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INVERSE OBSTACLE SCATTERING FOR ELASTIC WAVES IN THREE DIMENSIONS

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Abstract. Consider an exterior problem of the three-dimensional elastic wave equation, which models the scattering of a time-harmonic plane wave by a rigid obstacle. The scattering problem is reformulated into a boundary value problem by introducing a transparent boundary condition. Given the incident field, the direct problem is to determine the displacement of the wave field from the known obstacle; the inverse problem is to determine the obstacle's surface from the measurement of the displacement on an artificial boundary enclosing the obstacle. In this paper, we consider both the direct and inverse problems. The direct problem is shown to have a unique weak solution by examining its variational formulation. The domain derivative is studied and a frequency continuation method is developed for the inverse problem. Numerical experiments are presented to demonstrate the effectiveness of the proposed method.

1. **Introduction.** The obstacle scattering problem, which concerns the scattering of a time-harmonic incident wave by an impenetrable medium, is a fundamental problem in scattering theory [9]. It has played an important role in many scientific areas such as geophysical exploration, nondestructive testing, radar and sonar, and medical imaging. Given the incident field, the direct obstacle scattering problem is to determine the wave field from the known obstacle; the inverse obstacle scattering problem is to determine the obstacle from the measured wave field. Due to the wide applications and rich mathematics, the direct and inverse obstacle scattering problems have been extensively studied for acoustic and electromagnetic waves by numerous researchers in both the engineering and mathematical communities [10, 26, 28].

Recently, the scattering problems for elastic waves have received ever-increasing attention because of the significant applications in geophysics and seismology [1, 5, 22]. The propagation of elastic waves is governed by the Navier equation, which is complex due to the coupling of the compressional and shear waves with different wavenumbers. The inverse elastic obstacle scattering problem is investigated mathematically in [11, 15, 17] for uniqueness and numerically in [18, 23] for the shape reconstruction. We refer to [2, 6, 20, 27] on related direct and inverse scattering problems for elastic waves.

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In this paper, we consider the direct and inverse obstacle scattering problems for elastic waves in three dimensions. The goal is fourfold: (1) develop a transparent boundary condition to reduce the scattering problem into a boundary value problem; (2) establish the well-posedness of the solution for the direct problem by studying its variational formulation; (3) characterize the domain derivative of the wave field with respect to the variation of the obstacle's surface; (4) propose a frequency continuation method to reconstruct the obstacle's surface. This paper significantly extends the two-dimensional work [25]. We need to consider more complicated Maxwell's equation and associated spherical harmonics when studying the transparent boundary condition (TBC). Computationally, it is also more intensive.

The obstacle is assumed to be embedded in an open space filled with a homogeneous and isotropic elastic medium. The scattering problem is reduced into a boundary value problem by introducing a transparent boundary condition on a sphere enclosing the obstacle. The non-reflecting boundary conditions can also be found in [12, 13] for the two- and three-dimensional elastic wave equation. We show that the direct problem has a unique weak solution by examining its variational formulation. The proofs are based on asymptotic analysis of the boundary operators, the Helmholtz decomposition, and the Fredholm alternative theorem.

The calculation of domain derivatives, which characterize the variation of the wave field with respect to the perturbation of the boundary of an medium, is an essential step for inverse scattering problems. The domain derivatives have been discussed by many authors for the inverse acoustic and electromagnetic obstacle scattering problems [14, 21, 29]. Recently, the domain derivative is studied in [24] for the elastic wave by using boundary integral equations. Here we present a variational approach to show that it is the unique weak solution of some boundary value problem. We propose a frequency continuation method to solve the inverse problem. The method requires multi-frequency data and proceed with respect to the frequency. At each frequency, we apply the descent method with the starting point given by the output from the previous step, and create an approximation to the surface filtered at a higher frequency. Numerical experiments are presented to demonstrate the effectiveness of the proposed method. A topic review can be found in [3] for solving inverse scattering problems with multi-frequencies to increase the resolution and stability of reconstructions.

The paper is organized as follows. Section 2 introduces the formulation of the obstacle scattering problem for elastic waves. The direct problem is discussed in section 3 where well-posedness of the solution is established. Section 4 is devoted to the inverse problem. The domain derivative is studied and a frequency continuation method is introduced for the inverse problem. Numerical experiments are presented in section 5. The paper is concluded in section 6. To avoid distraction from the main results, we collect in the appendices some necessary notation and useful results on the spherical harmonics, functional spaces, and transparent boundary conditions.

2. **Problem formulation.**

Consider a three-dimensional elastically rigid obstacle D with a Lipschitz continuous boundary ∂D. Denote by $\boldsymbol{\nu} = (\nu_1, \nu_2, \nu_3)$ the unit normal vector on ∂D pointing towards the exterior of D. We assume that the open exterior domain $\mathbb{R}^3 \setminus \bar{D}$ is filled with a homogeneous and isotropic elastic medium with a unit mass density. Let $B_R = \{\boldsymbol{x} \in \mathbb{R}^3 : |\boldsymbol{x}| < R\}$ be a ball with radius $R > 0$ such that $\bar{D} \subset B_R$. Denote by $\Gamma_R = \{\boldsymbol{x} \in \mathbb{R}^3 : |\boldsymbol{x}| = R\}$ boundary of B_R. Let $\Omega = B_R \setminus \bar{D}$ be the bounded domain which is enclosed by ∂D and Γ_R.

Let the obstacle be illuminated by a time-harmonic plane wave

$$\boldsymbol{u}^{\rm inc} = \boldsymbol{d}e^{{\rm i}\kappa_{\rm p}\boldsymbol{x}\cdot\boldsymbol{d}} \quad \text{or} \quad \boldsymbol{u}^{\rm inc} = \boldsymbol{d}^{\perp}e^{{\rm i}\kappa_{\rm s}\boldsymbol{x}\cdot\boldsymbol{d}}, \tag{1}$$

where $\boldsymbol{d}$ is the unit incident direction vector and $\boldsymbol{d}^{\perp}$ is the unit polarization vector satisfying $\boldsymbol{d}\cdot\boldsymbol{d}^{\perp}$=0. In (1), the former is called the compressional plane wave while the latter is called the shear plane wave. Here

$$\kappa_{\rm p} = \frac{\omega}{(\lambda+2\mu)^{1/2}} \quad \text{and} \quad \kappa_{\rm s} = \frac{\omega}{\mu^{1/2}} \tag{2}$$

are known the compressional wavenumber and the shear wavenumber, respectively, where $\omega > 0$ is the angular frequency, μ and λ are the Lamé parameters satisfying $\mu > 0$ and $\lambda + \mu > 0$. It is easy to verify that both the compressional plane wave and the shear plane wave in (1) satisfy the three-dimensional Navier equation:

$$\mu\Delta\boldsymbol{u}^{\rm inc} + (\lambda+\mu)\nabla\nabla\cdot\boldsymbol{u}^{\rm inc} + \omega^2\boldsymbol{u}^{\rm inc} = 0 \quad \text{in } \mathbb{R}^3\setminus\bar{D}. \tag{3}$$

The displacement of the total wave field $\boldsymbol{u}$ also satisfies

$$\mu\Delta\boldsymbol{u} + (\lambda+\mu)\nabla\nabla\cdot\boldsymbol{u} + \omega^2\boldsymbol{u} = 0 \quad \text{in } \mathbb{R}^3\setminus\bar{D}. \tag{4}$$

Since the obstacle is elastically rigid, the total wave field vanishes on ∂D:

$$\boldsymbol{u} = 0 \quad \text{on } \partial D. \tag{5}$$

The total wave field $\boldsymbol{u}$ can be decomposed into the incident wave $\boldsymbol{u}^{\rm inc}$ and the scattered wave $\boldsymbol{v}$:

$$\boldsymbol{u} = \boldsymbol{u}^{\rm inc} + \boldsymbol{v}.$$

Subtracting (3) from (4) yields the Navier equation for the scattered wave $\boldsymbol{v}$:

$$\mu\Delta\boldsymbol{v} + (\lambda+\mu)\nabla\nabla\cdot\boldsymbol{v} + \omega^2\boldsymbol{v} = 0 \quad \text{in } \mathbb{R}^3\setminus\bar{D}. \tag{6}$$

An appropriate radiation condition is needed for the exterior scattering problem. For any solution $\boldsymbol{v}$ of (6), we introduce the Helmholtz decomposition:

$$\boldsymbol{v} = \nabla\phi + \nabla\times\boldsymbol{\psi}, \quad \nabla\cdot\boldsymbol{\psi} = 0, \tag{7}$$

where ϕ and $\boldsymbol{\psi}$ is called the scalar potential function and the vector potential function, respectively. Substituting (7) into (6) yields

$$\nabla\left[(\lambda+2\mu)\Delta\phi + \omega^2\phi\right] + \nabla\times(\mu\Delta\boldsymbol{\psi} + \omega^2\boldsymbol{\psi}) = 0,$$

which is fulfilled if ϕ and $\boldsymbol{\psi}$ satisfy the Helmholtz equation:

$$\Delta\phi + \kappa_{\rm p}^2\phi = 0, \quad \Delta\boldsymbol{\psi} + \kappa_{\rm s}^2\boldsymbol{\psi} = 0. \tag{8}$$

where $\kappa_{\rm p}$ and $\kappa_{\rm s}$ are defined in (2). Hence, we request that ϕ and $\boldsymbol{\psi}$ satisfy the Sommerfeld radiation condition:

$$\lim_{r\to\infty} r\left(\partial_r\phi - {\rm i}\kappa_{\rm p}\phi\right) = 0, \quad \lim_{r\to\infty} r\left(\partial_r\boldsymbol{\psi} - {\rm i}\kappa_{\rm s}\boldsymbol{\psi}\right) = 0, \quad r = |\boldsymbol{x}|. \tag{9}$$

Using the identity

$$\nabla\times(\nabla\times\boldsymbol{\psi}) = -\Delta\boldsymbol{\psi} + \nabla(\nabla\cdot\boldsymbol{\psi}),$$

we have from the Helmholtz equation (8) that $\boldsymbol{\psi}$ satisfies the Maxwell system:

$$\nabla\times(\nabla\times\boldsymbol{\psi}) - \kappa_{\rm s}^2\boldsymbol{\psi} = 0. \tag{10}$$

As is known, the Silver–Müller radition condition is commonly imposed as an appropriate radiation condition for Maxwell's equations. It is shown (cf. [10, Theorem

6.8]) that the Sommerfeld radiation for $\boldsymbol{\psi}$ in (9) is equivalent to the Silver–Müller radiation condition:

$$\lim_{r\to\infty}((\nabla\times\boldsymbol{\psi})\times\boldsymbol{x}-\mathrm{i}\kappa_{\mathrm{s}}r\boldsymbol{\psi})=0,\quad r=|\boldsymbol{x}|. \tag{11}$$

Given the incident field $\boldsymbol{u}^{\mathrm{inc}}$, the direct problem is to determine the displacement of the total field $\boldsymbol{u}$ for the known obstacle D; the inverse problem is to determine the obstacle's surface ∂D from the boundary measurement of the displacement $\boldsymbol{u}$ on Γ_R. The purpose of this paper is to study the well-posedness of the direct problem and develop a continuation method for the inverse problem. Hereafter, we take the notation of $a\lesssim b$ or $a\gtrsim b$ to stand for $a\leq Cb$ or $a\geq Cb$, where C is a positive constant. Some commonly used functional spaces, such as $\boldsymbol{H}^1_{\partial D}(\Omega)$, $\boldsymbol{H}^s(\Gamma_R)$, and $\boldsymbol{H}(\mathrm{curl},\Omega)$, are list in appendix B.

