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Inverse obstacle scattering for Maxwell's equations in an unbounded structure

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Abstract

This paper is concerned with the electromagnetic scattering of a point source by a perfectly electrically conducting obstacle which is embedded in a two-layered lossy medium separated by an unbounded rough surface. Given a dipole point source, the direct problem is to determine the electromagnetic wave field for the given obstacle and unbounded rough surface; the inverse problem is to reconstruct simultaneously the obstacle and unbounded rough surface from the reflected and transmitted fields measured on a plane surface which is above and below the unknown objects, respectively. For the direct problem, its well-posedness is established and a new boundary integral equation is proposed. The analysis is based on the exponential decay of the dyadic Green function for Maxwell's equations in a lossy medium. For the inverse problem, the global uniqueness is proved and a local stability is discussed. A crucial step in the proof of the stability is to obtain the existence and characterization of the domain derivative of the electric field with respect to the shape of the obstacle and unbounded rough surface.

Keywords: Maxwell's equations, inverse scattering problem, unbounded rough surface, domain derivative, uniqueness, local stability

(Some figures may appear in colour only in the online journal)

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1. Introduction

Consider the electromagnetic scattering of a dipole point source by an obstacle which is embedded in a two-layered medium separated by an unbounded rough surface in three dimensions. An obstacle is referred to as an impenetrable medium which has a bounded closed surface; an unbounded rough surface stands for a nonlocal perturbation of an infinite plane surface such that the perturbed surface lies within a finite distance of the original plane. Given a dipole point source, the direct problem is to determine the electromagnetic wave field for the known obstacle and unbounded rough surface; the inverse problem is to reconstruct both the obstacle and the unbounded rough surface, from the measured wave field. The scattering problems arise from diverse scientific areas such as radar and sonar, geophysical exploration, nondestructive testing, and medical imaging. In particular, the obstacle scattering in unbounded structures has significant applications in radar based object recognition above the sea surface and detection of underwater or underground mines.

As a fundamental problem in scattering theory, the classical obstacle scattering problem, where the obstacle is embedded in a homogeneous medium, has been examined extensively by numerous researchers. The details can be found in the monographs [6, 27] and [5, 7, 16] on the mathematical and numerical studies of the direct and inverse problems, respectively. The unbounded rough surface scattering problems have also been widely examined in both of the mathematical and engineering communities. We refer to [8, 12, 15, 25, 28–31, 33] for various solution methods including mathematical, computational, approximate, asymptotic, and statistical methods. The scattering problems in unbounded structures are quite challenging due to two major issues: the usual Silver–Müller radiation condition is no longer valid; the Fredholm alternative argument does not directly apply due to the unbounded rough surface. The mathematical analysis can be found in [10, 11, 18, 22, 32] and [13, 20, 23] on the well-posedness of the two-dimensional Helmholtz equation and the three-dimensional Maxwell equations, respectively. The inverse problems have also been considered mathematically and computationally for unbounded rough surfaces in [1–3, 24].

In this paper, we study the electromagnetic obstacle scattering for the three-dimensional Maxwell equations in an unbounded structure. Specifically, we consider the illumination of a time-harmonic electromagnetic wave, generated from a dipole point source, onto a perfectly electrically conducting obstacle which is embedded in a two-layered medium separated by an unbounded rough surface. The obstacle is located either above or below the unbounded rough surface and may have multiple disjoint components. For simplicity of presentation, we assume that the obstacle has only one component and is located above the surface. The free space above and below the unbounded rough surface is assumed to be filled with a homogeneous and lossy material accounting for the energy absorption, respectively. The problem has received much attention and many computational work have been done in the engineering community [14, 17, 19]. However, the rigorous analysis is very rare, especially for the three-dimensional Maxwell equations.

In this work, we introduce an energy decaying condition to replace the Silver–Müller radiation condition in order to ensure the uniqueness of the solution. The asymptotic behaviour of dyadic Green's function is analyzed and plays an important role in the analysis for the well-posedness of the direct problem. A new boundary integral equation is proposed for the associated boundary value problem. Based on some energy estimates, the uniqueness of the solution for the scattering problem is established. For the inverse problem, we intend to answer the following question: what information can we extract about the obstacle and the unbounded rough surface from the tangential trace of the electric field measured on the plane surface which is above and below the obstacle and unbounded rough surface, respectively? The first result is a global uniqueness theorem. We show that any two obstacles and unbounded rough surfaces are identical if they generate the same data. The proof is based on a combination of the Holmgren uniqueness and unique continuation. The second result is concerned with a local stability: if two obstacles are 'close' and two unbounded rough surfaces are also 'close', then for any $\delta > 0$, the measurements of the two tangential trace of the electric fields being δ-close implies that both of the two obstacles and the two unbounded rough surfaces are $\mathcal{O}(\delta)$-close. A crucial step in the stability proof is to obtain the existence and characterization of the domain derivative of the electric field with respect to the shape of the obstacle and unbounded rough surface.

The paper is organized as follows. In section 2, we introduce the model problem and present some asymptotic analysis for dyadic Green's function of the Maxwell equations. Section 3 is devoted to the well-posedness of the direct scattering problem. An equivalent integral representation is proposed for the boundary value problem. A new boundary integral equation is developed. In sections 4 and 5, we discuss the global uniqueness and local stability of the inverse problem, respectively. The domain derivative is studied. The paper is concluded with some general remarks in section 6.

2. Problem formulation

We begin with introducing the problem geometry, which is shown in figure 1. Let S be an infinite rough surface given by

$$S = \{\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 = f(x_1, x_2)\},$$

where $f \in C^2(\mathbb{R}^2)$. Hence the surface S divides the whole space $\mathbb{R}^3$ into the upper half space Ω_1^+ and the lower half space Ω_2, where

$$\Omega_1^+ = \{\boldsymbol{x} \in \mathbb{R}^3 : x_3 > f(x_1, x_2)\}, \qquad \Omega_2 = \{\boldsymbol{x} \in \mathbb{R}^3 : x_3 < f(x_1, x_2)\}.$$

Let D be a bounded obstacle with C^2 boundary Γ. The obstacle is assumed to be a perfect electrical conductor which is located either in Ω_1^+ or in Ω_2. For instance, we may assume that $D \subset\subset \Omega_1^+$. Define $\Omega_1 = \Omega_1^+ \setminus \overline{D}$. The domain Ω_j is assumed to be filled with some homogeneous, isotropic, and absorbing medium which may be characterized by the dielectric permittivity $\varepsilon_j > 0$, the magnetic permeability $\mu_j > 0$, and the electric conductivity $\sigma_j > 0,\ j = 1, 2$.

Let

$$f_- = \inf_{(x_1,x_2)\in\mathbb{R}^2} f(x_1, x_2), \qquad f_+ = \sup_{(x_1,x_2)\in\mathbb{R}^2} f(x_1, x_2).$$

Denote by $\Gamma_j = \{\boldsymbol{x} \in \mathbb{R}^3 : x_3 = h_j\}, j = 1, 2$ the plane surface above the obstacle and below the infinite rough surface, respectively, where the constants h_1, h_2 satisfy

$$-\infty < h_2 < f_- < f_+ < h_1 < \infty.$$

Define $R_1 = \{\boldsymbol{x} \in \mathbb{R}^3 : f(x_1, x_2) < x_3 < h_1\}$ and $R_2 = \{\boldsymbol{x} \in \mathbb{R}^3 : h_2 < x_3 < f(x_1, x_2)\}$. Let $R = R_1 \cup R_2 \cup S$.

In Ω_j, the electromagnetic waves satisfy the time-harmonic Maxwell equations (time dependence $e^{-i\omega t}$):

$$\begin{cases} \nabla \times \boldsymbol{E}_j = i\omega\mu_j \boldsymbol{H}_j, \\ \nabla \times \boldsymbol{H}_j = -i\omega\varepsilon_j \boldsymbol{E}_j + \boldsymbol{J}_j, \\ \nabla \cdot (\varepsilon_j \boldsymbol{E}_j) = \rho_j, \\ \nabla \cdot (\mu_j \boldsymbol{H}_j) = 0, \end{cases}$$

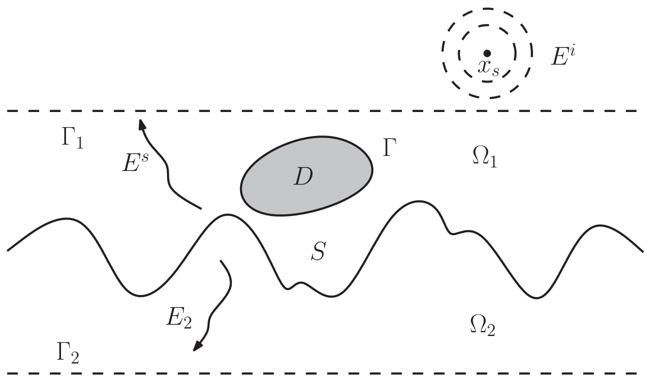

Figure 1. Problem geometry of the obstacle scattering in an unbounded structure.

where $\omega > 0$ is the angular frequency, $\boldsymbol{E}_j$, $\boldsymbol{H}_j$, $\boldsymbol{J}_j$ denote the electric field, the magnetic field, the electric current density, respectively, and $\rho_j = (\mathrm{i}\omega)^{-1}\nabla\cdot\boldsymbol{J}_j$ is the electric charge density. The external current source is assumed to be located in Ω_1. The relation between the electric current density and the electric field is given by

$$\begin{cases}\boldsymbol{J}_1 = \sigma_1\boldsymbol{E}_1 + \boldsymbol{J}_{\rm cs} & \text{in } \Omega_1,\\ \boldsymbol{J}_2 = \sigma_2\boldsymbol{E}_2 & \text{in } \Omega_2,\end{cases}$$

where $\boldsymbol{J}_{\rm cs}$ stands for the current source. Throughout, we also assume that the material is non-magnetic, i.e. $\mu_1 = \mu_2 = \mu$, where μ is a positive constant. Using the above constitutive relation, we obtain coupled systems

$$\begin{cases}\nabla\times\boldsymbol{E}_1 = \mathrm{i}\omega\mu\boldsymbol{H}_1,\\ \nabla\times\boldsymbol{H}_1 = -\mathrm{i}\omega\left(\varepsilon_1 + \mathrm{i}\frac{\sigma_1}{\omega}\right)\boldsymbol{E}_1 + \boldsymbol{J}_{\rm cs},\\ \left(\varepsilon_1 + \mathrm{i}\frac{\sigma_1}{\omega}\right)\nabla\cdot\boldsymbol{E}_1 = \frac{1}{\mathrm{i}\omega}\nabla\cdot\boldsymbol{J}_{\rm cs},\\ \nabla\cdot\boldsymbol{H}_1 = 0,\end{cases} \quad \text{in } \Omega_1, \tag{2.1}$$

and

$$\begin{cases}\nabla\times\boldsymbol{E}_2 = \mathrm{i}\omega\mu\boldsymbol{H}_2,\\ \nabla\times\boldsymbol{H}_2 = -\mathrm{i}\omega\left(\varepsilon_2 + \mathrm{i}\frac{\sigma_2}{\omega}\right)\boldsymbol{E}_2,\\ \left(\varepsilon_2 + \mathrm{i}\frac{\sigma_2}{\omega}\right)\nabla\cdot\boldsymbol{E}_2 = 0,\\ \nabla\cdot\boldsymbol{H}_2 = 0,\end{cases} \quad \text{in } \Omega_2. \tag{2.2}$$

Eliminating the magnetic field $\boldsymbol{H}_1$ in (2.1), we obtain a decoupled equation for the electric field $\boldsymbol{E}_1$:

$$\nabla\times(\nabla\times\boldsymbol{E}_1(\boldsymbol{x})) - \kappa_1^2\boldsymbol{E}_1(\boldsymbol{x}) = \mathrm{i}\omega\mu\boldsymbol{J}_{\rm cs}(\boldsymbol{x}), \quad \boldsymbol{x}\in\ \Omega_1. \tag{2.3}$$

Similarly, it follows from (2.2) that we may deduce a decoupled Maxwell system for the electric field $\boldsymbol{E}_2$:

$$\nabla\times(\nabla\times\boldsymbol{E}_2(\boldsymbol{x})) - \kappa_2^2\boldsymbol{E}_2(\boldsymbol{x}) = 0, \quad \boldsymbol{x}\in\ \Omega_2. \tag{2.4}$$

Here $\kappa_j = \omega\sqrt{\left(\varepsilon_j + \mathrm{i}\frac{\sigma_j}{\omega}\right)\mu}$ is the wave number in $\Omega_j, j = 1, 2$. Since $\varepsilon_j, \mu, \sigma_j$ are positive constants, κ_j satisfies

$$\Re(\kappa_j^2) > 0, \quad \Im(\kappa_j^2) > 0, \quad \Im(\kappa_j) > 0,$$

which accounts for the energy absorption.

By the perfect conductor assumption for the obstacle, it holds that

$$\boldsymbol{\nu}_\Gamma \times \boldsymbol{E}_1 = 0 \quad \text{on } \Gamma, \tag{2.5}$$

where $\boldsymbol{\nu}_\Gamma$ denotes the unit normal vector on the boundary Γ directed into the exterior of D. The usual continuity conditions need to be imposed, i.e. the tangential traces of the electric and magnetic fields are continuous across S:

$$\boldsymbol{\nu}_S \times \boldsymbol{E}_1 = \boldsymbol{\nu}_S \times \boldsymbol{E}_2, \quad \boldsymbol{\nu}_S \times \boldsymbol{H}_1 = \boldsymbol{\nu}_S \times \boldsymbol{H}_2 \quad \text{on } S, \tag{2.6}$$

which are equivalent to the continuity conditions

$$\boldsymbol{\nu}_S \times \boldsymbol{E}_1 = \boldsymbol{\nu}_S \times \boldsymbol{E}_2, \quad \boldsymbol{\nu}_S \times (\nabla \times \boldsymbol{E}_1) = \boldsymbol{\nu}_S \times (\nabla \times \boldsymbol{E}_2) \quad \text{on } S, \tag{2.7}$$

where $\boldsymbol{\nu}_S$ denotes the unit normal vector on S pointing from Ω_2 to Ω_1.

The incident electromagnetic fields $(\boldsymbol{E}^i, \boldsymbol{H}^i)$ satisfy Maxwell's equations

$$\begin{cases} \nabla \times (\nabla \times \boldsymbol{E}^i(\boldsymbol{x})) - \kappa_1^2 \boldsymbol{E}^i(\boldsymbol{x}) = \mathrm{i}\omega\mu \boldsymbol{J}_{\mathrm{cs}}(\boldsymbol{x}), \\ \nabla \times (\nabla \times \boldsymbol{H}^i(\boldsymbol{x})) - \kappa_1^2 \boldsymbol{H}^i(\boldsymbol{x}) = \nabla \times \boldsymbol{J}_{cs}(\boldsymbol{x}), \end{cases} \quad \boldsymbol{x} \in \Omega_1, \tag{2.8}$$

where $\boldsymbol{H}^i = \frac{1}{\mathrm{i}\omega\mu}(\nabla \times \boldsymbol{E}^i)$. In Ω_1, the total electromagnetic fields $(\boldsymbol{E}_1, \boldsymbol{H}_1)$ consist of the incident fields $(\boldsymbol{E}^i, \boldsymbol{H}^i)$ and the scattered fields $(\boldsymbol{E}^s, \boldsymbol{H}^s)$. In Ω_2, the electromagnetic fields $(\boldsymbol{E}_2, \boldsymbol{H}_2)$ are called the transmitted fields.

In addition, we propose an energy decaying condition

$$\lim_{r \to +\infty} \int_{\partial B_r^+} |\boldsymbol{E}^s|^2 \mathrm{d}s = 0, \quad \lim_{r \to +\infty} \int_{\partial B_r^+} |\nabla \times \boldsymbol{E}^s|^2 \mathrm{d}s = 0 \tag{2.9}$$

and

$$\lim_{r \to +\infty} \int_{\partial B_r^-} |\boldsymbol{E}_2|^2 \mathrm{d}s = 0, \quad \lim_{r \to +\infty} \int_{\partial B_r^-} |\nabla \times \boldsymbol{E}_2|^2 \mathrm{d}s = 0, \tag{2.10}$$

where $\partial B_r^{\pm}$ denotes the hemisphere of radius r above or below S.

For any tangential vector $\boldsymbol{u} = (u_1, u_2, 0)^\top$ on Γ_j, define the capacity operator T_j:

$$(T_j \boldsymbol{u})(x_1, x_2, h_j) = (v_1, v_2, 0)^\top,$$

where

$$\hat{v}_1 = \frac{1}{\omega\mu}\left[\beta_j \hat{u}_1 + \frac{\xi_1}{\beta_j}(\xi_1 \hat{u}_1 + \xi_2 \hat{u}_2)\right], \quad \hat{v}_2 = \frac{1}{\omega\mu}\left[\beta_j \hat{u}_2 + \frac{\xi_2}{\beta_j}(\xi_1 \hat{u}_1 + \xi_2 \hat{u}_2)\right],$$

with

$$\beta_j^2(\xi) = \kappa_j^2 - |\xi|^2, \quad \Im[\beta_j(\xi)] > 0.$$

Here $\hat{u}$ denotes the Fourier transform of u with respect to $\varrho = (x_1, x_2) \in \mathbb{R}^2$ and is defined by

$$\hat{u}(\xi) = \frac{1}{2\pi} \int_{\mathbb{R}^2} u(\varrho) \mathrm{e}^{-\mathrm{i}\varrho \cdot \xi} \mathrm{d}\varrho,$$

where $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$. It was shown in [20] that a transparent boundary condition can be imposed on Γ_j:

$$\boldsymbol{\nu}_{\Gamma_1} \times (\nabla \times \boldsymbol{E}_1) = -\mathrm{i}\omega\mu(T_1 \boldsymbol{E}_{\Gamma_1}) + \boldsymbol{g} \quad \text{on } \Gamma_1 \tag{2.11}$$

and

$$\boldsymbol{\nu}_{\Gamma_2} \times (\nabla \times \boldsymbol{E}_2) = -\mathrm{i}\omega\mu(T_2\boldsymbol{E}_{\Gamma_2}) \qquad \text{on } \Gamma_2, \tag{2.12}$$

where $\boldsymbol{\nu}_{\Gamma_j}$ is the unit normal vector on Γ_j, $\boldsymbol{E}_{\Gamma_j} = \boldsymbol{\nu}_{\Gamma_j} \times (\boldsymbol{E}_j \times \boldsymbol{\nu}_{\Gamma_j})$ is the tangential component of $\boldsymbol{E}_j$ on Γ_j, and the inhomogeneous term

$$\boldsymbol{g} = \mathrm{i}\omega\mu(T_1\boldsymbol{E}^i_{\Gamma_1}) + \boldsymbol{\nu}_{\Gamma_1} \times (\nabla \times \boldsymbol{E}^i).$$

The dyadic Green function is defined by the solution of the following equation

$$\nabla_{\boldsymbol{x}} \times (\nabla_{\boldsymbol{x}} \times \boldsymbol{G}_j(\boldsymbol{x}-\boldsymbol{y})) - \kappa_j^2\boldsymbol{G}_j(\boldsymbol{x}-\boldsymbol{y}) = \delta(\boldsymbol{x}-\boldsymbol{y})\boldsymbol{I} \quad \text{in } \Omega_j, \tag{2.13}$$

where $\boldsymbol{I}$ is the unitary dyadic and δ is the Dirac delta function. It is known that the dyadic Green function is given by

$$\boldsymbol{G}_j(\boldsymbol{x}-\boldsymbol{y}) = \left[\boldsymbol{I} + \frac{\nabla_{\boldsymbol{y}}\nabla_{\boldsymbol{y}}}{\kappa_j^2}\right]\frac{\exp(\mathrm{i}\kappa_j|\boldsymbol{x}-\boldsymbol{y}|)}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} = \left[\boldsymbol{I} + \frac{\nabla_{\boldsymbol{x}}\nabla_{\boldsymbol{x}}}{\kappa_j^2}\right]\frac{\exp(\mathrm{i}\kappa_j|\boldsymbol{x}-\boldsymbol{y}|)}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}. \tag{2.14}$$

We assume that the dipole point source is located at $\boldsymbol{x}_s \in R_1^+ = \{\boldsymbol{x} \in \mathbb{R}^3 : x_3 > h_1\}$ and has a polarization $\boldsymbol{q} \in \mathbb{R}^3$, $|\boldsymbol{q}| = 1$. Induced by this dipole point source, the incident electromagnetic fields are

$$\boldsymbol{E}^i(\boldsymbol{x}) = \boldsymbol{G}_1(\boldsymbol{x}-\boldsymbol{x}_s)\boldsymbol{q}, \quad \boldsymbol{H}^i(\boldsymbol{x}) = \frac{1}{\mathrm{i}\omega\mu}(\nabla \times \boldsymbol{E}^i(\boldsymbol{x})), \quad \boldsymbol{x} \in \Omega_1. \tag{2.15}$$

Hence the current source $\boldsymbol{J}_{\mathrm{cs}}$ satisfies

$$\mathrm{i}\omega\mu\boldsymbol{J}_{\mathrm{cs}}(\boldsymbol{x}) = \boldsymbol{q}\delta(\boldsymbol{x}-\boldsymbol{x}_s), \quad \boldsymbol{x} \in \Omega_1.$$

We next introduce some Banach spaces. For $Q \subset \mathbb{R}^3$, denote by $BC(Q)$ the set of bounded and continuous functions on Q, which is a Banach space under the norm

$$\|\phi\|_\infty = \sup_{\boldsymbol{x}\in Q}|\phi(\boldsymbol{x})|.$$

For $0 < \alpha \leqslant 1$, denote by $C^{0,\alpha}(Q)$ the Banach space of functions $\phi \in BC(Q)$ which are uniformly Hölder continuous with exponent α. The norm $\|\cdot\|_{C^{0,\alpha}(Q)}$ is defined by

$$\|\phi\|_{C^{0,\alpha}(Q)} = \|\phi\|_\infty + \sup_{\substack{\boldsymbol{x},\boldsymbol{y}\in Q\\ \boldsymbol{x}\neq\boldsymbol{y}}}\frac{|\phi(\boldsymbol{x})-\phi(\boldsymbol{y})|}{|\boldsymbol{x}-\boldsymbol{y}|^\alpha}.$$

Let $C^{1,\alpha}(Q) = \{\phi \in BC(Q) \cap C^1(Q) : \nabla\phi \in C^{0,\alpha}(Q)\}$, which is a Banach space under the norm

$$\|\phi\|_{C^{1,\alpha}(Q)} = \|\phi\|_\infty + \|\nabla\phi\|_{C^{0,\alpha}(Q)}.$$

Denote by $\mathcal{T}_j (j = 1, 2)$ the set of functions $\boldsymbol{\psi} \in C^2(\Omega_j) \cap C^{1,\alpha}(\overline{\Omega}_j)$. The direct scattering problem can be stated as follows.

