y|+\kappa_{\mathrm{s}}|y-z|)} - c_{\mathrm{p}}^3e^{\mathrm{i}\kappa_{\mathrm{p}}(|x-y|+|y-z|)}\Big)\frac{\boldsymbol{J}(x-y)\boldsymbol{M}(y)\boldsymbol{J}(y-z)\boldsymbol{f}(z)}{|x-y|^{\frac52}|y-z|^{\frac52}}dydz,
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{v}_2(x,\omega) = &\frac{e^{-\pi\mathrm{i}}}{16\mu}a_0^{(0)}a_0^{(1)}\int_D\int_D\Big(e^{\mathrm{i}\kappa_{\mathrm{s}}(|x-y|+|y-z|)} - c_{\mathrm{p}}^{\frac12}c_{\mathrm{s}}^{-\frac12}e^{\mathrm{i}(\kappa_{\mathrm{s}}|x-y|+\kappa_{\mathrm{p}}|y-z|)}\Big)\\
&\times\frac{\boldsymbol{M}(y)\boldsymbol{f}(z)}{|x-y|^{\frac12}|y-z|^{\frac32}}dydz + \frac{e^{-\pi\mathrm{i}}}{16\mu}a_0^{(0)}a_0^{(1)}\int_D\int_D\Big(e^{\mathrm{i}\kappa_{\mathrm{s}}(|x-y|+|y-z|)}\\
&-c_{\mathrm{p}}^{\frac12}c_{\mathrm{s}}^{-\frac12}e^{\mathrm{i}(\kappa_{\mathrm{p}}|x-y|+\kappa_{\mathrm{s}}|y-z|)}\Big)\frac{\boldsymbol{M}(y)\boldsymbol{f}(z)}{|x-y|^{\frac32}|y-z|^{\frac12}}dydz + \frac{e^{-2\pi\mathrm{i}}}{16}a_0^{(1)}a_0^{(2)}\\
&\times\int_D\int_D\Big(c_{\mathrm{s}}^2e^{\mathrm{i}\kappa_{\mathrm{s}}(|x-y|+|y-z|)} - c_{\mathrm{s}}^{\frac12}c_{\mathrm{p}}^{\frac32}e^{\mathrm{i}(\kappa_{\mathrm{s}}|x-y|+\kappa_{\mathrm{p}}|y-z|)}\\
&-c_{\mathrm{p}}^{\frac12}c_{\mathrm{s}}^{\frac32}e^{\mathrm{i}(\kappa_{\mathrm{p}}|x-y|+\kappa_{\mathrm{s}}|y-z|)} + c_{\mathrm{p}}^2e^{\mathrm{i}\kappa_{\mathrm{p}}(|x-y|+|y-z|)}\Big)\frac{\boldsymbol{M}(y)\boldsymbol{J}(y-z)\boldsymbol{f}(z)}{|x-y|^{\frac32}|y-z|^{\frac52}}dydz\\
&+\frac{e^{-2\pi\mathrm{i}}}{16}a_0^{(1)}a_0^{(2)}\int_D\int_D\Big(c_{\mathrm{s}}^2e^{\mathrm{i}\kappa_{\mathrm{s}}(|x-y|+|y-z|)} - c_{\mathrm{s}}^{\frac12}c_{\mathrm{p}}^{\frac32}e^{\mathrm{i}(\kappa_{\mathrm{p}}|x-y|+\kappa_{\mathrm{s}}|y-z|)}\\
&-c_{\mathrm{p}}^{\frac12}c_{\mathrm{s}}^{\frac32}e^{\mathrm{i}(\kappa_{\mathrm{s}}|x-y|+\kappa_{\mathrm{p}}|y-z|)} + c_{\mathrm{p}}^2e^{\mathrm{i}\kappa_{\mathrm{p}}(|x-y|+|y-z|)}\Big)\frac{\boldsymbol{J}(x-y)\boldsymbol{M}(y)\boldsymbol{f}(z)}{|x-y|^{\frac52}|y-z|^{\frac32}}dydz,
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{v}_3(x,\omega) = &\frac{e^{-\frac32\pi\mathrm{i}}}{16}a_0^{(1)^2}\int_D\int_D\Big(-c_{\mathrm{s}}e^{\mathrm{i}\kappa_{\mathrm{s}}(|x-y|+|y-z|)} + c_{\mathrm{s}}^{\frac12}c_{\mathrm{p}}^{\frac12}e^{\mathrm{i}(\kappa_{\mathrm{s}}|x-y|+\kappa_{\mathrm{p}}|y-z|)}\\
&+c_{\mathrm{s}}^{\frac12}c_{\mathrm{p}}^{\frac12}e^{\mathrm{i}(\kappa_{\mathrm{p}}|x-y|+\kappa_{\mathrm{s}}|y-z|)} - c_{\mathrm{p}}e^{\mathrm{i}\kappa_{\mathrm{p}}(|x-y|+|y-z|)}\Big)\frac{\boldsymbol{M}(y)\boldsymbol{f}(z)}{|x-y|^{\frac32}|y-z|^{\frac32}}dydz.
\end{aligned}$$

Here $\boldsymbol{J}(x-y) = (x-y)(x-y)^\top$ and $\boldsymbol{J}(y-z) = (y-z)(y-z)^\top$.