3. **Direct scattering problem.** In this section, we study the variational formulation for the direct problem and show that it admits a unique weak solution.

3.1. **Transparent boundary condition.** We derive a transparent boundary condition on Γ_R to reformulate the problem from the open domain $\mathbb{R}^3\setminus\bar{D}$ into the bounded domain Ω.

Given $\boldsymbol{v}\in\boldsymbol{L}^2(\Gamma_R)$, it follows from Appendix A that $\boldsymbol{v}$ has the Fourier expansion

$$\boldsymbol{v}(R,\theta,\varphi)=\sum_{n=0}^{\infty}\sum_{m=-n}^{n}v_{1n}^m\boldsymbol{T}_n^m(\theta,\varphi)+v_{2n}^m\boldsymbol{V}_n^m(\theta,\varphi)+v_{3n}^m\boldsymbol{W}_n^m(\theta,\varphi),$$

where the Fourier coefficients

$$v_{1n}^m=\int_{\Gamma_R}\boldsymbol{v}(R,\theta,\varphi)\cdot\bar{\boldsymbol{T}}_n^m(\theta,\varphi)\mathrm{d}\gamma,$$
$$v_{2n}^m=\int_{\Gamma_R}\boldsymbol{v}(R,\theta,\varphi)\cdot\bar{\boldsymbol{V}}_n^m(\theta,\varphi)\mathrm{d}\gamma,$$
$$v_{3n}^m=\int_{\Gamma_R}\boldsymbol{v}(R,\theta,\varphi)\cdot\bar{\boldsymbol{W}}_n^m(\theta,\varphi)\mathrm{d}\gamma.$$

Define a boundary operator

$$\mathscr{B}\boldsymbol{v}=\mu\partial_r\boldsymbol{v}+(\lambda+\mu)(\nabla\cdot\boldsymbol{v})\boldsymbol{e}_r\quad\text{on }\Gamma_R, \tag{12}$$

which is assumed to have the Fourier expansion:

$$(\mathscr{B}\boldsymbol{v})(R,\theta,\varphi)=\sum_{n=0}^{\infty}\sum_{m=-n}^{n}w_{1n}^m\boldsymbol{T}_n^m(\theta,\varphi)+w_{2n}^m\boldsymbol{V}_n^m(\theta,\varphi)+w_{3n}^m\boldsymbol{W}_n^m(\theta,\varphi). \tag{13}$$

Taking ∂_r of $\boldsymbol{v}$ in (60), evaluating it at $r=R$, and using the spherical Bessel differential equations [30], we get

$$\begin{aligned}\partial_r\boldsymbol{v}(R,\theta,\varphi)=\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\Bigg[&\frac{\sqrt{n(n+1)}\phi_n^m}{R^2}(z_n(\kappa_{\mathrm{p}}R)-1)-\frac{\psi_{2n}^m}{R^2}\Big(1+z_n(\kappa_{\mathrm{s}}R)\\&+(R\kappa_{\mathrm{s}})^2-n(n+1)\Big)\Bigg]\boldsymbol{T}_n^m+\left[\frac{\kappa_{\mathrm{s}}^2\psi_{3n}^m}{\sqrt{n(n+1)}}z_n(\kappa_{\mathrm{s}}R)\right]\boldsymbol{V}_n^m+\Bigg[\frac{\phi_n^m}{R^2}\Big(n(n+1)\\&-(R\kappa_{\mathrm{p}})^2-2z_n(\kappa_{\mathrm{p}}R)\Big)+\frac{\sqrt{n(n+1)}\psi_{2n}^m}{R^2}(z_n(\kappa_{\mathrm{s}}R)-1)\Bigg]\boldsymbol{W}_n^m,\end{aligned} \tag{14}$$

where $z_n(t) = th_n^{(1)'}(t)/h_n^{(1)}(t)$, $h_n^{(1)}$ is the spherical Hankel function of the first kind with order n, ϕ_n^m and ψ_{jn}^m are the Fourier coefficients for ϕ and $\boldsymbol{\psi}$ on Γ_R, respectively.

Noting (60) and using $\nabla\cdot\boldsymbol{v} = \Delta\phi = \frac{2}{r}\partial_r\phi + \partial_r^2\phi + \frac{1}{r}\Delta_{\Gamma_R}\phi$, we have

$$\nabla\cdot\boldsymbol{v}(r,\theta,\varphi) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\frac{\phi_n^m}{h_n^{(1)}(\kappa_{\rm p}R)}\bigg[\frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r}h_n^{(1)}(\kappa_{\rm p}r) + \frac{\mathrm{d}^2}{\mathrm{d}r^2}h_n^{(1)}(\kappa_{\rm p}r) - \frac{n(n+1)}{r^2}h_n^{(1)}(\kappa_{\rm p}r)\bigg]X_n^m, \tag{15}$$

where Δ_{Γ_R} is the Laplace–Beltrami operator on Γ_R.

Combining (12) and (14)–(15), we obtain

$$\begin{aligned}\mathscr{B}\boldsymbol{v} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}&\frac{\mu}{R^2}\Big[\sqrt{n(n+1)}(z_n(\kappa_{\rm p}R)-1)\phi_n^m - \big(1+z_n(\kappa_{\rm s}R)+(R\kappa_{\rm s})^2\\ &- n(n+1)\big)\psi_{2n}^m\Big]\boldsymbol{T}_n^m + \frac{\mu\kappa_{\rm s}^2}{\sqrt{n(n+1)}}z_n(\kappa_{\rm s}R)\psi_{3n}^m\boldsymbol{V}_n^m + \frac{1}{R^2}\Big[\mu\big(n(n+1)-(R\kappa_{\rm p})^2\\ &- 2z_n(\kappa_{\rm p}R)\big)\phi_n^m + \mu\sqrt{n(n+1)}(z_n(\kappa_{\rm s}R)-1)\psi_{2n}^m - (\lambda+\mu)(\kappa_{\rm p}R)^2\phi_n^m\Big]\boldsymbol{W}_n^m.\end{aligned} \tag{16}$$

Comparing (13) with (16), we have

$$(w_{1n}^m, w_{2n}^m, w_{3n}^m)^\top = \frac{1}{R^2}G_n(\phi_n^m, \psi_{2n}^m, \psi_{3n}^m)^\top, \tag{17}$$

where the matrix

$$G_n = \begin{bmatrix} 0 & 0 & G_{13}^{(n)} \\ G_{21}^{(n)} & G_{22}^{(n)} & 0 \\ G_{31}^{(n)} & G_{32}^{(n)} & 0 \end{bmatrix}.$$

Here

$$\begin{aligned} G_{13}^{(n)} &= \frac{\mu(\kappa_{\rm s}R)^2 z_n(\kappa_{\rm s}R)}{\sqrt{n(n+1)}}, \quad G_{21}^{(n)} = \mu\sqrt{n(n+1)}(z_n(\kappa_{\rm p}R)-1),\\ G_{22}^{(n)} &= \mu\left(n(n+1)-(\kappa_{\rm s}R)^2-1-z_n(\kappa_{\rm s}R)\right),\\ G_{31}^{(n)} &= \mu\left(n(n+1)-(\kappa_{\rm p}R)^2-2z_n(\kappa_{\rm p}R)\right)-(\lambda+\mu)(\kappa_{\rm p}R)^2,\\ G_{32}^{(n)} &= \mu\sqrt{n(n+1)}(z_n(\kappa_{\rm s}R)-1).\end{aligned}$$

Let $\boldsymbol{v}_n^m = (v_{1n}^m, v_{2n}^m, v_{3n}^m)^\top$, $\quad M_n\boldsymbol{v}_n^m = \boldsymbol{b}_n^m = (b_{1n}^m, b_{2n}^m, b_{3n}^m)^\top$, where the matrix

$$M_n = \begin{bmatrix} M_{11}^{(n)} & 0 & 0 \\ 0 & M_{22}^{(n)} & M_{23}^{(n)} \\ 0 & M_{32}^{(n)} & M_{33}^{(n)} \end{bmatrix}.$$

Here

$$\begin{aligned} M_{11}^{(n)} &= \left(\frac{\mu}{R}\right)z_n(\kappa_{\rm s}R), \quad M_{22}^{(n)} = -\left(\frac{\mu}{R}\right)\left(1+\frac{(\kappa_{\rm s}R)^2 z_n(\kappa_{\rm p}R)}{\Lambda_n}\right),\\ M_{23}^{(n)} &= \sqrt{n(n+1)}\left(\frac{\mu}{R}\right)\left(1+\frac{(\kappa_{\rm s}R)^2}{\Lambda_n}\right),\end{aligned}$$

$$M_{32}^{(n)} = \sqrt{n(n+1)}\left(\frac{\mu}{R} + \frac{(\lambda+2\mu)}{R}\frac{(\kappa_{\rm p}R)^2}{\Lambda_n}\right),$$

$$M_{33}^{(n)} = -\frac{(\lambda+2\mu)}{R}\frac{(\kappa_{\rm p}R)^2}{\Lambda_n}(1+z_n(\kappa_{\rm s}R)) - 2\left(\frac{\mu}{R}\right),$$

where $\Lambda_n = z_n(\kappa_{\rm p}R)(1+z_n(\kappa_{\rm s}R)) - n(n+1)$.

Using the above notation and combining (17) and (64), we derive the transparent boundary condition:

$$\mathscr{B}\boldsymbol{v} = \mathscr{T}\boldsymbol{v} := \sum_{n=0}^{\infty}\sum_{m=-n}^{n} b_{1n}^m \boldsymbol{T}_n^m + b_{2n}^m \boldsymbol{V}_n^m + b_{3n}^m \boldsymbol{W}_n^m \quad \text{on } \Gamma_R. \tag{18}$$

Lemma 3.1. *The matrix $\hat{M}_n = -\frac{1}{2}(M_n + M_n^*)$ is positive definite for sufficiently large n.*

Proof. Using the asymptotic expansions of the spherical Bessel functions [30], we may verify that

$$z_n(t) = -(n+1) + \frac{1}{16n}t^4 + \frac{1}{2n}t^2 + O\left(\frac{1}{n^2}\right),$$

$$\Lambda_n(t) = -\frac{1}{16}(\kappa_{\rm p}t)^4 - \frac{1}{16}(\kappa_{\rm s}t)^4 - \frac{1}{2}(\kappa_{\rm p}t)^2 - \frac{1}{2}(\kappa_{\rm s}t)^2 + O\left(\frac{1}{n}\right).$$