Problem 2.1. Given the incident field $\boldsymbol{E}^i$ in (2.15), the direct problem is to determine $\boldsymbol{E}^s \in \mathcal{T}_1$ and $\boldsymbol{E}_2 \in \mathcal{T}_2$ such that

(i) The electric fields $\boldsymbol{E}_1 = \boldsymbol{E}^s + \boldsymbol{E}^i$ and $\boldsymbol{E}_2$ satisfy (2.3) and (2.4), respectively;
(ii) The electric field $\boldsymbol{E}_1$ satisfies the boundary condition (2.5);
(iii) The electric fields $\boldsymbol{E}_1$ and $\boldsymbol{E}_2$ satisfy the continuity conditions (2.7);
(iv) The scattered fields $\boldsymbol{E}^s$ and the transmitted fields $\boldsymbol{E}_2$ satisfy the radiation conditions (2.9) and (2.10), respectively.

It requires to study the dyadic Green function in order to find the integral representation of the solution for the scattering problem. The details may be found in [4] on the general properties of the dyadic Green function.

Lemma 2.2. *For each fixed $\boldsymbol{y} \in \Omega_j, j = 1, 2$, the dyadic Green function $\boldsymbol{G}_j$ given in (2.14) admits the asymptotic behaviour*

$$|\boldsymbol{G}_j(\boldsymbol{x}-\boldsymbol{y})|,\ |\nabla_{\boldsymbol{x}} \times \boldsymbol{G}_j(\boldsymbol{x}-\boldsymbol{y})| \leqslant C\left(\frac{\exp\left(-\frac{1}{2}\Im(\kappa_j)|\boldsymbol{x}|\right)}{|\boldsymbol{x}|}\right), \quad |\boldsymbol{x}| \to \infty,$$

where C is a constant independent of $\boldsymbol{x}$ and $\boldsymbol{y}$.

Proof. For each fixed $\boldsymbol{y} \in \Omega_j, j = 1, 2$, since

$$|\boldsymbol{x}-\boldsymbol{y}| = \sqrt{|\boldsymbol{x}|^2 - 2\boldsymbol{x}\cdot\boldsymbol{y} + |\boldsymbol{y}|^2} = |\boldsymbol{x}| - \hat{\boldsymbol{x}}\cdot\boldsymbol{y} + \mathcal{O}\left(\frac{1}{|\boldsymbol{x}|}\right) \quad \text{as } |\boldsymbol{x}| \to \infty,$$

where $\hat{\boldsymbol{x}} = \boldsymbol{x}/|\boldsymbol{x}|$, we have

$$\frac{\exp(\mathrm{i}\kappa_j|\boldsymbol{x}-\boldsymbol{y}|)}{|\boldsymbol{x}-\boldsymbol{y}|} = \frac{\exp(\mathrm{i}\kappa_j|\boldsymbol{x}|)}{|\boldsymbol{x}|}\left\{\exp(-\mathrm{i}\kappa_j\hat{\boldsymbol{x}}\cdot\boldsymbol{y}) + \mathcal{O}\left(\frac{1}{|\boldsymbol{x}|}\right)\right\} \quad \text{as } |\boldsymbol{x}| \to \infty, \tag{2.16}$$

uniformly for each fixed $\boldsymbol{y}$ satisfying $|\boldsymbol{x}-\boldsymbol{y}| \to \infty$. By (2.16), for $\Im(\kappa_j) > 0$, we obtain for $|\boldsymbol{x}| \to \infty$ that

$$\begin{aligned}
\boldsymbol{G}_j(\boldsymbol{x}-\boldsymbol{y}) &= \left[\boldsymbol{I} + \frac{\nabla_{\boldsymbol{y}}\nabla_{\boldsymbol{y}}}{\kappa_j^2}\right]\frac{\exp(\mathrm{i}\kappa_j|\boldsymbol{x}|)}{4\pi|\boldsymbol{x}|}\left\{\exp(-\mathrm{i}\kappa_j\hat{\boldsymbol{x}}\cdot\boldsymbol{y}) + \mathcal{O}\left(\frac{1}{|\boldsymbol{x}|}\right)\right\}\\
&= \frac{\exp(\mathrm{i}\kappa_j|\boldsymbol{x}|)}{4\pi|\boldsymbol{x}|}\left\{[\boldsymbol{I} - \hat{\boldsymbol{x}}\hat{\boldsymbol{x}}]\exp(-\mathrm{i}\kappa_j\hat{\boldsymbol{x}}\cdot\boldsymbol{y}) + \mathcal{O}\left(\frac{1}{|\boldsymbol{x}|}\right)\hat{\boldsymbol{I}}\right\}\\
&= \frac{\exp\left(\frac{1}{2}\mathrm{i}\kappa_j|\boldsymbol{x}|\right)}{4\pi|\boldsymbol{x}|}\left\{[\boldsymbol{I} - \hat{\boldsymbol{x}}\hat{\boldsymbol{x}}]\exp\left(\mathrm{i}\kappa_j\left(\frac{|\boldsymbol{x}|}{2} - \hat{\boldsymbol{x}}\cdot\boldsymbol{y}\right)\right) + \mathcal{O}\left(\frac{1}{|\boldsymbol{x}|}\right)\hat{\boldsymbol{I}}\right\},
\end{aligned}$$

and

$$\begin{aligned}
\nabla_{\boldsymbol{x}} \times \boldsymbol{G}_j(\boldsymbol{x}-\boldsymbol{y}) &= -\nabla_{\boldsymbol{y}} \times \boldsymbol{G}_j(\boldsymbol{x}-\boldsymbol{y})\\
&= \frac{\exp(\mathrm{i}\kappa_j|\boldsymbol{x}|)}{4\pi|\boldsymbol{x}|}\left\{-\nabla_{\boldsymbol{y}} \times \left[(\boldsymbol{I} - \hat{\boldsymbol{x}}\hat{\boldsymbol{x}})\exp(-\mathrm{i}\kappa_j\hat{\boldsymbol{x}}\cdot\boldsymbol{y})\right] + \mathcal{O}\left(\frac{1}{|\boldsymbol{x}|}\right)\hat{\boldsymbol{I}}\right\}\\
&= \mathrm{i}\kappa_j\frac{\exp(\mathrm{i}\kappa_j|\boldsymbol{x}|)}{4\pi|\boldsymbol{x}|}\left\{\hat{\boldsymbol{x}} \times [(\boldsymbol{I} - \hat{\boldsymbol{x}}\hat{\boldsymbol{x}})\exp(-\mathrm{i}\kappa_j\hat{\boldsymbol{x}}\cdot\boldsymbol{y})] + \mathcal{O}\left(\frac{1}{|\boldsymbol{x}|}\right)\hat{\boldsymbol{I}}\right\}\\
&= \mathrm{i}\kappa_j\frac{\exp\left(\frac{1}{2}\mathrm{i}\kappa_j|\boldsymbol{x}|\right)}{4\pi|\boldsymbol{x}|}\left\{\hat{\boldsymbol{x}} \times \left[(\boldsymbol{I} - \hat{\boldsymbol{x}}\hat{\boldsymbol{x}})\exp\left(\mathrm{i}\kappa_j\left(\frac{|\boldsymbol{x}|}{2} - \hat{\boldsymbol{x}}\cdot\boldsymbol{y}\right)\right)\right] + \mathcal{O}\left(\frac{1}{|\boldsymbol{x}|}\right)\hat{\boldsymbol{I}}\right\},
\end{aligned}$$

where $\hat{\boldsymbol{I}} := \hat{\boldsymbol{e}}\hat{\boldsymbol{e}}$ and $\hat{\boldsymbol{e}} = (1, 1, 1)^\top$.

Note that for each fixed $\boldsymbol{y} \in \Omega_j, j = 1, 2$, we have $\hat{\boldsymbol{x}}\cdot\boldsymbol{y} < \frac{|\boldsymbol{x}|}{2}$ for $|\boldsymbol{x}| \to \infty$. Hence, we conclude from $\Im(\kappa_j) > 0$ that there exists a constant C independent of $\boldsymbol{x}, \boldsymbol{y}$ such that

$$|\boldsymbol{G}_j(\boldsymbol{x}-\boldsymbol{y})|,\ |\nabla_{\boldsymbol{x}} \times \boldsymbol{G}_j(\boldsymbol{x}-\boldsymbol{y})| \leqslant C\left(\frac{\exp\left(-\frac{1}{2}\Im(\kappa_j)|\boldsymbol{x}|\right)}{|\boldsymbol{x}|}\right),$$

which completes the proof. □

3. Well-posedness of the direct problem

In this section, we show the existence and uniqueness of the solution to problem 2.1 by using the boundary integral equation method. First we derive an integral representation for the solution of problem 2.1 using dyadic Green's theorem combined with the radiation conditions (2.9) and (2.10).

Theorem 3.1. *Let the fields* $(\boldsymbol{E}_1, \boldsymbol{E}_2)$ *be the solution of problem 2.1, then* $(\boldsymbol{E}_1, \boldsymbol{E}_2)$ *have the integral representations*

$$\begin{aligned}\boldsymbol{E}_1(\boldsymbol{x}) = \boldsymbol{E}^i(\boldsymbol{x}) + &\int_S \big\{[\boldsymbol{G}_1(\boldsymbol{x}-\boldsymbol{y})]\cdot[\boldsymbol{\nu}_S(\boldsymbol{y})\times(\nabla_{\boldsymbol{y}}\times\boldsymbol{E}_1(\boldsymbol{y}))] \\ &\quad + [\nabla_{\boldsymbol{x}}\times\boldsymbol{G}_1(\boldsymbol{x}-\boldsymbol{y})]\cdot[\boldsymbol{\nu}_S(\boldsymbol{y})\times\boldsymbol{E}_1(\boldsymbol{y})]\big\}\mathrm{d}s_{\boldsymbol{y}} \\ &+ \int_\Gamma \big\{[\boldsymbol{G}_1(\boldsymbol{x}-\boldsymbol{y})]\cdot[\boldsymbol{\nu}_\Gamma(\boldsymbol{y})\times(\nabla_{\boldsymbol{y}}\times\boldsymbol{E}_1(\boldsymbol{y}))]\big\}\mathrm{d}s_{\boldsymbol{y}}, \quad \boldsymbol{x}\in\Omega_1, \end{aligned} \tag{3.1}$$

and

$$\begin{aligned}\boldsymbol{E}_2(\boldsymbol{x}) = -&\int_S \big\{[\boldsymbol{G}_2(\boldsymbol{x}-\boldsymbol{y})]\cdot[\boldsymbol{\nu}_S(\boldsymbol{y})\times(\nabla_{\boldsymbol{y}}\times\boldsymbol{E}_1(\boldsymbol{y}))] \\ &+ [\nabla_{\boldsymbol{x}}\times\boldsymbol{G}_2(\boldsymbol{x}-\boldsymbol{y})]\cdot[\boldsymbol{\nu}_S(\boldsymbol{y})\times\boldsymbol{E}_1(\boldsymbol{y})]\big\}\mathrm{d}s_{\boldsymbol{y}}, \quad \boldsymbol{x}\in\Omega_2. \end{aligned} \tag{3.2}$$

Proof. Let $B_r = \{\boldsymbol{x}\in\mathbb{R}^3 : |\boldsymbol{x}| < r\}$. Denote $\Omega_r = B_r\cap\Omega_1$ with the boundary $\partial\Omega_r = \partial B_r^+\cup\Gamma\cup S_r$, where $\partial B_r^+ = \partial B_r\cap\Omega_1$ and $S_r = S\cap B_r$. For each fixed $\boldsymbol{x}\in\Omega_r$, applying the vector dyadic Green second theorem to $\boldsymbol{E}_1$ and $\boldsymbol{G}_1$ in the region Ω_r, we obtain

$$\begin{aligned}&\int_{\Omega_r}\{\boldsymbol{E}_1(\boldsymbol{y})\cdot[\nabla_{\boldsymbol{y}}\times(\nabla_{\boldsymbol{y}}\times\boldsymbol{G}_1(\boldsymbol{y}-\boldsymbol{x}))] - [\nabla_{\boldsymbol{y}}\times(\nabla_{\boldsymbol{y}}\times\boldsymbol{E}_1(\boldsymbol{y}))]\cdot\boldsymbol{G}_1(\boldsymbol{y}-\boldsymbol{x})\}\mathrm{d}\boldsymbol{y} \\ &= -\int_{\partial\Omega_r}\{[\boldsymbol{\nu}(\boldsymbol{y})\times(\nabla_{\boldsymbol{y}}\times\boldsymbol{E}_1(\boldsymbol{y}))]\cdot\boldsymbol{G}_1(\boldsymbol{y}-\boldsymbol{x}) + [\boldsymbol{\nu}(\boldsymbol{y})\times\boldsymbol{E}_1(\boldsymbol{y})]\cdot[\nabla_{\boldsymbol{y}}\times\boldsymbol{G}_1(\boldsymbol{y}-\boldsymbol{x})]\}\mathrm{d}s_{\boldsymbol{y}}, \end{aligned} \tag{3.3}$$

where $\boldsymbol{\nu} = \boldsymbol{\nu}(\boldsymbol{y})$ stands for the unit normal vector at $\boldsymbol{y}\in\partial\Omega_r$ pointing out of Ω_r.

It follows from (2.3) and (2.13) that

$$\begin{aligned}&\int_{\Omega_r}\{\boldsymbol{E}_1(\boldsymbol{y})\cdot[\nabla_{\boldsymbol{y}}\times(\nabla_{\boldsymbol{y}}\times\boldsymbol{G}_1(\boldsymbol{y}-\boldsymbol{x}))] - [\nabla_{\boldsymbol{y}}\times(\nabla_{\boldsymbol{y}}\times\boldsymbol{E}_1(\boldsymbol{y}))]\cdot\boldsymbol{G}_1(\boldsymbol{y}-\boldsymbol{x})\}\mathrm{d}\boldsymbol{y} \\ &= \int_{\Omega_r}[\boldsymbol{E}_1(\boldsymbol{y})]\cdot[\nabla_{\boldsymbol{y}}\times(\nabla_{\boldsymbol{y}}\times\boldsymbol{G}_1(\boldsymbol{y}-\boldsymbol{x})) - \kappa_1^2\boldsymbol{G}_1(\boldsymbol{y}-\boldsymbol{x})]\mathrm{d}\boldsymbol{y} \\ &\quad - \int_{\Omega_r}[\nabla_{\boldsymbol{y}}\times(\nabla_{\boldsymbol{y}}\times\boldsymbol{E}_1(\boldsymbol{y})) - \kappa_1^2\boldsymbol{E}_1(\boldsymbol{y})]\cdot[\boldsymbol{G}_1(\boldsymbol{y}-\boldsymbol{x})]\mathrm{d}\boldsymbol{y} \\ &= \int_{\Omega_r}[\boldsymbol{E}_1(\boldsymbol{y})\cdot(\delta(\boldsymbol{y}-\boldsymbol{x})\boldsymbol{I})]\mathrm{d}\boldsymbol{y} - \int_{\Omega_r}[\mathrm{i}\omega\mu\boldsymbol{J}_{\mathrm{cs}}(\boldsymbol{y})\cdot\boldsymbol{G}_1(\boldsymbol{y}-\boldsymbol{x})]\mathrm{d}\boldsymbol{y} \\ &= \boldsymbol{E}_1(\boldsymbol{x}) - \int_{\Omega_r}[\mathrm{i}\omega\mu\boldsymbol{J}_{\mathrm{cs}}(\boldsymbol{y})\cdot\boldsymbol{G}_1(\boldsymbol{y}-\boldsymbol{x})]\mathrm{d}\boldsymbol{y}, \end{aligned} \tag{3.4}$$

where

$$\lim_{r\to+\infty}\int_{\Omega_r}[\mathrm{i}\omega\mu\boldsymbol{J}_{\mathrm{cs}}(\boldsymbol{y})\cdot\boldsymbol{G}_1(\boldsymbol{y}-\boldsymbol{x})]\mathrm{d}\boldsymbol{y}=\int_{\Omega_1}[\boldsymbol{q}\delta(\boldsymbol{y}-\boldsymbol{x}_s)\cdot\boldsymbol{G}_1(\boldsymbol{y}-\boldsymbol{x})]\mathrm{d}\boldsymbol{y}$$
$$=\boldsymbol{G}_1(\boldsymbol{x}-\boldsymbol{x}_s)\boldsymbol{q}=\boldsymbol{E}^i(\boldsymbol{x}). \tag{3.5}$$

Hence, letting $r\to+\infty$, with the aid of (3.3)–(3.5), we have

$$\begin{aligned}\boldsymbol{E}_1(\boldsymbol{x})-\boldsymbol{E}^i(\boldsymbol{x})=&-\int_{\partial\Omega_1}\big\{[\boldsymbol{\nu}(\boldsymbol{y})\times(\nabla_{\boldsymbol{y}}\times\boldsymbol{E}_1(\boldsymbol{y}))]\cdot\boldsymbol{G}_1(\boldsymbol{y}-\boldsymbol{x})\\&+[\boldsymbol{\nu}(\boldsymbol{y})\times\boldsymbol{E}_1(\boldsymbol{y})]\cdot[\nabla_{\boldsymbol{y}}\times\boldsymbol{G}_1(\boldsymbol{y}-\boldsymbol{x})]\big\}\mathrm{d}s_{\boldsymbol{y}}\\=&-\left(\int_S+\int_\Gamma+\lim_{r\to+\infty}\int_{\partial B_r^+}\right)\big\{[\boldsymbol{\nu}(\boldsymbol{y})\times(\nabla_{\boldsymbol{y}}\times\boldsymbol{E}_1(\boldsymbol{y}))]\cdot\boldsymbol{G}_1(\boldsymbol{y}-\boldsymbol{x})\\&+[\boldsymbol{\nu}(\boldsymbol{y})\times\boldsymbol{E}_1(\boldsymbol{y})]\cdot[\nabla_{\boldsymbol{y}}\times\boldsymbol{G}_1(\boldsymbol{y}-\boldsymbol{x})]\big\}\mathrm{d}s_{\boldsymbol{y}}.\end{aligned}\tag{3.6}$$

Following lemma 2.2 and (2.9), we obtain for $r\to+\infty$ that

$$\begin{aligned}&\left|\int_{\partial B_r^+}\big\{[\boldsymbol{\nu}(\boldsymbol{y})\times(\nabla_{\boldsymbol{y}}\times\boldsymbol{E}^s(\boldsymbol{y}))]\cdot\boldsymbol{G}_1(\boldsymbol{y}-\boldsymbol{x})+[\boldsymbol{\nu}(\boldsymbol{y})\times\boldsymbol{E}^s(\boldsymbol{y})]\cdot[\nabla_{\boldsymbol{y}}\times\boldsymbol{G}_1(\boldsymbol{y}-\boldsymbol{x})]\big\}\mathrm{d}s_{\boldsymbol{y}}\right|\\&\leqslant\left[\int_{\partial B_r^+}|\nabla_{\boldsymbol{y}}\times\boldsymbol{E}^s(\boldsymbol{y})|^2\mathrm{d}s_{\boldsymbol{y}}\right]^{\frac12}\cdot\left[\int_{\partial B_r^+}|\boldsymbol{G}_1(\boldsymbol{y}-\boldsymbol{x})|^2\mathrm{d}s_{\boldsymbol{y}}\right]^{\frac12}\\&\quad+\left[\int_{\partial B_r^+}|\boldsymbol{E}^s(\boldsymbol{y})|^2\mathrm{d}s_{\boldsymbol{y}}\right]^{\frac12}\cdot\left[\int_{\partial B_r^+}|\nabla_{\boldsymbol{y}}\times\boldsymbol{G}_1(\boldsymbol{y}-\boldsymbol{x})|^2\mathrm{d}s_{\boldsymbol{y}}\right]^{\frac12}\to0.\end{aligned}\tag{3.7}$$