By (5.44) and the Cauchy–Schwarz inequality, we have

$$\int_1^Q \omega^{m+1}|\boldsymbol{u}_{1,r}(x,\omega)|^2 d\omega \lesssim \int_1^Q \big(\omega^{m-1}|\boldsymbol{v}_1(x,\omega)|^2 + \omega^{m-3}|\boldsymbol{v}_2(x,\omega)|^2 + \omega^{m-5}|\boldsymbol{v}_3(x,\omega)|^2\big) d\omega.$$

Noting the facts that $|x-y|$ has a positive lower bound for $x \in U$, $y \in D$, $\||y-z|^{-\frac{3}{2}}\|_{H^{\frac{1}{5},\frac{10}{9}}(D\times D)}$ is bounded from the above, $\boldsymbol{M}_{ij}(y) \in C_0^1(\overline{D})$, and $\|f_j(z)\|_{H^{-\frac{1}{5},10}(D)}$ is bounded from the assumption, we conclude that

$$|\boldsymbol{v}_2(x,\omega)| < \infty, \quad |\boldsymbol{v}_3(x,\omega)| < \infty, \quad x \in U, \ \omega \geq 1.$$

Hence, we have as $\omega \to \infty$ that

$$\frac{1}{Q-1}\int_1^Q \omega^{m-3}|\boldsymbol{v}_2(x,\omega)|^2 d\omega \lesssim \frac{1}{Q-1}\int_1^Q \omega^{m-3} d\omega \to 0,$$
$$\frac{1}{Q-1}\int_1^Q \omega^{m-5}|\boldsymbol{v}_3(x,\omega)|^2 d\omega \lesssim \frac{1}{Q-1}\int_1^Q \omega^{m-5} d\omega \to 0.$$

To prove (5.43), it suffices to prove that

$$\lim_{Q\to\infty} \frac{1}{Q-1}\int_1^Q \omega^{m-1}|\boldsymbol{v}_1(x,\omega)|^2 d\omega = 0. \tag{5.45}$$

We claim that in order to prove (5.45), it will be enough to show that

$$\int_1^\infty \omega^{m-2}|\boldsymbol{v}_1(x,\omega)|^2 d\omega < \infty, \quad \text{almost surely.} \tag{5.46}$$

To show this, we notice that

$$\frac{1}{Q}\int_1^Q \omega^{m-1}|\boldsymbol{v}_1(x,\omega)|^2 d\omega \leq \int_1^Q \frac{\omega}{Q}\omega^{m-2}|\boldsymbol{v}_1(x,\omega)|^2 d\omega \leq \int_1^\infty \min\left(1, \frac{\omega}{Q}\right)\omega^{m-2}|\boldsymbol{v}_1(x,\omega)|^2 d\omega.$$

From the dominated convergence theorem, the last integral in the above inequality converges almost surely to zero as $Q \to \infty$, so the claim follows. The remaining part of the proof will focus on (5.46). To this end, we define

$$g(x,\omega) = \int_D\int_D e^{\mathrm{i}\omega(c_1|x-y|+c_2|y-z|)} \frac{(x_1-y_1)^{p_1}(x_2-y_2)^{p_2}(y_1-z_1)^{p_3}(y_2-z_2)^{p_4}}{|x-y|^{l_1}|y-z|^{l_2}} q(y)\tilde{f}(z) dy dz, \tag{5.47}$$

where $c_1, c_2 > 0$, $p_1, \dots, l_2 \geq 0$, $\tilde{f}$ denotes a generalized Gaussian random field which equals to f_1 or f_2, and $q(y) \in C_0^1(\overline{D})$ stands for $\boldsymbol{M}_{ij}(y)$. From the formulation of $\boldsymbol{v}_1(x,\omega)$, we know that it is a linear combination of $g(x,\omega)$ for different

$(l_1, l_2, p_1, p_2, p_3, p_4) \in S$ which is given by

$$
\begin{aligned}
S =\Big\{ &\Big(\frac{1}{2}, \frac{1}{2}, 0,0,0,0\Big),\ \Big(\frac{1}{2}, \frac{5}{2}, 0,0,2,0\Big),\ \Big(\frac{1}{2}, \frac{5}{2}, 0,0,1,1\Big),\ \Big(\frac{1}{2}, \frac{5}{2}, 0,0,0,2\Big),\\
&\Big(\frac{5}{2}, \frac{1}{2}, 2,0,0,0\Big),\ \Big(\frac{5}{2}, \frac{1}{2}, 0,2,0,0\Big),\ \Big(\frac{5}{2}, \frac{1}{2}, 1,1,0,0\Big), \Big(\frac{5}{2}, \frac{5}{2}, 2,0,2,0\Big),\\
&\Big(\frac{5}{2}, \frac{5}{2}, 1,1,1,1\Big),\ \Big(\frac{5}{2}, \frac{5}{2}, 2,0,1,1\Big),\ \Big(\frac{5}{2}, \frac{5}{2}, 1,1,0,2\Big),\ \Big(\frac{5}{2}, \frac{5}{2}, 2,0,0,2\Big),\\
&\Big(\frac{5}{2}, \frac{5}{2}, 1,1,2,0\Big),\ \Big(\frac{5}{2}, \frac{5}{2}, 0,2,2,0\Big),\ \Big(\frac{5}{2}, \frac{5}{2}, 0,2,1,1\Big),\ \Big(\frac{5}{2}, \frac{5}{2}, 0,2,0,2\Big)\Big\}.
\end{aligned}
$$

To prove (5.46), it is enough to show that

$$\int_1^\infty \omega^{m-2}|g(x,\omega)|^2 d\omega < \infty, \quad \text{almost surely.} \tag{5.48}$$

In the following, we consider two cases.

Case 1. $m = 2$. In this case, Lemma 2.4 claims that $\tilde{f} \in H^{-\varepsilon,p}(D)$ almost surely for any $\varepsilon > 0$ and $1 < p < \infty$. In order to avoid the distribution dualities, we introduce the mollification $\tilde{f}_\delta := \tilde{f} * \rho_\delta$, where $\rho_\delta := \delta^{-2}\rho\left(\frac{x}{\delta}\right)$, $\rho \in C_0^\infty(\mathbb{R}^2)$ is a radially symmetric function satisfying $\int_{\mathbb{R}^2} \rho(x)dx = 1$. We denote g_δ by replacing $\tilde{f}$ by the standard mollification $\tilde{f}_\delta$ in (5.47). Let $M_\delta \tilde{f} := \tilde{f}_\delta$ be the mollification operator and C_δ be the covariance operator of $\tilde{f}_\delta$. Then it is easy to verify that $C_\delta = M_\delta C_{\tilde{f}} M_\delta$ and $g_\delta(x,\omega) \to g(x,\omega)$ as $\delta \to 0$. To prove (5.48), we claim that it is enough to show that

$$\sup_{\delta \in (0,1)} \int_1^\infty \mathbb{E}|g_\delta(x,\omega)|^2 d\omega < \infty. \tag{5.49}$$