It follows from straightforward calculations that

$$\hat{M}_n = \begin{bmatrix} \hat{M}_{11}^{(n)} & 0 & 0 \\ 0 & \hat{M}_{22}^{(n)} & \hat{M}_{23}^{(n)} \\ 0 & \hat{M}_{32}^{(n)} & \hat{M}_{33}^{(n)} \end{bmatrix},$$

where

$$\hat{M}_{11}^{(n)} = \left(\frac{\mu}{R}\right)(n+1) + O\left(\frac{1}{n}\right), \quad \hat{M}_{22}^{(n)} = -\left(\frac{\omega^2 R}{\Lambda_n}\right)(n+1) + O(1),$$

$$\hat{M}_{23}^{(n)} = -\left(\frac{\mu}{R} + \frac{\omega^2 R}{\Lambda_n}\right)\sqrt{n(n+1)} + O(1),$$

$$\hat{M}_{32}^{(n)} = -\left(\frac{\mu}{R} + \frac{\omega^2 R}{\Lambda_n}\right)\sqrt{n(n+1)} + O(1),$$

$$\hat{M}_{33}^{(n)} = \frac{2\mu}{R} + \frac{\omega^2 R}{\Lambda_n}(1+z_n(\kappa_{\rm s}R)) = -\left(\frac{\omega^2 R}{\Lambda_n}\right)n + O(1).$$

For sufficiently large n, we have

$$\hat{M}_{11}^{(n)} > 0 \quad \text{and} \quad \hat{M}_{22}^{(n)} > 0,$$

which gives

$$\det[(\hat{M}_n)_{(1:2,1:2)}] = \hat{M}_{11}^{(n)}\hat{M}_{22}^{(n)} > 0.$$

Since $\Lambda_n < 0$ for sufficiently large n, we have

$$\hat{M}_{22}^{(n)}\hat{M}_{33}^{(n)} - \left(\hat{M}_{23}^{(n)}\right)^2 = n(n+1)\left[\left(\frac{\omega^2 R}{\Lambda_n}\right)^2 - \left(\frac{\mu}{R} + \frac{\omega^2 R}{\Lambda_n}\right)^2\right] + O(n) > 0.$$

A simple calculation yields

$$\det[\hat{M}_n] = \hat{M}_{11}^{(n)}\left(\hat{M}_{22}^{(n)}\hat{M}_{33}^{(n)} - \left(\hat{M}_{23}^{(n)}\right)^2\right) > 0,$$

which completes the proof by applying Sylvester's criterion. $\square$

Lemma 3.2. *The boundary operator $\mathscr{T} : \boldsymbol{H}^{1/2}(\Gamma_R) \to \boldsymbol{H}^{-1/2}(\Gamma_R)$ is continuous, i.e.,*

$$\|\mathscr{T}\boldsymbol{u}\|_{\boldsymbol{H}^{-1/2}(\Gamma_R)} \lesssim \|\boldsymbol{u}\|_{\boldsymbol{H}^{1/2}(\Gamma_R)}, \quad \forall\, \boldsymbol{u} \in \boldsymbol{H}^{1/2}(\Gamma_R).$$

Proof. For any given $\boldsymbol{u} \in \boldsymbol{H}^{1/2}(\Gamma_R)$, it has the Fourier expansion

$$\boldsymbol{u}(R,\theta,\varphi) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} u_{1n}^m \boldsymbol{T}_n^m(\theta,\varphi) + u_{2n}^m \boldsymbol{V}_n^m(\theta,\varphi) + u_{3n}^m \boldsymbol{W}_n^m(\theta,\varphi).$$

Let $\boldsymbol{u}_n^m = (u_{1n}^m, u_{2n}^m, u_{3n}^m)^\top$. It is easy to verify from the definition of M_n and the asymptotic expansion of $z_n(t)$ that

$$|M_{i,j}^{(n)}| \lesssim (1+n(n+1))^{1/2}.$$

Hence we have

$$\begin{aligned}
\|\mathscr{T}\boldsymbol{u}\|^2_{\boldsymbol{H}^{1/2}(\Gamma_R)} &= \sum_{n=0}^{\infty}\sum_{m=-n}^{n} (1+n(n+1))^{-1/2}\, |M_n \boldsymbol{u}_n^m|^2 \\
&\lesssim \sum_{n=0}^{\infty}\sum_{m=-n}^{n} (1+n(n+1))^{1/2}\, |\boldsymbol{u}_n^m|^2 = \|\boldsymbol{u}\|^2_{\boldsymbol{H}^{1/2}(\Gamma_R)},
\end{aligned}$$

which completes the proof. $\square$

3.2. **Uniqueness.** It follows from the Dirichlet boundary condition (5) and the Helmholtz decomposition (7) that

$$\boldsymbol{v} = \nabla\phi + \nabla\times\boldsymbol{\psi} = -\boldsymbol{u}^{\rm inc} \quad \text{on } \partial D. \tag{19}$$

Taking the dot product and the cross product of (19) with the unit normal vector $\boldsymbol{\nu}$ on ∂D, respectively, we get

$$\partial_{\boldsymbol{\nu}}\phi + (\nabla\times\boldsymbol{\psi})\cdot\boldsymbol{\nu} = -u_1, \quad (\nabla\times\boldsymbol{\psi})\times\boldsymbol{\nu} + \nabla\phi\times\boldsymbol{\nu} = -\boldsymbol{u}_2,$$

where

$$u_1 = \boldsymbol{u}^{\rm inc}\cdot\boldsymbol{\nu}, \quad \boldsymbol{u}_2 = \boldsymbol{u}^{\rm inc}\times\boldsymbol{\nu}.$$

We obtain a coupled boundary value problem for the potential functions ϕ and $\boldsymbol{\psi}$:

$$\begin{cases}
\Delta\phi + \kappa_{\rm p}^2\phi = 0, \quad \nabla\times(\nabla\times\boldsymbol{\psi}) - \kappa_{\rm s}^2\boldsymbol{\psi} = 0, & \text{in } \Omega,\\
\partial_{\boldsymbol{\nu}}\phi + (\nabla\times\boldsymbol{\psi})\cdot\boldsymbol{\nu} = -u_1, \quad (\nabla\times\boldsymbol{\psi})\times\boldsymbol{\nu} + \nabla\phi\times\boldsymbol{\nu} = -\boldsymbol{u}_2 & \text{on } \partial D,\\
\partial_r\phi - \mathscr{T}_1\phi = 0, \quad (\nabla\times\boldsymbol{\psi})\times\boldsymbol{e}_r - {\rm i}\kappa_{\rm s}\mathscr{T}_2\boldsymbol{\psi}_{\Gamma_R} = 0 & \text{on } \Gamma_R.
\end{cases} \tag{20}$$

where $\mathscr{T}_1$ and $\mathscr{T}_2$ are the transparent boundary operators given in (46) and (54), respectively.

Multiplying test functions $(p,\boldsymbol{q}) \in H^1(\Omega)\times\boldsymbol{H}({\rm curl},\Omega)$, we arrive at the weak formulation of (20): To find $(\phi,\boldsymbol{\psi}) \in H^1(\Omega)\times\boldsymbol{H}({\rm curl},\Omega)$ such that

$$a(\phi,\boldsymbol{\psi};p,\boldsymbol{q}) = \langle u_1, p\rangle_{\partial D} + \langle \boldsymbol{u}_2, \boldsymbol{q}\rangle_{\partial D}, \quad \forall\,(p,\boldsymbol{q}) \in H^1(\Omega)\times\boldsymbol{H}({\rm curl},\Omega), \tag{21}$$

where the sesquilinear form

$$\begin{aligned}
a(\phi,\boldsymbol{\psi};p,\boldsymbol{q}) &= (\nabla\phi,\nabla p) + (\nabla\times\boldsymbol{\psi},\nabla\times\boldsymbol{q}) - \kappa_{\rm p}^2(\phi,p) - \kappa_{\rm s}^2(\boldsymbol{\psi},\boldsymbol{q})\\
&\quad - \langle(\nabla\times\boldsymbol{\psi})\cdot\boldsymbol{\nu}, p\rangle_{\partial D} - \langle\nabla\phi\times\boldsymbol{\nu},\boldsymbol{q}\rangle_{\partial D} - \langle\mathscr{T}_1\phi, p\rangle_{\Gamma_R} - {\rm i}\kappa_{\rm s}\langle\mathscr{T}_2\boldsymbol{\psi}_{\Gamma_R}, \boldsymbol{q}_{\Gamma_R}\rangle_{\Gamma_R}.
\end{aligned}$$

Theorem 3.3. *The variational problem (21) has at most one solution.*

Proof. It suffices to show that $\phi = 0, \boldsymbol{\psi} = 0$ in Ω if $u_1 = 0, \boldsymbol{u_2} = 0$ on ∂D. If $(\phi, \boldsymbol{\psi})$ satisfy the homogeneous variational problem (21), then we have

$$(\nabla\phi, \nabla\phi) + (\nabla \times \boldsymbol{\psi}, \nabla \times \boldsymbol{\psi}) - \kappa_{\rm p}^2(\phi, \phi) - \kappa_{\rm s}^2(\boldsymbol{\psi}, \boldsymbol{\psi}) - \langle(\nabla \times \boldsymbol{\psi}) \cdot \boldsymbol{\nu}, \phi\rangle_{\partial D}$$
$$-\langle\nabla\phi \times \boldsymbol{\nu}, \boldsymbol{\psi}\rangle_{\partial D} - \langle\mathscr{T}_1\phi, \phi\rangle_{\Gamma_R} - {\rm i}\kappa_{\rm s}\langle\mathscr{T}_2\boldsymbol{\psi}_{\Gamma_R}, \boldsymbol{\psi}_{\Gamma_R}\rangle_{\Gamma_R} = 0. \tag{22}$$

Using the integration by parts, we may verify that

$$\langle(\nabla \times \boldsymbol{\psi}) \cdot \boldsymbol{\nu}, \phi\rangle_{\partial D} = -\langle\boldsymbol{\psi}, \boldsymbol{\nu} \times \nabla\phi\rangle_{\partial D} = \langle\boldsymbol{\psi}, \nabla\phi \times \boldsymbol{\nu}\rangle_{\partial D},$$

which gives

$$\langle(\nabla \times \boldsymbol{\psi}) \cdot \boldsymbol{\nu}, \phi\rangle_{\partial D} + \langle\nabla\phi \times \boldsymbol{\nu}, \boldsymbol{\psi}\rangle_{\partial D} = 2{\rm Re}\langle\nabla\phi \times \boldsymbol{\nu}, \boldsymbol{\psi}\rangle_{\partial D}. \tag{23}$$

Taking the imaginary part of (22) and using (23), we obtain

$${\rm Im}\langle\mathscr{T}_1\phi, \phi\rangle_{\Gamma_R} + \kappa_{\rm s}{\rm Re}\langle\mathscr{T}_2\boldsymbol{\psi}_{\Gamma_R}, \boldsymbol{\psi}_{\Gamma_R}\rangle_{\Gamma_R} = 0,$$

which gives $\phi = 0, \boldsymbol{\psi} = 0$ on Γ_R, due to Lemma C.1 and Lemma C.2. Using (46) and (54), we have $\partial_r\phi = 0, (\nabla \times \boldsymbol{\psi}) \times \boldsymbol{e}_r = 0$ on Γ_R. By the Holmgren uniqueness theorem, we have $\phi = 0, \boldsymbol{\psi} = 0$ in $\mathbb{R}^3 \setminus \bar{B}$. A unique continuation result concludes that $\phi = 0, \boldsymbol{\psi} = 0$ in Ω. □