By lemma 2.2 and the definition of incident field $\boldsymbol{E}^i$, we have for $r\to+\infty$ that

$$\begin{aligned}&\left|\int_{\partial B_r^+}\big\{[\boldsymbol{\nu}(\boldsymbol{y})\times(\nabla_{\boldsymbol{y}}\times\boldsymbol{E}^i(\boldsymbol{y}))]\cdot\boldsymbol{G}_1(\boldsymbol{y}-\boldsymbol{x})+[\boldsymbol{\nu}(\boldsymbol{y})\times\boldsymbol{E}^i(\boldsymbol{y})]\cdot[\nabla_{\boldsymbol{y}}\times\boldsymbol{G}_1(\boldsymbol{y}-\boldsymbol{x})]\big\}\mathrm{d}s_{\boldsymbol{y}}\right|\\&\leqslant\left[\int_{\partial B_r^+}|\nabla_{\boldsymbol{y}}\times\boldsymbol{E}^i(\boldsymbol{y})|^2\mathrm{d}s_{\boldsymbol{y}}\right]^{\frac12}\cdot\left[\int_{\partial B_r^+}|\boldsymbol{G}_1(\boldsymbol{y}-\boldsymbol{x})|^2\mathrm{d}s_{\boldsymbol{y}}\right]^{\frac12}\\&\quad+\left[\int_{\partial B_r^+}|\boldsymbol{E}^i(\boldsymbol{y})|^2\mathrm{d}s_{\boldsymbol{y}}\right]^{\frac12}\cdot\left[\int_{\partial B_r^+}|\nabla_{\boldsymbol{y}}\times\boldsymbol{G}_1(\boldsymbol{y}-\boldsymbol{x})|^2\mathrm{d}s_{\boldsymbol{y}}\right]^{\frac12}\to0.\end{aligned}\tag{3.8}$$

Using (3.6)–(3.8) and conditions (ii), (iv) in problem 2.1, and letting $r\to+\infty$, we have for each fixed $\boldsymbol{x}\in\Omega_1$ that

$$\begin{aligned}\boldsymbol{E}_1(\boldsymbol{x})-\boldsymbol{E}^i(\boldsymbol{x})=&-\int_S\big\{[\boldsymbol{\nu}(\boldsymbol{y})\times(\nabla_{\boldsymbol{y}}\times\boldsymbol{E}_1(\boldsymbol{y}))]\cdot\boldsymbol{G}_1(\boldsymbol{y}-\boldsymbol{x})\\&+[\boldsymbol{\nu}(\boldsymbol{y})\times\boldsymbol{E}_1(\boldsymbol{y})]\cdot[\nabla_{\boldsymbol{y}}\times\boldsymbol{G}_1(\boldsymbol{y}-\boldsymbol{x})]\big\}\mathrm{d}s_{\boldsymbol{y}}\\&-\int_\Gamma\big\{[\boldsymbol{\nu}(\boldsymbol{y})\times(\nabla_{\boldsymbol{y}}\times\boldsymbol{E}_1(\boldsymbol{y}))]\cdot\boldsymbol{G}_1(\boldsymbol{y}-\boldsymbol{x})\big\}\mathrm{d}s_{\boldsymbol{y}}\\=&\int_S\big\{[\boldsymbol{G}_1(\boldsymbol{x}-\boldsymbol{y})]\cdot[\boldsymbol{\nu}_S(\boldsymbol{y})\times(\nabla_{\boldsymbol{y}}\times\boldsymbol{E}_1(\boldsymbol{y}))]\\&+[\nabla_{\boldsymbol{x}}\times\boldsymbol{G}_1(\boldsymbol{x}-\boldsymbol{y})]\cdot[\boldsymbol{\nu}_S(\boldsymbol{y})\times\boldsymbol{E}_1(\boldsymbol{y})]\big\}\mathrm{d}s_{\boldsymbol{y}}\\&+\int_\Gamma\big\{[\boldsymbol{G}_1(\boldsymbol{x}-\boldsymbol{y})]\cdot[\boldsymbol{\nu}_\Gamma(\boldsymbol{y})\times(\nabla_{\boldsymbol{y}}\times\boldsymbol{E}_1(\boldsymbol{y}))]\big\}\mathrm{d}s_{\boldsymbol{y}}.\end{aligned}$$

Similarly, for each fixed $\boldsymbol{x} \in \Omega_2$, using the continuity conditions (2.7), we have

$$\begin{aligned}\boldsymbol{E}_2(\boldsymbol{x}) &= -\int_S \big\{[\boldsymbol{G}_2(\boldsymbol{x}-\boldsymbol{y})]\cdot[\boldsymbol{\nu}_S(\boldsymbol{y})\times(\nabla_{\boldsymbol{y}}\times\boldsymbol{E}_2(\boldsymbol{y}))] \\ &\qquad + [\nabla_{\boldsymbol{x}}\times\boldsymbol{G}_2(\boldsymbol{x}-\boldsymbol{y})]\cdot[\boldsymbol{\nu}_S(\boldsymbol{y})\times\boldsymbol{E}_2(\boldsymbol{y})]\big\}\mathrm{d}s_{\boldsymbol{y}} \\ &= -\int_S \big\{[\boldsymbol{G}_2(\boldsymbol{x}-\boldsymbol{y})]\cdot[\boldsymbol{\nu}_S(\boldsymbol{y})\times(\nabla_{\boldsymbol{y}}\times\boldsymbol{E}_1(\boldsymbol{y}))] \\ &\qquad + [\nabla_{\boldsymbol{x}}\times\boldsymbol{G}_2(\boldsymbol{x}-\boldsymbol{y})]\cdot[\boldsymbol{\nu}_S(\boldsymbol{y})\times\boldsymbol{E}_1(\boldsymbol{y})]\big\}\mathrm{d}s_{\boldsymbol{y}},\end{aligned}$$

where

$$\begin{aligned}\boldsymbol{\nu}_S(\boldsymbol{y})\times\boldsymbol{E}_j(\boldsymbol{y}) &= \lim_{h\to+0}\boldsymbol{\nu}_S(\boldsymbol{y})\times\boldsymbol{E}_j(\boldsymbol{y}+(-1)^j h\boldsymbol{\nu}_S(\boldsymbol{y})),\\ \boldsymbol{\nu}_S(\boldsymbol{y})\times[\nabla_{\boldsymbol{y}}\times\boldsymbol{E}_j(\boldsymbol{y})] &= \lim_{h\to+0}\boldsymbol{\nu}_S(\boldsymbol{y})\times[\nabla_{\boldsymbol{y}}\times\boldsymbol{E}_j(\boldsymbol{y}+(-1)^j h\boldsymbol{\nu}_S(\boldsymbol{y}))]\end{aligned}$$

are to be understood in the sense of uniform convergence on S. □

The integral representation (3.1) and (3.2) can be used to derive the boundary integral equation for the direct scattering problem. Using the jump relations and the continuity conditions (2.7), we have from (3.1) and (3.2) that

$$\begin{aligned}&\boldsymbol{\nu}_\Gamma(\boldsymbol{x})\times(\nabla_{\boldsymbol{x}}\times\boldsymbol{E}_1(\boldsymbol{x}))\\ &= 2\boldsymbol{\nu}_\Gamma(\boldsymbol{x})\times(\nabla_{\boldsymbol{x}}\times\boldsymbol{E}^i(\boldsymbol{x}))\\ &\quad + 2\int_S\big\{[\boldsymbol{\nu}_\Gamma(\boldsymbol{x})\times(\nabla_{\boldsymbol{x}}\times\boldsymbol{G}_1(\boldsymbol{x}-\boldsymbol{y}))]\cdot[\boldsymbol{\nu}_S(\boldsymbol{y})\times(\nabla_{\boldsymbol{y}}\times\boldsymbol{E}_1(\boldsymbol{y}))]\\ &\qquad + [\boldsymbol{\nu}_\Gamma(\boldsymbol{x})\times(\kappa_1^2\boldsymbol{G}_1(\boldsymbol{x}-\boldsymbol{y}))]\cdot[\boldsymbol{\nu}_S(\boldsymbol{y})\times\boldsymbol{E}_1(\boldsymbol{y})]\big\}\mathrm{d}s_{\boldsymbol{y}}\\ &\quad + 2\int_\Gamma\big\{[\boldsymbol{\nu}_\Gamma(\boldsymbol{x})\times(\nabla_{\boldsymbol{x}}\times\boldsymbol{G}_1(\boldsymbol{x}-\boldsymbol{y}))]\cdot[\boldsymbol{\nu}_\Gamma(\boldsymbol{y})\times(\nabla_{\boldsymbol{y}}\times\boldsymbol{E}_1(\boldsymbol{y}))]\big\}\mathrm{d}s_{\boldsymbol{y}},\quad \boldsymbol{x}\in\Gamma,\end{aligned}\tag{3.9}$$

$$\begin{aligned}&\boldsymbol{\nu}_S(\boldsymbol{x})\times\boldsymbol{E}_1(\boldsymbol{x})\\ &= \boldsymbol{\nu}_S(\boldsymbol{x})\times\boldsymbol{E}^i(\boldsymbol{x})\\ &\quad + \int_S\big\{[\boldsymbol{\nu}_S(\boldsymbol{x})\times(\boldsymbol{G}_1(\boldsymbol{x}-\boldsymbol{y})-\boldsymbol{G}_2(\boldsymbol{x}-\boldsymbol{y}))]\cdot[\boldsymbol{\nu}_S(\boldsymbol{y})\times(\nabla_{\boldsymbol{y}}\times\boldsymbol{E}_1(\boldsymbol{y}))]\\ &\qquad + [\boldsymbol{\nu}_S(\boldsymbol{x})\times(\nabla_{\boldsymbol{x}}\times\boldsymbol{G}_1(\boldsymbol{x}-\boldsymbol{y})-\nabla_{\boldsymbol{x}}\times\boldsymbol{G}_2(\boldsymbol{x}-\boldsymbol{y}))]\cdot[\boldsymbol{\nu}_S(\boldsymbol{y})\times\boldsymbol{E}_1(\boldsymbol{y})]\big\}\mathrm{d}s_{\boldsymbol{y}}\\ &\quad + \int_\Gamma\big\{[\boldsymbol{\nu}_S(\boldsymbol{x})\times\boldsymbol{G}_1(\boldsymbol{x}-\boldsymbol{y})]\cdot[\boldsymbol{\nu}_\Gamma(\boldsymbol{y})\times(\nabla_{\boldsymbol{y}}\times\boldsymbol{E}_1(\boldsymbol{y}))]\big\}\mathrm{d}s_{\boldsymbol{y}},\quad \boldsymbol{x}\in S,\end{aligned}\tag{3.10}$$

and

$$\begin{aligned}&\boldsymbol{\nu}_S(\boldsymbol{x})\times(\nabla_{\boldsymbol{x}}\times\boldsymbol{E}_1(\boldsymbol{x}))\\ &= \boldsymbol{\nu}_S(\boldsymbol{x})\times(\nabla_{\boldsymbol{x}}\times\boldsymbol{E}^i(\boldsymbol{x}))\\ &\quad + \int_S\big\{[\boldsymbol{\nu}_S(\boldsymbol{x})\times(\nabla_{\boldsymbol{x}}\times\boldsymbol{G}_1(\boldsymbol{x}-\boldsymbol{y})-\nabla_{\boldsymbol{x}}\times\boldsymbol{G}_2(\boldsymbol{x}-\boldsymbol{y}))]\cdot[\boldsymbol{\nu}_S(\boldsymbol{y})\times(\nabla_{\boldsymbol{y}}\times\boldsymbol{E}_1(\boldsymbol{y}))]\\ &\qquad + [\boldsymbol{\nu}_S(\boldsymbol{x})\times(\kappa_1^2\boldsymbol{G}_1(\boldsymbol{x}-\boldsymbol{y})-\kappa_2^2\boldsymbol{G}_2(\boldsymbol{x}-\boldsymbol{y}))]\cdot[\boldsymbol{\nu}_S(\boldsymbol{y})\times\boldsymbol{E}_1(\boldsymbol{y})]\big\}\mathrm{d}s_{\boldsymbol{y}}\\ &\quad + \int_\Gamma\big\{[\boldsymbol{\nu}_S(\boldsymbol{x})\times(\nabla_{\boldsymbol{x}}\times\boldsymbol{G}_1(\boldsymbol{x}-\boldsymbol{y}))]\cdot[\boldsymbol{\nu}_\Gamma(\boldsymbol{y})\times(\nabla_{\boldsymbol{y}}\times\boldsymbol{E}_1(\boldsymbol{y}))]\big\}\mathrm{d}s_{\boldsymbol{y}},\quad \boldsymbol{x}\in S.\end{aligned}\tag{3.11}$$

To study the boundary integral equations (3.9)–(3.11) and to show the well-posedness of the direct problem, we introduce the normed subspace of continuous tangential fields

$$\mathfrak{T}(S) := \{\psi \in C(S) : \boldsymbol{\nu}_S \cdot \psi = 0\}, \quad \mathfrak{T}(\Gamma) := \{\psi \in C(\Gamma) : \boldsymbol{\nu}_\Gamma \cdot \psi = 0\}$$

and the normed space of uniformly Hölder continuous tangential fields

$$\mathfrak{T}^{0,\alpha}(S) := \{\psi \in \mathfrak{T}(S) |\ \psi \in C^{0,\alpha}(S)\}, \quad \mathfrak{T}^{0,\alpha}(\Gamma) := \{\psi \in \mathfrak{T}(\Gamma) |\ \psi \in C^{0,\alpha}(\Gamma)\},$$

where $0 < \alpha < 1$.

On the infinite rough surface S, we define the integral operators $\mathbf{T} : \mathfrak{T}^{0,\alpha}(S) \to \mathfrak{T}^{0,\alpha}(S)$ and $\mathbf{T}^* : \mathfrak{T}^{0,\alpha}(S) \to \mathfrak{T}^{0,\alpha}(S)$ by

$$(\mathbf{T}\boldsymbol{\Psi})(\boldsymbol{x}) = \int_S [\boldsymbol{\nu}_S(\boldsymbol{x}) \times (\boldsymbol{G}_1(\boldsymbol{x}-\boldsymbol{y}) - \boldsymbol{G}_2(\boldsymbol{x}-\boldsymbol{y}))] \cdot [\boldsymbol{\Psi}(\boldsymbol{y})]\mathrm{d}s_{\boldsymbol{y}}, \tag{3.12}$$

$$(\mathbf{T}^*\boldsymbol{\Psi})(\boldsymbol{x}) = \int_S [\boldsymbol{\nu}_S(\boldsymbol{x}) \times (\kappa_1^2\boldsymbol{G}_1(\boldsymbol{x}-\boldsymbol{y}) - \kappa_2^2\boldsymbol{G}_2(\boldsymbol{x}-\boldsymbol{y}))] \cdot [\boldsymbol{\Psi}(\boldsymbol{y})]\mathrm{d}s_{\boldsymbol{y}}, \tag{3.13}$$

and the integral operator $\mathbf{K} : \mathfrak{T}^{0,\alpha}(S) \to \mathfrak{T}^{0,\alpha}(S)$ defined by

$$(\mathbf{K}\boldsymbol{\Psi})(\boldsymbol{x}) = \int_S [\boldsymbol{\nu}_S(\boldsymbol{x}) \times (\nabla_{\boldsymbol{x}} \times \boldsymbol{G}_1(\boldsymbol{x}-\boldsymbol{y}) - \nabla_{\boldsymbol{x}} \times \boldsymbol{G}_2(\boldsymbol{x}-\boldsymbol{y}))] \cdot [\boldsymbol{\Psi}(\boldsymbol{y})]\mathrm{d}s_{\boldsymbol{y}}. \tag{3.14}$$

On Γ, we define the integral operator $\widetilde{\mathbf{K}} : \mathfrak{T}^{0,\alpha}(\Gamma) \to \mathfrak{T}^{0,\alpha}(\Gamma)$ by

$$(\widetilde{\mathbf{K}}\boldsymbol{\Psi})(\boldsymbol{x}) = 2\int_\Gamma [\boldsymbol{\nu}_\Gamma(\boldsymbol{x}) \times (\nabla_{\boldsymbol{x}} \times \boldsymbol{G}_1(\boldsymbol{x}-\boldsymbol{y}))] \cdot [\boldsymbol{\Psi}(\boldsymbol{y})]\mathrm{d}s_{\boldsymbol{y}}. \tag{3.15}$$

For each $n \in \mathbb{Z}^+$, define the truncated operators $\mathbf{T}_n : \mathfrak{T}^{0,\alpha}(S_n) \to \mathfrak{T}^{0,\alpha}(S)$ and $\mathbf{T}_n^* : \mathfrak{T}^{0,\alpha}(S_n) \to \mathfrak{T}^{0,\alpha}(S)$ by

$$(\mathbf{T}_n\boldsymbol{\Psi})(\boldsymbol{x}) = \int_{S_n} [\boldsymbol{\nu}_S(\boldsymbol{x}) \times (\boldsymbol{G}_1(\boldsymbol{x}-\boldsymbol{y}) - \boldsymbol{G}_2(\boldsymbol{x}-\boldsymbol{y}))] \cdot [\boldsymbol{\Psi}(\boldsymbol{y})]\mathrm{d}s_{\boldsymbol{y}}, \tag{3.16}$$

$$(\mathbf{T}_n^*\boldsymbol{\Psi})(\boldsymbol{x}) = \int_{S_n} [\boldsymbol{\nu}_S(\boldsymbol{x}) \times (\kappa_1^2\boldsymbol{G}_1(\boldsymbol{x}-\boldsymbol{y}) - \kappa_2^2\boldsymbol{G}_2(\boldsymbol{x}-\boldsymbol{y}))] \cdot [\boldsymbol{\Psi}(\boldsymbol{y})]\mathrm{d}s_{\boldsymbol{y}}, \tag{3.17}$$

and the operator $\mathbf{K}_n : \mathfrak{T}^{0,\alpha}(S_n) \to \mathfrak{T}^{0,\alpha}(S)$ by

$$(\mathbf{K}_n\boldsymbol{\Psi})(\boldsymbol{x}) = \int_{S_n} [\boldsymbol{\nu}_S(\boldsymbol{x}) \times (\nabla_{\boldsymbol{x}} \times \boldsymbol{G}_1(\boldsymbol{x}-\boldsymbol{y}) - \nabla_{\boldsymbol{x}} \times \boldsymbol{G}_2(\boldsymbol{x}-\boldsymbol{y}))] \cdot [\boldsymbol{\Psi}(\boldsymbol{y})]\mathrm{d}s_{\boldsymbol{y}}, \tag{3.18}$$

where $S_n = \{\boldsymbol{x} \in S : |x_j| \leqslant n,\ j = 1, 2\}$. Now, with the aid of (2.14), we have

$$\begin{aligned}
&\boldsymbol{G}_1(\boldsymbol{x}-\boldsymbol{y}) - \boldsymbol{G}_2(\boldsymbol{x}-\boldsymbol{y}) \\
&= \left[\boldsymbol{I} + \frac{\nabla_{\boldsymbol{x}}\nabla_{\boldsymbol{x}}}{\kappa_1^2}\right]\frac{\exp(\mathrm{i}\kappa_1|\boldsymbol{x}-\boldsymbol{y}|)}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} - \left[\boldsymbol{I} + \frac{\nabla_{\boldsymbol{x}}\nabla_{\boldsymbol{x}}}{\kappa_2^2}\right]\frac{\exp(\mathrm{i}\kappa_2|\boldsymbol{x}-\boldsymbol{y}|)}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} \\
&= \left[\frac{\exp(\mathrm{i}\kappa_1|\boldsymbol{x}-\boldsymbol{y}|)}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\boldsymbol{I} - \frac{\exp(\mathrm{i}\kappa_2|\boldsymbol{x}-\boldsymbol{y}|)}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\boldsymbol{I}\right] \\
&\quad + \nabla_{\boldsymbol{x}}\nabla_{\boldsymbol{x}}\left[\frac{\exp(\mathrm{i}\kappa_1|\boldsymbol{x}-\boldsymbol{y}|)}{4\pi\kappa_1^2|\boldsymbol{x}-\boldsymbol{y}|} - \frac{\exp(\mathrm{i}\kappa_2|\boldsymbol{x}-\boldsymbol{y}|)}{4\pi\kappa_2^2|\boldsymbol{x}-\boldsymbol{y}|}\right]
\end{aligned}$$

and

$$\begin{aligned}\nabla_{\boldsymbol{x}} \times \boldsymbol{G}_j(\boldsymbol{x}-\boldsymbol{y}) &= \nabla_{\boldsymbol{x}} \times \left\{ \left[\boldsymbol{I} + \frac{\nabla_{\boldsymbol{x}}\nabla_{\boldsymbol{x}}}{\kappa_j^2} \right] \frac{\exp\left(\mathrm{i}\kappa_j|\boldsymbol{x}-\boldsymbol{y}|\right)}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} \right\} \\ &= \nabla_{\boldsymbol{x}} \times \left[\frac{\exp\left(\mathrm{i}\kappa_j|\boldsymbol{x}-\boldsymbol{y}|\right)}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} \boldsymbol{I} \right] + \nabla_{\boldsymbol{x}} \times \left\{ \left[\frac{\nabla_{\boldsymbol{x}}\nabla_{\boldsymbol{x}}}{\kappa_j^2} \right] \frac{\exp\left(\mathrm{i}\kappa_j|\boldsymbol{x}-\boldsymbol{y}|\right)}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} \right\} \\ &= \nabla_{\boldsymbol{x}} \times \left[\frac{\exp\left(\mathrm{i}\kappa_j|\boldsymbol{x}-\boldsymbol{y}|\right)}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} \boldsymbol{I} \right].\end{aligned}$$

The above results imply that the kernels of the operators $\widetilde{\mathbf{K}}, \mathbf{T}_n, \mathbf{T}_n^*$ and $\mathbf{K}_n$ are weakly singular and decay exponentially. It follows from [6, theorem 1.11] and [6, theorem 2.7] that these integral operators are compact. Based on the compactness of the truncated operators, the integral operators $\mathbf{T}, \mathbf{T}^*$ and $\mathbf{K}$ are compact as described in the following theorem.