If (5.49) holds, then it follows from the Fubini theorem and Fatou's lemma that

$$\mathbb{E}\left(\int_1^\infty |g(x,\omega)|^2 d\omega\right) < \infty,$$

which shows that (5.48) holds immediately. So, we focus on proving (5.49) for this case. We look at the phase function $A(y,z) = c_1|x-y| + c_2|y-z|$ for some fixed $x \in U$. It is easy to see that $A(y,z)$ is smooth on $D \times D$ apart from the subset where $y = z$. Since the phase function $A(y,z)$ is not smooth at $y = z$, a stationary phase approach cannot be used in the analysis of $A(y,z)$. A direct computation shows

$$\nabla_y A(y,z) = c_1 \frac{y-x}{|y-x|} + c_2 \frac{y-z}{|y-z|}, \quad \nabla_z A(y,z) = c_2 \frac{z-y}{|z-y|}.$$

Hence,

$$|\nabla_y A(y,z)| \le c_1 + c_2, \quad |\nabla_z A(y,z)| \le c_2 \quad \forall (y,z) \in D \times D \text{ and } y \neq z.$$

Since

$$
\begin{aligned}
(y,z)\cdot \nabla A(y,z) &= c_1 \frac{y\cdot(y-x)}{|y-x|} + c_2|y-z| \\
&= c_1|y|\cos\theta + c_2|y-z| \ge c_0 > 0,
\end{aligned} \tag{5.50}
$$

where θ denotes the angle between y and $y-x$, noting the facts that the origin belongs to U and U is convex, we have that $(y,z)\cdot\nabla A(y,z)$ has a positive lower bound for $(y,z)\in D\times D$ and $y\neq z$. So

$$0<c_1'\leq|\nabla A(y,z)|\leq c_2'<\infty\quad\forall(y,z)\in D\times D\ \text{ and }\ y\neq z. \tag{5.51}$$

Our aim is to express $g_\delta(x,\omega)$ as a one-dimensional Fourier transform and get rid of the variable ω. Now, we define the following surface:

$$\Gamma_t':=\{(y,z)\in D\times D|A(y,z)=t\},\quad t>0.$$

It is easy to see that there exists smallest and largest values $T_0=T_0(x)$ and $T_1=T_1(x)$ such that Γ_t' is nonempty only for $t\in[T_0,T_1]$. Now we fix a $\tilde t\in[T_0,T_1]$, and then there exists $\eta=\eta(\tilde t)$ and an open cone $K=K(\tilde t)\subset\mathbb{R}^4$ with center at the origin such that for $t_0=\tilde t-\eta$ and $t_1=\tilde t+\eta$, we have

$$D\times D\cap\{t_0<A(y,z)<t_1\}\subset K\cap\{t_0<A(y,z)<t_1\}:=\Gamma.$$

Moreover, since D has a positive distance to the origin we may also choose η and K such that

$$|y|,|z|\geq c_3'>0\quad\forall(y,z)\in\Gamma.$$

Denote $\Gamma_t=\Gamma\cap\{(y,z):A(y,z)=t\}$. We obtain $\Gamma=\cup_{t_0\leq t\leq t_1}\Gamma_t$. By (5.50) and (5.51), we deduce that there is a radial stretch B_t yielding a bi-Lipschitz chart $B_t:F\to\Gamma_t$ over a subdomain F of the unit ball. The bi-Lipschitz constant of B_t is uniform over $t_0<t<t_1$ and each B_t is actually a local diffeomorphism apart from $y=z$. By (5.50) and (5.51), we may write B_t in the following form:

$$B_t(w_1,w_2)=\sigma(t,w_1,w_2)(w_1,w_2),$$

where the dependence $(w_1,w_2)\to\sigma(t,w_1,w_2)$ is Lipschitz with respect to t with a uniform Lipschitz constant with respect to w_1,w_2.

Letting h be a integrable Borel function on Γ and noting that $\Gamma=\cup_{t_0\leq t\leq t_1}\Gamma_t$, we get

$$\int_\Gamma h(y,z)dydz=\int_{t_0}^{t_1}\int_{\Gamma_t}h(y,z)\frac{1}{|\nabla A(y,z)|}d\mathcal{H}^3(y,z)dt, \tag{5.52}$$

where the inner integral is with respect to the three-dimensional Hausdorff measure on Γ_t. Using a change of variables, we have

$$\int_{\Gamma_t}h(y,z)d\mathcal{H}^3(y,z)=\int_F h(B_t(w_1,w_2))E_t(w_1,w_2)d\mathcal{H}^3(w_1,w_2). \tag{5.53}$$

By (5.50) and (5.51), the Jacobian E_t in (5.53) satisfies

$$0<c_4'\leq E_t(w_1,w_2):=\frac{|B_t(w_1,w_2)|^3|\nabla A(B_t(w_1,w_2))|}{|(w_1,w_2)\cdot\nabla A(B_t(w_1,w_2))|}\leq c_5'<\infty.$$

Since $B_t(w_1,w_2)$ is Lipschitz with respect to t, for our later purpose, we claim that the dependence $t \to E_t(w_1, w_2)$ is uniformly Lipschitz with respect to t. Using (5.52),

$$\begin{aligned} g_\delta(x,\omega) &= \int_D \int_D e^{\mathrm{i}\omega(c_1|x-y|+c_2|y-z|)} \frac{(x_1-y_1)^{p_1}(x_2-y_2)^{p_2}(y_1-z_1)^{p_3}(y_2-z_2)^{p_4}}{|x-y|^{l_1}|y-z|^{l_2}} \\ &\quad \times q(y)\tilde{f}_\delta(z)dydz \\ &= \int_\Gamma e^{\mathrm{i}\omega(c_1|x-y|+c_2|y-z|)} \frac{(x_1-y_1)^{p_1}(x_2-y_2)^{p_2}(y_1-z_1)^{p_3}(y_2-z_2)^{p_4}}{|x-y|^{l_1}|y-z|^{l_2}} \\ &\quad \times q(y)\tilde{f}_\delta(z)dydz \\ &= \int_{t_0}^{t_1} e^{\mathrm{i}\omega t} S_\delta(t)dt = [\mathcal{F}^{-1}S_\delta](-\omega), \end{aligned}$$

where S_δ is given by

$$\begin{aligned} S_\delta(t) = \int_{\Gamma_t} & \frac{(x_1-y_1)^{p_1}(x_2-y_2)^{p_2}(y_1-z_1)^{p_3}(y_2-z_2)^{p_4}}{|x-y|^{l_1}|y-z|^{l_2}} \\ & \times \frac{1}{|\nabla A(y,z)|} q(y)\tilde{f}_\delta(z)d\mathcal{H}^3(y,z). \end{aligned}$$