3.3. **Well-posedness.** Using the transparent boundary condition (18), we obtain a boundary value problem for $\boldsymbol{u}$:

$$\begin{cases} \mu\Delta\boldsymbol{u} + (\lambda + \mu)\nabla\nabla \cdot \boldsymbol{u} + \omega^2\boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} = 0 & \text{on } \partial D, \\ \mathscr{B}\boldsymbol{u} = \mathscr{T}\boldsymbol{u} + \boldsymbol{g} & \text{on } \Gamma_R, \end{cases} \tag{24}$$

where $\boldsymbol{g} = (\mathscr{B} - \mathscr{T})\boldsymbol{u}^{\rm inc}$. The variational problem of (24) is to find $\boldsymbol{u} \in \boldsymbol{H}^1_{\partial D}(\Omega)$ such that

$$b(\boldsymbol{u}, \boldsymbol{v}) = \langle\boldsymbol{g}, \boldsymbol{v}\rangle_{\Gamma_R}, \quad \forall\, \boldsymbol{v} \in \boldsymbol{H}^1_{\partial D}(\Omega), \tag{25}$$

where the sesquilinear form $b : \boldsymbol{H}^1_{\partial D}(\Omega) \times \boldsymbol{H}^1_{\partial D}(\Omega) \to \mathbb{C}$ is defined by

$$b(\boldsymbol{u}, \boldsymbol{v}) = \mu \int_\Omega \nabla\boldsymbol{u} : \nabla\bar{\boldsymbol{v}}\,{\rm d}\boldsymbol{x} + (\lambda + \mu)\int_\Omega (\nabla \cdot \boldsymbol{u})(\nabla \cdot \bar{\boldsymbol{v}})\,{\rm d}\boldsymbol{x}$$
$$-\omega^2\int_\Omega \boldsymbol{u} \cdot \bar{\boldsymbol{v}}\,{\rm d}\boldsymbol{x} - \langle\mathscr{T}\boldsymbol{u}, \boldsymbol{v}\rangle_{\Gamma_R}.$$

Here $A : B = {\rm tr}(AB^\top)$ is the Frobenius inner product of square matrices A and B.

The following result follows from the standard trace theorem of the Sobolev spaces. The proof is omitted for brevity.

Lemma 3.4. *It holds the estimate*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^{1/2}(\Gamma_R)} \lesssim \|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}, \quad \forall\, \boldsymbol{u} \in \boldsymbol{H}^1_{\partial D}(\Omega).$$

Lemma 3.5. *For any $\varepsilon > 0$, there exists a positive constant $C(\varepsilon)$ such that*

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Gamma_R)} \leq \varepsilon\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)} + C(\varepsilon)\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}, \quad \forall\, \boldsymbol{u} \in \boldsymbol{H}^1_{\partial D}(\Omega).$$

Proof. Let B' be the ball with radius $R' > 0$ such that $\bar{B}' \subset D$. Denote $\tilde{\Omega} = B \setminus \bar{B}'$. Given $\boldsymbol{u} \in \boldsymbol{H}^1_{\partial D}(\Omega)$, let $\tilde{\boldsymbol{u}}$ be the zero extension of $\boldsymbol{u}$ from Ω to $\tilde{\Omega}$, i.e.,

$$\tilde{\boldsymbol{u}}(\boldsymbol{x}) = \begin{cases} \boldsymbol{u}(\boldsymbol{x}), & \boldsymbol{x} \in \Omega, \\ 0, & \boldsymbol{x} \in \tilde{\Omega} \setminus \bar{\Omega}. \end{cases}$$

The extension of $\tilde{\boldsymbol{u}}$ has the Fourier expansion

$$\tilde{\boldsymbol{u}}(r,\theta,\varphi)=\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\tilde{u}_{1n}^{m}(r)\boldsymbol{T}_{n}^{m}(\theta,\varphi)+\tilde{u}_{2n}^{m}(r)\boldsymbol{V}_{n}^{m}(\theta,\varphi)+\tilde{u}_{3n}^{m}(r)\boldsymbol{W}_{n}^{m}(\theta,\varphi).$$

A simple calculation yields

$$\|\tilde{\boldsymbol{u}}\|_{\boldsymbol{L}^2(\Gamma_R)}^2=\sum_{n=0}^{\infty}\sum_{m=-n}^{n}|\tilde{u}_{1n}^{m}(R)|^2+|\tilde{u}_{2n}^{m}(R)|^2+|\tilde{u}_{3n}^{m}(R)|^2.$$

Since $\tilde{\boldsymbol{u}}(R',\theta,\varphi)=0$, we have $\tilde{u}_{jn}^{m}(R')=0$. For any given $\varepsilon>0$, it follows from Young's inequality that

$$\begin{aligned}|\tilde{u}_{jn}^{m}(R)|^2&=\int_{R'}^{R}\frac{\mathrm{d}}{\mathrm{d}r}|\tilde{u}_{jn}^{m}(r)|^2\mathrm{d}r\le\int_{R'}^{R}2|\tilde{u}_{jn}^{m}(r)|\left|\frac{\mathrm{d}}{\mathrm{d}r}\tilde{u}_{jn}^{m}(r)\right|\mathrm{d}r\\&\le(R'\varepsilon)^{-2}\int_{R'}^{R}|\tilde{u}_{jn}^{m}(r)|^2\mathrm{d}r+(R'\varepsilon)^2\int_{R'}^{R}\left|\frac{\mathrm{d}}{\mathrm{d}r}\tilde{u}_{jn}^{m}(r)\right|^2\mathrm{d}r,\end{aligned}$$

which gives

$$|\tilde{u}_{jn}^{m}(R)|^2\le C(\varepsilon)\int_{R'}^{R}|\tilde{u}_{jn}^{m}(r)|^2r^2\mathrm{d}r+\varepsilon^2\int_{R'}^{R}\left|\frac{\mathrm{d}}{\mathrm{d}r}\tilde{u}_{jn}^{m}(r)\right|^2r^2\mathrm{d}r.$$

The proof is completed by noting that

$$\|\tilde{\boldsymbol{u}}\|_{\boldsymbol{L}^2(\Gamma_R)}=\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Gamma_R)},\quad\|\tilde{\boldsymbol{u}}\|_{\boldsymbol{L}^2(\tilde{\Omega})}=\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)},\quad\|\tilde{\boldsymbol{u}}\|_{\boldsymbol{H}^1(\tilde{\Omega})}=\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}.$$

□

Lemma 3.6. *It holds the estimate*

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}\lesssim\|\nabla\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)},\quad\forall\,\boldsymbol{u}\in\boldsymbol{H}_{\partial D}^1(\Omega).$$

Proof. As is defined in the proof of Lemma 3.5, let $\tilde{\boldsymbol{u}}$ be the zero extension of $\boldsymbol{u}$ from Ω to $\tilde{\Omega}$. It follows from the Cauchy–Schwarz inequality that

$$|\tilde{\boldsymbol{u}}(r,\theta,\varphi)|^2=\left|\int_{R'}^{r}\partial_r\tilde{\boldsymbol{u}}(r,\theta,\varphi)\mathrm{d}r\right|^2\lesssim\int_{R'}^{R}|\partial_r\tilde{\boldsymbol{u}}(r,\theta,\varphi)|^2\,\mathrm{d}r.$$

Hence we have

$$\begin{aligned}\|\tilde{\boldsymbol{u}}\|_{\boldsymbol{L}^2(\tilde{\Omega})}^2&=\int_{R'}^{R}\int_0^{2\pi}\int_0^{\pi}|\tilde{\boldsymbol{u}}(r,\theta,\varphi)|^2r^2\mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi\\&\lesssim\int_{R'}^{R}\int_0^{2\pi}\int_0^{\pi}\int_{R'}^{R}|\partial_r\tilde{\boldsymbol{u}}(r,\theta,\varphi)|^2\mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi\mathrm{d}r\\&\lesssim\int_{R'}^{R}\int_0^{2\pi}\int_0^{\pi}|\partial_r\tilde{\boldsymbol{u}}(r,\theta,\varphi)|^2\mathrm{d}r\mathrm{d}\theta\mathrm{d}\varphi\lesssim\|\nabla\tilde{\boldsymbol{u}}\|_{\boldsymbol{L}^2(\tilde{\Omega})}^2.\end{aligned}$$

The proof is completed by noting that

$$\begin{aligned}&\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}=\|\tilde{\boldsymbol{u}}\|_{\boldsymbol{L}^2(\tilde{\Omega})},\quad\|\nabla\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}=\|\nabla\tilde{\boldsymbol{u}}\|_{\boldsymbol{L}^2(\tilde{\Omega})},\\&\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}^2=\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}^2+\|\nabla\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}^2.\end{aligned}$$

□

Theorem 3.7. *The variational problem* (25) *admits a unique weak solution* $\boldsymbol{u}\in\boldsymbol{H}_{\partial D}^1(\Omega)$.

Proof. Using the Cauchy–Schwarz inequality, Lemma 3.2, and Lemma 3.4, we have

$$\begin{aligned}|b(\boldsymbol{u},\boldsymbol{v})| \leq&\mu\|\nabla\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}\|\nabla\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)}+(\lambda+\mu)\|\nabla\cdot\boldsymbol{u}\|_{0,\Omega}\|\nabla\cdot\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)}\\ &+\omega^2\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}\|\boldsymbol{v}\|_{\boldsymbol{L}^2(\Omega)}+\|\mathscr{T}\boldsymbol{u}\|_{\boldsymbol{H}^{-1/2}(\Gamma_R)}\|\boldsymbol{v}\|_{\boldsymbol{H}^{1/2}(\Gamma_R)}\\ \lesssim&\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}\|\boldsymbol{v}\|_{\boldsymbol{H}^1(\Omega)},\end{aligned}$$

which shows that the sesquilinear form $b(\cdot,\cdot)$ is bounded.

It follows from Lemma 3.1 that there exists an $N_0\in\mathbb{N}$ such that $\hat{M}_n$ is positive definite for $n>N_0$. The sesquilinear form b can be written as

$$\begin{aligned}b(\boldsymbol{u},\boldsymbol{v}) =&\mu\int_\Omega(\nabla\boldsymbol{u}:\nabla\bar{\boldsymbol{v}})\,\mathrm{d}\boldsymbol{x}+(\lambda+\mu)\int_\Omega(\nabla\cdot\boldsymbol{u})(\nabla\cdot\bar{\boldsymbol{v}})\,\mathrm{d}\boldsymbol{x}-\omega^2\int_\Omega\boldsymbol{u}\cdot\bar{\boldsymbol{v}}\,\mathrm{d}\boldsymbol{x}\\ &-\sum_{|n|>N_0}\sum_{m=-n}^{n}\langle M_n\boldsymbol{u}_n^m,\boldsymbol{v}_n^m\rangle-\sum_{|n|\leq N_0}\sum_{m=-n}^{n}\langle M_n\boldsymbol{u}_n^m,\boldsymbol{v}_n^m\rangle.\end{aligned}$$

Taking the real part of b, and using Lemma 3.1, Lemma 3.6, Lemma 3.5, we obtain

$$\begin{aligned}\operatorname{Re} b(\boldsymbol{u},\boldsymbol{u}) =& \mu\|\nabla\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)}+(\lambda+\mu)\|\nabla\cdot\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)}+\sum_{|n|>N_0}\sum_{m=-n}^{n}\langle\hat{M}_n\boldsymbol{u}_n^m,\boldsymbol{u}_n^m\rangle\\ &-\omega^2\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}+\sum_{|n|\leq N_0}\sum_{m=-n}^{n}\langle\hat{M}_n\boldsymbol{u}_n^m,\boldsymbol{u}_n^m\rangle\\ \geq& C_1\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}-\omega^2\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}-C_2\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Gamma_R)}\\ \geq& C_1\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}-\omega^2\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}-C_2\varepsilon\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}-C(\varepsilon)\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}\\ =& (C_1-C_2\varepsilon)\|\boldsymbol{u}\|_{\boldsymbol{H}^1(\Omega)}-C_3\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}.\end{aligned}$$

Letting $\varepsilon>0$ to be sufficiently small, we have $C_1-C_2\varepsilon>0$ and thus Gårding's inequality. Since the injection of $\boldsymbol{H}^1_{\partial D}(\Omega)$ into $\boldsymbol{L}^2(\Omega)$ is compact, the proof is completed by using the Fredholm alternative (cf. [28, Theorem 5.4.5]) and the uniqueness result in Theorem 3.3. □

4. **Inverse scattering.** In this section, we study a domain derivative of the scattering problem and present a continuation method to reconstruct the surface.

