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A FRAMEWORK FOR SIMULATION OF MULTIPLE ELASTIC SCATTERING IN TWO DIMENSIONS*

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Abstract. Consider the elastic scattering of a time-harmonic wave by multiple well-separated rigid particles with smooth boundaries in two dimensions. Instead of using the complex Green's tensor of the elastic wave equation, we utilize the Helmholtz decomposition to convert the boundary value problem of the elastic wave equation into a coupled boundary value problem of the Helmholtz equation. Based on single, double, and combined layer potentials with the simpler Green's function of the Helmholtz equation, we present three different boundary integral equations for the coupled boundary value problem. The well-posedness of the new integral equations is established. Computationally, a scattering matrix based method is proposed to evaluate the elastic wave for arbitrarily shaped particles. The method uses the local expansion for the incident wave and the multipole expansion for the scattered wave. The linear system of algebraic equations is solved by GMRES with fast multipole method (FMM) acceleration. Numerical results show that the method is fast and highly accurate for solving elastic scattering problems with multiple particles.

Key words. elastic wave equations, elastic obstacle scattering, boundary integral equations, fast multiple method, Helmholtz decomposition

AMS subject classifications. 35P25, 45A05, 74J20, 74B05

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1. Introduction.

The scattering problems for elastic waves have recently received ever increasing attention in both the engineering and the mathematical communities for their important applications in geophysics and seismology [1, 3, 22, 35, 36, 42, 46]. The propagation of elastic waves is governed by the Navier equation, which is complex because of the coexistence of compressional and shear waves with different wavenumbers. In many applications, it is desirable to develop a computational model to simulate the wave propagation in a medium consisting of multiple particles [23, 24, 26, 38, 43], such as imaging a target in a cluttered environment [4] and designing composite materials with a specific wave response [16].

In this paper, we consider the two-dimensional elastic scattering problem of a time-harmonic wave by multiple smooth rigid obstacles which are embedded in a homogeneous and isotropic elastic medium. Compared to finite difference or finite element methods [5], the boundary integral method enjoys several intrinsic advantages: the solution is characterized solely in terms of surface distributions so that there are fewer unknowns, and the radiation condition is implicitly and exactly imposed so as to avoid the error that is introduced by using artificial radiation conditions [17, 19]. The well-posedness for the boundary integral formulations of elastic scattering problems can be found in [21, 31, 50]. However, the Green's function of the

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elastic wave equation is a second order tensor and is complicated to compute in the boundary integral equations. We refer the reader to [8, 11, 47] and references therein for some recent advances along this direction. To avoid this issue, we introduce two scalar potential functions and use the Helmholtz decomposition to split the displacement of the wave field into the compressional wave and the shear wave which satisfies the Helmholtz equation, respectively [51]. Therefore, the boundary value problem of the Navier equation is converted equivalently into a coupled boundary value problem of the Helmholtz equations for the potentials. By analyzing the properties of integral operators thoroughly and introducing appropriate regularizers, we prove the well-posedness for three different boundary integral formulations which are based on using the single, double, and combined layer potentials. The theoretical analysis lays a foundation on the numerical implementation of solving the elastic wave equation based on the Helmholtz decomposition.

In numerical practice, the advantages of boundary integral methods can be offset by the high computational cost incurred in evaluating the mutual interactions among all elements. Moreover, each interaction involves singular integrals whose analytical and/or numerical evaluation is expensive. In this work, we propose a fast and highly accurate numerical method for solving the elastic scattering problem with multiple particles. The method extends the classic multiple scattering theory for acoustic and electromagnetic waves to elastic waves. It can handle many particles that are arbitrarily shaped and randomly located in a homogeneous medium. The idea goes back to [16, 33, 32, 48, 49] for the electromagnetic scattering of multiple particles. For a given particle, we first use the integral formulation, which is based on the Helmholtz decomposition, to construct a scattering matrix. With this matrix precomputed, we then treat the outgoing scattering coefficients, instead of the discretization points on the boundary of particles, as the unknowns in our equation. Moreover, the resulting system based on outgoing coefficients can be preconditioned by the scattering matrix and the GMRES iterative solver becomes extremely efficient after the preconditioning. The algorithm is further accelerated by the fast multipole method (FMM) [44]. Numerical experiments show that the method is well suited for the elastic scattering problem with multiple particles. The idea of using the scattering matrix to replace the original scatterer is also called the T-matrix method in the literature [39]. The power of this method has been shown in the applications of computing the scattering in acoustics and electromagnetics from tens of thousands particles in both the two- and three-dimensional cases [7, 13, 14, 20, 28, 52]. In particular, a stable way was proposed in [12] to compute the scattering matrix for elongated particles. Theoretical analysis on how to choose the appropriate number of truncation terms in a scattering matrix was also studied in [15]. In this paper, we restrict ourselves to particles which do not have high aspect ratio in the two-dimensional case. Extensions to more general shaped particles and the three-dimensional case will be investigated in the future.

The paper is organized as follows. In section 2, we introduce the model equation for the elastic scattering by multiple obstacles. In particular, the Helmholtz decomposition is utilized to convert the elastic wave equation into a coupled Helmholtz system. Section 3 gives some preliminaries for boundary integral operators. Section 4 is devoted to three different boundary integral formulations for the coupled Helmholtz system. Their well-posedness is proved based on the regularization theory and the Fredholm alternative. In section 5, the scattering matrix based numerical method is proposed for solving the coupled integral equation. Numerical experiments are presented in section 6 to show the performance of the proposed method. The paper is concluded with some general remarks in section 7.

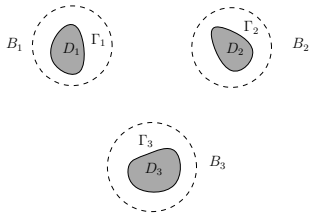

FIG. 1. *Problem geometry of the elastic scattering by multiple obstacles.*

2. Problem formulation. Let us first specify the problem geometry which is shown in Figure 1. Consider the scattering problem for many two-dimensional elastically rigid obstacles, the union of which is represented by a bounded domain D with boundary Γ. The infinite exterior domain $\mathbb{R}^2 \setminus \overline{D}$ is assumed to be filled with a homogeneous and isotropic elastic medium. In particular, we assume that the domain D consists of M inclusions D_j, $j = 1, \ldots, M$, which are bounded with smooth boundaries Γ_j, i.e., $D = \cup_{j=1}^{M} D_j$ and $\Gamma_j = \cup_{j=1}^{M} \Gamma_j$. Moreover, the obstacles are assumed to be well separated; i.e., there exist balls B_j such that $\overline{D}_j \subset B_j$, $j = 1, \ldots, M$, and $B_i \cap B_j = \emptyset$ for $i \neq j$. Denote by $\nu = (\nu_1, \nu_2)$ and $\tau = (\tau_1, \tau_2)$ the unit normal and tangential vectors on Γ, respectively, where $\tau_1 = -\nu_2$ and $\tau_2 = \nu_1$.

Let the obstacles be illuminated by a time-harmonic plane wave $\mathbf{u}^{\text{inc}}$, which satisfies the two-dimensional Navier equation

$$\mu \Delta \mathbf{u}^{\text{inc}} + (\lambda + \mu) \nabla \nabla \cdot \mathbf{u}^{\text{inc}} + \omega^2 \mathbf{u}^{\text{inc}} = 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{D},$$

where $\omega > 0$ is the angular frequency and μ, λ are the Lamé constants satisfying $\mu > 0, \lambda + \mu > 0$.

The displacement of the total wave field $\mathbf{u}$ also satisfies the Navier equation

$$\mu \Delta \mathbf{u} + (\lambda + \mu) \nabla \nabla \cdot \mathbf{u} + \omega^2 \mathbf{u} = 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{D}.$$

By assuming that each obstacle is impenetrable and rigid, we have

$$\mathbf{u} = 0 \quad \text{on } \Gamma.$$

The total field $\mathbf{u}$ consists of the incident field $\mathbf{u}^{\text{inc}}$ and the scattered field $\mathbf{v}$:

$$\mathbf{u} = \mathbf{u}^{\text{inc}} + \mathbf{v}.$$

It is easy to verify that the scattered field $\mathbf{v}$ satisfies the Navier equation

$$\mu \Delta \mathbf{v} + (\lambda + \mu) \nabla \nabla \cdot \mathbf{v} + \omega^2 \mathbf{v} = 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{D} \tag{2.1}$$

and the boundary condition

$$\mathbf{v} = -\mathbf{u}^{\text{inc}} \quad \text{on } \Gamma. \tag{2.2}$$

Given a vector function $\mathbf{w} = (w_1, w_2)$ and a scalar function w, define the scalar and vector curl operators

$$\text{curl}\mathbf{w} = \partial_{x_1} w_2 - \partial_{x_1} w_1, \quad \mathbf{curl} w = (\partial_{x_2} w, -\partial_{x_1} w).$$

For any solution $\mathbf{v}$ of the elastic wave equation (2.1), we introduce the Helmholtz decomposition

$$\mathbf{v} = \nabla\phi + \mathbf{curl}\psi, \tag{2.3}$$

where the scalar functions ϕ and ψ are called potentials. Substituting (2.3) into (2.1) yields

$$\nabla((\lambda+2\mu)\Delta\phi + \omega^2\phi) + \mathbf{curl}(\mu\Delta\psi + \omega^2\psi) = 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{D},$$

which is fulfilled if ϕ, ψ satisfy the Helmholtz equation

$$\Delta\phi + k_p^2\phi = 0, \quad \Delta\psi + k_s^2\psi = 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{D},$$

where $k_p = \omega/(\lambda+2\mu)^{1/2}$ and $k_s = \omega/\mu^{1/2}$ are the compressional wavenumber and the shear wavenumber, respectively. In addition, the potentials ϕ, ψ are required to satisfy the Sommerfeld radiation condition [45]

$$\partial_\rho\phi - \mathrm{i}k_p\phi = o(\rho^{-1/2}), \quad \partial_\rho\psi - \mathrm{i}k_s\psi = o(\rho^{-1/2}).$$

Combining (2.2) and (2.3) yields the boundary condition

$$\mathbf{v} = \nabla\phi + \mathbf{curl}\psi = -\mathbf{u}^{\mathrm{inc}} \quad \text{on } \Gamma.$$