Lemma 3.2. *The integral operators* $\mathbf{T}$, $\mathbf{T}^*$ *and* $\mathbf{K}$ *are compact.*

Proof. Since the proofs are similar for $\mathbf{T}$, $\mathbf{T}^*$ and $\mathbf{K}$, we shall only show the details for the operator $\mathbf{T}$. For each fixed $\boldsymbol{x} \in S$, it follows from (3.12) and (3.16) that

$$\begin{aligned}&(\mathbf{T}\boldsymbol{\Psi})(\boldsymbol{x}) - (\mathbf{T}_n\boldsymbol{\Psi})(\boldsymbol{x}) \\ &= \left(\int_n^{+\infty}\int_{-\infty}^{+\infty} + \int_{-\infty}^{-n}\int_{-\infty}^{+\infty} + \int_{-n}^{+n}\int_{-\infty}^{-n} + \int_{-n}^{+n}\int_{n}^{+\infty} \right) \varphi(\boldsymbol{x}, y_1, y_2)\mathrm{d}y_1\mathrm{d}y_2 \\ &= I_1 + I_2 + I_3 + I_4,\end{aligned} \tag{3.19}$$

where

$$\varphi(\boldsymbol{x}, y_1, y_2) = [(\boldsymbol{\nu}_S(\boldsymbol{x}) \times (\boldsymbol{G}_1(\boldsymbol{x}-\boldsymbol{y}) - \boldsymbol{G}_2(\boldsymbol{x}-\boldsymbol{y}))) \cdot \boldsymbol{\Psi}(\boldsymbol{y})|_{y_3=f(y_1,y_2)}](1+f_{y_1}^2+f_{y_2}^2)^{1/2}.$$

By lemma 2.2, for each fixed $\boldsymbol{x} \in S$, when $n \to +\infty$, we have

$$\begin{aligned}|I_1| &\leqslant \int_n^{+\infty}\int_{-\infty}^{+\infty} |\varphi(\boldsymbol{x}, y_1, y_2)|\mathrm{d}y_1\mathrm{d}y_2 \\ &\leqslant C\int_n^{+\infty}\int_{-\infty}^{+\infty} [|\boldsymbol{G}_1(\boldsymbol{x}-\boldsymbol{y}) - \boldsymbol{G}_2(\boldsymbol{x}-\boldsymbol{y})| \cdot |\boldsymbol{\Psi}(\boldsymbol{y})|_{y_3=f(y_1,y_2)}]\mathrm{d}y_1\mathrm{d}y_2 \\ &\leqslant C\|\boldsymbol{\Psi}\|_{C^{0,\alpha}(S)} \int_n^{+\infty}\int_{-\infty}^{+\infty} \left(\frac{\exp\left(-\frac{1}{2}\Im(\kappa_1)|\boldsymbol{y}|\right)}{|\boldsymbol{y}|} + \frac{\exp\left(-\frac{1}{2}\Im(\kappa_2)|\boldsymbol{y}|\right)}{|\boldsymbol{y}|} \right)\bigg|_{y_3=f(y_1,y_2)} \mathrm{d}y_1\mathrm{d}y_2 \\ &\leqslant C\|\boldsymbol{\Psi}\|_{C^{0,\alpha}(S)} \int_0^{+\infty} \exp\left(-\frac{1}{4}\hat{\kappa}y_1\right)\mathrm{d}y_1 \int_n^{+\infty} \frac{\exp\left(-\frac{1}{4}\hat{\kappa}y_2\right)}{y_2}\mathrm{d}y_2 \\ &= C\|\boldsymbol{\Psi}\|_{C^{0,\alpha}(S)} \left(\frac{4}{\hat{\kappa}}\right)^2 \left(\frac{1}{n}\exp\left(-\frac{n}{4}\hat{\kappa}\right)\right) \to 0 \qquad \text{as } n \to +\infty\end{aligned} \tag{3.20}$$

where $\hat{\kappa} = \min\{\Im(\kappa_1), \Im(\kappa_2)\} > 0$, C is a positive constant and may change from step to step. Similarly, we may show for $j = 2, 3, 4$ that

$$|I_j| \leqslant C\|\boldsymbol{\Psi}\|_{C^{0,\alpha}(S)} \left(\frac{1}{n}\exp\left(-\frac{n}{4}\hat{\kappa}\right)\right) \to 0 \quad \text{as } n \to +\infty. \tag{3.21}$$

Combining (3.19)–(3.21) leads to

$$|(\mathbf{T}\boldsymbol{\Psi})(\boldsymbol{x}) - (\mathbf{T}_n\boldsymbol{\Psi})(\boldsymbol{x})| \leqslant \sum_{j=1}^{4} |I_j| \leqslant C\|\boldsymbol{\Psi}\|_{C^{0,\alpha}(S)} \left(\frac{1}{n}\exp\left(-\frac{n}{4}\hat{\kappa}\right)\right) \to 0 \quad \text{as } n \to +\infty.$$

Hence we have

$$\|(\mathbf{T} - \mathbf{T}_n)\boldsymbol{\Psi}\|_\infty \leqslant C\|\boldsymbol{\Psi}\|_{C^{0,\alpha}(S)} \left(\frac{1}{n}\exp\left(-\frac{n}{4}\hat{\kappa}\right)\right) \to 0 \quad \text{as } n \to +\infty. \tag{3.22}$$

For each fixed $\boldsymbol{x}, \widetilde{\boldsymbol{x}} \in S$ and $\boldsymbol{x} \neq \widetilde{\boldsymbol{x}}$, it follows from (3.12) and (3.16) that

$$\begin{aligned} &((\mathbf{T} - \mathbf{T}_n)\boldsymbol{\Psi})(\boldsymbol{x}) - ((\mathbf{T} - \mathbf{T}_n)\boldsymbol{\Psi})(\widetilde{\boldsymbol{x}}) \\ &= \left(\int_n^{+\infty}\int_{-\infty}^{+\infty} + \int_{-\infty}^{-n}\int_{-\infty}^{+\infty} + \int_{-n}^{+n}\int_{-\infty}^{-n} + \int_{-n}^{+n}\int_n^{+\infty}\right)[\varphi(\boldsymbol{x}, y_1, y_2) - \varphi(\widetilde{\boldsymbol{x}}, y_1, y_2)]\mathrm{d}y_1\mathrm{d}y_2 \\ &= I_5 + I_6 + I_7 + I_8. \end{aligned} \tag{3.23}$$

From lemma 2.2 and the mean value theorem, we get

$$|\boldsymbol{G}_j(\boldsymbol{x} - \boldsymbol{y}) - \boldsymbol{G}_j(\widetilde{\boldsymbol{x}} - \boldsymbol{y})| \leqslant C\frac{\exp\left(-\frac{1}{2}\Im(\kappa_j)|\boldsymbol{y}|\right)}{|\boldsymbol{y}|}|\boldsymbol{x} - \widetilde{\boldsymbol{x}}|, \qquad j = 1, 2.$$

Therefore

$$\begin{aligned} |I_5| &\leqslant \int_n^{+\infty}\int_{-\infty}^{+\infty} |\varphi(\boldsymbol{x}, y_1, y_2) - \varphi(\widetilde{\boldsymbol{x}}, y_1, y_2)|\mathrm{d}y_1\mathrm{d}y_2 \\ &\leqslant C\int_n^{+\infty}\int_{-\infty}^{+\infty} [(|\boldsymbol{G}_1(\boldsymbol{x} - \boldsymbol{y}) - \boldsymbol{G}_1(\widetilde{\boldsymbol{x}} - \boldsymbol{y})| + |\boldsymbol{G}_2(\boldsymbol{x} - \boldsymbol{y}) - \boldsymbol{G}_2(\widetilde{\boldsymbol{x}} - \boldsymbol{y})|) \cdot |\boldsymbol{\Psi}(\boldsymbol{y})|_{y_3 = f(y_1, y_2)}]\mathrm{d}y_1\mathrm{d}y_2 \\ &\leqslant C(|\boldsymbol{x} - \widetilde{\boldsymbol{x}}|)\sup_{\boldsymbol{y}\in S}|\boldsymbol{\Psi}(\boldsymbol{y})| \int_n^{+\infty}\int_{-\infty}^{+\infty} \left(\frac{\exp\left(-\frac{1}{2}\hat{\kappa}|\boldsymbol{y}|\right)}{|\boldsymbol{y}|}\Bigg|_{y_3 = f(y_1, y_2)}\right)\mathrm{d}y_1\mathrm{d}y_2 \\ &\leqslant C(|\boldsymbol{x} - \widetilde{\boldsymbol{x}}|)\|\boldsymbol{\Psi}\|_{C^{0,\alpha}(S)} \left(\frac{4}{\hat{\kappa}}\right)^2 \left(\frac{1}{n}\exp\left(-\frac{n}{4}\hat{\kappa}\right)\right). \end{aligned} \tag{3.24}$$

Similarly, for $j = 6, 7, 8$, we also have

$$|I_j| \leqslant C(|\boldsymbol{x} - \widetilde{\boldsymbol{x}}|)\|\boldsymbol{\Psi}\|_{C^{0,\alpha}(S)} \left(\frac{1}{n}\exp\left(-\frac{n}{4}\hat{\kappa}\right)\right). \tag{3.25}$$

Combining (3.23)–(3.25) and noting $0 < \alpha < 1$, we obtain

$$\begin{aligned} &\frac{|((\mathbf{T} - \mathbf{T}_n)\boldsymbol{\Psi})(\boldsymbol{x}) - ((\mathbf{T} - \mathbf{T}_n)\boldsymbol{\Psi})(\widetilde{\boldsymbol{x}})|}{|\boldsymbol{x} - \widetilde{\boldsymbol{x}}|^\alpha} \\ &\leqslant \sum_{j=5}^{8}|I_j| \leqslant C(|\boldsymbol{x} - \widetilde{\boldsymbol{x}}|^{1-\alpha})\|\boldsymbol{\Psi}\|_{C^{0,\alpha}(S)} \left(\frac{1}{n}\exp\left(-\frac{n}{4}\hat{\kappa}\right)\right) \\ &\leqslant C(n^{1-\alpha})\|\boldsymbol{\Psi}\|_{C^{0,\alpha}(S)} \left(\frac{1}{n}\exp\left(-\frac{n}{4}\hat{\kappa}\right)\right) \\ &= C\|\boldsymbol{\Psi}\|_{C^{0,\alpha}(S)} \left(n^{-\alpha}\exp\left(-\frac{n}{4}\hat{\kappa}\right)\right) \to 0 \quad \text{as } n \to +\infty. \end{aligned} \tag{3.26}$$

For $0 < \alpha < 1$, it can be deduced from (3.22) and (3.26) that

$$\begin{aligned}
&\|\mathbf{T}-\mathbf{T}_n\|_{C^{0,\alpha}(S)} \\
&= \sup_{\|\boldsymbol{\Psi}\|_{C^{0,\alpha}(S)}\neq 0} \frac{\|(\mathbf{T}-\mathbf{T}_n)\boldsymbol{\Psi}\|_{C^{0,\alpha}(S)}}{\|\boldsymbol{\Psi}\|_{C^{0,\alpha}(S)}} \\
&= \sup_{\|\boldsymbol{\Psi}\|_{C^{0,\alpha}(S)}\neq 0} \frac{1}{\|\boldsymbol{\Psi}\|_{C^{0,\alpha}(S)}} \Bigg[\|(\mathbf{T}-\mathbf{T}_n)\boldsymbol{\Psi}\|_\infty \\
&\qquad + \sup_{\substack{\boldsymbol{x},\tilde{\boldsymbol{x}}\in S\\ \boldsymbol{x}\neq\tilde{\boldsymbol{x}}}} \frac{|((\mathbf{T}-\mathbf{T}_n)\boldsymbol{\Psi})(\boldsymbol{x}) - ((\mathbf{T}-\mathbf{T}_n)\boldsymbol{\Psi})(\tilde{\boldsymbol{x}})|}{|\boldsymbol{x}-\tilde{\boldsymbol{x}}|^\alpha}\Bigg] \\
&\leqslant C\left(n^{-\alpha}\exp\left(-\frac{n}{4}\hat{\kappa}\right)\right) \to 0 \quad \text{as } n\to+\infty,
\end{aligned}$$

which shows that the operator $\mathbf{T}$ is compact on $\mathfrak{T}^{0,\alpha}(S)$. Similarly, it can be shown that operators $\mathbf{T}^*$ and $\mathbf{K}$ are also compact on $\mathfrak{T}^{0,\alpha}(S)$. □

By lemma 3.2, the system of the boundary integral equations (3.9)–(3.11) is of the Fredholm type, i.e. the existence of the solution follows immediately from the uniqueness of the solution.

Theorem 3.3. *Let $\boldsymbol{E}^s \in \mathcal{T}_1, \boldsymbol{E}_2 \in \mathcal{T}_2$ have the integral representations (3.1) and (3.2) and satisfy the boundary integral equations (3.9)–(3.11). Then $(\boldsymbol{E}_1, \boldsymbol{E}_2)$ are the solutions of problem 2.1.*

Proof. We only show the proof for the field $\boldsymbol{E}_1$. If the field $\boldsymbol{E}^s \in \mathcal{T}_1$ has the integral representation (3.1), then we have

$$\begin{aligned}
\boldsymbol{E}^s(\boldsymbol{x}) = &\int_S \big\{[\boldsymbol{G}_1(\boldsymbol{x}-\boldsymbol{y})]\cdot[\boldsymbol{\nu}_S(\boldsymbol{y})\times(\nabla_{\boldsymbol{y}}\times\boldsymbol{E}_1(\boldsymbol{y}))] + [\nabla_{\boldsymbol{x}}\times\boldsymbol{G}_1(\boldsymbol{x}-\boldsymbol{y})]\cdot[\boldsymbol{\nu}_S(\boldsymbol{y})\times\boldsymbol{E}_1(\boldsymbol{y})]\big\}\mathrm{d}s_{\boldsymbol{y}} \\
&+ \int_\Gamma \big\{[\boldsymbol{G}_1(\boldsymbol{x}-\boldsymbol{y})]\cdot[\boldsymbol{\nu}_\Gamma(\boldsymbol{y})\times(\nabla_{\boldsymbol{y}}\times\boldsymbol{E}_1(\boldsymbol{y}))]\big\}\mathrm{d}s_{\boldsymbol{y}}, \quad \boldsymbol{x}\in\Omega_1.
\end{aligned} \tag{3.27}$$

Noting that for any $\boldsymbol{x}\in\Omega_1$ and $\boldsymbol{y}\in S\cup\Gamma$, we have $\boldsymbol{x}\neq\boldsymbol{y}$. Taking double curl of (3.27), multiplying (3.27) by $-\kappa_1^2 = -\omega^2\mu(\varepsilon_1 + \mathrm{i}\frac{\sigma_1}{\omega})$, and adding the resulting two equations with the aid of (2.13), we obtain

$$\begin{aligned}
&\nabla\times(\nabla\times\boldsymbol{E}^s(\boldsymbol{x})) - \kappa_1^2\boldsymbol{E}^s(\boldsymbol{x}) \\
&= \int_S \big\{[\nabla_{\boldsymbol{x}}\times(\nabla_{\boldsymbol{x}}\times\boldsymbol{G}_1(\boldsymbol{x}-\boldsymbol{y})) - \kappa_1^2\boldsymbol{G}_1(\boldsymbol{x}-\boldsymbol{y})]\cdot[\boldsymbol{\nu}_S(\boldsymbol{y})\times(\nabla_{\boldsymbol{y}}\times\boldsymbol{E}_1(\boldsymbol{y}))] \\
&\quad + [\nabla_{\boldsymbol{x}}\times(\nabla_{\boldsymbol{x}}\times(\nabla_{\boldsymbol{x}}\times\boldsymbol{G}_1(\boldsymbol{x}-\boldsymbol{y})) - \kappa_1^2\boldsymbol{G}_1(\boldsymbol{x}-\boldsymbol{y}))]\cdot[\boldsymbol{\nu}_S(\boldsymbol{y})\times\boldsymbol{E}_1(\boldsymbol{y})]\big\}\mathrm{d}s_{\boldsymbol{y}} \\
&\quad + \int_\Gamma \big\{[\nabla_{\boldsymbol{x}}\times(\nabla_{\boldsymbol{x}}\times\boldsymbol{G}_1(\boldsymbol{x}-\boldsymbol{y})) - \kappa_1^2\boldsymbol{G}_1(\boldsymbol{x}-\boldsymbol{y})]\cdot[\boldsymbol{\nu}_\Gamma(\boldsymbol{y})\times(\nabla_{\boldsymbol{y}}\times\boldsymbol{E}_1(\boldsymbol{y}))]\big\}\mathrm{d}s_{\boldsymbol{y}} \\
&=0, \quad \boldsymbol{x}\in\Omega_1.
\end{aligned} \tag{3.28}$$

It follows from (2.8) and (3.28) that

$$
\begin{aligned}
&\nabla\times(\nabla\times\boldsymbol{E}_1(\boldsymbol{x}))-\kappa_1^2\boldsymbol{E}_1(\boldsymbol{x})\\
=&[\nabla\times(\nabla\times\boldsymbol{E}^s(\boldsymbol{x}))-\kappa_1^2\boldsymbol{E}^s(\boldsymbol{x})]+[\nabla\times(\nabla\times\boldsymbol{E}^i(\boldsymbol{x}))-\kappa_1^2\boldsymbol{E}^i(\boldsymbol{x})]\\
=&\mathrm{i}\omega\mu\boldsymbol{J}_{\mathrm{cs}}(\boldsymbol{x}),\quad \boldsymbol{x}\in\ \Omega_1.
\end{aligned}
$$

Furthermore, with the help of lemma 3.2 and (3.27), we deduce that

$$
\begin{aligned}
|\boldsymbol{E}^s(\boldsymbol{x})|\leqslant& C\bigg[\int_S|\boldsymbol{G}_1(\boldsymbol{x}-\boldsymbol{y})|\cdot|\boldsymbol{\nu}_S(\boldsymbol{y})\times(\nabla_{\boldsymbol{y}}\times\boldsymbol{E}_1(\boldsymbol{y}))|\mathrm{d}s_{\boldsymbol{y}}\\
&+\int_S|\nabla_{\boldsymbol{x}}\times\boldsymbol{G}_1(\boldsymbol{x}-\boldsymbol{y})|\cdot|\boldsymbol{\nu}_S(\boldsymbol{y})\times\boldsymbol{E}_1(\boldsymbol{y})|\mathrm{d}s_{\boldsymbol{y}}\\
&+\int_\Gamma|\boldsymbol{G}_1(\boldsymbol{x}-\boldsymbol{y})|\cdot|\boldsymbol{\nu}_\Gamma(\boldsymbol{y})\times(\nabla_{\boldsymbol{y}}\times\boldsymbol{E}_1(\boldsymbol{y}))|\mathrm{d}s_{\boldsymbol{y}}\bigg]\\
\leqslant& C\bigg[\|\nabla\times\boldsymbol{E}_1\|_{C^{0,\alpha}(S)}\int_S|\boldsymbol{G}_1(\boldsymbol{x}-\boldsymbol{y})|\mathrm{d}s_{\boldsymbol{y}}+\|\boldsymbol{E}_1\|_{C^{0,\alpha}(S)}\int_S|\nabla_{\boldsymbol{x}}\times\boldsymbol{G}_1(\boldsymbol{x}-\boldsymbol{y})|\mathrm{d}s_{\boldsymbol{y}}\\
&+\|\nabla\times\boldsymbol{E}_1\|_{C^{0,\alpha}(\Gamma)}\int_\Gamma|\boldsymbol{G}_1(\boldsymbol{x}-\boldsymbol{y})|\mathrm{d}s_{\boldsymbol{y}}\bigg]\\
\leqslant& C\bigg[\lim_{n\to+\infty}\int_{S_n}|\boldsymbol{G}_1(\boldsymbol{x}-\boldsymbol{y})|\mathrm{d}s_{\boldsymbol{y}}+\lim_{n\to+\infty}\int_{S_n}|\nabla_{\boldsymbol{x}}\times\boldsymbol{G}_1(\boldsymbol{x}-\boldsymbol{y})|\mathrm{d}s_{\boldsymbol{y}}\\
&+\int_\Gamma|\boldsymbol{G}_1(\boldsymbol{x}-\boldsymbol{y})|\mathrm{d}s_{\boldsymbol{y}}\bigg].
\end{aligned}
\tag{3.29}
$$