4.1. **Domain derivative.** We assume that the obstacle has a C^2 boundary, i.e., $\partial D\in C^2$. Given a sufficiently small number $h>0$, define a perturbed domain Ω_h which is surrounded by ∂D_h and Γ_R, where

$$\partial D_h=\{\boldsymbol{x}+h\boldsymbol{p}(\boldsymbol{x}):\boldsymbol{x}\in\partial D\}.$$

Here the function $\boldsymbol{p}\in\boldsymbol{C}^2(\partial D)$.

Consider the variational formulation for the direct problem in the perturbed domain Ω_h: To find $\boldsymbol{u}_h\in\boldsymbol{H}^1_{\partial D_h}(\Omega_h)$ such that

$$b^h(\boldsymbol{u}_h,\boldsymbol{v}_h)=\langle\boldsymbol{g},\boldsymbol{v}_h\rangle_{\Gamma_R},\quad\forall\,\boldsymbol{v}_h\in\boldsymbol{H}^1_{\partial D_h}(\Omega_h),\tag{26}$$

where the sesquilinear form $b^h:\boldsymbol{H}^1_{\partial D_h}(\Omega_h)\times\boldsymbol{H}^1_{\partial D_h}(\Omega_h)\to\mathbb{C}$ is defined by

$$\begin{aligned}b^h(\boldsymbol{u}_h,\boldsymbol{v}_h)=\mu\int_{\Omega_h}\nabla\boldsymbol{u}_h:\nabla\bar{\boldsymbol{v}}_h\,\mathrm{d}\boldsymbol{y}+(\lambda+\mu)\int_{\Omega_h}(\nabla\cdot\boldsymbol{u}_h)(\nabla\cdot\bar{\boldsymbol{v}}_h)\,\mathrm{d}\boldsymbol{y}\\ -\omega^2\int_{\Omega_h}\boldsymbol{u}_h\cdot\bar{\boldsymbol{v}}_h\,\mathrm{d}\boldsymbol{y}-\langle\mathscr{T}\boldsymbol{u}_h,\boldsymbol{v}_h\rangle\Gamma_R.\end{aligned}\tag{27}$$

Similarly, we may follow the proof of Theorem 3.7 to show that the variational problem (26) has a unique weak solution $\boldsymbol{u}_h \in \boldsymbol{H}^1_{\partial D_h}(\Omega_h)$ for any $h > 0$.

Since the variational problem (3.7) is well-posed, we introduce a nonlinear scattering operator:

$$\mathscr{S} : \partial D_h \to \boldsymbol{u}_h|_{\Gamma_R},$$

which maps the obstacle's surface to the displacement of the wave field on Γ_R. Let $\boldsymbol{u}_h$ and $\boldsymbol{u}$ be the solution of the direct problem in the domain Ω_h and Ω, respectively. Define the domain derivative of the scattering operator $\mathscr{S}$ on ∂D along the direction $\boldsymbol{p}$ as

$$\mathscr{S}'(\partial D; \boldsymbol{p}) := \lim_{h\to 0} \frac{\mathscr{S}(\partial D_h) - \mathscr{S}(\partial D)}{h} = \lim_{h\to=0} \frac{\boldsymbol{u}_h|_{\Gamma_R} - \boldsymbol{u}|_{\Gamma_R}}{h}.$$

For a given $\boldsymbol{p} \in \boldsymbol{C}^2(\partial D)$, we extend its domain to $\bar{\Omega}$ by requiring that $\boldsymbol{p} \in \boldsymbol{C}^2(\Omega) \cap \boldsymbol{C}(\bar{\Omega}), \boldsymbol{p} = 0$ on Γ_R, and $\boldsymbol{y} = \boldsymbol{\xi}^h(\boldsymbol{x}) = \boldsymbol{x} + h\boldsymbol{p}(\boldsymbol{x})$ maps Ω to Ω_h. It is clear to note that $\boldsymbol{\xi}^h$ is a diffeomorphism from Ω to Ω_h for sufficiently small h. Denote by $\boldsymbol{\eta}^h(\boldsymbol{y}) : \Omega_h \to \Omega$ the inverse map of $\boldsymbol{\xi}^h$.

Define $\breve{\boldsymbol{u}}(\boldsymbol{x}) = (\breve{u}_1, \breve{u}_2, \breve{u}_3) := (\boldsymbol{u}_h \circ \boldsymbol{\xi}^h)(\boldsymbol{x})$. Using the change of variable $\boldsymbol{y} = \xi^h(\boldsymbol{x})$, we have from straightforward calculations that

$$\int_{\Omega_h} (\nabla \boldsymbol{u}_h : \nabla \bar{\boldsymbol{v}}_h)\, \mathrm{d}\boldsymbol{y} = \sum_{j=1}^{3} \int_{\Omega} \nabla \breve{u}_j J_{\boldsymbol{\eta}^h} J_{\boldsymbol{\eta}^h}^{\top} \nabla \bar{\breve{v}}_j \det(J_{\boldsymbol{\xi}^h})\, \mathrm{d}\boldsymbol{x},$$

$$\int_{\Omega_h} (\nabla \cdot \boldsymbol{u}_h)(\nabla \cdot \bar{\boldsymbol{v}}_h)\, \mathrm{d}\boldsymbol{y} = \int_{\Omega} (\nabla \breve{\boldsymbol{u}} : J_{\boldsymbol{\eta}^h}^{\top})(\nabla \bar{\breve{\boldsymbol{v}}} : J_{\boldsymbol{\eta}^h}^{\top}) \det(J_{\boldsymbol{\xi}^h})\, \mathrm{d}\boldsymbol{x},$$

$$\int_{\Omega_h} \boldsymbol{u}_h \cdot \bar{\boldsymbol{v}}_h\, \mathrm{d}\boldsymbol{y} = \int_{\Omega} \breve{\boldsymbol{u}} \cdot \bar{\breve{\boldsymbol{v}}} \det(J_{\boldsymbol{\xi}^h})\, \mathrm{d}\boldsymbol{x},$$

where $\breve{\boldsymbol{v}}(\boldsymbol{x}) = (\breve{v}_1, \breve{v}_2, \breve{v}_3) := (\boldsymbol{v}_h \circ \boldsymbol{\xi}^h)(\boldsymbol{x})$, $J_{\boldsymbol{\eta}^h}$ and $J_{\boldsymbol{\xi}^h}$ are the Jacobian matrices of the transforms $\boldsymbol{\eta}^h$ and $\boldsymbol{\xi}^h$, respectively.

For a test function $\boldsymbol{v}_h$ in the domain Ω_h, it follows from the transform that $\breve{\boldsymbol{v}}$ is a test function in the domain Ω. Therefore, the sesquilinear form b^h in (27) becomes

$$b^h(\breve{\boldsymbol{u}}, \boldsymbol{v}) = \sum_{j=1}^{3} \mu \int_{\Omega} \nabla \breve{u}_j J_{\boldsymbol{\eta}^h} J_{\boldsymbol{\eta}^h}^{\top} \nabla \bar{v}_j \det(J_{\boldsymbol{\xi}^h})\, \mathrm{d}\boldsymbol{x} + (\lambda + \mu) \int_{\Omega} (\nabla \breve{\boldsymbol{u}} : J_{\boldsymbol{\eta}^h}^{\top})(\nabla \bar{\boldsymbol{v}} : J_{\boldsymbol{\eta}^h}^{\top})$$

$$\times \det(J_{\boldsymbol{\xi}^h})\, \mathrm{d}\boldsymbol{x} - \omega^2 \int_{\Omega} \breve{\boldsymbol{u}} \cdot \bar{\boldsymbol{v}} \det(J_{\boldsymbol{\xi}^h})\, \mathrm{d}\boldsymbol{x} - \langle \mathscr{T} \breve{\boldsymbol{u}}, \boldsymbol{v} \rangle_{\Gamma_R},$$

which gives an equivalent variational formulation of (26):

$$b^h(\breve{\boldsymbol{u}}, \boldsymbol{v}) = \langle \boldsymbol{g}, \boldsymbol{v} \rangle_{\Gamma_R}, \quad \forall\, \boldsymbol{v} \in \boldsymbol{H}^1_{\partial D}(\Omega).$$

A simple calculation yields

$$b(\breve{\boldsymbol{u}} - \boldsymbol{u}, \boldsymbol{v}) = b(\breve{\boldsymbol{u}}, \boldsymbol{v}) - \langle \boldsymbol{g}, \boldsymbol{v} \rangle_{\Gamma_R} = b(\breve{\boldsymbol{u}}, \boldsymbol{v}) - b^h(\breve{\boldsymbol{u}}, \boldsymbol{v}) = b_1 + b_2 + b_3,$$

where

$$b_1 = \sum_{j=1}^{3} \mu \int_{\Omega} \nabla \breve{u}_j \left(I - J_{\boldsymbol{\eta}^h} J_{\boldsymbol{\eta}^h}^{\top} \det(J_{\boldsymbol{\xi}^h}) \right) \nabla \bar{v}_j\, \mathrm{d}\boldsymbol{x}, \tag{28}$$

$$b_2 = (\lambda + \mu) \int_{\Omega} (\nabla \cdot \breve{\boldsymbol{u}})(\nabla \cdot \bar{\boldsymbol{v}}) - (\nabla \breve{\boldsymbol{u}} : J_{\boldsymbol{\eta}^h}^{\top})(\nabla \bar{\boldsymbol{v}} : J_{\boldsymbol{\eta}^h}^{\top}) \det(J_{\boldsymbol{\xi}^h})\, \mathrm{d}\boldsymbol{x}, \tag{29}$$

$$b_3 = \omega^2 \int_{\Omega} \breve{\boldsymbol{u}} \cdot \bar{\boldsymbol{v}} \left(\det(J_{\boldsymbol{\xi}^h}) - 1 \right) \mathrm{d}\boldsymbol{x}. \tag{30}$$

Here I is the identity matrix. Following the definitions of the Jacobian matrices, we may easily verify that

$$\begin{aligned}\det(J_{\boldsymbol{\xi}^h}) &= 1 + h\nabla\cdot\boldsymbol{p} + O(h^2),\\ J_{\boldsymbol{\eta}^h} &= J_{\boldsymbol{\xi}^h}^{-1}\circ\boldsymbol{\eta}^h = I - hJ_{\boldsymbol{p}} + O(h^2),\\ J_{\boldsymbol{\eta}^h}J_{\boldsymbol{\eta}^h}^\top\det(J_{\boldsymbol{\xi}^h}) &= I - h(J_{\boldsymbol{p}} + J_{\boldsymbol{p}}^\top) + h(\nabla\cdot\boldsymbol{p})I + O(h^2),\end{aligned}$$

where the matrix $J_{\boldsymbol{p}} = \nabla\boldsymbol{p}$.