Taking the dot product of the above equation with ν and τ, respectively, and noting that $\tau_1 = -\nu_2, \tau_2 = \nu_1$, we obtain a coupled boundary condition for ϕ, ψ on Γ:

$$\partial_\nu\phi + \partial_\tau\psi = f, \quad \partial_\tau\phi - \partial_\nu\psi = g, \tag{2.4}$$

where

$$f = -\nu \cdot \mathbf{u}^{\mathrm{inc}}, \quad g = -\tau \cdot \mathbf{u}^{\mathrm{inc}}.$$

Hence the obstacle scattering problem for elastic waves can be reduced equivalently to the coupled boundary value problem of the Helmholtz equations:

$$\begin{cases} \Delta\phi + k_p^2\phi = 0, \quad \Delta\psi + k_s^2\psi = 0 & \text{in } \mathbb{R}^2 \setminus \overline{D}, \\ \partial_\nu\phi + \partial_\tau\psi = f, \quad \partial_\tau\phi - \partial_\nu\psi = g & \text{on } \Gamma, \\ \partial_\rho\phi - \mathrm{i}k_p\phi = o(\rho^{-1/2}), \quad \partial_\rho\psi - \mathrm{i}k_s\psi = o(\rho^{-1/2}) & \text{as } \rho \to \infty. \end{cases} \tag{2.5}$$

The proof can be found in [37] for the well-posedness of the above scattering problem (2.5) by using the variational approach. In this work, our goal is to develop a new and well-posed boundary integral equation and propose a fast numerical method to the scattering problem (2.5). We first claim that the boundary value problem (2.5) has a unique solution.

THEOREM 2.1. *The coupled Helmholtz system* (2.5) *has at most one solution for* $k_s > 0$ *and* $k_p > 0$.

The proof is standard and given in the supplementary materials, linked from the main article webpage.

3. Preliminaries of integral operators. Let $\Gamma \subset \mathbb{R}^2$ be a smooth closed curve. Consider the integral operator of the form

$$F(x) = \int_\Gamma K(x, x-y)\phi(y)ds(y) \quad \text{for } x \in \Gamma \tag{3.1}$$

and its adjoint with respect to $L^2(\Gamma)$

$$G(x) = \int_\Gamma K(y, y-x)\phi(y)ds(y) \quad \text{for } x \in \Gamma, \tag{3.2}$$

where K is an integral kernel and ϕ is called the density. The following theorem can be found in [40].

THEOREM 3.1. *Let $\alpha = (\alpha_1, \alpha_2)$ be a multi-index and β be a positive integer. Assume that the kernel K in (3.1)–(3.2) is given by*

$$K(x, y) = h(x)y^\alpha |y|^{2\beta} \ln |y|,$$

where $h(x)$ is a smooth function defined on Γ. Then the kernel is of class $m = -(|\alpha| + 2\beta + 1)$. The integral operator in (3.1)–(3.2) associated with the kernel K is continuous from $H^r(\Gamma)$ into $H^{r+m}(\Gamma)$ for any real r.

Consider Green's function of the two-dimensional Helmholtz equation

$$\Phi_k(x, y) = \frac{\mathrm{i}}{4} H_0^{(1)}(k|x-y|),$$

where $H_0^{(1)}$ is the Hankel function of the first kind with order zero. Given a bounded domain $D \subset \mathbb{R}^2$ with smooth boundary Γ, let ν and τ be the exterior unit normal vector and the unit tangential vector of Γ, respectively. For $x \notin \Gamma$, define the single and double layer potentials

$$\mathcal{S}_k\phi(x) = \int_\Gamma \Phi_k(x, y)\phi(y)ds(y),$$
$$\mathcal{D}_k\phi(x) = \int_\Gamma \frac{\partial \Phi_k(x, y)}{\partial \nu(y)}\phi(y)ds(y)$$

and the tangential boundary layer potential

$$\mathcal{H}_k\phi(x) = \int_\Gamma \frac{\partial \Phi_k(x, y)}{\partial \tau(y)}\phi(y)ds(y).$$

For $k = 0$, these potentials denote the layer potentials corresponding to the two-dimensional Laplace equation where the Green function is $\Phi_0(x, y) = -\frac{1}{2\pi} \ln |x-y|$. Moreover, these potentials satisfy the well-known jump relations [10]:

$$\lim_{x \to \Gamma^\pm} \mathcal{S}_k\phi(x) = S_k\phi(x) = \int_\Gamma \Phi_k(x, y)\phi(y)ds(y), \tag{3.3a}$$

$$\lim_{x \to \Gamma^\pm} \mathcal{D}_k\phi(x) = \left(\pm\frac{1}{2} + D_k \right)\phi(x) = \pm\frac{1}{2}\phi(x) + \int_\Gamma \frac{\partial \Phi_k(x, y)}{\partial \nu(y)}\phi(y)ds(y), \tag{3.3b}$$

$$\lim_{x \to \Gamma^\pm} \frac{\partial \mathcal{S}_k\phi(x)}{\partial \nu(x)} = \left(\mp\frac{1}{2} + D_k' \right)\phi(x) = \mp\frac{1}{2}\phi(x) + \int_\Gamma \frac{\partial \Phi_k(x, y)}{\partial \nu(x)}\phi(y)ds(y), \tag{3.3c}$$

$$\lim_{x \to \Gamma^\pm} \mathcal{H}_k\phi(x) = H_k\phi(x) = \int_\Gamma \frac{\partial \Phi_k(x, y)}{\partial \tau(y)}\phi(y)ds(y), \tag{3.3d}$$

where the plus sign means that x approaches Γ from the exterior and the minus sign means that x approaches Γ from the interior. The boundary operators D_k, D_k', and H_k are defined in the sense of the Cauchy principal value. In $L^2(\Gamma)$, S_k is self-adjoint, i.e., $S_k = S_k'$, and D_k' is the adjoint of D_k. The adjoint of H_k is given by

$$H_k'\phi(x) = \frac{\partial S_k\phi(x)}{\partial\tau(x)} = \int_\Gamma \frac{\partial\Phi_k(x,y)}{\partial\tau(x)}\phi(y)ds(y).$$

For further investigation, it is indispensable to study the regularity of all these boundary operators. We begin with the asymptotic form of the Green function $\Phi_k(x,y)$ which can be found in [41].

LEMMA 3.2. *When $k > 0$, the Green function Φ_k has the expansion*

$$\Phi_k(x,y) = \Phi_0(x,y) - \frac{k^2|x-y|^2}{4}\Phi_0(x,y) + |x-y|^4 p_1(|x-y|^2)\Phi_0(x,y) + p_2(|x-y|^2),$$

where $p_1(x)$ and $p_2(x)$ are analytic functions.

The next lemma follows from the property of Cauchy integrals [30].

LEMMA 3.3. *Let $\Gamma \subset \mathbb{R}^2$ be a smooth curve. Then*

$$D_0^2 - H_0^2 = \frac{I}{4}, \quad H_0D_0 = -D_0H_0, \quad {D_0'}^2 - {H_0'}^2 = \frac{I}{4}, \quad H_0'D_0' = -D_0'H_0',$$

where I is the identity operator.

Proof. It was shown in [30, Theorem 7.12] that

$$(H_0 + \mathrm{i}D_0)^2 = -\frac{I}{4}. \tag{3.4}$$

Taking the real and imaginary parts of (3.4) gives the first and second identities, respectively. The remaining two identities follow from the duality argument. ∎

Combining Theorem 3.1 and Lemmas 3.2 and 3.3, we obtain several useful properties for the integral operators. The following results are related to the regularity of boundary operators D_k, H_k and their adjoint.

COROLLARY 3.4. *The following operators are bounded:*

$$\begin{aligned}
&D_0, D_0' : H^r(\Gamma) \to H^{r+s}(\Gamma),\\
&H_0, H_0' : H^r(\Gamma) \to H^r(\Gamma),\\
&D_k, D_k' : H^r(\Gamma) \to H^{r+3}(\Gamma),\\
&H_k, H_k' : H^r(\Gamma) \to H^r(\Gamma),
\end{aligned}$$

where r is an arbitrary real number and s is an arbitrary positive real number.

Proof. We first give the proof for the integral operators D_k and D_k'. It follows from Lemma 3.2 that the kernel D_k satisfies

$$\begin{aligned}
\frac{\partial\Phi_k(x,y)}{\partial\nu(y)} &= \frac{\partial\Phi_0(x,y)}{\partial\nu(y)} - \frac{|k|^2}{4}\frac{\partial(|x-y|^2\Phi_0(x,y))}{\partial\nu(y)} + O\left(\frac{\partial(|x-y|^4\Phi_0(x,y))}{\partial\nu(y)}\right)\\
&= K_0(x,y) - \frac{|k|^2}{4}K_1(x,y) + K_2(x,y),
\end{aligned}$$

where K_0 is the kernel for the integral operator D_0 and is of class $-\infty$, and K_2 is of class at most -4 by Theorem 3.1. A simple calculation yields

$$K_1(x,y) = -2(\nu(y)\cdot(x-y))\Phi_0(x,y) + K_0(x,y)|x-y|^2.$$

For a smooth curve Γ, it is shown in [10] that

$$\nu(y)\cdot(x-y) = O(|x-y|^2).$$

Hence it follows from Theorem 3.1 that K_1 is a kernel of class -3 and D_k is a bounded operator from $H^r(\Gamma)$ to $H^{r+3}(\Gamma)$. Similarly, we can show that the kernel of D_k' is also of class -3. The boundedness for H_0, H_0' follows from Lemma 3.3 directly. We complete the proof by noticing that for H_k and H_k', their differences with H_0 and H_0', respectively, are compact from $H^r(\Gamma)$ to $H^r(\Gamma)$. ∎

The following results are related to the properties of difference of boundary operators. The proof is similar to that for Corollary 3.4, so we omit it.

COROLLARY 3.5. *The following mappings are bounded:*

$$D_k - D_0, D_k' - D_0' : H^r(\Gamma) \to H^{r+3}(\Gamma),$$
$$H_k - H_0, H_k' - H_0 : H^r(\Gamma) \to H^{r+2}(\Gamma),$$

where r is an arbitrary real number.