For each fixed $n\geqslant 1$, as $|\boldsymbol{x}|\to+\infty$, by lemma 2.2, we have

$$
\begin{aligned}
\int_{S_n}|\boldsymbol{G}_1(\boldsymbol{x}-\boldsymbol{y})|\mathrm{d}s_{\boldsymbol{y}}\leqslant& C\int_{S_n}\left|\frac{\exp\left(\frac12\mathrm{i}\kappa_1|\boldsymbol{x}-\boldsymbol{y}|\right)}{|\boldsymbol{x}-\boldsymbol{y}|}\exp\left(\frac12\mathrm{i}\kappa_1|\boldsymbol{x}-\boldsymbol{y}|\right)\right|\mathrm{d}s_{\boldsymbol{y}}\\
\leqslant& C\frac{\exp\left(-\frac14\Im(\kappa_1)|\boldsymbol{x}|\right)}{|\boldsymbol{x}|}\int_{S_n}\exp\left(-\frac12\Im(\kappa_1)|\boldsymbol{y}|\right)\mathrm{d}s_{\boldsymbol{y}}\\
\leqslant& C\frac{\exp\left(-\frac14\Im(\kappa_1)|\boldsymbol{x}|\right)}{|\boldsymbol{x}|}\left(\int_0^n\exp\left(-\frac14\Im(\kappa_1)y_1\right)\mathrm{d}y_1\right)^2\\
\leqslant& C\frac{\exp\left(-\frac14\Im(\kappa_1)|\boldsymbol{x}|\right)}{|\boldsymbol{x}|}\left(1-\exp\left(-\frac{n}{4}\Im(\kappa_1)\right)\right)^2,
\end{aligned}
\tag{3.30}
$$

and

$$
\int_{S_n}|\nabla_{\boldsymbol{x}}\times\boldsymbol{G}_1(\boldsymbol{x}-\boldsymbol{y})|\mathrm{d}s_{\boldsymbol{y}}\leqslant C\frac{\exp\left(-\frac14\Im(\kappa_1)|\boldsymbol{x}|\right)}{|\boldsymbol{x}|}\left(1-\exp\left(-\frac{n}{4}\Im(\kappa_1)\right)\right)^2.
\tag{3.31}
$$

Similarly, we can obtain

$$
\begin{aligned}
\int_\Gamma|\boldsymbol{G}_1(\boldsymbol{x}-\boldsymbol{y})|\mathrm{d}s_{\boldsymbol{y}}\leqslant& C\frac{\exp\left(-\Im(\kappa_1)|\boldsymbol{x}|\right)}{|\boldsymbol{x}|}\int_\Gamma\left|\exp\left(-\mathrm{i}\kappa_j\hat{\boldsymbol{x}}\cdot\boldsymbol{y}\right)[\boldsymbol{I}-\hat{\boldsymbol{x}}\hat{\boldsymbol{x}}]+\mathcal{O}\left(\frac{1}{|\boldsymbol{x}|}\right)\hat{\boldsymbol{I}}\right|\mathrm{d}s_{\boldsymbol{y}}\\
\leqslant& C\frac{\exp\left(-\Im(\kappa_1)|\boldsymbol{x}|\right)}{|\boldsymbol{x}|}.
\end{aligned}
\tag{3.32}
$$

Combining (3.29)–(3.32), we have for $\Im(\kappa_1)>0$ that

$$|\boldsymbol{E}^s(\boldsymbol{x})| \leqslant C\left(\frac{\exp\left(-\frac{1}{4}\Im(\kappa_1)|\boldsymbol{x}|\right)}{|\boldsymbol{x}|}\right) \quad \text{as } |\boldsymbol{x}| \to +\infty$$

and

$$\begin{aligned}\int_{\partial B_r^+} |\boldsymbol{E}^s|^2 \mathrm{d}s_{\boldsymbol{x}} &\leqslant C\int_{\partial B_r^+} \frac{\exp\left(-\frac{1}{2}\Im(\kappa_1)|\boldsymbol{x}|\right)}{|\boldsymbol{x}|^2}\mathrm{d}s_{\boldsymbol{x}} \\ &\leqslant C\left(\frac{\exp\left(-\frac{1}{2}\Im(\kappa_1)r\right)}{r^2}4\pi r^2\right) = C\exp\left(-\frac{1}{2}\Im(\kappa_1)r\right) \to 0 \quad \text{as } r = |\boldsymbol{x}| \to +\infty,\end{aligned}$$

where C is a positive constant independent of r.

Similarly, we can also show that

$$\lim_{r\to+\infty}\int_{\partial B_r^+} |\nabla\times\boldsymbol{E}^s|^2\mathrm{d}s_{\boldsymbol{x}} = \lim_{r\to+\infty}\int_{\partial B_r^-} |\boldsymbol{E}_2|^2\mathrm{d}s_{\boldsymbol{x}} = \lim_{r\to+\infty}\int_{\partial B_r^-} |\nabla\times\boldsymbol{E}_2|^2\mathrm{d}s_{\boldsymbol{x}} = 0.$$

Furthermore, since $\boldsymbol{E}_1$ satisfies (2.3) and the radiation conditions, applying the vector dyadic Green second theorem to $\boldsymbol{E}_1$ and $\boldsymbol{G}_1$ in the region Ω_1, we have

$$\begin{aligned}\boldsymbol{E}_1(\boldsymbol{x}) &= \boldsymbol{E}^i(\boldsymbol{x}) \\ &\quad+\int_S \left\{[\boldsymbol{G}_1(\boldsymbol{x}-\boldsymbol{y})]\cdot[\boldsymbol{\nu}_S(\boldsymbol{y})\times(\nabla_{\boldsymbol{y}}\times\boldsymbol{E}_1(\boldsymbol{y}))] + [\nabla_{\boldsymbol{x}}\times\boldsymbol{G}_1(\boldsymbol{x}-\boldsymbol{y})]\cdot[\boldsymbol{\nu}_S(\boldsymbol{y})\times\boldsymbol{E}_1(\boldsymbol{y})]\right\}\mathrm{d}s_{\boldsymbol{y}} \\ &\quad+\int_\Gamma \left\{[\boldsymbol{G}_1(\boldsymbol{x}-\boldsymbol{y})]\cdot[\boldsymbol{\nu}_\Gamma(\boldsymbol{y})\times(\nabla_{\boldsymbol{y}}\times\boldsymbol{E}_1(\boldsymbol{y}))]\right. \\ &\qquad\left.+[\nabla_{\boldsymbol{x}}\times\boldsymbol{G}_1(\boldsymbol{x}-\boldsymbol{y})]\cdot[\boldsymbol{\nu}_\Gamma(\boldsymbol{y})\times\boldsymbol{E}_1(\boldsymbol{y})]\right\}\mathrm{d}s_{\boldsymbol{y}}, \qquad \boldsymbol{x}\in\Omega_1.\end{aligned} \tag{3.33}$$

Then, from (3.1) and (3.33), it is easy to verify that $\boldsymbol{\nu}_\Gamma(\boldsymbol{y})\times\boldsymbol{E}_1(\boldsymbol{y})\big|_\Gamma = 0$, i.e. $\boldsymbol{E}_1 = \boldsymbol{E}^s + \boldsymbol{E}^i$ satisfies the boundary condition (ii) of problem 2.1.

Taking $\boldsymbol{\nu}_S(\boldsymbol{x})\times$ of (3.1) and (3.2), respectively, using the jump conditions of the single- and double-layer potentials, we get the boundary integral equations

$$\begin{aligned}\frac{1}{2}\boldsymbol{\nu}_S(\boldsymbol{x})\times\boldsymbol{E}_1(\boldsymbol{x}) &= \boldsymbol{\nu}_S(\boldsymbol{x})\times\boldsymbol{E}^i(\boldsymbol{x}) \\ &\quad+\int_S \left\{[\boldsymbol{\nu}_S(\boldsymbol{x})\times\boldsymbol{G}_1(\boldsymbol{x}-\boldsymbol{y})]\cdot[\boldsymbol{\nu}_S(\boldsymbol{y})\times(\nabla_{\boldsymbol{y}}\times\boldsymbol{E}_1(\boldsymbol{y}))]\right. \\ &\qquad\left.+[\boldsymbol{\nu}_S(\boldsymbol{x})\times(\nabla_{\boldsymbol{x}}\times\boldsymbol{G}_1(\boldsymbol{x}-\boldsymbol{y}))]\cdot[\boldsymbol{\nu}_S(\boldsymbol{y})\times\boldsymbol{E}_1(\boldsymbol{y})]\right\}\mathrm{d}s_{\boldsymbol{y}} \\ &\quad+\int_\Gamma \left\{[\boldsymbol{\nu}_S(\boldsymbol{x})\times\boldsymbol{G}_1(\boldsymbol{x}-\boldsymbol{y})]\cdot[\boldsymbol{\nu}_\Gamma(\boldsymbol{y})\times(\nabla_{\boldsymbol{y}}\times\boldsymbol{E}_1(\boldsymbol{y}))]\right\}\mathrm{d}s_{\boldsymbol{y}}, \quad \boldsymbol{x}\in S,\end{aligned} \tag{3.34}$$

and

$$\begin{aligned}&\boldsymbol{\nu}_S(\boldsymbol{x})\times\boldsymbol{E}_2(\boldsymbol{x}) - \frac{1}{2}\boldsymbol{\nu}_S(\boldsymbol{x})\times\boldsymbol{E}_1(\boldsymbol{x}) \\ &= -\int_S \left\{[\boldsymbol{\nu}_S(\boldsymbol{x})\times\boldsymbol{G}_2(\boldsymbol{x}-\boldsymbol{y})]\cdot[\boldsymbol{\nu}_S(\boldsymbol{y})\times(\nabla_{\boldsymbol{y}}\times\boldsymbol{E}_1(\boldsymbol{y}))]\right. \\ &\qquad\left.+[\boldsymbol{\nu}_S(\boldsymbol{x})\times(\nabla_{\boldsymbol{x}}\times\boldsymbol{G}_2(\boldsymbol{x}-\boldsymbol{y}))]\cdot[\boldsymbol{\nu}_S(\boldsymbol{y})\times\boldsymbol{E}_1(\boldsymbol{y})]\right\}\mathrm{d}s_{\boldsymbol{y}}, \quad \boldsymbol{x}\in S.\end{aligned} \tag{3.35}$$

Now adding (3.34) and (3.35) gives

$$\begin{aligned}
&\boldsymbol{\nu}_S(\boldsymbol{x}) \times \boldsymbol{E}_2(\boldsymbol{x})\\
&= \boldsymbol{\nu}_S(\boldsymbol{x}) \times \boldsymbol{E}^i(\boldsymbol{x})\\
&\quad + \int_S \big\{[\boldsymbol{\nu}_S(\boldsymbol{x}) \times (\boldsymbol{G}_1(\boldsymbol{x}-\boldsymbol{y}) - \boldsymbol{G}_2(\boldsymbol{x}-\boldsymbol{y}))] \cdot [\boldsymbol{\nu}_S(\boldsymbol{y}) \times (\nabla_{\boldsymbol{y}} \times \boldsymbol{E}_1(\boldsymbol{y}))]\\
&\qquad + [\boldsymbol{\nu}_S(\boldsymbol{x}) \times (\nabla_{\boldsymbol{x}} \times \boldsymbol{G}_1(\boldsymbol{x}-\boldsymbol{y}) - \nabla_{\boldsymbol{x}} \times \boldsymbol{G}_2(\boldsymbol{x}-\boldsymbol{y}))] \cdot [\boldsymbol{\nu}_S(\boldsymbol{y}) \times \boldsymbol{E}_1(\boldsymbol{y})]\big\} \mathrm{d}s_{\boldsymbol{y}}\\
&\quad + \int_\Gamma \big\{[\boldsymbol{\nu}_S(\boldsymbol{x}) \times \boldsymbol{G}_1(\boldsymbol{x}-\boldsymbol{y})] \cdot [\boldsymbol{\nu}_\Gamma(\boldsymbol{y}) \times (\nabla_{\boldsymbol{y}} \times \boldsymbol{E}_1(\boldsymbol{y}))]\big\} \mathrm{d}s_{\boldsymbol{y}}, \quad \boldsymbol{x} \in S.
\end{aligned} \tag{3.36}$$

From (3.10) and (3.36), it is easy to verify that $\boldsymbol{\nu}_S \times \boldsymbol{E}_1|_S = \boldsymbol{\nu}_S \times \boldsymbol{E}_2|_S$.

Taking $\boldsymbol{\nu}_S(\boldsymbol{x}) \times \nabla_{\boldsymbol{x}} \times$ of (3.1) and (3.2), respectively, using the jump conditions of the single- and double-layer potentials, we get the boundary integral equations

$$\begin{aligned}
&\frac{1}{2}\boldsymbol{\nu}_S(\boldsymbol{x}) \times (\nabla_{\boldsymbol{x}} \times \boldsymbol{E}_1(\boldsymbol{x}))\\
&= \boldsymbol{\nu}_S(\boldsymbol{x}) \times (\nabla_{\boldsymbol{x}} \times \boldsymbol{E}^i(\boldsymbol{x}))\\
&\quad + \int_S \big\{[\boldsymbol{\nu}_S(\boldsymbol{x}) \times (\nabla_{\boldsymbol{x}} \times \boldsymbol{G}_1(\boldsymbol{x}-\boldsymbol{y}))] \cdot [\boldsymbol{\nu}_S(\boldsymbol{y}) \times (\nabla_{\boldsymbol{y}} \times \boldsymbol{E}_1(\boldsymbol{y}))]\\
&\qquad + [\boldsymbol{\nu}_S(\boldsymbol{x}) \times (\kappa_1^2 \boldsymbol{G}_1(\boldsymbol{x}-\boldsymbol{y}))] \cdot [\boldsymbol{\nu}_S(\boldsymbol{y}) \times \boldsymbol{E}_1(\boldsymbol{y})]\big\} \mathrm{d}s_{\boldsymbol{y}}\\
&\quad + \int_\Gamma \big\{[\boldsymbol{\nu}_S(\boldsymbol{x}) \times (\nabla_{\boldsymbol{x}} \times \boldsymbol{G}_1(\boldsymbol{x}-\boldsymbol{y}))] \cdot [\boldsymbol{\nu}_\Gamma(\boldsymbol{y}) \times (\nabla_{\boldsymbol{y}} \times \boldsymbol{E}_1(\boldsymbol{y}))]\big\} \mathrm{d}s_{\boldsymbol{y}}, \quad \boldsymbol{x} \in S,
\end{aligned} \tag{3.37}$$

and

$$\begin{aligned}
&\boldsymbol{\nu}_S(\boldsymbol{x}) \times (\nabla_{\boldsymbol{x}} \times \boldsymbol{E}_2(\boldsymbol{x})) - \frac{1}{2}\boldsymbol{\nu}_S(\boldsymbol{x}) \times (\nabla_{\boldsymbol{x}} \times \boldsymbol{E}_1(\boldsymbol{x}))\\
&= -\int_S \big\{[\boldsymbol{\nu}_S(\boldsymbol{x}) \times (\nabla_{\boldsymbol{x}} \times \boldsymbol{G}_2(\boldsymbol{x}-\boldsymbol{y}))] \cdot [\boldsymbol{\nu}_S(\boldsymbol{y}) \times (\nabla_{\boldsymbol{y}} \times \boldsymbol{E}_1(\boldsymbol{y}))]\\
&\qquad + [\boldsymbol{\nu}_S(\boldsymbol{x}) \times (\kappa_2^2 \boldsymbol{G}_2(\boldsymbol{x}-\boldsymbol{y}))] \cdot [\boldsymbol{\nu}_S(\boldsymbol{y}) \times \boldsymbol{E}_1(\boldsymbol{y})]\big\} \mathrm{d}s_{\boldsymbol{y}}, \quad \boldsymbol{x} \in S.
\end{aligned} \tag{3.38}$$

Now adding (3.37) and (3.38) gives

$$\begin{aligned}
&\boldsymbol{\nu}_S(\boldsymbol{x}) \times (\nabla_{\boldsymbol{x}} \times \boldsymbol{E}_2(\boldsymbol{x}))\\
&= \boldsymbol{\nu}_S(\boldsymbol{x}) \times (\nabla_{\boldsymbol{x}} \times \boldsymbol{E}^i(\boldsymbol{x}))\\
&\quad + \int_S \big\{[\boldsymbol{\nu}_S(\boldsymbol{x}) \times (\nabla_{\boldsymbol{x}} \times \boldsymbol{G}_1(\boldsymbol{x}-\boldsymbol{y}) - \nabla_{\boldsymbol{x}} \times \boldsymbol{G}_2(\boldsymbol{x}-\boldsymbol{y}))] \cdot [\boldsymbol{\nu}_S(\boldsymbol{y}) \times (\nabla_{\boldsymbol{y}} \times \boldsymbol{E}_1(\boldsymbol{y}))]\\
&\qquad + [\boldsymbol{\nu}_S(\boldsymbol{x}) \times (\kappa_1^2 \boldsymbol{G}_1(\boldsymbol{x}-\boldsymbol{y}) - \kappa_2^2 \boldsymbol{G}_2(\boldsymbol{x}-\boldsymbol{y}))] \cdot [\boldsymbol{\nu}_S(\boldsymbol{y}) \times \boldsymbol{E}_1(\boldsymbol{y})]\big\} \mathrm{d}s_{\boldsymbol{y}}\\
&\quad + \int_\Gamma \big\{[\boldsymbol{\nu}_S(\boldsymbol{x}) \times (\nabla_{\boldsymbol{x}} \times \boldsymbol{G}_1(\boldsymbol{x}-\boldsymbol{y}))] \cdot [\boldsymbol{\nu}_\Gamma(\boldsymbol{y}) \times (\nabla_{\boldsymbol{y}} \times \boldsymbol{E}_1(\boldsymbol{y}))]\big\} \mathrm{d}s_{\boldsymbol{y}}, \quad \boldsymbol{x} \in S.
\end{aligned} \tag{3.39}$$

From (3.11) and (3.39), it is easy to verify that $\boldsymbol{\nu}_S \times (\nabla_{\boldsymbol{x}} \times \boldsymbol{E}_1)|_S = \boldsymbol{\nu}_S \times (\nabla_{\boldsymbol{x}} \times \boldsymbol{E}_2)|_S$. □

To prove the uniqueness, it suffices to show that $\boldsymbol{E}_1 = \boldsymbol{E}^s$ and $\boldsymbol{E}_2$ vanish identically in $\overline{\Omega}_1$ and $\overline{\Omega}_2$ if $\boldsymbol{E}^i = 0$. For the sake of brevity for the proof, we consider the homogeneous Maxwell's equations

$$\nabla \times (\nabla \times \boldsymbol{E}_j) - \kappa_j^2 \boldsymbol{E}_j = 0 \quad \text{in } \Omega_j, \tag{3.40}$$

along with the boundary condition

$$\boldsymbol{\nu}_\Gamma \times \boldsymbol{E}_1 = 0 \quad \text{on } \Gamma, \tag{3.41}$$

and the continuity conditions

$$\boldsymbol{\nu}_S \times \boldsymbol{E}_1 = \boldsymbol{\nu}_S \times \boldsymbol{E}_2, \quad \boldsymbol{\nu}_S \times (\nabla \times \boldsymbol{E}_1) = \boldsymbol{\nu}_S \times (\nabla \times \boldsymbol{E}_2) \quad \text{on } S, \tag{3.42}$$

and the radiation conditions

$$\begin{aligned} &\lim_{r\to+\infty} \int_{\partial B_r^+} |\boldsymbol{E}_1|^2 \mathrm{d}s = \lim_{r\to+\infty} \int_{\partial B_r^+} |\nabla \times \boldsymbol{E}_1|^2 \mathrm{d}s \\ &= \lim_{r\to+\infty} \int_{\partial B_r^-} |\boldsymbol{E}_2|^2 \mathrm{d}s = \lim_{r\to+\infty} \int_{\partial B_r^-} |\nabla \times \boldsymbol{E}_2|^2 \mathrm{d}s = 0. \end{aligned} \tag{3.43}$$

Theorem 3.4. *Let $(\boldsymbol{E}_1, \boldsymbol{E}_2)$ be the solutions of the problem (3.40)–(3.43). Then $(\boldsymbol{E}_1, \boldsymbol{E}_2)$ vanish identically.*

Proof. Denote $\Omega_r = B_r \cap \Omega_1$ with boundary $\partial\Omega_r = \partial B_r^+ \cup \Gamma \cup S_r$, where $\partial B_r^+ = \partial B_r \cap \Omega_1$ and $S_r = S \cap B_r$. Taking the dot product of the Maxwell equation $\nabla \times (\nabla \times \boldsymbol{E}_1(\boldsymbol{x})) - \kappa_1^2 \boldsymbol{E}_1(\boldsymbol{x}) = 0$ with the complex conjugate of $\boldsymbol{E}_1$, integrating over Ω_r, and using the integration by parts, we get

$$\begin{aligned} &\int_\Omega |\nabla \times \boldsymbol{E}_1|^2 \mathrm{d}\boldsymbol{x} - \kappa_1^2 \int_\Omega |\boldsymbol{E}_1|^2 \mathrm{d}\boldsymbol{x} \\ &= \int_{\partial\Omega_r} \boldsymbol{\nu} \cdot [\overline{\boldsymbol{E}}_1 \times (\nabla \times \boldsymbol{E}_1)] \mathrm{d}s_{\boldsymbol{x}} = - \int_{\partial\Omega_r} \overline{\boldsymbol{E}}_1 \cdot [\boldsymbol{\nu} \times (\nabla \times \boldsymbol{E}_1)] \mathrm{d}s_{\boldsymbol{x}} \\ &= - \int_{\partial\Omega_r} (\boldsymbol{\nu} \times (\nabla \times \boldsymbol{E}_1)) \cdot ((\boldsymbol{\nu} \times \overline{\boldsymbol{E}}_1) \times \boldsymbol{\nu}) \mathrm{d}s_{\boldsymbol{x}} \\ &= - \left(\int_{\partial B_r^+} + \int_\Gamma + \int_{S_r} \right) (\boldsymbol{\nu} \times (\nabla \times \boldsymbol{E}_1)) \cdot ((\boldsymbol{\nu} \times \overline{\boldsymbol{E}}_1) \times \boldsymbol{\nu}) \mathrm{d}s_{\boldsymbol{x}}, \end{aligned} \tag{3.44}$$

where $\boldsymbol{\nu} = \boldsymbol{\nu}(\boldsymbol{x})$ stands for the unit normal vector at $\boldsymbol{x} \in \partial\Omega_r$ pointing out of Ω_r. Letting $r \to +\infty$, we have from (3.41), (3.43) and (3.44) that

$$\int_\Omega |\nabla \times \boldsymbol{E}_1|^2 \mathrm{d}\boldsymbol{x} - \kappa_1^2 \int_\Omega |\boldsymbol{E}_1|^2 \mathrm{d}\boldsymbol{x} = \int_S (\boldsymbol{\nu}_S \times (\nabla \times \boldsymbol{E}_1)) \cdot ((\boldsymbol{\nu}_S \times \overline{\boldsymbol{E}}_1) \times \boldsymbol{\nu}_S) \mathrm{d}s_{\boldsymbol{x}}, \tag{3.45}$$

where $\boldsymbol{\nu}_S = \boldsymbol{\nu}_S(\boldsymbol{x})$ denotes the unit normal vector at $\boldsymbol{x} \in S$ pointing from region Ω_2 to region Ω_1. By taking the imaginary part of (3.45) that

$$-\Im \int_S (\boldsymbol{\nu}_S \times (\nabla \times \boldsymbol{E}_1)) \cdot ((\boldsymbol{\nu}_S \times \overline{\boldsymbol{E}}_1) \times \boldsymbol{\nu}_S) \mathrm{d}s_{\boldsymbol{x}} = \Im(\kappa_1^2) \int_{\Omega_1} |\boldsymbol{E}_1|^2 \mathrm{d}\boldsymbol{x} \geqslant 0. \tag{3.46}$$

Similarly, we may show that

$$\Im \int_S (\boldsymbol{\nu}_S \times (\nabla \times \boldsymbol{E}_2)) \cdot ((\boldsymbol{\nu}_S \times \overline{\boldsymbol{E}}_2) \times \boldsymbol{\nu}_S) \mathrm{d}s_{\boldsymbol{x}} = \Im(\kappa_2^2) \int_{\Omega_2} |\boldsymbol{E}_2|^2 \mathrm{d}\boldsymbol{x} \geqslant 0. \tag{3.47}$$

Noting the continuity conditions (3.42) and $\Im(\kappa_j^2) > 0$, we obtain from (3.46) and (3.47) that

$$\int_{\Omega_1} |\boldsymbol{E}_1|^2 \mathrm{d}\boldsymbol{x} = \int_{\Omega_2} |\boldsymbol{E}_2|^2 \mathrm{d}\boldsymbol{x} = 0,$$

which implies that $\boldsymbol{E}_1 = 0$ in Ω_1 and $\boldsymbol{E}_2 = 0$ in Ω_2. □

4. Uniqueness of the inverse problem

This section addresses the uniqueness of the inverse surface scattering problem. For the given incident field, we show that the obstacle and the unbounded rough surface can be uniquely determined by the tangential trace of the electric field $\boldsymbol{\nu}_{\Gamma_j} \times \boldsymbol{E}_j|_{\Gamma_j}$, $j = 1, 2$.