Substituting the above estimates into (28)–(30), we obtain

$$\begin{aligned}b_1 &= \sum_{j=1}^3 \mu\int_\Omega \nabla\breve{u}_j\left(h(J_{\boldsymbol{p}} + J_{\boldsymbol{p}}^\top) - h(\nabla\cdot\boldsymbol{p})I + O(h^2)\right)\nabla\bar{v}_j\,\mathrm{d}\boldsymbol{x},\\ b_2 &= (\lambda+\mu)\int_\Omega h(\nabla\cdot\breve{\boldsymbol{u}})(\nabla\bar{\boldsymbol{v}} : J_{\boldsymbol{p}}^\top) + h(\nabla\cdot\bar{\boldsymbol{v}})(\nabla\breve{\boldsymbol{u}} : J_{\boldsymbol{p}}^\top)\\ &\qquad\qquad - h(\nabla\cdot\boldsymbol{p})(\nabla\cdot\breve{\boldsymbol{u}})(\nabla\cdot\bar{\boldsymbol{v}}) + O(h^2)\,\mathrm{d}\boldsymbol{x},\\ b_3 &= \omega^2\int_\Omega \breve{\boldsymbol{u}}\cdot\bar{\boldsymbol{v}}\,\left(h\nabla\cdot\boldsymbol{p} + O(h^2)\right)\,\mathrm{d}\boldsymbol{x}.\end{aligned}$$

Hence we have

$$b\left(\frac{\breve{\boldsymbol{u}} - \boldsymbol{u}}{h}, \boldsymbol{v}\right) = g_1(\boldsymbol{p})(\breve{\boldsymbol{u}},\boldsymbol{v}) + g_2(\boldsymbol{p})(\breve{\boldsymbol{u}},\boldsymbol{v}) + g_3(\boldsymbol{p})(\breve{\boldsymbol{u}},\boldsymbol{v}) + O(h), \tag{31}$$

where

$$\begin{aligned}g_1 &= \sum_{j=1}^3 \mu\int_\Omega \nabla\breve{u}_j\left((J_{\boldsymbol{p}} + J_{\boldsymbol{p}}^\top) - (\nabla\cdot\boldsymbol{p})I\right)\nabla\bar{v}_j\,\mathrm{d}\boldsymbol{x},\\ g_2 &= (\lambda+\mu)\int_\Omega (\nabla\cdot\breve{\boldsymbol{u}})(\nabla\bar{\boldsymbol{v}} : J_{\boldsymbol{p}}^\top) + (\nabla\cdot\bar{\boldsymbol{v}})(\nabla\breve{\boldsymbol{u}} : J_{\boldsymbol{p}}^\top) - (\nabla\cdot\boldsymbol{p})(\nabla\cdot\breve{\boldsymbol{u}})(\nabla\cdot\bar{\boldsymbol{v}})\,\mathrm{d}\boldsymbol{x},\\ g_3 &= \omega^2\int_\Omega (\nabla\cdot\boldsymbol{p})\breve{\boldsymbol{u}}\cdot\bar{\boldsymbol{v}}\,\mathrm{d}\boldsymbol{x}.\end{aligned}$$

Theorem 4.1. *Given $\boldsymbol{p}\in\boldsymbol{C}^2(\partial D)$, the domain derivative of the scattering operator $\mathscr{S}$ is $\mathscr{S}'(\partial D;\boldsymbol{p}) = \boldsymbol{u}'|_{\Gamma_R}$, where $\boldsymbol{u}'$ is the unique weak solution of the boundary value problem:*

$$\begin{cases}\mu\Delta\boldsymbol{u}' + (\lambda+\mu)\nabla\nabla\cdot\boldsymbol{u}' + \omega^2\boldsymbol{u}' = 0 & \text{in } \Omega,\\ \boldsymbol{u}' = -(\boldsymbol{p}\cdot\boldsymbol{\nu})\partial_{\boldsymbol{\nu}}\boldsymbol{u} & \text{on } \partial D,\\ \mathscr{B}\boldsymbol{u}' = \mathscr{T}\boldsymbol{u}' & \text{on } \Gamma_R,\end{cases} \tag{32}$$

and $\boldsymbol{u}$ is the solution of the variational problem (25) corresponding to the domain Ω.

Proof. Given $\boldsymbol{p}\in\boldsymbol{C}^2(\partial D)$, we extend its definition to the domain $\bar{\Omega}$ as before. It follows from the well-posedness of the variational problem (25) that $\breve{\boldsymbol{u}}\to\boldsymbol{u}$ in $\boldsymbol{H}^1_{\partial D}(\Omega)$ as $h\to 0$. Taking the limit $h\to 0$ in (31) gives

$$b\left(\lim_{h\to 0}\frac{\breve{\boldsymbol{u}} - \boldsymbol{u}}{h}, \boldsymbol{v}\right) = g_1(\boldsymbol{p})(\boldsymbol{u},\boldsymbol{v}) + g_2(\boldsymbol{p})(\boldsymbol{u},\boldsymbol{v}) + g_3(\boldsymbol{p})(\boldsymbol{u},\boldsymbol{v}), \tag{33}$$

which shows that $(\breve{\boldsymbol{u}} - \boldsymbol{u})/h$ is convergent in $\boldsymbol{H}^1_{\partial D}(\Omega)$ as $h\to 0$. Denote the limit by $\dot{\boldsymbol{u}}$ and rewrite (33) as

$$b(\dot{\boldsymbol{u}},\boldsymbol{v}) = g_1(\boldsymbol{p})(\boldsymbol{u},\boldsymbol{v}) + g_2(\boldsymbol{p})(\boldsymbol{u},\boldsymbol{v}) + g_3(\boldsymbol{p})(\boldsymbol{u},\boldsymbol{v}). \tag{34}$$

First we compute $g_1(\boldsymbol{p})(\boldsymbol{u},\boldsymbol{v})$. Noting $\boldsymbol{p}=0$ on ∂B and using the identity

$$\begin{aligned}\nabla u\left((J_{\boldsymbol{p}}+J_{\boldsymbol{p}}^\top)-(\nabla\cdot\boldsymbol{p})I\right)\nabla\bar v =&\nabla\cdot\left[(\boldsymbol{p}\cdot\nabla u)\nabla\bar v+(\boldsymbol{p}\cdot\nabla\bar v)\nabla u-(\nabla u\cdot\nabla\bar v)\boldsymbol{p}\right]\\ &-(\boldsymbol{p}\cdot\nabla u)\Delta\bar v-(\boldsymbol{p}\cdot\nabla\bar v)\Delta u,\end{aligned}$$

we obtain from the divergence theorem that

$$\begin{aligned}g_1(\boldsymbol{p})(\boldsymbol{u},\boldsymbol{v})=&-\sum_{j=1}^3\mu\int_{\partial D}(\boldsymbol{p}\cdot\nabla u_j)(\boldsymbol{\nu}\cdot\nabla\bar v_j)+(\boldsymbol{p}\cdot\nabla\bar v_j)(\boldsymbol{\nu}\cdot\nabla u_j)\,\mathrm{d}\gamma\\ &+\sum_{j=1}^3\mu\int_{\partial D}(\boldsymbol{p}\cdot\boldsymbol{\nu})(\nabla u_j\cdot\nabla\bar v_j)\,\mathrm{d}\gamma\\ &-\sum_{j=1}^3\mu\int_\Omega(\boldsymbol{p}\cdot\nabla u_j)\Delta\bar v_j+(\boldsymbol{p}\cdot\nabla\bar v_j)\Delta u_j\,\mathrm{d}\boldsymbol{x}\\ =&-\mu\int_{\partial D}(\boldsymbol{p}\cdot\nabla\boldsymbol{u})\cdot(\boldsymbol{\nu}\cdot\nabla\bar{\boldsymbol{v}})+(\boldsymbol{p}\cdot\nabla\bar{\boldsymbol{v}})\cdot(\boldsymbol{\nu}\cdot\nabla\boldsymbol{u})\\ &+\mu\int_{\partial D}(\boldsymbol{p}\cdot\boldsymbol{\nu})(\nabla\boldsymbol{u}:\nabla\bar{\boldsymbol{v}})\,\mathrm{d}\gamma\\ &-\mu\int_\Omega(\boldsymbol{p}\cdot\nabla\boldsymbol{u})\cdot\Delta\bar{\boldsymbol{v}}+(\boldsymbol{p}\cdot\nabla\bar{\boldsymbol{v}})\cdot\Delta\boldsymbol{u}\,\mathrm{d}\boldsymbol{x}.\end{aligned}$$

Noting

$$\mu\Delta\boldsymbol{u}+(\lambda+\mu)\nabla\nabla\cdot\boldsymbol{u}+\omega^2\boldsymbol{u}=0\quad\text{in }\Omega,$$

we have from the integration by parts that

$$\begin{aligned}&\mu\int_\Omega(\boldsymbol{p}\cdot\nabla\bar{\boldsymbol{v}})\cdot\Delta\boldsymbol{u}\,\mathrm{d}\boldsymbol{x}=-(\lambda+\mu)\int_\Omega(\boldsymbol{p}\cdot\nabla\bar{\boldsymbol{v}})\cdot(\nabla\nabla\cdot\boldsymbol{u})\,\mathrm{d}\boldsymbol{x}-\omega^2\int_\Omega(\boldsymbol{p}\cdot\nabla\bar{\boldsymbol{v}})\cdot\boldsymbol{u}\,\mathrm{d}\boldsymbol{x}\\ &=(\lambda+\mu)\int_\Omega(\nabla\cdot\boldsymbol{u})\nabla\cdot(\boldsymbol{p}\cdot\nabla\bar{\boldsymbol{v}})\,\mathrm{d}\boldsymbol{x}+(\lambda+\mu)\int_{\partial D}(\nabla\cdot\boldsymbol{u})(\boldsymbol{\nu}\cdot(\boldsymbol{p}\cdot\nabla\bar{\boldsymbol{v}}))\,\mathrm{d}\gamma\\ &\qquad-\omega^2\int_\Omega(\boldsymbol{p}\cdot\nabla\bar{\boldsymbol{v}})\cdot\boldsymbol{u}\,\mathrm{d}\boldsymbol{x}.\end{aligned}$$

Using the integration by parts again yields

$$\mu\int_\Omega(\boldsymbol{p}\cdot\nabla\boldsymbol{u})\cdot\Delta\bar{\boldsymbol{v}}\,\mathrm{d}\boldsymbol{x}=-\mu\int_\Omega\nabla(\boldsymbol{p}\cdot\nabla\boldsymbol{u}):\nabla\bar{\boldsymbol{v}}\,\mathrm{d}\boldsymbol{x}+\mu\int_{\partial D}(\boldsymbol{p}\cdot\nabla\boldsymbol{u})\cdot(\boldsymbol{\nu}\cdot\nabla\bar{\boldsymbol{v}})\,\mathrm{d}\gamma.$$