In this paper, we mainly focus on functions in $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$, which are the trace spaces of $H^1(D)$ and $L^2(D)$, respectively. We also denote the vector function space with each component in $H^{1/2}(\Gamma)$ by $H^{1/2}(\Gamma)^2$. Similar notation applies to $H^s(\Gamma)$ for any real s. It is well known that the two-dimensional single layer boundary operator S_0, which is bounded from $H^s(\Gamma)$ to $H^{s+1}(\Gamma)$, is not invertible in general. However, we have the following result, which can be found in [30].

LEMMA 3.6. *There exists a constant $c > 0$, which only depends on the curve Γ, such that the operator $\overline{S}_0$, defined by*

$$(\overline{S}_0\phi)(x) = \int_\Gamma (\Phi_0(x,y) + c)\phi(y)ds(y) = S_0\phi(x) + \int_\Gamma c\phi(y)ds(y),$$

is invertible from $H^s(\Gamma)$ to $H^{s+1}(\Gamma)$ for any real s.

To this end, we denote the operator

$$\overline{\mathbf{S}}_0\overline{\mathbf{S}}_0 = \begin{bmatrix} \overline{S}_0\overline{S}_0 & 0 \\ 0 & \overline{S}_0\overline{S}_0 \end{bmatrix}$$

for a vector function $\mathbf{w} = (w_1, w_2) \in H^s(\Gamma)^2$ by $\overline{\mathbf{S}}_0\overline{\mathbf{S}}_0$. By Lemma 3.6, the operator $\overline{\mathbf{S}}_0\overline{\mathbf{S}}_0$ is invertible from $H^s(\Gamma)^2$ to $H^{s+2}(\Gamma)^2$.

4. Boundary integral equations. In this section, we derive boundary integral equations for the scattering problem (2.5) and show the well-posedness of the proposed boundary integral equations. For clarity, we restrict our discussion to the scattering of a single particle, which is still denoted by D with boundary Γ.

Define two single layer potentials corresponding to the compressional and shear wavenumbers:

$$\phi(x) = \mathcal{S}_{k_p}\alpha(x), \quad \psi(x) = \mathcal{S}_{k_s}\beta(x), \quad x \in \mathbb{R}^2\backslash\overline{D},$$

where $(\alpha(x), \beta(x)) \in H^{-1/2}(\Gamma)^2$ are densities. Let $x \to \Gamma$ from the exterior of D. Using the boundary condition (2.4) and the jump relations (3.3), we obtain the integral equation

$$A \begin{bmatrix} \alpha(x) \\ \beta(x) \end{bmatrix} = \begin{bmatrix} -\frac{I}{2} + D'_{k_p} & H'_{k_s} \\ H'_{k_p} & \frac{I}{2} - D'_{k_s} \end{bmatrix} \begin{bmatrix} \alpha(x) \\ \beta(x) \end{bmatrix} = \begin{bmatrix} f(x) \\ g(x) \end{bmatrix}, \tag{4.1}$$

where I is the identity operator. We first state the following existence result for (4.1).

THEOREM 4.1. *Assume that neither k_s nor k_p is the eigenvalue of the interior Dirichlet problem for the Helmholtz equation in D. Then the integral equation* (4.1) *has a unique solution in $H^{-1/2}(\Gamma)^2$.*

Remark 4.2. It is easy to see that $A = A_0 + K$, where

$$A_0 = \begin{bmatrix} -\frac{I}{2} & H'_0 \\ H'_0 & \frac{I}{2} \end{bmatrix} \tag{4.2}$$

is bounded in $H^{-1/2}(\Gamma)^2$ and K is a compact operator in $H^{-1/2}(\Gamma)^2$. If A_0 is invertible, the Fredholm alternative can be directly applied to show the invertibility of A. However, A_0 is degenerated in the sense that

$$A_0^2 = \begin{bmatrix} {D'}_0^2 & 0 \\ 0 & {D'}_0^2 \end{bmatrix}, \tag{4.3}$$

where D'_0 is a smooth operator by Corollary 3.4.

The existence result also holds for the integral representation by using double layer potentials

$$\phi(x) = \mathcal{D}_{k_p}\alpha(x), \quad \psi(x) = \mathcal{D}_{k_s}\beta(x), \quad x \in \mathbb{R}^2 \backslash \overline{D}.$$

Using the jump relations (3.3), we obtain the integral equation

$$M \begin{bmatrix} \alpha(x) \\ \beta(x) \end{bmatrix} = \begin{bmatrix} T_{k_p} & \frac{1}{2}\partial_\tau + \partial_\tau D_{k_s} \\ \frac{1}{2}\partial_\tau + \partial_\tau D_{k_p} & -T_{k_s} \end{bmatrix} \begin{bmatrix} \alpha(x) \\ \beta(x) \end{bmatrix} = \begin{bmatrix} f(x) \\ g(x) \end{bmatrix}, \tag{4.4}$$

where $T_k = \partial_\nu D_k$. The following existence result holds.

THEOREM 4.3. *If neither k_s nor k_p is the eigenvalue of the interior Neumann problem for the Helmholtz equation in D, the integral equation* (4.4) *has a unique solution in $H^{1/2}(\Gamma)^2$.*

To remove the assumption of Theorem 4.1 or 4.3, we propose a combined double and single layer representation to obtain a uniquely solvable integral system for any k_s and k_p. Consider the combined layer potentials

$$\phi(x) = (\mathcal{D}_{k_p} - \mathrm{i}\mathcal{S}_{k_p})\alpha(x), \quad \psi(x) = (\mathcal{D}_{k_s} - \mathrm{i}\mathcal{S}_{k_s})\beta(x), \quad x \in \mathbb{R}^2 \backslash \overline{D},$$

which results in a combined integral equation

$$(M - \mathrm{i}A) \begin{bmatrix} \alpha(x) \\ \beta(x) \end{bmatrix} = \left(\begin{bmatrix} T_{k_p} & \frac{1}{2}\partial_\tau + \partial_\tau D_{k_s} \\ \frac{1}{2}\partial_\tau + \partial_\tau D_{k_p} & -T_{k_s} \end{bmatrix} - \mathrm{i} \begin{bmatrix} -\frac{I}{2} + D'_{k_p} & H'_{k_s} \\ H'_{k_p} & \frac{I}{2} - D'_{k_s} \end{bmatrix} \right) \begin{bmatrix} \alpha(x) \\ \beta(x) \end{bmatrix} = \begin{bmatrix} f(x) \\ g(x) \end{bmatrix}. \tag{4.5}$$

We have the following existence result.

THEOREM 4.4. *For any $k_p > 0$ and $k_s > 0$, the integral equation* (4.5) *admits a unique solution in $H^{1/2}(\Gamma)^2$.*

Remark 4.5. In general, we may consider the following combined potentials:

$$\phi(x) = (\mathcal{D}_{k_p} - \mathrm{i}\eta\mathcal{S}_{k_p})\alpha(x), \quad \psi(x) = (\mathcal{D}_{k_s} - \mathrm{i}\eta\mathcal{S}_{k_s})\beta(x), \quad x \in \mathbb{R}^2\backslash\overline{D},$$

where $\eta > 0$ is a given coupling parameter. Depending on the wavenumbers k_p and k_s, the coupling parameter $\eta = 1$ may not be the optimal choice in terms of the condition number in the discretized system. We will not elaborate on this issue in this work, and we refer the reader to [6, 9, 29] for how to choose the optimal η for the acoustic scattering problems.

In what follows, we discuss the proof of Theorem 4.1 in detail and give the proofs of Theorems 4.3 and 4.4 in the supplementary materials.

4.1. Proof of Theorem 4.1. We first construct an appropriate regularizer for the operator A by considering the coupled Helmholtz system obtained from an interior Dirichlet elastic scattering problem:

$$\begin{cases} \Delta\phi + k_p^2\phi = 0, \quad \Delta\psi + k_s^2\psi = 0 & \text{in } D, \\ -\partial_\nu\phi + \partial_\tau\psi = f, \quad \partial_\tau\phi + \partial_\nu\psi = g & \text{on } \Gamma. \end{cases} \tag{4.6}$$

Compared with the system (2.5), the boundary condition here can be understood, as the compressional wavenumber k_p and shear wavenumber k_s are interchanged. It is constructed in order to regularize the operator A. The solutions ϕ and ψ of (4.6) are assumed to have the integral representations

$$\phi(x) = S_{k_p}\alpha(x), \quad (x)\psi = S_{k_s}\beta(x), \quad x \in \mathbb{R}^2\backslash\overline{D}.$$

Using the boundary condition, we obtain the following integral equation when $x \to \Gamma$:

$$B\begin{bmatrix}\alpha\\ \beta\end{bmatrix} = \begin{bmatrix}f\\ g\end{bmatrix}, \text{ where } B = \begin{bmatrix} -\frac{I}{2} - D'_{k_p} & H'_{k_s} \\ H'_{k_p} & \frac{I}{2} + D'_{k_s} \end{bmatrix}.$$

To prove Theorem 4.1, we also need to derive the adjoint operator of A in $L^2(\Gamma)^2$. By applying Green's identity to (4.6) and using the boundary condition in (4.6), we have

$$\left(-\frac{I}{2} + D_{k_p}\right)\phi - S_{k_p}\partial_\tau\psi = -S_{k_p}f,$$
$$\left(-\frac{I}{2} + D_{k_s}\right)\psi + S_{k_s}\partial_\tau\phi = S_{k_s}g.$$

Noting that $H_k\phi = -S_k\partial_\tau\phi$, we obtain

$$A'\begin{bmatrix}\phi\\ \psi\end{bmatrix} = \begin{bmatrix} -\frac{I}{2} + D_{k_p} & H_{k_p} \\ H_{k_s} & \frac{I}{2} - D_{k_s} \end{bmatrix}\begin{bmatrix}\phi\\ \psi\end{bmatrix} = \begin{bmatrix}-S_{k_p}f\\ -S_{k_s}g\end{bmatrix}.$$

It is easy to check that the operator A' is the adjoint of the operator A with respect to the bilinear form in $L^2(\Gamma)^2$ given by

$$\langle \mathbf{u}, \mathbf{v}\rangle = \int_\Gamma \big(u_1v_1 + u_2v_2\big)ds,$$

where $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$.