Since Γ_t is only nonempty for $t \in [T_0, T_1]$, $S_\delta(t)$ is compactly supported inside $[T_0, T_1]$. For fixed $x \in U$, let $L(x,y)$ be a smooth cutoff of the function $\frac{(x_1-y_1)^{p_1}(x_2-y_2)^{p_2}}{|x-y|^{l_1}}$ that vanishes outside D, and hence, $L(x,\cdot) \in C_0^\infty(\mathbb{R}^2)$. Thus, we can rewrite $S_\delta(t)$ as

$$S_\delta(t) = \int_{\Gamma_t} \frac{(y_1-z_1)^{p_3}(y_2-z_2)^{p_4}}{|y-z|^{l_2}} \frac{L(x,y)}{|\nabla A(y,z)|} q(y)\tilde{f}_\delta(z)d\mathcal{H}^3(y,z). \tag{5.54}$$

Recall that our aim is to prove $\sup_{\delta\in(0,1)} \int_1^\infty \mathbb{E}|g_\delta(x,\omega)|^2 d\omega < \infty$. It is sufficient to show that for each $\tilde{t} \in [T_0, T_1]$, there exists a finite constant $M = M(\tilde{t}) < \infty$ such that

$$\mathbb{E}|S_\delta(t)|^2 \le M \quad \forall \delta \in (0,1) \text{ and } t \in [t_0(\tilde{t}), t_1(\tilde{t})]. \tag{5.55}$$

This can be seen by the following facts: by compactness, we can choose finitely many $\tilde{t} \in [T_0, T_1]$ such that the union set of $[t_0(\tilde{t}), t_1(\tilde{t})]$ for these $\tilde{t}$ can cover $[T_0, T_1]$. Hence, for any $t \in [T_0, T_1]$, we have $\mathbb{E}|S_\delta(t)|^2 \le M'$. The Parseval formula yields

$$\sup_{\delta\in(0,1)} \int_1^\infty \mathbb{E}|g_\delta(x,\omega)|^2 d\omega \lesssim \sup_{\delta\in(0,1)} \int_{T_0}^{T_1} \mathbb{E}|S_\delta(t)|^2 dt \le M'(T_1 - T_0) < \infty.$$

It remains to show (5.55). By (5.54), we have

$$\begin{aligned} \mathbb{E}|S_\delta(t)|^2 = \int_{\Gamma_t\times\Gamma_t} & \frac{(y_1-z_1)^{p_3}(y_2-z_2)^{p_4}}{|y-z|^{l_2}} \frac{(y_1'-z_1')^{p_3}(y_2'-z_2')^{p_4}}{|y'-z'|^{l_2}} \\ & \times \frac{L(x,y)}{|\nabla A(y,z)|} \frac{L(x,y')}{|\nabla A(y',z')|} q(y)q(y')\mathbb{E}(\tilde{f}_\delta(z)\tilde{f}_\delta(z'))d\mathcal{H}^3(y,z)d\mathcal{H}^3(y',z'). \end{aligned}$$

Noting that $\mathbb{E}(\tilde{f}_\delta(z)\tilde{f}_\delta(z')) = C_\delta(z,z')$ and $C_\delta = M_\delta C_{\tilde{f}} M_\delta$, we obtain from Lemma 2.5 that for any given $\beta > 0$, there is a finite constant C'_β such that $C_\delta(z,z') \le C'_\beta |z-z'|^{-\beta}$ for any $\delta \in (0,1)$ and $(z,z') \in D \times D$. Since $q \in C_0^1(\overline{D})$, an application of Hölder's inequality arrives at

$$\sup_{\delta\in(0,1)} \mathbb{E}|S_\delta(t)|^2 \lesssim \int_{\Gamma_t\times\Gamma_t} |z-z'|^{-\beta}(|y-z||y'-z'|)^{-(l_2-p_3-p_4)} d\mathcal{H}^3(y,z)d\mathcal{H}^3(y',z')$$
$$\lesssim \left[\int_{\Gamma_t\times\Gamma_t} |z-z'|^{-2\beta} d\mathcal{H}^3(y,z)d\mathcal{H}^3(y',z')\right]^{\frac12}$$
$$\times\left[\int_{\Gamma_t}|y-z|^{-1}d\mathcal{H}^3(y,z)\int_{\Gamma_t}|y'-z'|^{-1}d\mathcal{H}^3(y',z')\right]^{\frac12},$$

where we use the fact $l_2-p_3-p_4=\frac12$ for $(l_1,l_2,p_1,p_2,p_3,p_4)\in S$. To show the integral in the right-hand side of the above inequality is bounded, we need the following result [33, Lemma 6]).

LEMMA 5.7. *Given $\gamma\in(0,2)$, there is a finite constant c such that for every $t\in[t_0,t_1]$ we have*

$$\int_{\Gamma_t}|y-z|^{-\gamma}d\mathcal{H}^3(y,z)\le c,\quad \int_{\Gamma_t\times\Gamma_t}|\tilde y-\tilde z|^{-\gamma}d\mathcal{H}^3(y,z)d\mathcal{H}^3(y',z')\le c$$

for $(\tilde y,\tilde z)=(y,y'),(y,z'),(z,y'),(z,z')$.

Choosing $\beta=\frac12$ and applying Lemma 5.7 gives (5.55). So Theorem 5.1 holds for the case $m=2$.