Let $\widetilde{S} \in C^2$ be an infinite rough surface which divides $\mathbb{R}^3$ into the upper half space $\widetilde{\Omega}_1^+$ and the lower half space $\widetilde{\Omega}_2$. Let $\widetilde{D} \subset\subset \widetilde{\Omega}_1^+$ be a bounded domain with the boundary $\widetilde{\Gamma} \in C^2$. Define $\widetilde{\Omega}_1 = \widetilde{\Omega}_1^+ \setminus \overline{\widetilde{D}}$. Let $(\widetilde{\boldsymbol{E}}_1, \widetilde{\boldsymbol{E}}_2)$ be the unique solutions of problem 2.1 with the surfaces (D, S) replaced by $(\widetilde{D}, \widetilde{S})$ but for the same incident field $\boldsymbol{E}^i$ satisfying (2.15). Recall that the point dipole source is assumed to be located at $\boldsymbol{x}_s \in R_1^+$.

Lemma 4.1. *Let $\Omega \subset \mathbb{R}^3$ be a Lipschitz domain. Consider the boundary value problem*

$$\begin{cases} \nabla \times (\nabla \times \boldsymbol{E}(\boldsymbol{x})) - \kappa_1^2 \boldsymbol{E}(\boldsymbol{x}) = 0, \quad \nabla \times (\nabla \times \breve{\boldsymbol{E}}(\boldsymbol{x})) - \kappa_2^2 \breve{\boldsymbol{E}}(\boldsymbol{x}) = 0 & \text{in } \Omega, \\ \boldsymbol{\nu} \times \boldsymbol{E} = \boldsymbol{\nu} \times \breve{\boldsymbol{E}}, \quad \boldsymbol{\nu} \times (\nabla \times \boldsymbol{E}) = \boldsymbol{\nu} \times (\nabla \times \breve{\boldsymbol{E}}) & \text{on } \partial\Omega, \end{cases} \tag{4.1}$$

where $\boldsymbol{\nu}$ denotes the unit normal vector on the boundary $\partial\Omega$ directed into the exterior of Ω. Then $\boldsymbol{E} = \breve{\boldsymbol{E}} = 0$ in Ω.

Proof. Consider an extension $\breve{\boldsymbol{E}}^e$ of $\breve{\boldsymbol{E}}$ to the exterior domain $\Omega^e = \mathbb{R}^3 \setminus \overline{\Omega}$, where $\breve{\boldsymbol{E}}^e$ satisfies

$$\begin{cases} \nabla \times (\nabla \times \breve{\boldsymbol{E}}^e(\boldsymbol{x})) - \kappa_2^2 \breve{\boldsymbol{E}}^e(\boldsymbol{x}) = 0 & \text{in } \Omega^e, \\ \boldsymbol{\nu} \times \breve{\boldsymbol{E}}^e = \boldsymbol{\nu} \times \breve{\boldsymbol{E}}, \quad \boldsymbol{\nu} \times (\nabla \times \breve{\boldsymbol{E}}^e) = \boldsymbol{\nu} \times (\nabla \times \breve{\boldsymbol{E}}) & \text{on } \partial\Omega, \end{cases}$$

and the radiation conditions

$$\lim_{r\to\infty} \int_{\partial B_r} |\breve{\boldsymbol{E}}^e|^2 \mathrm{d}s = \lim_{r\to\infty} \int_{\partial B_r} |\nabla \times \breve{\boldsymbol{E}}^e|^2 \mathrm{d}s = 0.$$

Taking the dot product of the Maxwell equation $\nabla \times (\nabla \times \boldsymbol{E}(\boldsymbol{x})) - \kappa_1^2 \boldsymbol{E}(\boldsymbol{x}) = 0$ with the complex conjugate of $\boldsymbol{E}$, integrating over Ω, and using the integration by parts, we get

$$\int_\Omega |\nabla \times \boldsymbol{E}|^2 \mathrm{d}\boldsymbol{x} - \kappa_1^2 \int_\Omega |\boldsymbol{E}|^2 \mathrm{d}\boldsymbol{x} = -\int_{\partial\Omega} (\boldsymbol{\nu} \times (\nabla \times \boldsymbol{E})) \cdot ((\boldsymbol{\nu} \times \overline{\boldsymbol{E}}) \times \boldsymbol{\nu}) \mathrm{d}s_{\boldsymbol{x}}. \tag{4.2}$$

On the other hand, taking the dot product of the equation $\nabla \times (\nabla \times \breve{\boldsymbol{E}}^e(\boldsymbol{x})) - \kappa_2^2 \breve{\boldsymbol{E}}^e(\boldsymbol{x}) = 0$ with the complex conjugate of $\breve{\boldsymbol{E}}^e$, integrating over Ω^e, using the integration by parts and the radiation condition, we have

$$\int_{\Omega^e} |\nabla \times \breve{\boldsymbol{E}}^e|^2 \mathrm{d}\boldsymbol{x} - \kappa_2^2 \int_{\Omega^e} |\breve{\boldsymbol{E}}^e|^2 \mathrm{d}\boldsymbol{x} = \int_{\partial\Omega} (\boldsymbol{\nu} \times (\nabla \times \breve{\boldsymbol{E}}^e)) \cdot ((\boldsymbol{\nu} \times \overline{\breve{\boldsymbol{E}}^e}) \times \boldsymbol{\nu}) \mathrm{d}s_{\boldsymbol{x}}. \tag{4.3}$$

Since

$$\boldsymbol{\nu} \times \boldsymbol{E} = \boldsymbol{\nu} \times \breve{\boldsymbol{E}} = \boldsymbol{\nu} \times \breve{\boldsymbol{E}}^e, \quad \boldsymbol{\nu} \times (\nabla \times \boldsymbol{E}) = \boldsymbol{\nu} \times (\nabla \times \breve{\boldsymbol{E}}) = \boldsymbol{\nu} \times (\nabla \times \breve{\boldsymbol{E}}^e),$$

we add (4.2) and (4.3) and get

$$\int_{\Omega} |\nabla \times \boldsymbol{E}|^2 \mathrm{d}\boldsymbol{x} + \int_{\Omega^e} |\nabla \times \breve{\boldsymbol{E}}^e|^2 \mathrm{d}\boldsymbol{x} - \kappa_1^2 \int_{\Omega} |\boldsymbol{E}|^2 \mathrm{d}\boldsymbol{x} - \kappa_2^2 \int_{\Omega^e} |\breve{\boldsymbol{E}}^e|^2 \mathrm{d}\boldsymbol{x} = 0.$$

Noting $\Im(\kappa_j^2) > 0$ and taking the imaging part of the above equation yields that $\boldsymbol{E} = 0$ in Ω and $\breve{\boldsymbol{E}}^e = 0$ in Ω^e, which implies immediately that $\boldsymbol{E} = \breve{\boldsymbol{E}} = 0$ in Ω. □

Remark 4.2. In lemma 4.1, the domain Ω does not have to be a bounded and the integration by parts still holds in the proof of lemma 4.1 due to $\Im(\kappa_j^2) > 0$ and the radiation conditions. We refer to [20] for a closely related problem on the electromagnetic scattering by unbounded rough surfaces, where the variational problem is discussed in an unbounded domain.

Remark 4.3. The result still holds for $\kappa_1 = \kappa_2$ in lemma 4.1. In this case, the problem (4.1) is equivalent to the following scattering problem: to find $\boldsymbol{E}$ such that it satisfies the Maxwell equation

$$\nabla \times (\nabla \times \boldsymbol{E}(\boldsymbol{x})) - \kappa_1^2 \boldsymbol{E}(\boldsymbol{x}) = 0 \quad \text{in } \mathbb{R}^3$$

and the radiation condition

$$\lim_{r\to\infty} \int_{\partial B_r} |\boldsymbol{E}|^2 \mathrm{d}s = \lim_{r\to\infty} \int_{\partial B_r} |\nabla \times \boldsymbol{E}|^2 \mathrm{d}s = 0.$$

It is clear to note that the above scattering problem has a unique solution $\boldsymbol{E} = 0$ in $\mathbb{R}^3$ due to $\Im(\kappa_1^2) > 0$.

Lemma 4.4. *Let $\boldsymbol{E}_1$ be the solution of problem 2.1. Then $\boldsymbol{\nu}_S \times \boldsymbol{E}_1 \neq 0$ on S.*

Proof. We prove it by contradiction. First we assume that $\boldsymbol{\nu}_S \times \boldsymbol{E}_1 = 0$ on S. Since $\boldsymbol{\nu}_S \times \boldsymbol{E}_1 = \boldsymbol{\nu}_S \times \boldsymbol{E}_2$ on S, we may consider the following problem

$$\begin{cases} \nabla \times (\nabla \times \boldsymbol{E}_2(\boldsymbol{x})) - \kappa_2^2 \boldsymbol{E}_2(\boldsymbol{x}) = 0 & \text{in } \Omega_2, \\ \boldsymbol{\nu}_S \times \boldsymbol{E}_2 = 0 & \text{on } S. \end{cases} \tag{4.4}$$

In addition, $\boldsymbol{E}_2$ is required to satisfy the radiation condition (2.10). Taking the dot product of the Maxwell equation in (4.4) with the complex conjugate of $\boldsymbol{E}_2$, integrating over Ω_2, and using the boundary condition and the radiation condition, we obtain

$$\int_{\Omega_2} |\nabla \times \boldsymbol{E}_2|^2 \mathrm{d}\boldsymbol{x} - \kappa_2^2 \int_{\Omega_2} |\boldsymbol{E}_2|^2 \mathrm{d}\boldsymbol{x} = 0,$$

which implies $\boldsymbol{E}_2 = 0$ in Ω_2 due to $\Im(\kappa_2^2) > 0$. Hence $\boldsymbol{\nu}_S \times (\nabla \times \boldsymbol{E}_2) = 0$ on S. Since $\boldsymbol{\nu}_S \times \boldsymbol{E}_1 = \boldsymbol{\nu}_S \times \boldsymbol{E}_2 = 0$ and $\boldsymbol{\nu}_S \times (\nabla \times \boldsymbol{E}_1) = \boldsymbol{\nu}_S \times (\nabla \times \boldsymbol{E}_2) = 0$ on S, it follows from the Holmgren uniqueness theorem for the Maxwell system in [5, theorem 6.5] that we obtain $\boldsymbol{E}_1 = 0$ in R_1. In fact, $\boldsymbol{E}_1$ can be extended to $\Omega_1 \cup \Omega_2$ as follows

$$\breve{\boldsymbol{E}}_1 := \begin{cases} \boldsymbol{E}_1 & \text{in } \Omega_1, \\ \boldsymbol{E}_1^e & \text{in } \Omega_2, \end{cases}$$

where $\boldsymbol{E}_1^e$ satisfies

$$\begin{cases} \nabla \times (\nabla \times \boldsymbol{E}_1^e(\boldsymbol{x})) - \kappa_1^2 \boldsymbol{E}_1^e(\boldsymbol{x}) = 0 & \text{in } \Omega_2, \\ \boldsymbol{\nu}_S \times \boldsymbol{E}_1^e = \boldsymbol{\nu}_S \times \boldsymbol{E}_1, \quad \boldsymbol{\nu}_S \times (\nabla \times \boldsymbol{E}_1^e) = \boldsymbol{\nu}_S \times (\nabla \times \boldsymbol{E}_1) & \text{on } S, \end{cases} \tag{4.5}$$

and the radiation condition (2.10). Clearly the problem (4.5) has a unique solution $\boldsymbol{E}_1^e = 0$ in Ω_2. By the unique continuation, we have $\boldsymbol{E}_1 = \breve{\boldsymbol{E}}_1 = 0$ in R_1, which contradicts the transparent boundary condition (2.11). □

Theorem 4.5. *Assume that $\boldsymbol{\nu}_{\Gamma_j} \times \boldsymbol{E}_j|_{\Gamma_j} = \boldsymbol{\nu}_{\Gamma_j} \times \widetilde{\boldsymbol{E}}_j|_{\Gamma_j}$ for the given the incident field $\boldsymbol{E}^i$. Then $D = \widetilde{D}, S = \widetilde{S}$.*

Proof. We prove it by contradiction and assume that $D \neq \widetilde{D}$, $S \neq \widetilde{S}$. The schematic of the domains (D, S) and $(\widetilde{D}, \widetilde{S})$ is shown in figure 2. Let $\boldsymbol{F}_j = \boldsymbol{E}_j - \widetilde{\boldsymbol{E}}_j$, then $\boldsymbol{F}_j$ satisfies Maxwell's equation

$$\nabla \times (\nabla \times \boldsymbol{F}_j) - \kappa_j^2 \boldsymbol{F}_j = 0 \quad \text{in } \Omega_j \cap \widetilde{\Omega}_j$$

and the radiation condition. By the assumption $\boldsymbol{\nu}_{\Gamma_j} \times \boldsymbol{F}_j|_{\Gamma_j} = \boldsymbol{\nu}_{\Gamma_j} \times \boldsymbol{E}_j|_{\Gamma_j} - \boldsymbol{\nu}_{\Gamma_j} \times \widetilde{\boldsymbol{E}}_j|_{\Gamma_j} = 0$ and the uniqueness result for the direct scattering problem, it follows that $\boldsymbol{E}_j(\boldsymbol{x}) = \widetilde{\boldsymbol{E}}_j(\boldsymbol{x}), j = 1, 2$ in $R_1^+ = \{\boldsymbol{x} \in \mathbb{R}^3 : x_3 > h_1\}$ and $R_2^+ = \{\boldsymbol{x} \in \mathbb{R}^3 : x_3 < h_2\}$, respectively. Since $\boldsymbol{F}_j \in C^2(\Omega_j \cap \widetilde{\Omega}_j) \cap C^{1,\alpha}(\overline{\Omega_j \cap \widetilde{\Omega}_j})$, by the unique continuation, we get that

$$\boldsymbol{F}_j(\boldsymbol{x}) = \boldsymbol{E}_j(\boldsymbol{x}) - \widetilde{\boldsymbol{E}}_j(\boldsymbol{x}) = 0, \quad \boldsymbol{x} \in \overline{\Omega_j \cap \widetilde{\Omega}_j}, \tag{4.6}$$

and

$$\nabla \times \boldsymbol{F}_j(\boldsymbol{x}) = \nabla \times \boldsymbol{E}_j(\boldsymbol{x}) - \nabla \times \widetilde{\boldsymbol{E}}_j(\boldsymbol{x}) = 0, \quad \boldsymbol{x} \in \overline{\Omega_j \cap \widetilde{\Omega}_j}. \tag{4.7}$$

First, we prove that the obstacle can be uniquely determined. In the case when $D \neq \widetilde{D}$ which include $D \cap \widetilde{D} \neq \emptyset$ and $D \cap \widetilde{D} = \emptyset$, without loss of generality, we assume that $Q = D \setminus \overline{(D \cap \widetilde{D})} \neq \emptyset$ with the boundary $\partial Q = \Gamma_p \cup \widetilde{\Gamma}_p$, where $\Gamma_p \subset \Gamma$ and $\widetilde{\Gamma}_p \subset \widetilde{\Gamma}$ denotes the part of the boundary on Γ and $\widetilde{\Gamma}$, respectively. Thus, from (4.6) and (2.5), we obtain

$$\boldsymbol{\nu}_{\Gamma_p} \times \widetilde{\boldsymbol{E}}_1\big|_{\Gamma_p} = \boldsymbol{\nu}_{\widetilde{\Gamma}_p} \times \widetilde{\boldsymbol{E}}_1\big|_{\widetilde{\Gamma}_p} = 0. \tag{4.8}$$

Consider the boundary value problem

$$\begin{cases} \nabla \times (\nabla \times \widetilde{\boldsymbol{E}}_1) - \kappa_1^2 \widetilde{\boldsymbol{E}}_1 = 0 & \text{in } Q, \\ \boldsymbol{\nu} \times \widetilde{\boldsymbol{E}}_1 = 0 & \text{on } \partial Q, \end{cases}$$

where $\boldsymbol{\nu}$ is the unit normal on ∂Q. Multiplying the Maxwell equation $\nabla\times(\nabla\times\widetilde{\boldsymbol{E}}_1)-\kappa_1^2\widetilde{\boldsymbol{E}}_1=0$ by the complex conjugate of $\widetilde{\boldsymbol{E}}_1$, integrating over Q, using the integration by parts, we obtain

$$\int_Q|\nabla\times\widetilde{\boldsymbol{E}}_1|^2\mathrm{d}\boldsymbol{x}-\kappa_1^2\int_Q|\widetilde{\boldsymbol{E}}_1|^2\mathrm{d}\boldsymbol{x}=0,$$

which implies that $\widetilde{\boldsymbol{E}}_1=0$ in Q since $\Im(\kappa_1^2)>0$. An application of the unique continuation gives $\widetilde{\boldsymbol{E}}_1=0$ in R_1. But this contradicts the transparent boundary condition (2.11) on Γ_1 Hence, $D=\widetilde{D}$.