THEOREM 4.6. *For any vector function* $\mathbf{f} \in H^{-1/2}(\Gamma)^2$*, the operators* A, B *satisfy*

$$(AB)\mathbf{f} = \left(-\frac{(k_s^2+k_p^2)}{2}\overline{\mathbf{S}}_0\overline{\mathbf{S}}_0 + K_1\right)\mathbf{f},$$
$$(BA)\mathbf{f} = \left(-\frac{(k_s^2+k_p^2)}{2}\overline{\mathbf{S}}_0\overline{\mathbf{S}}_0 + K_2\right)\mathbf{f},$$

where K_1, K_2 *are compact operators from* $H^{-1/2}(\Gamma)^2$ *to* $H^{3/2}(\Gamma)^2$.

Proof. It follows from a straightforward calculation that

$$AB = \begin{bmatrix} -\frac{I}{2} + D'_{k_p} & H'_{k_s} \\ H'_{k_p} & \frac{I}{2} - D'_{k_s} \end{bmatrix} \begin{bmatrix} -\frac{I}{2} - D'_{k_p} & H'_{k_s} \\ H'_{k_p} & \frac{I}{2} + D'_{k_s} \end{bmatrix}$$
$$= \begin{bmatrix} \frac{I}{4} - (D'_{k_p})^2 + H'_{k_s}H'_{k_p} & D'_{k_p}H'_{k_s} + H'_{k_s}D'_{k_s} \\ -D'_{k_s}H'_{k_p} - H'_{k_p}D'_{k_p} & \frac{I}{4} - (D'_{k_s})^2 + H'_{k_p}H'_{k_s} \end{bmatrix}.$$

We first look at the off-diagonal elements. It can be verified that

$$D'_{k_p}H'_{k_s} + H'_{k_s}D'_{k_s} = (D'_{k_p} - D'_0)H'_0 + D'_{k_p}(H'_{k_s} - H'_0) + H'_{k_s}(D'_{k_s} - D'_0) + (H'_{k_s} - H'_0)D'_0,$$

where $D'_0H'_0 + H'_0D'_0$ vanishes due to Lemma 3.3. It follows from Corollaries 3.4 and 3.5 that $(D'_{k_p} - D'_0)H'_0, H'_{k_s}(D'_{k_s} - D'_0)$ are bounded operators from $H^{-1/2}(\Gamma)$ to $H^{-1/2+3}(\Gamma)$ and $D'_{k_p}(H'_{k_s} - H'_0), (H'_{k_s} - H'_0)D'_0$ are bounded operators from $H^{-1/2}(\Gamma)$ to $H^{-1/2+5}(\Gamma)$. Therefore, $D'_{k_p}H'_{k_s} + H'_{k_s}D'_{k_s}$ is a compact operator from $H^{-1/2}(\Gamma)$ to $H^{3/2}(\Gamma)$. Similarly, we can show that $-D'_{k_s}H'_{k_p} - H'_{k_p}D'_{k_p}$ is also a compact operator from $H^{-1/2}(\Gamma)$ to $H^{3/2}(\Gamma)$.

Next we check the diagonal elements. Using Lemma 3.3, we obtain

$$\frac{I}{4} - (D'_{k_p})^2 + H'_{k_s}H'_{k_p} = (H'_{k_s} - H'_0)H'_0 + H'_{k_s}(H'_{k_p} - H'_0) - (D'_{k_p})^2 + (D'_0)^2.$$

From Corollaries 3.4 and 3.5, the operators $(D'_{k_p})^2$, $(D'_0)^2$ are bounded from $H^{-1/2}(\Gamma)$ to $H^{-1/2+6}(\Gamma)$. Hence they are both compact from $H^{-1/2}(\Gamma)$ to $H^{3/2}(\Gamma)$. Consider the operator

$$(H'_{k_s} - H'_0)H'_0 + H'_{k_s}(H'_{k_p} - H'_0). \tag{4.7}$$

Clearly, it is bounded from $H^{-1/2}(\Gamma)$ to $H^{3/2}(\Gamma)$. Using the asymptotic form in Lemma 3.2, we have the following decomposition:

$$(H'_{k_s} - H'_0)H'_0\phi(x) = -\frac{\partial}{\partial\tau(x)}\int_\Gamma \frac{|k_s(x-y)|^2}{4}\Phi_0(x,y) \times \frac{\partial}{\partial\tau(y)}\int_\Gamma \Phi_0(y,z)\phi(z)ds(z)ds(y) + K_1\phi(x),$$

where K_1 is a compact operator from $H^{-1/2}(\Gamma)$ to $H^{3/2}(\Gamma)$. For the first operator in the right-hand side of $(H'_{k_s} - H'_0)H'_0$, we note that

$$\begin{aligned}
&-\frac{\partial}{\partial\tau(x)}\int_\Gamma \frac{|k_s(x-y)|^2}{4}\Phi_0(x,y)\frac{\partial}{\partial\tau(y)}\int_\Gamma \Phi_0(y,z)\phi(z)ds(z)ds(y)\\
&=\int_\Gamma\left(\frac{\partial}{\partial\tau(x)}+\frac{\partial}{\partial\tau(y)}\right)\left(\frac{\partial}{\partial\tau(y)}\frac{|k_s(x-y)|^2}{4}\Phi_0(x,y)\right)\int_\Gamma \Phi_0(y,z)\phi(z)ds(z)ds(y)\\
&\quad-\int_\Gamma\frac{\partial}{\partial\tau(y)}\left(\frac{\partial}{\partial\tau(y)}\frac{|k_s(x-y)|^2}{4}\Phi_0(x,y)\right)\int_\Gamma \Phi_0(y,z)\phi(z)ds(z)ds(y)\\
&=M\phi(x)+N\phi(x),
\end{aligned}$$

where M denotes the first operator and N denotes the second one.

We show that M is a compact operator from $H^{-1/2}(\Gamma)$ to $H^{3/2}(\Gamma)$. In fact, it holds that

$$\begin{aligned}
&\left(\frac{\partial}{\partial\tau(x)}+\frac{\partial}{\partial\tau(y)}\right)\left(\frac{\partial}{\partial\tau(y)}\frac{|k_s(x-y)|^2}{4}\Phi_0(x,y)\right)\\
&=-\frac{k_s^2}{4\pi}(1-\tau(x)\tau(y))\ln(|x-y|)+O((x-y)\ln(|x-y|))\\
&=O((x-y)\ln(|x-y|)).
\end{aligned}$$

By Theorem 3.1, M is bounded from $H^{-1/2}(\Gamma)$ to $H^{5/2}(\Gamma)$, which implies that M is compact from $H^{-1/2}(\Gamma)$ to $H^{3/2}(\Gamma)$. For the operator N, it is clear to note that

$$\begin{aligned}
&-\frac{\partial}{\partial\tau(y)}\left(\frac{\partial}{\partial\tau(y)}\frac{|k_s(x-y)|^2}{4}\Phi_0(x,y)\right)\\
&=-\frac{k_s^2}{2}\Phi_0(x-y)+O((x-y)\ln(|x-y|)).
\end{aligned}$$

Therefore,

$$N\phi(x)=-\frac{k_s^2}{2}S_0S_0\phi(x)+K\phi(x), \tag{4.8}$$

where K is compact from $H^{-1/2}(\Gamma)$ to $H^{3/2}(\Gamma)$. Similarly, we may show that

$$\left(H'_{k_s}(H'_{k_p}-H'_0)\right)\phi(x)=-\frac{k_p^2}{2}S_0S_0\phi(x)+K\phi(x).$$

Combining (4.7)–(4.8) yields that

$$\left(\frac{I}{4}-(D'_{k_p})^2+H'_{k_s}H'_{k_p}\right)\phi=\left(-\frac{(k_s^2+k_p^2)}{2}S_0S_0+K\right)\phi,$$

where K is a compact operator from $H^{-1/2}(\Gamma)$ to $H^{3/2}(\Gamma)$. Following the same argument, we have

$$\left(\frac{I}{4}-(D'_{k_s})^2+H'_{k_p}H'_{k_s}\right)\phi=\left(-\frac{(k_s^2+k_p^2)}{2}S_0S_0+K\right)\phi,$$

which proves the first part of the theorem since $\overline{S}_0$ and S_0 only differ by a smooth operator.

For the second part, we have from straightforward calculations that

$$BA = \begin{bmatrix} -\frac{I}{2} - D'_{k_p} & H'_{k_s} \\ H'_{k_p} & \frac{I}{2} + D'_{k_s} \end{bmatrix} \begin{bmatrix} -\frac{I}{2} + D'_{k_p} & H'_{k_s} \\ H'_{k_p} & \frac{I}{2} - D'_{k_s} \end{bmatrix}$$
$$= \begin{bmatrix} \frac{I}{4} - (D'_{k_p})^2 + H'_{k_p} H'_{k_s} & -D'_{k_p} H'_{k_s} - H'_{k_s} D'_{k_s} \\ D'_{k_s} H'_{k_p} + H'_{k_p} D'_{k_p} & \frac{I}{4} - (D'_{k_s})^2 + H'_{k_p} H'_{k_s} \end{bmatrix}.$$

The rest of the proof is the same as the first part and is omitted here. ∎

Next we consider the adjoint operator A' and introduce the operator

$$B' = \begin{bmatrix} -\frac{I}{2} - D_{k_p} & H_{k_p} \\ H_{k_s} & \frac{I}{2} + D_{k_s} \end{bmatrix},$$

which is the adjoint of operator B in $L^2(\Gamma)^2$. Following exactly the same argument, we have the following result.