Let $\boldsymbol{\tau}_1(\boldsymbol{x}),\boldsymbol{\tau}_2(\boldsymbol{x})$ be any two linearly independent unit tangent vectors on ∂D. Since $\boldsymbol{u}=\boldsymbol{v}=0$ on ∂D, we have

$$\partial_{\boldsymbol{\tau}_1}u_j=\partial_{\boldsymbol{\tau}_2}u_j=\partial_{\boldsymbol{\tau}_1}v_j=\partial_{\boldsymbol{\tau}_2}v_j=0.$$

Using the identities

$$\begin{aligned}\nabla u_j&=\boldsymbol{\tau}_1\partial_{\boldsymbol{\tau}_1}u_j+\boldsymbol{\tau}_2\partial_{\boldsymbol{\tau}_2}u_j+\boldsymbol{\nu}\partial_{\boldsymbol{\nu}}u_j=\boldsymbol{\nu}\partial_{\boldsymbol{\nu}}u_j,\\ \nabla v_j&=\boldsymbol{\tau}_1\partial_{\boldsymbol{\tau}_1}v_j+\boldsymbol{\tau}_2\partial_{\boldsymbol{\tau}_2}v_j+\boldsymbol{\nu}\partial_{\boldsymbol{\nu}}v_j=\boldsymbol{\nu}\partial_{\boldsymbol{\nu}}v_j,\end{aligned}$$

we have

$$(\boldsymbol{p}\cdot\nabla\bar v_j)(\boldsymbol{\nu}\cdot\nabla u_j)=(\boldsymbol{p}\cdot\boldsymbol{\nu}\partial_{\boldsymbol{\nu}}\bar v_j)(\boldsymbol{\nu}\cdot\boldsymbol{\nu}\partial_{\boldsymbol{\nu}}u_j)=(\boldsymbol{p}\cdot\boldsymbol{\nu})(\partial_{\boldsymbol{\nu}}\bar v_j\partial_{\boldsymbol{\nu}}u_j),$$

which gives

$$\int_{\partial D}(\boldsymbol{p}\cdot\nabla\bar{\boldsymbol{v}})\cdot(\boldsymbol{\nu}\cdot\nabla\boldsymbol{u})-(\boldsymbol{p}\cdot\boldsymbol{\nu})(\nabla\boldsymbol{u}:\nabla\bar{\boldsymbol{v}})\,\mathrm{d}\gamma=0.$$

Noting $\boldsymbol{v} = 0$ on ∂D and

$$(\nabla \cdot \boldsymbol{p})(\boldsymbol{u} \cdot \bar{\boldsymbol{v}}) + (\boldsymbol{p} \cdot \nabla \bar{\boldsymbol{v}}) \cdot \boldsymbol{u} = \nabla \cdot ((\boldsymbol{u} \cdot \bar{\boldsymbol{v}})\boldsymbol{p}) - (\boldsymbol{p} \cdot \nabla \boldsymbol{u}) \cdot \bar{\boldsymbol{v}},$$

we obtain by the divergence theorem that

$$\int_\Omega (\nabla \cdot \boldsymbol{p})(\boldsymbol{u} \cdot \bar{\boldsymbol{v}}) + (\boldsymbol{p} \cdot \nabla \bar{\boldsymbol{v}}) \cdot \boldsymbol{u} \,\mathrm{d}\boldsymbol{x} = -\int_\Omega (\boldsymbol{p} \cdot \nabla \boldsymbol{u}) \cdot \bar{\boldsymbol{v}} \,\mathrm{d}\boldsymbol{x}.$$

Combining the above identities, we conclude that

$$\begin{aligned} & g_1(\boldsymbol{p})(\boldsymbol{u}, \boldsymbol{v}) + g_3(\boldsymbol{p})(\boldsymbol{u}, \boldsymbol{v}) \\ &= \mu \int_\Omega \nabla(\boldsymbol{p} \cdot \nabla \boldsymbol{u}) : \nabla \bar{\boldsymbol{v}} \,\mathrm{d}\boldsymbol{x} - (\lambda + \mu) \int_\Omega (\nabla \cdot \boldsymbol{u}) \nabla \cdot (\boldsymbol{p} \cdot \nabla \bar{\boldsymbol{v}}) \,\mathrm{d}\boldsymbol{x} \\ & \quad - \omega^2 \int_\Omega (\boldsymbol{p} \cdot \nabla \boldsymbol{u}) \cdot \bar{\boldsymbol{v}} \,\mathrm{d}\boldsymbol{x} + (\lambda + \mu) \int_{\partial D} (\nabla \cdot \boldsymbol{u})(\boldsymbol{\nu} \cdot (\boldsymbol{p} \cdot \nabla \bar{\boldsymbol{v}})) \,\mathrm{d}\gamma. \end{aligned} \tag{35}$$

Next we compute $g_2(\boldsymbol{p})(\boldsymbol{u}, \boldsymbol{v})$. It is easy to verify that

$$\begin{aligned} \int_\Omega (\nabla \cdot \boldsymbol{u})(\nabla \bar{\boldsymbol{v}} : J_{\boldsymbol{p}}^\top) + (\nabla \cdot \bar{\boldsymbol{v}})(\nabla \boldsymbol{u} : J_{\boldsymbol{p}}^\top) \,\mathrm{d}\boldsymbol{x} &= \int_\Omega (\nabla \cdot \boldsymbol{u}) \nabla \cdot (\boldsymbol{p} \cdot \nabla \bar{\boldsymbol{v}}) \,\mathrm{d}\boldsymbol{x} \\ - \int_\Omega (\nabla \cdot \boldsymbol{u})(\boldsymbol{p} \cdot (\nabla \cdot (\nabla \bar{\boldsymbol{v}})^\top)) \,\mathrm{d}\boldsymbol{x} &+ \int_\Omega (\nabla \cdot \bar{\boldsymbol{v}}) \nabla \cdot (\boldsymbol{p} \cdot \nabla \boldsymbol{u}) \,\mathrm{d}\boldsymbol{x} \\ &- \int_\Omega (\nabla \cdot \bar{\boldsymbol{v}})(\boldsymbol{p} \cdot (\nabla \cdot (\nabla \boldsymbol{u})^\top)) \,\mathrm{d}\boldsymbol{x}. \end{aligned}$$

Using the integration by parts, we obtain

$$\begin{aligned} \int_\Omega (\nabla \cdot \boldsymbol{p})(\nabla \cdot \boldsymbol{u})(\nabla \cdot \bar{\boldsymbol{v}}) \,\mathrm{d}\boldsymbol{x} &= -\int_\Omega \boldsymbol{p} \cdot \nabla((\nabla \cdot \boldsymbol{u})(\nabla \cdot \bar{\boldsymbol{v}})) \,\mathrm{d}\boldsymbol{x} \\ & \qquad - \int_{\partial D} (\nabla \cdot \boldsymbol{u})(\nabla \cdot \bar{\boldsymbol{v}})(\boldsymbol{\nu} \cdot \boldsymbol{p}) \,\mathrm{d}\gamma \\ = -\int_\Omega (\nabla \cdot \bar{\boldsymbol{v}})(\boldsymbol{p} \cdot (\nabla \cdot (\nabla \boldsymbol{u})^\top)) \,\mathrm{d}\boldsymbol{x} &- \int_\Omega (\nabla \cdot \boldsymbol{u})(\boldsymbol{p} \cdot (\nabla \cdot (\nabla \boldsymbol{v})^\top)) \,\mathrm{d}\boldsymbol{x} \\ & \qquad - \int_{\partial D} (\nabla \cdot \boldsymbol{u})(\nabla \cdot \bar{\boldsymbol{v}})(\boldsymbol{\nu} \cdot \boldsymbol{p}) \,\mathrm{d}\gamma. \end{aligned}$$

Let $\boldsymbol{\tau}_1 = (-\nu_3, 0, \nu_1)^\top, \boldsymbol{\tau}_2 = (0, -\nu_3, \nu_2)^\top, \boldsymbol{\tau}_3 = (-\nu_2, \nu_1, 0)^\top$. It follows from $\boldsymbol{\tau}_j \cdot \boldsymbol{\nu} = 0$ that $\boldsymbol{\tau}_j$ are tangent vectors on ∂D. Since $\boldsymbol{v} = 0$ on ∂D, we have $\partial_{\boldsymbol{\tau}_j} \boldsymbol{v} = 0$, which yields that

$$\begin{aligned} \nu_1 \partial_{x_3} v_1 &= \nu_3 \partial_{x_1} v_1, & \nu_1 \partial_{x_3} v_2 &= \nu_3 \partial_{x_1} v_2, & \nu_1 \partial_{x_2} v_1 &= \nu_2 \partial_{x_1} v_1, \\ \nu_1 \partial_{x_3} v_3 &= \nu_3 \partial_{x_1} v_3, & \nu_1 \partial_{x_2} v_2 &= \nu_2 \partial_{x_1} v_2, & \nu_1 \partial_{x_2} v_3 &= \nu_2 \partial_{x_1} v_3, \\ \nu_2 \partial_{x_3} v_1 &= \nu_3 \partial_{x_2} v_1, & \nu_2 \partial_{x_3} v_2 &= \nu_3 \partial_{x_2} v_2, & \nu_2 \partial_{x_3} v_3 &= \nu_3 \partial_{x_2} v_3. \end{aligned}$$

Hence we get

$$\int_{\partial D} (\nabla \cdot \boldsymbol{u})(\nabla \cdot \bar{\boldsymbol{v}})(\boldsymbol{\nu} \cdot \boldsymbol{p}) \,\mathrm{d}\gamma = \int_{\partial D} (\nabla \cdot \boldsymbol{u})(\boldsymbol{\nu} \cdot (\boldsymbol{p} \cdot \nabla \bar{\boldsymbol{v}})) \,\mathrm{d}\gamma.$$

Combining the above identities gives

$$\begin{aligned} g_2(\boldsymbol{p})(\boldsymbol{u}, \boldsymbol{v}) = (\lambda + \mu) \int_\Omega (\nabla \cdot \boldsymbol{u}) \nabla \cdot (\boldsymbol{p} \cdot \nabla \bar{\boldsymbol{v}}) \,\mathrm{d}\boldsymbol{x} + (\lambda + \mu) \int_\Omega \nabla \cdot (\boldsymbol{p} \cdot \nabla \boldsymbol{u})(\nabla \cdot \bar{\boldsymbol{v}}) \,\mathrm{d}\boldsymbol{x} \\ - (\lambda + \mu) \int_{\partial D} (\nabla \cdot \boldsymbol{u})(\nu \cdot (\boldsymbol{p} \cdot \nabla \bar{\boldsymbol{v}})) \,\mathrm{d}\gamma. \end{aligned} \tag{36}$$

Noting (34), adding (35) and (36), we obtain

$$b(\dot{\boldsymbol{u}},\boldsymbol{v})=\mu\int_\Omega \nabla(\boldsymbol{p}\cdot\nabla\boldsymbol{u}):\nabla\bar{\boldsymbol{v}}\,\mathrm{d}\boldsymbol{x}+(\lambda+\mu)\int_\Omega \nabla\cdot(\boldsymbol{p}\cdot\nabla\boldsymbol{u})(\nabla\cdot\bar{\boldsymbol{v}})\,\mathrm{d}\boldsymbol{x}-\omega^2\int_\Omega(\boldsymbol{p}\cdot\nabla\boldsymbol{u})\cdot\bar{\boldsymbol{v}}\,\mathrm{d}\boldsymbol{x}.$$