Case 2. $m\in(2,5/2)$. By Lemma 2.4, we know that in this case the realizations of $\tilde f$ are Hölder continuous with probability one. So it is not necessary to introduce the mollification, and we define

$$S(t)=\int_{\Gamma_t}\frac{(y_1-z_1)^{p_3}(y_2-z_2)^{p_4}}{|y-z|^{l_2}}\frac{L(x,y)}{|\nabla A(y,z)|}q(y)\tilde f(z)d\mathcal{H}^3(y,z).$$

In order to prove (5.48), i.e., $\int_1^\infty\omega^{m-2}|g(x,\omega)|^2d\omega<\infty$, by $g(x,\omega)=[\mathcal{F}^{-1}S](-\omega)$, it suffices to prove that $S(t)\in H^{\frac{m-2}{2}}_{\rm homog}(\mathbb{R})$, which denotes the homogeneous Sobolev space. By compactness, it is enough to show that $S(t)\in H^{\frac{m-2}{2}}_{\rm homog}(t_0(\tilde t),t_1(\tilde t))$ for each $\tilde t\in[T_0,T_1]$. According to the Besov characterization of the homogeneous Sobolev space, it is sufficient to show

$$\mathbb{E}\int_{t_0}^{t_1}\int_{t_0}^{t_1}\frac{|S(t)-S(t')|^2}{|t-t'|^{m-1}}dtdt'<\infty. \tag{5.56}$$

The Fubini theorem shows that (5.56) holds if there exists a positive constant M such that the following estimate holds:

$$\mathbb{E}|S(t)-S(t')|^2\le M|t-t'|^{\frac{m-1}{2}}\quad\forall t,t'\in[t_0(\tilde t),t_1(\tilde t)]. \tag{5.57}$$

We can rewrite $S(t)$ by

$$S(t)=\int_{\Gamma_t}N(y,z)L(x,y)\frac{1}{|\nabla A(y,z)|}q(y)\tilde f(z)d\mathcal{H}^3(y,z). \tag{5.58}$$

Recall that the bi-Lipschitz chart $B_t:F\to\Gamma_t$ is given by

$$B_t(w_1,w_2)=\sigma(t,w_1,w_2)(w_1,w_2):=(y_t(w_1,w_2),z_t(w_1,w_2)).$$

Denote

$$N_t(y,z)=\frac{(y_1-z_1)^{p_3}(y_2-z_2)^{p_4}}{|y-z|^{l_2}}.$$

By (5.58), we can rewrite $S(t)$ as

$$S(t)=\int_F N_t(y_t,z_t)T_t(w_1,w_2)q(y_t)\tilde{f}(z_t)d\mathcal{H}^3(w_1,w_2),$$

where the function

$$T_t(w_1,w_2)=E_t(w_1,w_2)\frac{L(x,y_t)}{|\nabla A(y_t,z_t)|}$$

is uniformly bounded and Lipschitz continuous with respect to t. Since

$$\begin{aligned}S(t)-S(t')&=S_1(t)-S_1(t')\\&\quad+\int_F N_{t'}(y_{t'},z_{t'})[T_t(w_1,w_2)-T_{t'}(w_1,w_2)]q(y_t)\tilde{f}(z_t)d\mathcal{H}^3(w_1,w_2),\end{aligned}$$

where

$$S_1(t)=\int_F N_t(y_t,z_t)T(w_1,w_2)q(y_t)\tilde{f}(z_t)d\mathcal{H}^3(w_1,w_2),\quad T(w_1,w_2)=T_t(w_1,w_2),$$

we have

$$\begin{aligned}\|S(t)-S(t')\|_{L^2(\Omega)}&\lesssim\|S_1(t)-S_1(t')\|_{L^2(\Omega)}\\&\quad+|t-t'|\int_F|q(y_{t'})|\|\tilde{f}(z_{t'})\|_{L^2(\Omega)}|N_{t'}(y_{t'},z_{t'})|d\mathcal{H}^3(w_1,w_2)\\&\lesssim\|S_1(t)-S_1(t')\|_{L^2(\Omega)}+|t-t'|.\end{aligned}$$

Since $|t-t'|=|t-t'|^{\frac{m-1}{2}}|t-t'|^{\frac{3-m}{2}}\lesssim|t-t'|^{\frac{m-1}{2}}$, it suffices to estimate $\|S_1(t)-S_1(t')\|_{L^2(\Omega)}$. Similarly, we have

$$\begin{aligned}S_1(t)-S_1(t')&=S_2(t)-S_2(t')\\&\quad+\int_F[N_t(y_t,z_t)-N_{t'}(y_{t'},z_{t'})]T(w_1,w_2)q(y_t)\tilde{f}(z_t)d\mathcal{H}^3(w_1,w_2),\end{aligned}$$

where

$$S_2(t)=\int_F N(w_1,w_2)T(w_1,w_2)q(y_t)\tilde{f}(z_t)d\mathcal{H}^3(w_1,w_2),\quad N(w_1,w_2)=N_t(w_1,w_2).$$

Note that

$$\begin{aligned}&|N_t(y_t,z_t)-N_{t'}(y_{t'},z_{t'})|\\&=\left|\frac{(y_1(t)-z_1(t))^{p_3}(y_2(t)-z_2(t))^{p_4}}{|y(t)-z(t)|^{l_2}}-\frac{(y_1(t')-z_1(t'))^{p_3}(y_2(t')-z_2(t'))^{p_4}}{|y(t')-z(t')|^{l_2}}\right|\\&=\left|\frac{\sigma_t^{p_3}(w_1^{(1)}-w_2^{(1)})^{p_3}\sigma_t^{p_4}(w_1^{(2)}-w_2^{(2)})^{p_4}}{\sigma_t^{l_2}|w_1-w_2|^{l_2}}-\frac{\sigma_{t'}^{p_3}(w_1^{(1)}-w_2^{(1)})^{p_3}\sigma_{t'}^{p_4}(w_1^{(2)}-w_2^{(2)})^{p_4}}{\sigma_{t'}^{l_2}|w_1-w_2|^{l_2}}\right|\\&\le|\sigma_t^{-\frac12}-\sigma_{t'}^{-\frac12}||w_1-w_2|^{-\frac12}\lesssim|t-t'||w_1-w_2|^{-\frac12}.\end{aligned}$$