THEOREM 4.7. *For any vector function* $\mathbf{f} \in H^{1/2}(\Gamma)^2$*, the operators* A', B' *satisfy*

$$(A'B')\mathbf{f} = \left(-\frac{(k_s^2 + k_p^2)}{2} \overline{\mathbf{S}}_0 \overline{\mathbf{S}}_0 + K_1 \right) \mathbf{f},$$
$$(B'A')\mathbf{f} = \left(-\frac{(k_s^2 + k_p^2)}{2} \overline{\mathbf{S}}_0 \overline{\mathbf{S}}_0 + K_2 \right) \mathbf{f},$$

where K_1, K_2 *are compact operators from* $H^{1/2}(\Gamma)^2$ *to* $H^{5/2}(\Gamma)^2$*.*

Since $\overline{\mathbf{S}}_0 \overline{\mathbf{S}}_0$ is invertible from $H^s(\Gamma)^2$ to $H^{s+2}(\Gamma)^2$ with $s \in \mathbb{R}$, by the Fredholm alternative, the operators A and A' have finite-dimensional null spaces and their ranges are given by

$$\mathrm{Ran}(A) = \{\mathbf{f} \in H^{-1/2}(\Gamma)^2 : \langle \mathbf{f}, \mathbf{g} \rangle = 0,\ \mathbf{g} \in \mathrm{Ker}(A')\},$$
$$\mathrm{Ran}(A') = \{\mathbf{f} \in H^{1/2}(\Gamma)^2 : \langle \mathbf{f}, \mathbf{h} \rangle = 0,\ \mathbf{h} \in \mathrm{Ker}(A)\}.$$

The kernels of A and A' are given in the following theorem.

THEOREM 4.8. *If neither* k_s *nor* k_p *is the eigenvalue of the interior Dirichlet problem for the Helmholtz equation in* D*, then* $\mathrm{Ker}(A) = \mathrm{Ker}(A') = \{0\}$*.*

Proof. Assume $(\alpha(x), \beta(x)) \in H^{-1/2}(\Gamma)^2$ satisfies

$$A \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Let

$$\phi(x) = \mathcal{S}_{k_p} \alpha(x), \quad \psi(x) = \mathcal{S}_{k_s} \beta(x), \quad x \in \mathbb{R}^2 \setminus \Gamma.$$

Then (ϕ, ψ) satisfies (2.5) with $f = 0$, $g = 0$. By the uniqueness result in Theorem 2.1, it holds that

$$\phi(x) = \psi(x) = 0, \quad x \in \mathbb{R}^2 \setminus \overline{D}.$$

It follows from the continuity of the single layer potential that $\phi(x) = \psi(x) = 0$ for $x \in \Gamma^-$. Since neither k_s nor k_p is the eigenvalue of the interior Dirichlet problem in

D, we have $\phi(x) = \psi(x) = 0$ for $x \in D$. Using the jump relation of the double layer potential, we obtain $\alpha(x) = \beta(x) = 0$, which implies $\mathrm{Ker}(A) = \{0\}$.

Now assume $(\alpha(x), \beta(x)) \in \mathrm{Ker}(A')$. Let $x \in \mathbb{R}^2 \setminus \Gamma$, and consider

$$\phi(x) = \mathcal{D}_{k_p}\alpha(x) - \mathcal{S}_{k_p}\partial_\tau\beta(x),$$
$$\psi(x) = \mathcal{D}_{k_s}\beta(x) + \mathcal{S}_{k_s}\partial_\tau\alpha(x).$$

Since $\phi(x) = \psi(x) = 0$ when x approaches Γ from the interior, by assumption, it holds that $\phi(x) = \psi(x) = 0$ for $x \in D$. By Green's theorem, when x approaches Γ from the exterior, i.e., $x \to \Gamma^+$, we have

$$\phi(x) = \alpha(x), \quad \psi(x) = \beta(x), \tag{4.9a}$$
$$\partial_\nu\phi(x) = \partial_\tau\beta(x), \quad \partial_\nu\psi(x) = -\partial_\tau\alpha(x), \tag{4.9b}$$

which shows that ϕ and ψ satisfy (2.5) with $f = g = 0$. Therefore, by the uniqueness of the scattering problem, $\phi(x) = \psi(x) = 0$ in $\mathbb{R}^2 \setminus \overline{D}$. Following (4.9), it yields that $\alpha(x) = \beta(x) = 0$, which completes the proof. ∎

The well-posedness of the integral equation (4.1) follows immediately from the Fredholm alternative, which completes the proof of Theorem 4.1.

5. Numerical method. Multiple scattering of small particles, including mineral particles, liquid cloud particles, and biological microorganisms, is an important research topic in material sciences, climatology, and biomedical engineering. Classic multiple scattering theory, which will be mentioned below, is restricted to circular shaped particles. In practice, particles may be arbitrarily shaped and highly disordered. In this section, we introduce a fast numerical method for the elastic obstacle scattering with multiparticles that are noncircular and randomly located in a homogeneous and isotropic elastic background medium. Numerical methods can be found in [16, 32, 33] for the acoustic and electromagnetic scattering problems involving multiparticles.

5.1. Scattering of a single disk. Consider a rigid disk located in a homogeneous medium with Lamé constants given by λ and μ and the angular frequency given by ω. The corresponding compressional wavenumber is k_p, and the shear wavenumber is k_s. Let the disk be centered at the origin with radius R. Given an incident compressional wave u_p^{inc} and shear wave u_s^{inc}, one can expand them in terms of the Bessel functions, which is also called the local expansion:

$$u_p^{\mathrm{inc}}(r,\theta) = \sum_{n=-\infty}^{\infty} a_n J_n(k_p r) e^{\mathrm{i}n\theta}, \tag{5.1a}$$
$$u_s^{\mathrm{inc}}(r,\theta) = \sum_{n=-\infty}^{\infty} b_n J_n(k_s r) e^{\mathrm{i}n\theta}, \tag{5.1b}$$

where J_n is the Bessel function of order n. By the classic Mie theory, the exterior elastic scattered compressional and shear wave fields can be expanded by Hankel functions, which is also called the multipole expansion:

$$u_p^{\mathrm{s}}(r,\theta) = \sum_{n=-\infty}^{\infty} c_n H_n^{(1)}(k_p r) e^{\mathrm{i}n\theta}, \tag{5.2a}$$
$$u_s^{\mathrm{s}}(r,\theta) = \sum_{n=-\infty}^{\infty} d_n H_n^{(1)}(k_s r) e^{\mathrm{i}n\theta}, \tag{5.2b}$$

where $H_n^{(1)}$ is the Hankel function of the first kind with order n. Given the expansion coefficients $\{a_n\}$ and $\{b_n\}$ of the incident wave and the boundary conditions

$$\partial_\nu(u_p^{\text{inc}} + u_p^{\text{s}})|_{r=R} + \partial_\tau(u_s^{\text{inc}} + u_s^{\text{s}})|_{r=R} = 0,$$
$$\partial_\tau(u_p^{\text{inc}} + u_p^{\text{s}})|_{r=R} - \partial_\nu(u_s^{\text{inc}} + u_s^{\text{s}})|_{r=R} = 0,$$

we can easily find the expansion coefficients $\{c_n\}$ and $\{d_n\}$ of the scattered fields by solving a 2×2 linear system for each n:

$$\begin{bmatrix} k_p H_n^{(1)'}(k_p R) & \mathrm{i}n H_n^{(1)}(k_s R) \\ \mathrm{i}n H_n^{(1)}(k_p R) & -k_s H_n^{(1)'}(k_s R) \end{bmatrix} \begin{bmatrix} c_n \\ d_n \end{bmatrix} = - \begin{bmatrix} a_n k_p J_n'(k_p R) + \mathrm{i}n b_n J_n(k_s R) \\ \mathrm{i}n a_n J_n(k_p R) - b_n k_s J_n'(k_s R) \end{bmatrix}.$$

Explicitly, we have

$$\begin{bmatrix} c_n \\ d_n \end{bmatrix} = \mathscr{S}_n \begin{bmatrix} a_n \\ b_n \end{bmatrix},$$

where

$$\mathscr{S}_n = - \begin{bmatrix} k_p H_n^{(1)'}(k_p R) & \mathrm{i}n H_n^{(1)}(k_s R) \\ \mathrm{i}n H_n^{(1)}(k_p R) & -k_s H_n^{(1)'}(k_s R) \end{bmatrix}^{-1} \begin{bmatrix} k_p J_n'(k_p R) & \mathrm{i}n J_n(k_s R) \\ \mathrm{i}n J_n(k_p R) & -k_s J_n'(k_s R) \end{bmatrix}.$$

Note that the uniqueness result in Theorem 2.1 guarantees that the denominator part of $\mathscr{S}_n$ is always invertible if k_p and k_s are both positive, which implies $\mathscr{S}_n$ is well-defined. Therefore, we can give the following definition.

DEFINITION 5.1. *The mapping between the incoming coefficients $\{a_n\}$ and $\{b_n\}$ and outgoing coefficients $\{c_n\}$ and $\{d_n\}$ is referred to as the scattering matrix for the disk and is denoted by $\mathscr{S}$, i.e.,*

$$\begin{bmatrix} \{c_n\} \\ \{d_n\} \end{bmatrix} = \mathscr{S} \begin{bmatrix} \{a_n\} \\ \{b_n\} \end{bmatrix}.$$

5.2. Scattering of multiple disks. Now let's consider M $(M > 1)$ rigid disks with the same radius R. A global expansion for the exterior field which is done for a single disk does not hold anymore. However, if we assume that the disks are well separated, i.e., there exists a positive distance between any two disks, then the Mie series expansion still holds in the vicinity of each disk. For the mth disk, the field around it can be expanded in terms of the Hankel functions (5.2) with expansion coefficients $\{c_n^m\}$ and $\{d_n^m\}$. The incoming field has two components: the first one is the external incident field, as the case for a single disk, and the second one is the scattered field of the other disks. Therefore, in order to find $\{c_n^m\}$ and $\{d_n^m\}$, we need to solve the linear system

$$\begin{bmatrix} \mathscr{S}^{-1} & \mathscr{T}^{12} & \cdots & \mathscr{T}^{1M} \\ \mathscr{T}^{21} & \mathscr{S}^{-1} & \cdots & \mathscr{T}^{2M} \\ \vdots & \vdots & \ddots & \vdots \\ \mathscr{T}^{M1} & \mathscr{T}^{M2} & \cdots & \mathscr{S}^{-1} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \{c_n^1\} \\ \{d_n^1\} \end{bmatrix} \\ \begin{bmatrix} \{c_n^2\} \\ \{d_n^2\} \end{bmatrix} \\ \vdots \\ \begin{bmatrix} \{c_n^M\} \\ \{d_n^M\} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} \{a_n^1\} \\ \{b_n^1\} \end{bmatrix} \\ \begin{bmatrix} \{a_n^2\} \\ \{b_n^2\} \end{bmatrix} \\ \vdots \\ \begin{bmatrix} \{a_n^M\} \\ \{b_n^M\} \end{bmatrix} \end{bmatrix}, \tag{5.3}$$

where the matrix $\mathscr{T}^{ml}$, $m = 1, \ldots, M$, $l = 1, \ldots, M$, which maps the outgoing coefficients $\{c_n^l\}$ and $\{d_n^l\}$ of the lth disk to the incoming coefficients of the mth disk, is constructed based on the Graf addition theorem [44].