Next is show that the infinite rough surface S can be uniquely determined by the wave fields $\boldsymbol{\nu}_{\Gamma_1}\times\boldsymbol{E}_1$ and $\boldsymbol{\nu}_{\Gamma_2}\times\boldsymbol{E}_2$ measured on Γ_1 and Γ_2, respectively. Assume that $S\neq\widetilde{S}$, where $\widetilde{S}=\{\boldsymbol{x}\in\mathbb{R}^3: x_3=\tilde{f}(x_1,x_2)\}$ with $\tilde{f}\neq f$. Let $S_p\subseteq S$ and $\widetilde{S}_p\subseteq\widetilde{S}$. Noting remark 4.2, we may assume without loss of generality that $S_p\subset S$ is located above $\widetilde{S}_p\subset\widetilde{S}_p$. Thus, we have a domain Ω which is bounded by $\partial\Omega=S_p\cup\widetilde{S}_p$. The schematic of the domain Ω is also shown in figure 2. By (2.6), (4.6) and (4.7), we have

$$\boldsymbol{\nu}_{S_p}\times\boldsymbol{E}_1=\boldsymbol{\nu}_{S_p}\times\widetilde{\boldsymbol{E}}_1,\quad \boldsymbol{\nu}_{S_p}\times(\nabla\times\boldsymbol{E}_1)=\boldsymbol{\nu}_{S_p}\times(\nabla\times\widetilde{\boldsymbol{E}}_1)\quad\text{on } S_p,$$

and

$$\boldsymbol{\nu}_{\widetilde{S}_p}\times\boldsymbol{E}_2=\boldsymbol{\nu}_{\widetilde{S}_p}\times\widetilde{\boldsymbol{E}}_2,\quad \boldsymbol{\nu}_{\widetilde{S}_p}\times(\nabla\times\boldsymbol{E}_2)=\boldsymbol{\nu}_{\widetilde{S}_p}\times(\nabla\times\widetilde{\boldsymbol{E}}_2)\quad\text{on } \widetilde{S}_p.$$

It follows from the continuity conditions

$$\boldsymbol{\nu}_{S_p}\times\boldsymbol{E}_1=\boldsymbol{\nu}_{S_p}\times\boldsymbol{E}_2,\quad \boldsymbol{\nu}_{S_p}\times(\nabla\times\boldsymbol{E}_1)=\boldsymbol{\nu}_{S_p}\times(\nabla\times\boldsymbol{E}_2)\quad\text{on } S_p,$$

and

$$\boldsymbol{\nu}_{\widetilde{S}_p}\times\widetilde{\boldsymbol{E}}_1=\boldsymbol{\nu}_{\widetilde{S}_p}\times\widetilde{\boldsymbol{E}}_2,\quad \boldsymbol{\nu}_{\widetilde{S}_p}\times(\nabla\times\widetilde{\boldsymbol{E}}_1)=\boldsymbol{\nu}_{\widetilde{S}_p}\times(\nabla\times\widetilde{\boldsymbol{E}}_2)\quad\text{on } \widetilde{S}_p.$$

Combining the above equations yields that

$$\boldsymbol{\nu}_{\partial\Omega}\times\widetilde{\boldsymbol{E}}_1=\boldsymbol{\nu}_{\partial\Omega}\times\boldsymbol{E}_2,\quad \boldsymbol{\nu}_{\partial\Omega}\times(\nabla\times\widetilde{\boldsymbol{E}}_1)=\boldsymbol{\nu}_{\partial\Omega}\times(\nabla\times\boldsymbol{E}_2)\quad\text{on } \partial\Omega=S_p\cup\widetilde{S}_p.$$

We consider the following boundary value problem

$$\begin{cases}\nabla\times(\nabla\times\widetilde{\boldsymbol{E}}_1)-\kappa_1^2\widetilde{\boldsymbol{E}}_1=0,\quad \nabla\times(\nabla\times\boldsymbol{E}_2)-\kappa_2^2\boldsymbol{E}_2=0 & \text{in } \Omega,\\ \boldsymbol{\nu}_{\partial\Omega}\times\widetilde{\boldsymbol{E}}_1=\boldsymbol{\nu}_{\partial\Omega}\times\boldsymbol{E}_2,\quad \boldsymbol{\nu}_{\partial\Omega}\times(\nabla\times\widetilde{\boldsymbol{E}}_1)=\boldsymbol{\nu}_{\partial\Omega}\times(\nabla\times\boldsymbol{E}_2) & \text{on } \partial\Omega.\end{cases}\tag{4.9}$$

It follows from lemma 4.1 that $\widetilde{\boldsymbol{E}}_1=\boldsymbol{E}_2=0$ in Ω. An application of the unique continuation gives $\widetilde{\boldsymbol{E}}_1=0$ in R_1, which contradicts the transparent boundary condition (2.11). Hence, $S=\widetilde{S}$. □

5. Local stability

In this section, we present a local stability result. Let us begin with the calculation of domain derivative which plays an important role in the stability analysis.

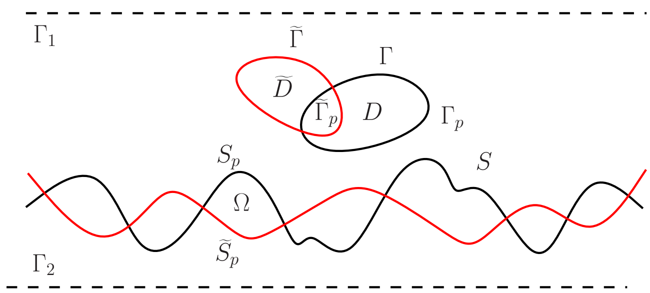

Figure 2. Schematic of domains for the proof of uniqueness.

Let $\mathcal{I}:\mathbb{R}^3\to\mathbb{R}^3$ be the identity mapping and let $\theta:\Gamma\cup S\to\mathbb{R}^3$ be an admissible perturbation, where θ is assumed to be an admissible perturbation in $C^2(\Gamma\cup S,\mathbb{R}^3)$ and has a compact support. For $\theta\in C^2(\Gamma\cup S,\mathbb{R}^3)$, we can extend the definition of function $\theta(\boldsymbol{x})$ to $\overline{\Omega}_j$ by satisfying: $\theta(\boldsymbol{x})\in C^2(\Omega_j,\mathbb{R}^3)\cap C(\overline{\Omega}_j)$; $\mathcal{I}+\theta:\Omega_j\to\Omega_{j,\theta}, j=1,2$. Here the region $\Omega_{j,\theta}$ bounded by Γ_θ and S_θ, where

$$\Gamma_\theta=\{\boldsymbol{x}+\theta(\boldsymbol{x}):\boldsymbol{x}\in\Gamma\},\quad S_\theta=\{\boldsymbol{x}+\theta(\boldsymbol{x}):\boldsymbol{x}\in S\}.$$

Let $\theta(\boldsymbol{x})=(\theta_1(\boldsymbol{x}),\theta_2(\boldsymbol{x}),\theta_3(\boldsymbol{x}))^\top$. Clearly, $\Omega_{j,\theta}$ is an admissible perturbed configuration of the reference region Ω_j. Note that $\Omega_{j,\boldsymbol{0}}=\Omega_j$, $\Gamma_{\boldsymbol{0}}=\Gamma$, and $S_{\boldsymbol{0}}=S$. According to theorem 3.4, there exist the unique solutions $(\boldsymbol{E}_{1,\theta},\boldsymbol{E}_{2,\theta})$ to problem 2.1 corresponding to the region $\Omega_{j,\theta}$ for any small enough θ. Note that this function $\boldsymbol{E}_{j,\theta}=\boldsymbol{E}_j(\theta,\boldsymbol{x})$ cannot be differentiated with respect to θ in the classical sense. For this reason, we adopt the following concept of a domain derivative.

Denote by

$$\boldsymbol{E}_j'=\frac{\partial \boldsymbol{E}_{j,\theta}}{\partial\theta}(0)\boldsymbol{p}$$

the domain derivative of $\boldsymbol{E}_{j,\theta}$ at $\theta=0$ in the direction $\boldsymbol{p}(\boldsymbol{x})=(p_1(\boldsymbol{x}),p_2(\boldsymbol{x}),p_3(\boldsymbol{x}))^\top\in C^2(\Gamma\cup S,\mathbb{R}^3)$. Define a nonlinear map

$$Y:\Gamma_\theta\cup S_\theta\to\boldsymbol{\nu}_{\Gamma_H}\times\boldsymbol{E}_{1,\theta}|_{\Gamma_1}.$$

The domain derivative of the operator Y on the boundary $\Gamma\cup S$ along the direction $\boldsymbol{p}$ is defined by

$$Y'(\Gamma\cup S,\boldsymbol{p}):=\boldsymbol{\nu}_{\Gamma_1}\times\boldsymbol{E}_1'|_{\Gamma_H}.$$

We introduce the notations

$$\boldsymbol{V}_{\Gamma_\tau}=\boldsymbol{\nu}_\Gamma\times(\boldsymbol{V}\times\boldsymbol{\nu}_\Gamma),\quad \boldsymbol{V}_{\Gamma_\nu}=\boldsymbol{\nu}_\Gamma\cdot\boldsymbol{V},\quad \boldsymbol{V}_{S_\tau}=\boldsymbol{\nu}_S\times(\boldsymbol{V}\times\boldsymbol{\nu}_S),\quad \boldsymbol{V}_{S_\nu}=\boldsymbol{\nu}_S\cdot\boldsymbol{V},$$

which are the tangential and the normal components of a vector $\boldsymbol{V}$ on the boundary Γ and S, respectively. It is clear to note that $\boldsymbol{V}=\boldsymbol{V}_{\Gamma_\tau}+\boldsymbol{V}_{\Gamma_\nu}\boldsymbol{\nu}_\Gamma$ on Γ and $\boldsymbol{V}=\boldsymbol{V}_{S_\tau}+\boldsymbol{V}_{S_\nu}\boldsymbol{\nu}_S$ on S. Denote by ∇_{Γ_τ} and ∇_{S_τ} the surface gradient on Γ and S, and denote by $\partial_{\boldsymbol{\nu}_\Gamma}$ and $\partial_{\boldsymbol{\nu}_S}$ the normal derivative on Γ and S, respectively.

Define the jump

$$[\boldsymbol{E}]=\lim_{\substack{\boldsymbol{a}_1\to 0\\ \boldsymbol{x}+\boldsymbol{a}_1\in\Omega_1}}\boldsymbol{E}_1(\boldsymbol{x}+\boldsymbol{a}_1)-\lim_{\substack{\boldsymbol{a}_2\to 0\\ \boldsymbol{x}+\boldsymbol{a}_2\in\Omega_2}}\boldsymbol{E}_2(\boldsymbol{x}+\boldsymbol{a}_2),\quad \boldsymbol{x}\in S,\tag{5.1}$$

of the continuous extension of a function $\boldsymbol{E}$ to the boundary from Ω_1 and Ω_2, respectively.

Theorem 5.1. *Let $(\boldsymbol{E}_1, \boldsymbol{E}_2)$ be the solutions of problem 2.1. Given $\boldsymbol{p} \in C^2(\Gamma \cup S, \mathbb{R}^3)$, the domain derivatives $(\boldsymbol{E}_1', \boldsymbol{E}_2')$ of $(\boldsymbol{E}_1, \boldsymbol{E}_2)$ are the radiation solutions of the following problem:*

$$\begin{cases} \nabla \times \nabla \times \boldsymbol{E}_1' - \kappa_1^2 \boldsymbol{E}_1' = 0 & \text{in } \Omega_1, \\ \nabla \times \nabla \times \boldsymbol{E}_2' - \kappa_2^2 \boldsymbol{E}_2' = 0 & \text{in } \Omega_2, \\ \boldsymbol{\nu}_\Gamma \times \boldsymbol{E}_1' = \left[\boldsymbol{p}_{\Gamma_\nu} (\partial_{\boldsymbol{\nu}_\Gamma} \boldsymbol{E}_{1,\Gamma_\tau}) + \boldsymbol{E}_{1,\Gamma_\nu} (\nabla_{\Gamma_\tau} \boldsymbol{p}_{\Gamma_\nu}) \right] \times \boldsymbol{\nu}_\Gamma & \text{on } \Gamma, \\ [\boldsymbol{\nu}_S \times \boldsymbol{E}'] = -[\boldsymbol{\nu}_S \times (\nabla_{S_\tau} (\boldsymbol{p}_{S_\nu} \boldsymbol{E}_{S_\nu}))] & \text{on } S, \\ [\boldsymbol{\nu}_S \times (\nabla \times \boldsymbol{E}')] = -\omega^2 \mu \left[\left(\varepsilon + \mathrm{i} \frac{\sigma}{\omega} \right) \boldsymbol{E}_{S_\tau} \right] \boldsymbol{p}_{S_\nu} - [\boldsymbol{\nu}_S \times (\nabla_{S_\tau} (\boldsymbol{p}_{S_\nu} (\nabla \times \boldsymbol{E})_{S_\nu}))] & \text{on } S. \end{cases} \tag{5.2}$$

Proof. Define the operator $\mathcal{A} = \nabla \times (\nabla \times) - \kappa_1^2 \mathcal{I}$ and let

$$\boldsymbol{\omega}_\theta = \mathcal{A} \boldsymbol{E}_{1,\theta}, \tag{5.3}$$

where $\boldsymbol{E}_{j,\theta}$ is a solution of problem 2.1 corresponding to the region $\Omega_{j,\theta}, j = 1, 2$ for sufficiently small θ. Then, we have

$$\boldsymbol{\omega}_\theta = \boldsymbol{q} \delta \quad \text{in } \Omega_{1,\theta} \tag{5.4}$$

and

$$\boldsymbol{\omega}_\theta (\mathcal{I} + \theta) = \boldsymbol{q} \delta \quad \text{in } \Omega_1. \tag{5.5}$$

Since $\mathcal{A}$ is a linear and continuous operator from $H(\mathrm{curl}, \Omega_1) = \{ \boldsymbol{u} \in L^2(\Omega_1)^3 : \nabla \times \boldsymbol{u} \in L^2(\Omega_1)^3 \}$ into $\mathcal{D}'(\Omega_1)$, $\mathcal{A}$ is differentiable in the distribution sense, i.e. $\boldsymbol{v} \mapsto \langle \mathcal{A} \boldsymbol{v}, \boldsymbol{\psi} \rangle$ is differentiable for each $\boldsymbol{\psi} \in \mathcal{D}(\Omega_1)$ and

$$\frac{\partial \mathcal{A}}{\partial \boldsymbol{v}} = \mathcal{A}. \tag{5.6}$$

Here $\mathcal{D}(\Omega_1)$ is the standard space of infinitely differentiable functions with compact support in Ω_1 and $\mathcal{D}'(\Omega_1)$ is the standard space of distributions. Therefore, it follows from the differentiability of $\theta \mapsto \boldsymbol{E}_{1,\theta}(\mathcal{I} + \theta)$ and $\theta \mapsto \boldsymbol{E}_{1,\theta}$ that $\theta \mapsto \boldsymbol{\omega}_\theta(\mathcal{I} + \theta)$ is continuously Fréchet differentiable at $\theta = 0$ in the direction $\boldsymbol{p} \in C^2(\Gamma \cup S, \mathbb{R}^3)$. Moreover, for an admissible perturbation θ, their derivatives satisfy

$$\frac{\partial}{\partial \theta} (\boldsymbol{\omega}_\theta (\mathcal{I} + \theta))(0) \boldsymbol{p} = \frac{\partial \boldsymbol{\omega}_\theta}{\partial \theta}(0) \boldsymbol{p} + (\boldsymbol{p} \cdot \nabla) \boldsymbol{\omega} \quad \text{in } \Omega_1. \tag{5.7}$$

We deduce from (5.3)–(5.5) and (5.7) that

$$\begin{aligned} \frac{\partial \boldsymbol{\omega}_\theta}{\partial \theta}(0) \boldsymbol{p} &= \frac{\partial \mathcal{A}}{\partial \boldsymbol{E}_{1,\theta}} \frac{\partial \boldsymbol{E}_{1,\theta}}{\partial \theta}(0) \boldsymbol{p} = \frac{\partial \mathcal{A}}{\partial \boldsymbol{E}_1} \boldsymbol{E}_1' \\ &= \frac{\partial}{\partial \theta} (\boldsymbol{\omega}_\theta (\mathcal{I} + \theta))(0) \boldsymbol{p} - (\boldsymbol{p} \cdot \nabla) \boldsymbol{\omega} \\ &= (\boldsymbol{p} \cdot \nabla) \boldsymbol{q} \delta - (\boldsymbol{p} \cdot \nabla) \boldsymbol{q} \delta = 0 \quad \text{in } \Omega_1. \end{aligned} \tag{5.8}$$

It follows from (5.6) and (5.8) that

$$\mathcal{A} \boldsymbol{E}_1' = \nabla \times (\nabla \times \boldsymbol{E}_1') - \kappa_1^2 \boldsymbol{E}_1' = 0 \quad \text{in } \Omega_1.$$

For the boundary condition, we may follow the same steps as those in [21] and obtain

$$\boldsymbol{\nu}_\Gamma \times \boldsymbol{E}_1' = \big[\boldsymbol{p}_{\Gamma_\nu}(\partial_{\boldsymbol{\nu}_\Gamma}\boldsymbol{E}_{1,\Gamma_\tau}) + \boldsymbol{E}_{1,\Gamma_\nu}(\nabla_{\Gamma_\tau}\boldsymbol{p}_{\Gamma_\nu})\big] \times \boldsymbol{\nu}_\Gamma \quad \text{on } \Gamma.$$

Furthermore, for every perturbation $\theta \in C^2(\Gamma \cup S, \mathbb{R}^3)$, the tangential traces of the electric fields are assumed to be continuous across S, i.e.

$$\boldsymbol{\nu}_\theta \times \boldsymbol{E}_{1,\theta} = \boldsymbol{\nu}_\theta \times \boldsymbol{E}_{2,\theta} \quad \text{on } S_\theta. \tag{5.9}$$

Hence, we have

$$[\boldsymbol{\nu}_\theta(\mathcal{I}+\theta)] \times [\boldsymbol{E}_{1,\theta}(\mathcal{I}+\theta)] = [\boldsymbol{\nu}_\theta(\mathcal{I}+\theta)] \times [\boldsymbol{E}_{2,\theta}(\mathcal{I}+\theta)] \quad \text{on } S. \tag{5.10}$$

Moreover, it follows from [9, lemma 3] and [26, lemma 4.8] that

$$\boldsymbol{\nu}_\theta(\mathcal{I}+\theta) = \frac{1}{\|g(\theta)\boldsymbol{\nu}_S\|_{L^2(S)}} g(\theta)\boldsymbol{\nu}_S \quad \text{on } S, \tag{5.11}$$

where the matrix $g(\theta) = (\boldsymbol{I} + \frac{\partial\theta}{\partial\boldsymbol{x}})^{-\top}$ satisfies

$$g(\boldsymbol{0}) = \boldsymbol{I}, \quad \frac{\partial g(\theta)}{\partial\theta}(0)\boldsymbol{p} = -(\nabla\boldsymbol{p})^\top.$$

By (5.10) and (5.11), we have

$$[g(\theta)\boldsymbol{\nu}_S] \times [\boldsymbol{E}_{1,\theta}(\mathcal{I}+\theta)] = [g(\theta)\boldsymbol{\nu}_S] \times [\boldsymbol{E}_{2,\theta}(\mathcal{I}+\theta)] \quad \text{on } S \tag{5.12}$$

and

$$\frac{\partial}{\partial\theta}\{[g(\theta)\boldsymbol{\nu}_S] \times [\boldsymbol{E}_{1,\theta}(\mathcal{I}+\theta)]\}(0)\boldsymbol{p} = \frac{\partial}{\partial\theta}\{[g(\theta)\boldsymbol{\nu}_S] \times [\boldsymbol{E}_{2,\theta}(\mathcal{I}+\theta)]\}(0)\boldsymbol{p} \quad \text{on } S. \tag{5.13}$$

Using the chain rule, we deduce from (5.13) that

$$\begin{aligned}
&\frac{\partial}{\partial\theta}\{[g(\theta)\boldsymbol{\nu}_S] \times [\boldsymbol{E}_{j,\theta}(\mathcal{I}+\theta)]\}(0)\boldsymbol{p} \\
&= \left[\left(\frac{\partial g(\theta)}{\partial\theta}(0)\boldsymbol{p}\right)\boldsymbol{\nu}_S\right] \times \boldsymbol{E}_j + \boldsymbol{\nu}_S \times \left[\frac{\partial}{\partial\theta}(\boldsymbol{E}_{j,\theta}(\mathcal{I}+\theta))(0)\boldsymbol{p}\right] \\
&= -((\nabla\boldsymbol{p})^\top\boldsymbol{\nu}_S) \times \boldsymbol{E}_j + \boldsymbol{\nu}_S \times [\boldsymbol{E}_j' + (\boldsymbol{p}\cdot\nabla)\boldsymbol{E}_j] \quad \text{on } S, \quad j = 1, 2.
\end{aligned} \tag{5.14}$$

Since on S we have

$$\begin{aligned}
((\nabla\boldsymbol{p})^\top\boldsymbol{\nu}_S) \times \boldsymbol{E}_j &= [\boldsymbol{\nu}_S \times (\nabla \times \boldsymbol{p}) + (\boldsymbol{\nu}_S\cdot\nabla)\boldsymbol{p}] \times \boldsymbol{E}_j \\
&= [\boldsymbol{\nu}_S \times (\nabla \times \boldsymbol{p})] \times \boldsymbol{E}_j + [(\boldsymbol{\nu}_S\cdot\nabla)\boldsymbol{p}] \times \boldsymbol{E}_j \\
&= -\boldsymbol{\nu}_S \times [\boldsymbol{E}_j \times (\nabla \times \boldsymbol{p})] - (\nabla \times \boldsymbol{p}) \times (\boldsymbol{\nu}_S \times \boldsymbol{E}_j) + [(\boldsymbol{\nu}_S\cdot\nabla)\boldsymbol{p}] \times \boldsymbol{E}_j \\
&= -\boldsymbol{\nu}_S \times [\boldsymbol{E}_j \times (\nabla \times \boldsymbol{p})] - \boldsymbol{\nu}_S \times [(\boldsymbol{E}_j\cdot\nabla)\boldsymbol{p}] \\
&\quad - (\nabla\boldsymbol{p})(\boldsymbol{\nu}_S \times \boldsymbol{E}_j) + (\nabla\cdot\boldsymbol{p})(\boldsymbol{\nu}_S \times \boldsymbol{E}_j), \quad j = 1, 2.
\end{aligned} \tag{5.15}$$