Hence

$$\|S_1(t) - S_1(t')\|_{L^2(\Omega)} \lesssim \|S_2(t) - S_2(t')\|_{L^2(\Omega)} + |t - t'|.$$

Now we estimate $\|S_2(t) - S_2(t')\|_{L^2(\Omega)}$, which can be rewritten in a double integral as

$$\begin{aligned}\|S_2(t) - S_2(t')|_{L^2(\Omega)} &= \mathbb{E}\int_F [q(y_t)\tilde{f}(z_t) - q(y_{t'})\tilde{f}(z_{t'})]R(w_1, w_2)d\mathcal{H}^3(w_1, w_2)\\ &\quad\times \int_F [q(s_t)\tilde{f}(u_t) - q(s_{t'})\tilde{f}(u_{t'})]R(v_1, v_2)d\mathcal{H}^3(v_1, v_2)\\ &= \int_{F\times F} G(w_1, w_2, v_1, v_2)R(w_1, w_2)R(v_1, v_2)d\mathcal{H}^3(w_1, w_2)d\mathcal{H}^3(v_1, v_2),\end{aligned}$$

where $(y_t, z_t) = \sigma_t(w_1, w_2)$, $(y_{t'}, z_{t'}) = \sigma_{t'}(w_1, w_2)$, $(s_t, u_t) = \sigma_t(v_1, v_2)$, $(s_{t'}, u_{t'}) = \sigma_{t'}(v_1, v_2)$,

$$R(w_1, w_2) = N(w_1, w_2)T(w_1, w_2), \quad R(v_1, v_2) = N(v_1, v_2)T(v_1, v_2),$$

and

$$\begin{aligned}G(w_1, w_2, v_1, v_2) &= \mathbb{E}[q(y_t)\tilde{f}(z_t) - q(y_{t'})\tilde{f}(z_{t'})][q(s_t)\tilde{f}(u_t) - q(s_{t'})\tilde{f}(u_{t'})]\\ &= q(y_t)q(s_t)C_{\tilde{f}}(z_t, u_t) - q(y_t)q(s_{t'})C_{\tilde{f}}(z_t, u_{t'})\\ &\quad - q(y_{t'})q(s_t)C_{\tilde{f}}(z_{t'}, u_t) + q(y_{t'})q(s_{t'})C_{\tilde{f}}(z_{t'}, u_{t'})\\ &= q(y_t)q(s_t)[C_{\tilde{f}}(z_t, u_t) - C_{\tilde{f}}(z_t, u_{t'})] + q(y_t)[q(s_t) - q(s_{t'})]C_{\tilde{f}}(z_t, u_{t'})\\ &\quad + q(y_{t'})q(s_{t'})[C_{\tilde{f}}(z_{t'}, u_{t'}) - C_{\tilde{f}}(z_{t'}, u_t)] + q(y_{t'})[q(s_{t'}) - q(s_t)]C_{\tilde{f}}(z_{t'}, u_t).\end{aligned}$$

Recall that the covariance function has the form

$$C_{\tilde{f}}(y, z) = c_0(y, z)|y - z|^{m-2} + r_1(y, z),$$

where $c_0 \in C_0^\infty(D \times D)$ and $r_1 \in C_0^\alpha(D \times D)$ for any $\alpha < 1$. Combining the fact $q \in C_0^1(\overline{D})$ yields immediately that

$$|G(w_1, w_2, v_1, v_2)| \lesssim |t - t'|^{m-2}. \tag{5.59}$$

Denoting $d = |z_t - u_t| = |\sigma_t(w_2 - v_2)|$ and $\delta = |u_t - u_{t'}| = |(\sigma_t - \sigma_{t'})v_2|$, if $\frac{\delta}{d} < 1$, we have

$$\begin{aligned}\big||z_t - u_t|^{m-2} - |z_t - u_{t'}|^{m-2}\big| &\le \big|(d+\delta)^{m-2} - d^{m-2}\big| = d^{m-2}\left|\left(1 + \frac{\delta}{d}\right)^{m-2} - 1\right|\\ &\le d^{m-2}(m-2)\frac{\delta}{d} = (m-2)d^{m-3}\delta \lesssim \delta^{\frac{m-1}{2}} \lesssim |t - t'|^{\frac{m-1}{2}}.\end{aligned}$$

Hence, if $|t - t'| < c|w_2 - v_2|$ for some small enough $c > 0$, we have

$$\big||z_t - u_t|^{m-2} - |z_t - u_{t'}|^{m-2}\big| \lesssim |t - t'|^{\frac{m-1}{2}}.$$

Similarly, we have that

$$\big||z_{t'} - u_{t'}|^{m-2} - |z_{t'} - u_t|^{m-2}\big| \lesssim |t - t'|^{\frac{m-1}{2}}$$

holds if $|t - t'| < c|w_2 - v_2|$ for some small enough $c > 0$. Thus, if we define a set

$$P := \{(w_1, w_2, v_1, v_2) \in F \times F : |w_2 - v_2| \le C|t - t'| \quad \text{for some large enough } C > 0\},$$

then we have

$$|G(w_1,w_2,v_1,v_2)| \lesssim |t-t'|^{\frac{m-1}{2}} \quad \text{for } (w_1,w_2,v_1,v_2) \in F\times F \setminus P. \tag{5.60}$$

Dividing integration on $F\times F$ over the sets $P\cap F\times F$ and $(F\times F)\setminus P$, we obtain

$$\begin{aligned}
&\|S_2(t)-S_2(t')\|_{L^2(\Omega)}\\
&= \int_{F\times F\cap P} G(w_1,w_2,v_1,v_2)R(w_1,w_2)R(v_1,v_2)d\mathcal{H}^3(w_1,w_2)d\mathcal{H}^3(v_1,v_2)\\
&\quad + \int_{(F\times F)\setminus P} G(w_1,w_2,v_1,v_2)R(w_1,w_2)R(v_1,v_2)d\mathcal{H}^3(w_1,w_2)d\mathcal{H}^3(v_1,v_2)\\
&:= I_1 + I_2.
\end{aligned}$$