From Lemma 3.1 in [33], we see that the translation matrix $\mathscr{T}^{ml}$ has the form

$$\mathscr{T}^{ml} = \begin{bmatrix} \{a_{ij}^{ml}\}_{i,j\in\mathbb{Z}} & 0 \\ 0 & \{b_{ij}^{ml}\}_{i,j\in\mathbb{Z}} \end{bmatrix},$$

where

$$a_{ij}^{ml} = H_{i-j}^{(1)}(k_p|x_l - x_m|)e^{-\mathrm{i}(i-j)(\theta_{ml}-\pi)}, \quad b_{ij}^{ml} = H_{i-j}^{(1)}(k_s|x_l - x_m|)e^{-\mathrm{i}(i-j)(\theta_{ml}-\pi)}.$$

Since the scattering matrix $\mathscr{S}$ is ill-conditioned, it introduces large numerical errors if inverting $\mathscr{S}$ directly. In particular, $\mathscr{S}$ is not invertible when k_pR and k_sR happen to be the zero points of Bessel function J_n. The condition number also deteriorates when the particle has high aspect ratio [12]. A better way to solve the linear system (5.3) is to first introduce a block diagonal matrix, whose diagonal blocks are the scattering matrix $\mathscr{S}$, as the preconditioner of (5.3). Hence, instead of solving (5.3), we solve the preconditioned linear system

$$\begin{bmatrix} I & \mathscr{S}\mathscr{T}^{12} & \cdots & \mathscr{S}\mathscr{T}^{1M} \\ \mathscr{S}\mathscr{T}^{21} & I & \cdots & \mathscr{S}\mathscr{T}^{2M} \\ \vdots & \vdots & \ddots & \vdots \\ \mathscr{S}\mathscr{T}^{M1} & \mathscr{S}\mathscr{T}^{M2} & \cdots & I \end{bmatrix} \begin{bmatrix} \begin{bmatrix} \{c_n^1\} \\ \{d_n^1\} \end{bmatrix} \\ \begin{bmatrix} \{c_n^2\} \\ \{d_n^2\} \end{bmatrix} \\ \vdots \\ \begin{bmatrix} \{c_n^M\} \\ \{d_n^M\} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \mathscr{S}\begin{bmatrix} \{a_n^1\} \\ \{b_n^1\} \end{bmatrix} \\ \mathscr{S}\begin{bmatrix} \{a_n^2\} \\ \{b_n^2\} \end{bmatrix} \\ \vdots \\ \mathscr{S}\begin{bmatrix} \{a_n^M\} \\ \{b_n^M\} \end{bmatrix} \end{bmatrix}. \tag{5.4}$$

Due to the existence of positive distance between any two disks, the translation operators $\mathscr{T}^{ml}$, $m = 1, \ldots, M$, $l = 1, \ldots, M$, are compact and the scattering matrix $\mathscr{S}$ is bounded, which implies the system (5.4) is much better conditioned than the original system (5.3). Therefore, one can apply an iterative solver, such as GMRES, to the system (5.4) and expect a fast convergence rate. In particular, the system (5.4) is well-defined regardless of the invertibility of $\mathscr{S}$. For numerical purposes, all the infinite series $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, and $\{d_n\}$ need to be truncated to a finite number of terms with $N > 0$. Analysis on how to choose the right number of terms can be found in [15]. Since the linear matrix in (5.4) is dense, the direct matrix-vector product in each iteration takes a computational complexity on the order of $O(M^2)$ if the truncation number N is relatively small. In this case, the FMM can be utilized to reduce the complexity to $O(M)$ in each iteration and greatly accelerate the computation [18]. More complicated and effective preconditioning methods can be found in [25, 13].

5.3. Scattering of arbitrarily shaped multiple obstacles. The theory described above for the elastic scattering of multiple disks is based on the classic acoustic multiple scattering theory, which is efficient to find the scattering field for a large number of disks. It is not easy, however, to extend to the scattering of noncircular shaped particles. Here, we propose a fast algorithm for the scattering of a large number of arbitrarily shaped multiparticles, which are assumed to be well separated in the sense that each particle is included in a disk, and all the disks do not overlap. Given such an assumption, we construct the scattering matrix $\mathscr{S}$ for each particle based on the disk

that includes the particle and extend the multiple scattering theory to noncircular particles.

More specifically, given M randomly located particles D_j, $j = 1, \ldots, M$, each particle is included by a nonoverlapping disk B_j, $j = 1, \ldots, M$. We sample the incoming field on the disk B_j rather than D_j in the form of (5.1). In particular, for each $n \in \{-N, \ldots, N\}$, let α_n and β_n denote the solution to the integral equation (4.1) with its right-hand side given by (5.1), where we choose $a_n = 1$ or $b_n = 1$ with n sequentially being $-N, -N+1, \ldots, N-1, N$ and $a_m = 0$ or $b_m = 0$ for any $-N \le m \le N$ and $m \ne n$. Then we precompute the multipole expansion (5.2) from these source distributions, where

$$c_l^n = \int_{\Gamma_j} J_l(k_p|y|)e^{-\mathrm{i}l\theta_j(y)}\,\alpha_n(y)ds(y), \quad d_l^n = \int_{\Gamma_j} J_l(k_s|y|)e^{-\mathrm{i}l\theta_j(y)}\,\beta_n(y)ds(y),$$

$l = -N, \ldots, N$. Here, y is the location of a point on Γ_j with respect to the center of the disk B_j and $\theta_j(y)$ is the polar angle subtended with respect to the center of disk B_j. The formulas for c_l^n and d_l^n are standard [44] and derived from Graf's addition theorem [41]. Note that we only have to solve the integral equation (4.1) by the LU factorization or any other direct solver once and apply it to different hand sides. Once the computation is done for each $n \in \{-N, \ldots, N\}$, we obtain the scattering matrix $\mathscr{S}_j$ for D_j.

Remark 5.2. Here, we construct the scattering matrix for a given particle based on the integral formulation (4.1) under the assumption of Theorem 4.8. If this assumption is not satisfied, we can use either (4.4) or (4.5) to find the scattering matrix.

When the scattering matrix $\mathscr{S}_j$ for each particle is available, we plug them into the linear system (5.4) to find the elastic scattered field. The advantage of using the scattering matrix, instead of points that discretize each particle directly, is that the number of unknowns represented by multipole expansion coefficients is usually much less than the one represented by points, especially for particles with complicated geometries. Moreover, by using the scattering matrix, we obtain a much better conditioned system and GMRES can find the solution rapidly. In addition, if all the particles are identical up to a rotation, we only have to compute the scattering matrix $\mathscr{S}$ for one particle and apply it to all the other particles.

6. Numerical experiments. In this section, we test our algorithm by evaluating the elastic scattered field for a large number of identical particles embedded in a homogeneous and isotropic background. Particles tested in all the examples, up to a rotation and shift, are parametrized by

$$x(\theta) = (a + b\cos(c\theta))\cos\theta, \quad y(\theta) = (a + b\cos(c\theta))\sin\theta, \tag{6.1}$$

where $\theta \in [0, 2\pi)$ and the parameters a, b, c will be specified in each example. For simplicity, we fix the Lamé constants to be $\lambda = 3.88$ and $\mu = 2.56$ in all examples and change the angular frequency ω only. The choice of λ and μ is made such that $\sqrt{\lambda + 2\mu}$ and $\sqrt{\mu}$ are rational numbers, but any other choice can be made as long as $\lambda + 2\mu > 0$ and $\mu > 0$ are satisfied. In order to discretize the singular integral accurately, we use the Nyström discretization for the system of equations (4.1) based on the high order hybrid Gauss-trapezoidal rule of Alpert [2].

The following notations are given in Tables 1–4 to illustrate the results:

- ω: the angular frequency,
- N_{pts}: the number of points to discretize a single particle,

- $N_{particle}$: the total number of particles,
- N_{term}: the highest order used in the local and multipole expansions of a single particle, i.e., the local and multipole expansions have $2N_{term}+1$ terms,
- N_{tot}: the total number of unknowns in the linear equation. If the unknowns are given by points, then it is equal to $2N_{pts}N_{particle}$. If the unknowns are given by the coefficients of multipole expansion, then it is equal to $2(2N_{term}+1)N_{particle}$,
- N_{iter}: the number of GMRES iterations,
- T_{solve}: the time (secs.) to solve the linear system by GMRES,
- E_{error}: the relative L^2 error of the elastic field measured at 20 random points. They are chosen from a circle centered at $(0,-8)$ with radius 15.

All experiments were implemented in FORTRAN 90 and carried out on a laptop with an Intel CPU and 16 GB of memory. We made use of the simple LU factorization for matrix inversion when constructing the scattering matrix given in section 5.3. The accuracy for GMRES was chosen to be 1E-9. No further acceleration was explored during the GMRES iteration except for using the FMM.