With the aid of (5.14) and (5.15), we obtain

$$\begin{aligned}
&\frac{\partial}{\partial\theta}\{[g(\theta)\boldsymbol{\nu}_S]\times[\boldsymbol{E}_{j,\theta}(\mathcal{I}+\theta)]\}(0)\boldsymbol{p}\\
&=-\{-\boldsymbol{\nu}_S\times[\boldsymbol{E}_j\times(\nabla\times\boldsymbol{p})]-\boldsymbol{\nu}_S\times[(\boldsymbol{E}_j\cdot\nabla)\boldsymbol{p}]-(\nabla\boldsymbol{p})(\boldsymbol{\nu}_S\times\boldsymbol{E}_j)\\
&\qquad+(\nabla\cdot\boldsymbol{p})(\boldsymbol{\nu}_S\times\boldsymbol{E}_j)\}+\boldsymbol{\nu}_S\times\boldsymbol{E}_j'+\boldsymbol{\nu}_S\times[(\boldsymbol{p}\cdot\nabla)\boldsymbol{E}_j]\\
&=\{\boldsymbol{\nu}_S\times[\boldsymbol{E}_j\times(\nabla\times\boldsymbol{p})]+\boldsymbol{\nu}_S\times[(\boldsymbol{E}_j\cdot\nabla)\boldsymbol{p}]+\boldsymbol{\nu}_S\times[(\boldsymbol{p}\cdot\nabla)\boldsymbol{E}_j]\}\\
&\qquad+\boldsymbol{\nu}_S\times\boldsymbol{E}_j'+(\nabla\boldsymbol{p})(\boldsymbol{\nu}_S\times\boldsymbol{E}_j)-(\nabla\cdot\boldsymbol{p})(\boldsymbol{\nu}_S\times\boldsymbol{E}_j)\\
&=\boldsymbol{\nu}_S\times[\boldsymbol{E}_j\times(\nabla\times\boldsymbol{p})+(\boldsymbol{E}_j\cdot\nabla)\boldsymbol{p}+(\boldsymbol{p}\cdot\nabla)\boldsymbol{E}_j]\\
&\qquad+\boldsymbol{\nu}_S\times\boldsymbol{E}_j'+(\nabla\boldsymbol{p})(\boldsymbol{\nu}_S\times\boldsymbol{E}_j)-(\nabla\cdot\boldsymbol{p})(\boldsymbol{\nu}_S\times\boldsymbol{E}_j)\\
&=\boldsymbol{\nu}_S\times[(\nabla\times\boldsymbol{E}_j)\times\boldsymbol{p}]+\boldsymbol{\nu}_S\times[\boldsymbol{p}\times(\nabla\times\boldsymbol{E}_j)+\boldsymbol{E}_j\times(\nabla\times\boldsymbol{p})+(\boldsymbol{E}_j\cdot\nabla)\boldsymbol{p}\\
&\qquad+(\boldsymbol{p}\cdot\nabla)\boldsymbol{E}_j]+\boldsymbol{\nu}_S\times\boldsymbol{E}_j'+(\nabla\boldsymbol{p})(\boldsymbol{\nu}_S\times\boldsymbol{E}_j)-(\nabla\cdot\boldsymbol{p})(\boldsymbol{\nu}_S\times\boldsymbol{E}_j)\\
&=\boldsymbol{\nu}_S\times[(\nabla\times\boldsymbol{E}_j)\times\boldsymbol{p}]+\boldsymbol{\nu}_S\times[\nabla(\boldsymbol{p}\cdot\boldsymbol{E}_j)]+\boldsymbol{\nu}_S\times\boldsymbol{E}_j'\\
&\qquad+(\nabla\boldsymbol{p})(\boldsymbol{\nu}_S\times\boldsymbol{E}_j)-(\nabla\cdot\boldsymbol{p})(\boldsymbol{\nu}_S\times\boldsymbol{E}_j)\quad\text{on } S,\ j=1,2.
\end{aligned}\tag{5.16}$$

By taking into account of the continuous conditions (2.7) and $\boldsymbol{p}\in C^2(\Gamma\cup S,\mathbb{R}^3)$, from (5.1) and (5.16), the jump relations read

$$[\boldsymbol{\nu}_S\times\boldsymbol{E}']=-[\boldsymbol{\nu}_S\times((\nabla\times\boldsymbol{E})\times\boldsymbol{p})]-[\boldsymbol{\nu}_S\times(\nabla(\boldsymbol{p}\cdot\boldsymbol{E}))].\tag{5.17}$$

For the first term of in the right hand side of (5.17), we conclude from the continuous conditions (2.7) and the jump condition $[(\nabla\times\boldsymbol{E})_{S_\nu}]=0$ that

$$\begin{aligned}
[\boldsymbol{\nu}_S\times((\nabla\times\boldsymbol{E})\times\boldsymbol{p})]&=[(\nabla\times\boldsymbol{E})(\boldsymbol{\nu}_S\cdot\boldsymbol{p})-\boldsymbol{p}(\boldsymbol{\nu}_S\cdot(\nabla\times\boldsymbol{E}))]\\
&=[((\nabla\times\boldsymbol{E})_{S_\tau}+(\nabla\times\boldsymbol{E})_{S_\nu}\boldsymbol{\nu}_S)\boldsymbol{p}_{S_\nu}-(\boldsymbol{p}_{S_\tau}+\boldsymbol{p}_{S_\nu}\boldsymbol{\nu}_S)(\nabla\times\boldsymbol{E})_{S_\nu}]\\
&=[(\nabla\times\boldsymbol{E})_{S_\tau}\boldsymbol{p}_{S_\nu}-(\nabla\times\boldsymbol{E})_{S_\nu}\boldsymbol{p}_{S_\tau}]\\
&=[(\nabla\times\boldsymbol{E})_{S_\tau}]\boldsymbol{p}_{S_\nu}-[(\nabla\times\boldsymbol{E})_{S_\nu}]\boldsymbol{p}_{S_\tau}\\
&=[(\nabla\times\boldsymbol{E})_{S_\tau}]\boldsymbol{p}_{S_\nu}=0\quad\text{on } S.
\end{aligned}\tag{5.18}$$

It follows from $[\boldsymbol{\nu}_S\times\boldsymbol{E}]=[\boldsymbol{\nu}_S\times\boldsymbol{E}_{S_\tau}]=0$ and the definition of the surface gradient ∇_{S_τ} that we obtain $[\boldsymbol{\nu}_S\times(\nabla_{S_\tau}(\boldsymbol{p}_{S_\tau}\cdot\boldsymbol{E}_{S_\tau}))]=0$. Thus, the second term in the right hand side of (5.17) reduces to

$$\begin{aligned}
[\boldsymbol{\nu}_S\times(\nabla(\boldsymbol{p}\cdot\boldsymbol{E}))]&=[\boldsymbol{\nu}_S\times(\nabla_{S_\tau}(\boldsymbol{p}\cdot\boldsymbol{E}))]\\
&=[\boldsymbol{\nu}_S\times(\nabla_{S_\tau}((\boldsymbol{p}_{S_\tau}+\boldsymbol{p}_{S_\nu}\boldsymbol{\nu}_S)\cdot(\boldsymbol{E}_{S_\tau}+\boldsymbol{E}_{S_\nu}\boldsymbol{\nu}_S)))]\\
&=[\boldsymbol{\nu}_S\times(\nabla_{S_\tau}(\boldsymbol{p}_{S_\tau}\cdot\boldsymbol{E}_{S_\tau}+\boldsymbol{p}_{S_\nu}\boldsymbol{E}_{S_\nu}))]\\
&=[\boldsymbol{\nu}_S\times(\nabla_{S_\tau}(\boldsymbol{p}_{S_\nu}\boldsymbol{E}_{S_\nu}))]\quad\text{on } S.
\end{aligned}\tag{5.19}$$

Finally, by (5.17)–(5.19), we have the boundary condition

$$[\boldsymbol{\nu}_S\times\boldsymbol{E}']=-[\boldsymbol{\nu}_S\times(\nabla_{S_\tau}(\boldsymbol{p}_{S_\nu}\boldsymbol{E}_{S_\nu}))]\quad\text{on } S.$$

We define $\boldsymbol{H}'=\frac{1}{\mathrm{i}\omega\mu}(\nabla\times\boldsymbol{E}')$, and similarly, we can obtain

$$\begin{aligned}
[\boldsymbol{\nu}_S\times\boldsymbol{H}']&=-[\boldsymbol{\nu}_S\times((\nabla\times\boldsymbol{H})\times\boldsymbol{p})]-[\boldsymbol{\nu}_S\times(\nabla(\boldsymbol{p}\cdot\boldsymbol{H}))]\\
&=-[(\nabla\times\boldsymbol{H})_{S_\tau}]\boldsymbol{p}_{S_\nu}-[\boldsymbol{\nu}_S\times(\nabla_{S_\tau}(\boldsymbol{p}_{S_\nu}\boldsymbol{H}_{S_\nu}))]\\
&=\mathrm{i}\omega\left[\left(\varepsilon+\mathrm{i}\frac{\sigma}{\omega}\right)\boldsymbol{E}_{S_\tau}\right]\boldsymbol{p}_{S_\nu}-[\boldsymbol{\nu}_S\times(\nabla_{S_\tau}(\boldsymbol{p}_{S_\nu}\boldsymbol{H}_{S_\nu}))]\quad\text{on } S,
\end{aligned}\tag{5.20}$$

which implies that

$$\begin{aligned}[\boldsymbol{\nu}_S\times(\nabla\times\boldsymbol{E}')] &= \mathrm{i}\omega\mu\left(\mathrm{i}\omega\left[\left(\varepsilon+\mathrm{i}\frac{\sigma}{\omega}\right)\boldsymbol{E}_{S_\tau}\right]\boldsymbol{p}_{S_\nu}-\frac{1}{\mathrm{i}\omega\mu}[\boldsymbol{\nu}_S\times(\nabla_{S_\tau}(\boldsymbol{p}_{S_\nu}(\nabla\times\boldsymbol{E})_{S_\nu}))]\right)\\ &= -\omega^2\mu\left[\left(\varepsilon+\mathrm{i}\frac{\sigma}{\omega}\right)\boldsymbol{E}_{S_\tau}\right]\boldsymbol{p}_{S_\nu}-[\boldsymbol{\nu}_S\times(\nabla_{S_\tau}(\boldsymbol{p}_{S_\nu}(\nabla\times\boldsymbol{E})_{S_\nu}))]\quad \text{on } S.\end{aligned}\tag{5.21}$$

Based on the existence of the domain derivatives $\boldsymbol{E}_j'$, the proof of the the integral representations for $\boldsymbol{E}_j'$ follow in the same manner as for the the integral representation of $\boldsymbol{E}_j$. Therefore, the asymptotic behavior to the domain derivative $\boldsymbol{E}_j'$ has the same form as $\boldsymbol{E}_j$. This means that the domain derivatives $(\boldsymbol{E}_1', \boldsymbol{E}_2')$ are the radiation solutions of the problem (5.2). □

Introduce the domain $\Omega_{1,h}$ bounded by Γ_h and S_h, where

$$\Gamma_h=\{\boldsymbol{x}+hq(\boldsymbol{x})\boldsymbol{\nu}_\Gamma:\boldsymbol{x}\in\Gamma\},\quad S_h=\{\boldsymbol{x}+hq(\boldsymbol{x})\boldsymbol{\nu}_S:\boldsymbol{x}\in S\}$$

where $q\in C^2(\mathbb{R}^3,\mathbb{R})$ and $h>0$. For any two domains Ω_1 and $\Omega_{1,h}$ in $\mathbb{R}^3$, define the Hausdorff distance

$$\mathrm{dist}(\Omega_1,\Omega_{1,h})=\max\{\rho(\Omega_{1,h},\Omega_1),\rho(\Omega_1,\Omega_{1,h})\},$$

where

$$\rho(\Omega_1,\Omega_{1,h})=\sup_{\boldsymbol{x}\in\Omega_1}\inf_{\boldsymbol{y}\in\Omega_{1,h}}|\boldsymbol{x}-\boldsymbol{y}|.$$

It can be easily seen that the Hausdorff distance between $\Omega_{1,h}$ and Ω_1 is of the order h, i.e. $\mathrm{dist}(\Omega_1,\Omega_{1,h})=\mathcal{O}(h)$. We have the following local stability result.

Theorem 5.2. *If $q\in C^2(\Gamma\cup S,\mathbb{R})$ and $h>0$ is sufficiently small, then*

$$\mathrm{dist}(\Omega_1,\Omega_{1,h})\leqslant C\|\boldsymbol{\nu}_{\Gamma_1}\times\boldsymbol{E}_{1,h}-\boldsymbol{\nu}_{\Gamma_1}\times\boldsymbol{E}_1\|_{C^{0,\alpha}(\Gamma_1)},$$

where $\boldsymbol{E}_{1,h}$ and $\boldsymbol{E}_1$ is the solution of problem 2.1 corresponding to the domain $\Omega_{1,h}$ and Ω_1, respectively, and C is a positive constant independent of h.

Proof. Assume by contradiction that there exists a subsequence from $\{\boldsymbol{E}_{1,h}\}$, which is still denoted as $\{\boldsymbol{E}_{1,h}\}$ for simplicity, such that

$$\lim_{h\to0}\left\|\frac{\boldsymbol{\nu}_{\Gamma_1}\times\boldsymbol{E}_{1,h}-\boldsymbol{\nu}_{\Gamma_1}\times\boldsymbol{E}_1}{h}\right\|_{C^{0,\alpha}(\Gamma_1)}=\|\boldsymbol{\nu}_{\Gamma_1}\times\boldsymbol{E}_1'\|_{C^{0,\alpha}(\Gamma_1)}=0\quad\text{as } h\to0,$$

which yields $\boldsymbol{\nu}_{\Gamma_H}\times\boldsymbol{E}_1'=0$ on Γ_1. Following a similar proof of theorem 3.4, we can show the uniqueness of the solution for problem (5.2). An application of the uniqueness for problem (5.2) yields that $\boldsymbol{E}_j'=0$ in $\overline{\Omega}_j, j=1,2$. Furthermore, we also have $\nabla\times\boldsymbol{E}_j'=0$ in $\overline{\Omega}_j, j=1,2$.

Taking $\boldsymbol{p}(\boldsymbol{x})=q(\boldsymbol{x})\boldsymbol{\nu}_\Gamma$ on Γ in problem (5.2), we have from the boundary condition of $\boldsymbol{E}_1'$ in problem (5.2) that

$$\begin{aligned}\boldsymbol{\nu}_\Gamma\times\boldsymbol{E}_1' &= [(q(\boldsymbol{x})\boldsymbol{\nu}_\Gamma)_{\Gamma_\nu}(\partial_{\boldsymbol{\nu}_\Gamma}\boldsymbol{E}_{1,\Gamma_\tau})+\boldsymbol{E}_{1,\Gamma_\nu}(\nabla_{\Gamma_\tau}(q(\boldsymbol{x})\boldsymbol{\nu}_\Gamma)_{\Gamma_\nu})]\times\boldsymbol{\nu}_\Gamma\\ &= [q(\partial_{\boldsymbol{\nu}_\Gamma}\boldsymbol{E}_{1,\Gamma_\tau})+\boldsymbol{E}_{1,\Gamma_\nu}(\nabla_{\Gamma_\tau}q)]\times\boldsymbol{\nu}_\Gamma=0\quad\text{on }\Gamma.\end{aligned}\tag{5.22}$$

Since q is arbitrary in (5.22), we have

$$\begin{aligned}\partial_{\boldsymbol{\nu}_\Gamma}\boldsymbol{E}_{1,\Gamma_\tau} &= \partial_{\boldsymbol{\nu}_\Gamma}[\boldsymbol{\nu}_\Gamma \times (\boldsymbol{E}_1 \times \boldsymbol{\nu}_\Gamma)] \\ &= \partial_{\boldsymbol{\nu}_\Gamma}\boldsymbol{E}_1 - \partial_{\boldsymbol{\nu}_\Gamma}[(\boldsymbol{\nu}_\Gamma \cdot \boldsymbol{E}_1)\boldsymbol{\nu}_\Gamma] = 0 \quad \text{on } \Gamma,\end{aligned} \tag{5.23}$$

and

$$\boldsymbol{E}_{1,\Gamma_\nu} = \boldsymbol{\nu}_\Gamma \cdot \boldsymbol{E}_1 = 0 \quad \text{on } \Gamma. \tag{5.24}$$

It follows from (5.23) and (5.24) that

$$\partial_{\boldsymbol{\nu}_\Gamma}\boldsymbol{E}_1 = 0 \quad \text{on } \Gamma. \tag{5.25}$$

With the aid of $\boldsymbol{\nu}_\Gamma \times \boldsymbol{E}_1|_\Gamma = 0$ and $\boldsymbol{\nu}_\Gamma \cdot \boldsymbol{E}_1|_\Gamma = 0$, we have

$$\boldsymbol{E}_1 = 0 \quad \text{on } \Gamma. \tag{5.26}$$

Therefore, combining (5.25) and (5.26), we infer by unique continuation that

$$\boldsymbol{E}_1 = 0 \quad \text{in } R_1,$$

which contradicts the transparent boundary condition (2.11).

We next consider the perturbation on S, take $\boldsymbol{p}(x) = q(x)\boldsymbol{\nu}_S$ on S in problem (5.2), from $\nabla \times \boldsymbol{E}'_j = 0$ in $\overline{\Omega}_j$, one can get

$$\begin{aligned}0 = [\boldsymbol{\nu}_S \times (\nabla \times \boldsymbol{E}')] &= -\omega^2\mu\Big[\Big(\varepsilon + \mathrm{i}\frac{\sigma}{\omega}\Big)\boldsymbol{E}_{S_\tau}\Big]\boldsymbol{p}_{S_\nu} - [\boldsymbol{\nu}_S \times (\nabla_{S_\tau}(\boldsymbol{p}_{S_\nu}(\nabla \times \boldsymbol{E})_{S_\nu}))] \\ &= -\omega^2\mu\Big[\Big(\varepsilon + \mathrm{i}\frac{\sigma}{\omega}\Big)\boldsymbol{E}_{S_\tau}\Big]q - [\boldsymbol{\nu}_S \times (\nabla_{S_\tau}(q(\nabla \times \boldsymbol{E})_{S_\nu}))] \quad \text{on } S.\end{aligned} \tag{5.27}$$

Since q is arbitrary in (5.27) and $\boldsymbol{E}_{1,S_\tau} = [\boldsymbol{\nu}_S \times (\boldsymbol{E}_1 \times \boldsymbol{\nu}_S)] = [\boldsymbol{\nu}_S \times (\boldsymbol{E}_2 \times \boldsymbol{\nu}_S)] = \boldsymbol{E}_{2,S_\tau}$, we have

$$0 = \Big[\Big(\varepsilon + \mathrm{i}\frac{\sigma}{\omega}\Big)\boldsymbol{E}_{S_\tau}\Big] = \Big[\Big(\varepsilon_1 + \mathrm{i}\frac{\sigma_1}{\omega}\Big) - \Big(\varepsilon_2 + \mathrm{i}\frac{\sigma_2}{\omega}\Big)\Big]\boldsymbol{E}_{2,S_\tau} \quad \text{on } S. \tag{5.28}$$

For $\varepsilon_1 \neq \varepsilon_2$ and $\sigma_1 \neq \sigma_2$, from (5.28) that

$$\boldsymbol{E}_{1,S_\tau} = \boldsymbol{E}_{2,S_\tau} = 0 \quad \text{on } S. \tag{5.29}$$

Taking the dot product of the Maxwell equation (2.4) with the complex conjugate of $\boldsymbol{E}_2$, integrating over Ω_2, and using (5.29) and the radiation condition (2.10), we obtain

$$\int_{\Omega_2}|\nabla \times \boldsymbol{E}_2|^2\mathrm{d}\boldsymbol{x} - \kappa_2^2\int_{\Omega_2}|\boldsymbol{E}_2|^2\mathrm{d}\boldsymbol{x} = -\int_S(\boldsymbol{\nu}_S \times (\nabla \times \boldsymbol{E}_2)) \cdot ((\boldsymbol{\nu}_S \times \overline{\boldsymbol{E}}_2) \times \boldsymbol{\nu}_S)\mathrm{d}s_{\boldsymbol{x}} = 0, \tag{5.30}$$

which implies $\boldsymbol{E}_2 = 0$ in Ω_2 due to $\Im(\kappa_2^2) > 0$. Hence from (2.7), we have $\boldsymbol{\nu}_S \times \boldsymbol{E}_1 = \boldsymbol{\nu}_S \times \boldsymbol{E}_2 = 0$ on S, which is impossible by lemma 4.4. □

The result indicates that for small h, if the boundary measurements are $\mathcal{O}(h)$ close to each other, then the corresponding domains are also $\mathcal{O}(h)$ close to each other in the Hausdorff distance.

6. Conclusion

In this paper, we have studied the direct and inverse electromagnetic obstacle scattering problems for the three-dimensional Maxwell equations in an unbounded structure. We present an equivalent integral equation to the boundary value problem and show that it has a unique solution. For the inverse problem, we prove that the obstacle and unbounded rough surface can be uniquely determined by the tangential component of the electric field measured on two plane surfaces which enclose the unknown obstacle and unbounded rough surface. The local stability shows that the Hausdorff distance of the two regions, corresponding to small perturbations of the obstacle and the unbounded rough surface, is bounded by the distance of corresponding tangential trace of the electric fields if they are close enough. To prove the stability, the domain derivative of the electric field with respect to the change of the shape of the obstacle and unbounded rough surface is examined. In particular, we deduce that the domain derivative satisfies a boundary value problem of the Maxwell equations, which is similar to the model equation of the direct problem. The results hold for multiple obstacles which are located either above or below the unbounded rough surface.

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