Observing that $|R(w_1,w_2)| \lesssim |w_1-w_2|^{-\frac{1}{2}}$ and $|R(v_1,v_2)| \lesssim |v_1-v_2|^{-\frac{1}{2}}$, using (5.59), the Hölder inequality, and Lemma 5.7, we have

$$\begin{aligned}
I_1 &\lesssim |t-t'|^{m-2} \int_{F\times F\cap P} |w_1-w_2|^{-\frac{1}{2}}|v_1-v_2|^{-\frac{1}{2}} d\mathcal{H}^3(w_1,w_2)d\mathcal{H}^3(v_1,v_2)\\
&\lesssim |t-t'|^{m-2} \int_{F\times F\cap P} |w_2-v_2|^{\frac{1}{2}}|w_2-v_2|^{-\frac{1}{2}}|w_1-w_2|^{-\frac{1}{2}}\\
&\qquad \times |v_1-v_2|^{-\frac{1}{2}} d\mathcal{H}^3(w_1,w_2)d\mathcal{H}^3(v_1,v_2)\\
&\lesssim |t-t'|^{m-\frac{3}{2}} \int_{F\times F\cap P} |w_2-v_2|^{-\frac{1}{2}}|w_1-w_2|^{-\frac{1}{2}}|v_1-v_2|^{-\frac{1}{2}} d\mathcal{H}^3(w_1,w_2)d\mathcal{H}^3(v_1,v_2)\\
&\lesssim |t-t'|^{\frac{m-1}{2}+\frac{m-2}{2}} \left(\int_{F\times F\cap P} |w_2-v_2|^{-\frac{3}{2}} d\mathcal{H}^3(w_1,w_2)d\mathcal{H}^3(v_1,v_2)\right)^{\frac{1}{3}}\\
&\qquad \times \left(\int_{F\times F\cap P} |w_1-w_2|^{-\frac{3}{2}} d\mathcal{H}^3(w_1,w_2)\right)^{\frac{1}{3}} \left(\int_{F\times F\cap P} |v_1-v_2|^{-\frac{3}{2}} d\mathcal{H}^3(v_1,v_2)\right)^{\frac{1}{3}}\\
&\lesssim |t-t'|^{\frac{m-1}{2}}.
\end{aligned}$$

For I_2, we have from (5.60) that

$$\begin{aligned}
I_2 &\lesssim |t-t'|^{\frac{m-1}{2}} \int_{(F\times F)\setminus P} |w_1-w_2|^{-\frac{1}{2}}|v_1-v_2|^{-\frac{1}{2}} d\mathcal{H}^3(w_1,w_2)d\mathcal{H}^3(v_1,v_2)\\
&\lesssim |t-t'|^{\frac{m-1}{2}} \left(\int_{(F\times F)\setminus P} |w_1-w_2|^{-1} d\mathcal{H}^3(w_1,w_2)\right)^{\frac{1}{2}}\\
&\qquad \times \left(\int_{(F\times F)\setminus P} |v_1-v_2|^{-1} d\mathcal{H}^3(v_1,v_2)\right)^{\frac{1}{2}}\\
&\lesssim |t-t'|^{\frac{m-1}{2}},
\end{aligned}$$

where we use the Hölder inequality along with Lemma 5.7. Hence, we arrive at

$$\|S_2(t)-S_2(t')\|_{L^2(\Omega)} \lesssim |t-t'|^{\frac{m-1}{2}},$$

which shows that (5.57) holds true. By the previous argument we have that (5.48) holds for this case. The proof is completed. ∎

With Lemmas 5.2, 5.3, and 5.6, we are able to prove Theorem 5.1.

Proof. Noting Lemmas 5.2 and 5.3, we have

$$|\boldsymbol{u}_1(x,\omega) - \boldsymbol{u}_{1,r}(x,\omega)| \lesssim \omega^{-2+\varepsilon},$$

which gives

$$\frac{1}{Q-1}\int_1^Q \omega^{m+1}|\boldsymbol{u}_1(x,\omega)|^2 d\omega \lesssim \frac{2}{Q-1}\int_1^Q \omega^{m+1}|\boldsymbol{u}_{1,r}(x,\omega)|^2 d\omega + \frac{2}{Q-1}\int_1^Q \omega^{m-3+2\varepsilon} d\omega.$$

It is easy to note that

$$\frac{2}{Q-1}\int_1^Q \omega^{m-3+2\varepsilon} d\omega \to 0$$

for $m \in [2, 5/2)$ and small enough ε. By Lemma 5.6, we get

$$\lim_{Q\to\infty} \frac{1}{Q-1}\int_1^Q \omega^{m+1}|\boldsymbol{u}_{1,r}(x,\omega)|^2 d\omega = 0, \quad x \in U.$$

Hence

$$\lim_{Q\to\infty} \frac{1}{Q-1}\int_1^Q \omega^{m+1}|\boldsymbol{u}_1(x,\omega)|^2 d\omega = 0, \quad x \in U.$$

The proof is completed. ∎

In the proof of Theorem 5.1, (5.2) corresponding to $\boldsymbol{u}_1$ involves $\boldsymbol{u}_{1,l}, \boldsymbol{u}_{1,r}$, $I_{jkl}^{(i)}$, $\{g_0, g_1, g_2\}$, $\{g_{11}, g_{12}\}$,$\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$, $g(x,\omega)$, $g_\delta(x,\omega)$, $S_\delta(t)$, $\{S_1, S_2\}$, and $\{I_1, I_2\}$. In the following, we present a chart to summarize the major steps of the proof.

Step 1: $\boldsymbol{u}_1 \xrightarrow{\textit{replace the left } G \textit{ in } \boldsymbol{u}_1 \textit{ by } G_0} \boldsymbol{u}_{1,l} \xrightarrow{\textit{replace the right } G \textit{ in } \boldsymbol{u}_{1,l} \textit{ by } G_0} \boldsymbol{u}_{1,r}$.