6.1. Example 1: Scattering with an analytic solution. In this example, we consider the elastic scattering of 10 particles, denoted by D_i, $i=1,\dots,10$. Two methods are used for comparison. One method is to discretize all particles by points and apply the Nyström discretization to the integral equation directly. Since the number of unknowns is large, we do not explicitly assemble the matrix but solve it by the GMRES with the FMM acceleration. We call it a direct method. Another one is the proposed method by constructing a scattering matrix first and then solving for the coefficients of multipole expansion, which is called the scattering matrix based method. To verify the accuracy of these two methods, we construct an artificial solution by letting the field outside the particles be generated by a point source inside the first particle. In particular, we choose the exterior elastic field to be

$$\mathbf{u}(x)=\nabla u_p^s+\mathbf{curl}u_s^s=\begin{bmatrix}\partial_{x_1}u_p^s\\ \partial_{x_2}u_p^s\end{bmatrix}+\begin{bmatrix}\partial_{x_2}u_s^s\\ -\partial_{x_1}u_s^s\end{bmatrix},$$

where

$$u_p^s(x)=H_0^{(1)}(k_p|x-x_0|),\quad u_s^s(x)=H_0^{(1)}(k_s|x-x_0|),\quad x\in\mathbb{R}^2\setminus\cup_{j=1}^{10}D_j, \tag{6.2}$$

where $x_0\in D_1$. Due to the uniqueness, the solution can be recovered by enforcing a boundary condition on Γ_j, $j=1,\dots,10$, that is consistent with the given $\mathbf{u}$. More specifically, we choose the boundary conditions on Γ_j to be

$$\begin{cases} f_j(x)=\partial_\nu H_0^{(1)}(k_p|x-x_0|)+\partial_\tau H_0^{(1)}(k_s|x-x_0|),\\ g_j(x)=\partial_\tau H_0^{(1)}(k_p|x-x_0|)-\partial_\nu H_0^{(1)}(k_s|x-x_0|),\end{cases}\quad x\in\Gamma_j,\quad j=1,\dots,10.$$

With such boundary conditions, it is easy to see that solution (6.2) satisfies (2.5) and is the only solution by the uniqueness result given in Theorem 2.1.

We first check the accuracy for 10 disks with $a=1$, $b=0$, $c=0$ in the parametrization equation (6.1) and $\omega=4\pi$. Nine digit accuracy is obtained with 200 discretization points for each disk by comparing with the analytic solution. We then test the geometries with $b\neq 0$ and $c\neq 0$. Results for various angular frequencies are shown in Figure 2 and Tables 1–2. Table 1 shows the results of the direct method. It can be seen that the number of iterations grows rapidly when the number of discretization points increases. Since the integral equation (4.1) is not the second kind

in L^2 space, convergence rate based on the direct method is very slow due to the ill-conditioning of the matrix. As shown in Table 1, if each particle is discretized by 200 points, more than 1000 iterations are required for the GMRES to converge in all cases. Even with the FMM acceleration, the CPU time is on the order of hundreds of seconds, which suggests that the direct matrix factorization may be more efficient than the iterative method in this case. On the other hand, our scattering matrix based method always achieves a quick convergence in various cases. One important feature is that the convergence rate is almost constant for different N_{term} used in the multipole expansion if all the other factors are unchanged. If we ignore the cost for precomputation of the scattering matrix, which is constructed by solving the integral equation (4.1) with 200 discretization points, our solver is more than 1000 times faster than the direct method for the same accuracy.

Figure 2 shows the error of the computed elastic field compared with the analytic solution when $\omega = 4\pi$ by the scattering matrix based method. More specifically, the field is evaluated by

$$\begin{cases} u_p^{\mathrm{s}}(x) = \sum_{j=1}^{10} \sum_{n=-\infty}^{\infty} c_n^j H_n^{(1)}(k_p r_j) e^{\mathrm{i}n\theta_j}, \\ u_s^{\mathrm{s}}(x) = \sum_{j=1}^{10} \sum_{n=-\infty}^{\infty} d_n^j H_n^{(1)}(k_s r_j) e^{\mathrm{i}n\theta_j}, \end{cases} \quad x \in \mathbb{R}^2 \setminus \cup_{j=1}^{10} B_j, \tag{6.3}$$

where (r_j, θ_j) are the polar coordinates of x with respect to the center of B_j, and B_j is the disk that encloses D_j. Due to the near field singularity, the scattered field in $B_j \backslash D_j$ is evaluated by the QBX [27]; i.e., the field near the boundary of each particle is evaluated by the use of local expansions formed by the FMM. Overall, the error is less than 1E-7. We also show the comparison of convergence rate between the direct method and our method in Figures 2(c)–2(d). Obviously, one can see a much faster convergence for the scattering matrix based method.

6.2. Example 2: Point source incidence. In this example, we test our algorithm on a large number of rigid particles with point source incidence. The point source is given by the form of (6.2) with $x_0 = (5, 5)$, and all the particles are randomly located in the lower half plane. To ensure that the particles are well separated but confined in a fixed region, we use a bin sorting algorithm to construct the random distribution; i.e., we begin with particles located on a regular grid and then perturb their positions randomly several times. The details can be found in [33].

We construct the scattering matrix by solving the integral equation (4.1) on a single particle with 200 discretization points. The number of terms in the multipole expansion is chosen to be $N_{term} = 20$. To verify the accuracy of the computed solution, we compare it with the solution obtained by choosing $N_{term} = 40$. Numerical results for various angular frequencies are shown in Table 3 and Figure 3. From Table 3, we can see that the number of iterations grows roughly linearly with respect to the angular frequency ω for a fixed number of particles. If ω is fixed, the number of iterations increases sublinearly with respect to the number of particles. Another observation is that the field is mainly affected by the size of a particle, not by the detailed geometry, since the number of iterations is almost constant when we change the value of c, which controls how many "leaves" that a particle has. The total field plotted in Figure 3 for the scattering of 1000 particles also confirms this observation. We have to note, however, that this conclusion may only hold when the size of each particle is in a subwavelength regime for a given incident field.

6.3. Example 3: Plane incidence wave. For the third example, we evaluate the elastic scattered field of a large number of particles by a plane wave incidence,

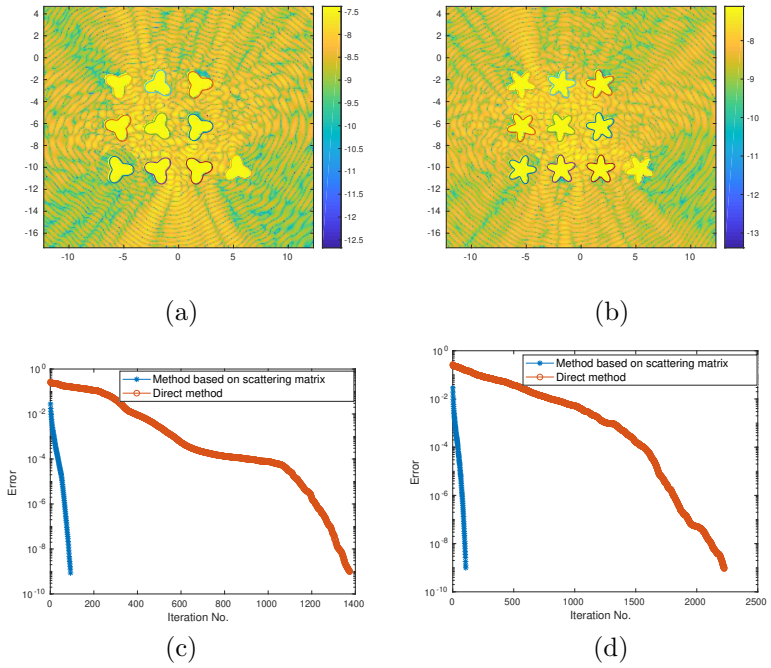

FIG. 2. *Elastic scattering of* 10 *particles at* $\omega = 4\pi$. (a) *The logarithmic error of the computed field when* $a = 1$, $b = 1/3$, $c = 3$. (b) *The logarithmic error of the computed field when* $a = 1$, $b = 1/3$, $c = 5$. (c) *Comparison of the GMRES convergence rate when* $a = 1$, $b = 1/3$, $c = 3$. (d) *Comparison of the GMRES convergence rate when* $a = 1$, $b = 1/3$, $c = 5$. *More details are given in the text of Example* 1.

TABLE 1

Example 1: *Results for the elastic scattering of* 10 *particles based on the direct method with the FMM acceleration.*

			$a=1$	$b=\frac{1}{3}$	$c=3$	$a=1$	$b=\frac{1}{3}$	$c=5$
ω	N_{pts}	N_{tot}	N_{iter}	T_{solve}	E_{err}	N_{iter}	T_{solve}	E_{err}
π	50	1000	930	7.96E1	5.79E-4	943	7.84E1	2.12E-2
	100	2000	1071	1.79E2	1.36E-6	1613	2.75E2	2.17E-7
	200	4000	2243	7.23E2	4.47E-10	2814	1.01E3	2.56E-9
2π	50	1000	930	7.92E1	5.79E-4	933	8.02E1	2.75E-3
	100	2000	1410	2.43E2	8.08E-7	1588	2.78E2	2.63E-5
	200	4000	2243	7.26E2	4.47E-10	2327	8.15E2	7.94E-9
4π	50	1000	847	7.42E1	2.12E-3	993	8.92E1	1.72E-2
	100	2000	939	1.61E2	1.04E-6	1670	3.01E2	5.27E-5
	200	4000	1374	4.29E2	8.11E-10	2225	8.22E2	2.47E-8

which is given by

$$\mathbf{u}^{\text{inc}}(x) = de^{\mathrm{i}k_p x\cdot d} + d^{\perp}e^{\mathrm{i}k_s x\cdot d},$$

where d is the propagation direction and $d^{\perp}$ is orthogonal to d. In our test, we choose $d = (\cos(-\frac{\pi}{3}), \sin(-\frac{\pi}{3}))$ and $d^{\perp} = (\sin(\frac{\pi}{3}), \cos(\frac{\pi}{3}))$. The locations of particles are randomly distributed in a fixed region which is the same as that in Example 2. The transformation of plane wave into the local expansion (5.1) is given by the Jacobi–Anger identity [41]. Numerical results for the plane wave incidence are given in Figure 4 and Table 4. Comparing the results between Tables 2 and 4, we find that

TABLE 2

Example 1: *Results for the elastic scattering of* 10 *particles by using the scattering matrix based method with the FMM acceleration.*

			$a=1$	$b=\frac{1}{3}$	$c=3$	$a=1$	$b=\frac{1}{3}$	$c=5$
ω	N_{term}	N_{tot}	N_{iter}	T_{solve}	E_{err}	N_{iter}	T_{solve}	E_{err}
π	10	420	63	4.01E-2	8.35E-6	67	4.02E-2	6.59E-6
	20	820	62	8.81E-2	4.71E-9	67	9.61E-2	6.17E-9
	40	1620	61	2.39E-1	5.75E-9	67	2.72E-1	6.84E-10
2π	10	420	74	4.81E-2	3.39E-5	88	6.01E-2	8.41E-5
	20	820	73	1.07E-1	6.49E-9	87	1.24E-1	5.32E-9
	40	1620	73	2.91E-1	2.09E-9	87	3.61E-1	2.31E-9
4π	10	420	94	6.39E-2	1.84E-2	104	7.21E-2	1.70E-2
	20	820	94	1.41E-1	7.85E-7	108	1.64E-1	1.72E-7
	40	1620	93	3.84E-1	3.42E-9	107	4.47E-1	1.12E-8

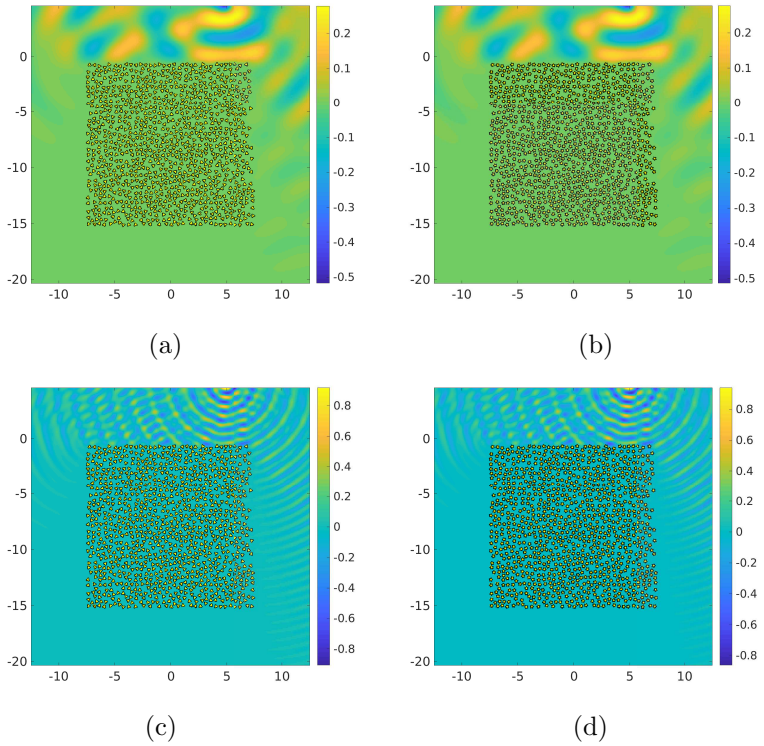

FIG. 3. *The elastic scattering of* 1000 *particles by point source illumination in Example* 2. *Here, we show the real part of the first component of the total elastic field.* (a) *Field for* $a=\frac{1}{8}$, $b=\frac{1}{24}$, $c=3$, $\omega=\pi$. (b) *Field for* $a=\frac{1}{8}$, $b=\frac{1}{24}$, $c=5$, $\omega=\pi$. (c) *Field for* $a=\frac{1}{8}$, $b=\frac{1}{24}$, $c=3$, $\omega=4\pi$. (d) *Field for* $a=\frac{1}{8}$, $b=\frac{1}{24}$, $c=5$, $\omega=4\pi$.

the number of iterations for the plane wave incidence is similar to the one with the point source incidence. In particular, the results for both the point source incidence and the plane wave incidence show that the number of iterations for GMRES does depend on the size of particles but is almost independent of the shape of particles. This fact is further illustrated in Figure 4 since the fields for two different kinds of particles look almost identical. Again, the conclusion may only hold if we restrict ourselves in the subwavelength regime. Another observation from Figure 4 is that

TABLE 3

Example 2: *Results for the elastic scattering of multiple particles by using the scattering matrix based method with the FMM acceleration.*

			$a=\frac{1}{8}$	$b=\frac{1}{24}$	$c=3$	$a=\frac{1}{8}$	$b=\frac{1}{24}$	$c=5$
ω	$N_{particle}$	N_{tot}	N_{iter}	T_{solve}	E_{err}	N_{iter}	T_{solve}	E_{err}
π	100	8400	57	3.51E0	1.38E-9	59	3.41E0	1.62E-10
	500	42000	130	5.28E1	9.39E-10	131	5.21E1	2.21E-9
	1000	82000	210	1.34E2	9.62E-10	212	1.36E2	2.81E-9
2π	100	8400	96	6.09E0	1.64E-9	97	5.86E0	4.24E-9
	500	42000	249	1.05E2	4.10E-9	251	1.06E2	3.71E-9
	1000	84000	347	2.48E2	1.03E-9	353	2.53E2	3.34E-9
4π	100	8400	271	1.79E1	4.22E-9	255	1.65E1	6.31E-9
	500	42000	614	3.09E2	1.63E-9	667	3.43E2	2.54E-9
	1000	84000	1197	1.31E3	7.69E-10	1211	1.34E3	7.42E-9

TABLE 4

Example 3: *Results for the elastic scattering of multiple particles by using the scattering matrix based method with the FMM acceleration.*

			$a=\frac{1}{8}$	$b=\frac{1}{24}$	$c=3$	$a=\frac{1}{8}$	$b=\frac{1}{24}$	$c=5$
ω	$N_{particle}$	N_{tot}	N_{iter}	T_{solve}	E_{err}	N_{iter}	T_{solve}	E_{err}
π	100	8400	64	4.49E0	1.75E-9	66	3.68E0	8.66E-10
	500	42000	140	5.62E1	2.21E-9	142	5.79E1	3.33E-9
	1000	82000	221	1.47E2	2.44E-9	224	1.55E2	4.97E-9
2π	100	8400	101	7.36E0	7.76E-10	103	6.34E0	3.85E-9
	500	42000	271	1.17E2	1.32E-9	273	1.19E2	7.06E-9
	1000	84000	384	2.93E2	2.09E-9	391	2.93E2	1.01E-8
4π	100	8400	287	1.91E1	1.05E-8	270	1.89E1	3.48E-8
	500	42000	693	3.70E2	6.21E-9	727	4.10E2	1.01E-8
	1000	84000	1459	1.95E3	1.03E-9	1433	1.78E3	2.62E-8

when the average distance among particles is small, the scattered field acts as if there exists a large obstacle. How to quantify such an equivalence will be explored in our future investigation.

7. Conclusion. In this paper, we have studied the elastic scattering problem with multiple rigid particles by using the Helmholtz decomposition. Three different integral formulations are presented for the coupled Helmholtz system. Their well-posedness is studied by using appropriate regularizers. A fast numerical method is proposed for the elastic scattering of multiple arbitrarily shaped obstacles. The idea is to construct the scattering matrix based on the proposed integral formulation for a single particle and then extend the multiple scattering theory from acoustic waves to elastic waves. In the end, the resulting linear equation is solved by the GMRES with the FMM acceleration. Numerical results show that our algorithm is much faster than the one that directly discretizes particles by points. In particular, we show that the method can achieve high order accuracy even for the scattering of up to 1000 elastic particles. One limitation of our solver is that the number of GMRES iteration grows linearly when the number of particles increases. A successful approach to overcome this difficulty was introduced in [25] for the Laplace equation, which is based on the combination of the fast direct solver and GMRES. In addition, we also observe that the iteration number grows as we increase the wavenumber. This is a well-known phenomenon since higher frequency wave leads to stronger interactions among particles, in which case the fast direct solver [7] can be applied to avoid too many iterations in GMRES. Another factor that will increase the iteration number is the shape of particles, particularly for the ones with high aspect ratio. One remedy

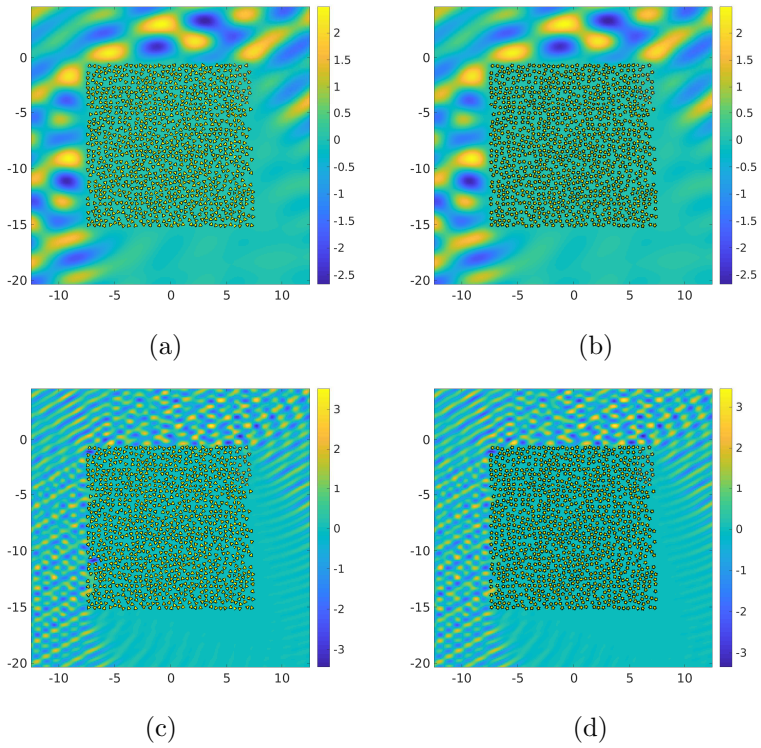

FIG. 4. *The elastic scattering of* 1000 *particles by the plane wave incidence in Example* 3. *Here, we show the real part of the first component of the total elastic field.* (a) *Field for* $a = \frac{1}{8}$, $b = \frac{1}{24}$, $c = 3$, $\omega = \pi$. (b) *Field for* $a = \frac{1}{8}$, $b = \frac{1}{24}$, $c = 5$, $\omega = \pi$. (c) *Field for* $a = \frac{1}{8}$, $b = \frac{1}{24}$, $c = 3$, $\omega = 4\pi$. (d) *Field for* $a = \frac{1}{8}$, $b = \frac{1}{24}$, $c = 5$, $\omega = 4\pi$.

for this issue was discussed in [12].

The method can be extended to the three-dimensional elastic wave scattering problem where the Helmholtz decomposition involves a scalar potential function and a vector potential function. The scalar function satisfies the acoustic equation, and the vector function satisfies the Maxwell equation. They are coupled at the boundary. The analysis for the well-posedness of the coupled system is more involved than the two-dimensional counterpart. A numerical method also requires high order quadrature for discretizing the singular surface integral [34]. The progress will be reported elsewhere in the future.

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