Step 2: $\boldsymbol{u}_{1,l}(x,\omega) - \boldsymbol{u}_{1,r}(x,\omega) = \left(\sum_{j,k,l=1}^2 I_{jkl}^{(1)}, \sum_{j,k,l=1}^2 I_{jkl}^{(2)}\right)^\top$.

Step 3: $I_{111}^{(1)} := \int_D\int_D \boldsymbol{G}_{0,11}(x,y,\omega)\boldsymbol{M}_{11}(y)\left(\boldsymbol{G}_{11}(y,z,\omega) - \boldsymbol{G}_{0,11}(y,z,\omega)\right) f_1(z)dydz$.

Step 4: $G_{11} - G_{0,11} = g_0 + g_1 + g_2, \quad g_1 = g_{11} + g_{12}$.

Step 5: $\boldsymbol{u}_{1,r} = \omega^{-1}\boldsymbol{v_1} + \omega^{-2}\boldsymbol{v_2} + \omega^{-3}\boldsymbol{v_3}$.

Step 6: $\boldsymbol{v}_1$ is a linear combination of g with

$$g(x,\omega) = \int_D\int_D e^{\mathrm{i}\omega(c_1|x-y|+c_2|y-z|)} \times \frac{(x_1-y_1)^{p_1}(x_2-y_2)^{p_2}(y_1-z_1)^{p_3}(y_2-z_2)^{p_4}}{|x-y|^{l_1}|y-z|^{l_2}} q(y)\tilde{f}(z)dydz.$$

Case 1: $m = 2$, $g \xrightarrow{mollification} g_\delta \xrightarrow{Fourier\ \ transform} S_\delta$;

Case 2: $m \in (2, \frac{5}{2})$, $g \xrightarrow{Fourier\ transform} S \xrightarrow{replace\ T_t\ by\ T} S_1 \xrightarrow{replace\ N_t\ by\ N} S_2$.

Step 7: $\|S_2(t) - S_2(t')\|_{L^2(\Omega)} = I_1 + I_2$.

With the convergence of the Born approximation, using Theorems 4.2 and 5.1, we are ready to show the proof of Theorem 1.4.

Proof. Recall the convergence of the Born approximation

$$\boldsymbol{u}(x,\omega) = \boldsymbol{u}_0(x,\omega) + \boldsymbol{u}_1(x,\omega) + \boldsymbol{b}(x,\omega),$$

where $\boldsymbol{b}(x,\omega)=\sum_{n=2}^{\infty}\boldsymbol{u}_n(x,\omega)$. It follows from (4.4) that

$$\|\boldsymbol{b}(x,\omega)\|_{L^\infty(U)^2}\lesssim\omega^{-2+\varepsilon'}$$

for some small enough $\varepsilon'>0$. So

$$\frac{1}{Q-1}\int_1^Q\omega^{m+1}|\boldsymbol{b}(x,\omega)|^2\mathrm{d}\omega\lesssim\frac{1}{Q-1}\int_1^Q\omega^{m-3+2\varepsilon'}d\omega\to 0 \tag{5.61}$$

as $Q\to\infty$, where we use the fact $m\in(2,5/2)$. Recalling Theorems 4.2 and 5.1, we have

$$\lim_{Q\to\infty}\frac{1}{Q-1}\int_1^Q\omega^{m+1}|\boldsymbol{u}_0(x,\omega)|^2d\omega=a\int_{\mathbb{R}^2}\frac{1}{|x-y|}\phi(y)dy, \tag{5.62}$$

$$\lim_{Q\to\infty}\frac{1}{Q-1}\int_1^Q\omega^{m+1}|\boldsymbol{u}_1(x,\omega)|^2d\omega=0 \tag{5.63}$$

hold almost surely, where a is a constant given in Theorem 1.4. Since

$$\begin{aligned}|\boldsymbol{u}(x,\omega)|^2&=|\boldsymbol{u}_0(x,\omega)|^2+|\boldsymbol{u}_1(x,\omega)|^2+|\boldsymbol{b}(x,\omega)|^2\\&\quad+\ 2\Re[\boldsymbol{u}_0(x,\omega)\overline{\boldsymbol{u}_1(x,\omega)}]+2\Re[\boldsymbol{u}_0(x,\omega)\overline{\boldsymbol{b}(x,\omega)}]+2\Re[\boldsymbol{u}_1(x,\omega)\overline{\boldsymbol{b}(x,\omega)}],\end{aligned}$$

along with (5.61)–(5.63) and the Cauchy–Schwartz inequality, it is to easy to verify that

$$\lim_{Q\to\infty}\frac{1}{Q-1}\int_1^Q\omega^{m+1}|\boldsymbol{u}(x,\omega)|^2d\omega=a\int_{\mathbb{R}^2}\frac{1}{|x-y|}\phi(y)dy.$$

By Lemma 3.8 in [30], we know that the integral $\int_{\mathbb{R}^2}\frac{1}{|x-y|}\phi(y)dy$ for all $x\in U$ can uniquely determine the function ϕ. The proof is completed. ∎

6. Conclusion. We have studied the inverse random source scattering problem for the two-dimensional elastic wave equation with a linear load. The source is modeled as a generalized Gaussian random function and its covariance operator is described as a classical pseudodifferential operator. Both the direct and the inverse problems are considered. The direct problem is equivalently formulated as a Lippmann–Schwinger integral equation which is shown to have a unique solution. Combining the Born approximation and microlocal analysis, we deduce a relationship between the principal symbol of the covariance operator for the random source and the amplitude of the displacement generated from a single realization of the random source. Based on this connection, we obtain the uniqueness for the reconstruction of the principal symbol of the random source. In this paper, the linear load is considered to be a smooth deterministic matrix. An ongoing project is to study the direct and inverse scattering problems when both the source and the linear load are random. Another challenging problem is to study the random source scattering problem for the three-dimensional elastic wave equation. We hope to be able to report the progress elsewhere in the future.

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