Define $\boldsymbol{u}'=\dot{\boldsymbol{u}}-\boldsymbol{p}\cdot\nabla\boldsymbol{u}$. It is clear to note that $\boldsymbol{p}\cdot\nabla\boldsymbol{u}=0$ on Γ_R since $\boldsymbol{p}=0$ on Γ_R. Hence, we have

$$b(\boldsymbol{u}',\boldsymbol{v})=0,\quad \forall\,\boldsymbol{v}\in\boldsymbol{H}^1_{\partial D}(\Omega), \tag{37}$$

which shows that $\boldsymbol{u}'$ is the weak solution of the boundary value problem (32). To verify the boundary condition of $\boldsymbol{u}'$ on ∂D, we recall the definition of $\boldsymbol{u}'$ and have from $\breve{\boldsymbol{u}}=\boldsymbol{u}=0$ on ∂D that

$$\boldsymbol{u}'=\lim_{h\to 0}\frac{\breve{\boldsymbol{u}}-\boldsymbol{u}}{h}-\boldsymbol{p}\cdot\nabla\boldsymbol{u}=-\boldsymbol{p}\cdot\nabla\boldsymbol{u}\quad\text{on }\partial D.$$

Noting $\boldsymbol{u}=0$ on ∂D, we have

$$\boldsymbol{p}\cdot\nabla\boldsymbol{u}=(\boldsymbol{p}\cdot\boldsymbol{\nu})\partial_{\boldsymbol{\nu}}\boldsymbol{u}, \tag{38}$$

which completes the proof by combining (37) and (38). □

4.2. **Reconstruction method.** Consider a parametric equation for the surface:

$$\partial D=\{\boldsymbol{r}(\theta,\varphi)=(r_1(\theta,\varphi),r_2(\theta,\varphi),r_3(\theta,\varphi))^\top,\ \theta\in(0,\pi),\ \varphi\in(0,2\pi)\},$$

where r_j are biperiodic functions of (θ,φ) and have the Fourier series expansions:

$$r_j(\theta,\phi)=\sum_{n=0}^{\infty}\sum_{m=-n}^{n}a_{jn}^m\mathrm{Re}Y_n^m(\theta,\varphi)+b_{jn}^m\mathrm{Im}Y_n^m(\theta,\varphi),$$

where Y_n^m are the spherical harmonics of order n. It suffices to determine a_{jn}^m, b_{jn}^m in order to reconstruct the surface. In practice, a cut-off approximation is needed:

$$r_{j,N}(\theta,\varphi)=\sum_{n=0}^{N}\sum_{m=-n}^{n}a_{jn}^m\mathrm{Re}Y_n^m(\theta,\varphi)+b_{jn}^m\mathrm{Im}Y_n^m(\theta,\varphi).$$

Denote by D_N the approximated obstacle with boundary ∂D_N, which has the parametric equation

$$\partial D_N=\{\boldsymbol{r}_N(\theta,\varphi)=(r_{1,N}(\theta,\varphi),r_{2,N}(\theta,\varphi),r_{3,N}(\theta,\varphi))^\top,\ \theta\in(0,\pi),\ \phi\in(0,2\pi)\}.$$

Let $\Omega_N=B_R\setminus\bar{D}_N$ and

$$\boldsymbol{a}_j=(a_{j0}^0,\cdots,a_{jn}^m,\cdots,a_{jN}^N),\quad \boldsymbol{b}_j=(b_{j0}^0,\cdots,b_{jn}^m,\cdots,b_{jN}^N),$$

where $n=0,1,\dots,N,\ m=-n,\dots,n$. Denote the vector of Fourier coefficients

$$\boldsymbol{C}=(\boldsymbol{a}_1,\boldsymbol{b}_1,\boldsymbol{a}_2,\boldsymbol{b}_2,\boldsymbol{a}_3,\boldsymbol{b}_3)^\top=(c_1,c_2,\dots,c_{6(N+1)^2})^\top\in\mathbb{R}^{6(N+1)^2}$$

and a vector of scattering data

$$\boldsymbol{U}=(\boldsymbol{u}(\boldsymbol{x}_1),\dots,\boldsymbol{u}(\boldsymbol{x}_K))^\top\in\mathbb{C}^{3K},$$

where $\boldsymbol{x}_k\in\Gamma_R, k=1,\dots,K$. Then the inverse problem can be formulated to solve an approximate nonlinear equation:

$$\mathscr{F}(\boldsymbol{C})=\boldsymbol{U},$$

where the operator $\mathscr{F}$ maps a vector in $\mathbb{R}^{6(N+1)^2}$ into a vector in $\mathbb{C}^{3K}$.

Theorem 4.2. *Let $\boldsymbol{u}_N$ be the solution of (25) corresponding to the obstacle D_N. The operator $\mathscr{F}$ is differentiable and its derivatives are*

$$\frac{\partial \mathscr{F}_k(\boldsymbol{C})}{\partial c_i} = \boldsymbol{u}_i'(\boldsymbol{x}_k), \quad i = 1, \dots, 6(N+1)^2,\ k = 1, \dots, K,$$

where $\boldsymbol{u}_i'$ is the unique weak solution of the boundary value problem

$$\begin{cases} \mu\Delta\boldsymbol{u}_i' + (\lambda+\mu)\nabla\nabla\cdot\boldsymbol{u}_i' + \omega^2\boldsymbol{u}_i' = 0 & \text{in } \Omega_N, \\ \boldsymbol{u}_i' = -q_i\partial_{\boldsymbol{\nu}_N}\boldsymbol{u}_N & \text{on } \partial D_N. \\ \mathscr{B}\boldsymbol{u}_i' = \mathscr{T}\boldsymbol{u}_i' & \text{on } \Gamma_R. \end{cases} \tag{39}$$

Here $\boldsymbol{\nu}_N = (\nu_{N1}, \nu_{N2}, \nu_{N3})^\top$ is the unit normal vector on ∂D_N and

$$q_i(\theta,\varphi) = \begin{cases} \nu_{N1}\mathrm{Re}Y_n^m(\theta,\varphi), & i = n^2+n+m+1, \\ \nu_{N1}\mathrm{Im}Y_n^m(\theta,\varphi), & i = (N+1)^2+n^2+n+m+1, \\ \nu_{N2}\mathrm{Re}Y_n^m(\theta,\varphi), & i = 2(N+1)^2+n^2+n+m+1, \\ \nu_{N2}\mathrm{Im}Y_n^m(\theta,\varphi), & i = 3(N+1)^2+n^2+n+m+1, \\ \nu_{N3}\mathrm{Re}Y_n^m(\theta,\varphi), & i = 4(N+1)^2+n^2+n+m+1, \\ \nu_{N3}\mathrm{Im}Y_n^m(\theta,\varphi), & i = 5(N+1)^2+n^2+n+m+1, \end{cases}$$

where $n = 0, 1, \dots, N, m = -n, \dots, n$.

Proof. Fix $i \in \{1, \dots, 6(N+1)^2\}$ and $k \in \{1, \dots, K\}$, and let $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_{6(N+1)^2}\}$ be the set of natural basis vectors in $\mathbb{R}^{6(N+1)^2}$. By definition, we have

$$\frac{\partial \mathscr{F}_k(\boldsymbol{C})}{\partial c_i} = \lim_{h\to 0}\frac{\mathscr{F}_k(\boldsymbol{C}+h\boldsymbol{e}_i) - \mathscr{F}_k(\boldsymbol{C})}{h}.$$

A direct application of Theorem 4.1 shows that the above limit exists and the limit is the unique weak solution of the boundary value problem (39). □

Consider the objective function

$$f(\boldsymbol{C}) = \frac{1}{2}\|\mathscr{F}(\boldsymbol{C}) - \boldsymbol{U}\|^2 = \frac{1}{2}\sum_{k=1}^K |\mathscr{F}_k(\boldsymbol{C}) - \boldsymbol{u}(\boldsymbol{x}_k)|^2.$$

The inverse problem can be formulated as the minimization problem:

$$\min_{\boldsymbol{C}} f(\boldsymbol{C}), \quad \boldsymbol{C} \in \mathbb{R}^{6(N+1)^2}.$$

To apply the descend method, we compute the gradient of the objective function:

$$\nabla f(\boldsymbol{C}) = \left(\frac{\partial f(\boldsymbol{C})}{\partial c_1}, \dots, \frac{f(\boldsymbol{C})}{\partial c_{6(N+1)^2}}\right)^\top.$$

We have from Theorem 4.2 that

$$\frac{\partial f(\boldsymbol{C})}{\partial c_i} = \mathrm{Re}\sum_{k=1}^K \boldsymbol{u}_i'(\boldsymbol{x}_k)\cdot(\bar{\mathscr{F}}_k(\boldsymbol{C}) - \bar{\boldsymbol{u}}(\boldsymbol{x}_k)).$$

We assume that the scattering data $\boldsymbol{U}$ is available over a range of frequencies $\omega \in [\omega_{\min}, \omega_{\max}]$, which may be divided into $\omega_{\min} = \omega_0 < \omega_1 < \cdots < \omega_J = \omega_{\max}$. We now propose an algorithm to reconstruct the Fourier coefficients $c_i, i = 1, \dots, 6(N+1)^2$.

Algorithm: Frequency continuation algorithm for surface reconstruction.

1. **Initialization**: take an initial guess $c_2 = -c_4 = 1.44472R_0$ and $c_{3(N+1)^2+2} = c_{3(N+1)^2+4} = 1.44472R_0$, $c_{4(N+1)^2+3} = 2.0467R_0$ and $c_i = 0$ otherwise. The initial guess is a ball with radius R_0 under the spherical harmonic functions;
2. **First approximation**: begin with ω_0, let $k_0 = [\omega_0]$, seek an approximation to the functions $r_{j,N}$:
$$r_{j,k_0} = \sum_{n=0}^{k_0}\sum_{m=-n}^{n} a_{jn}^m \mathrm{Re}Y_n^m(\theta,\phi) + b_{jn}^m \mathrm{Im}Y_n^m(\theta,\phi).$$
Denote $\boldsymbol{C}_{k_0}^{(1)} = (c_1, c_2, \ldots, c_{6(k_0+1)^2})^\top$ and consider the iteration:
$$\mathbf{C}_{k_0}^{(l+1)} = \mathbf{C}_{k_0}^{(l)} - \tau\nabla f(\mathbf{C}_{k_0}^{(l)}), \quad l = 1, \ldots, L, \tag{40}$$
where $\tau > 0$ and $L > 0$ are the step size and the number of iterations for every fixed frequency, respectively.
3. **Continuation**: increase to ω_1, let $k_1 = [\omega_1]$, repeat Step 2 with the previous approximation to $r_{j,N}$ as the starting point. More precisely, approximate $r_{j,N}$ by
$$r_{j,k_1} = \sum_{n=0}^{k_1}\sum_{m=-n}^{n} a_{jn}^m \mathrm{Re}Y_n^m(\theta,\phi) + b_{jn}^m \mathrm{Im}Y_n^m(\theta,\phi),$$
and determine the coefficients $\tilde{c}_i, i = 1, \ldots, 6(k_1+1)^2$ by using the descent method starting from the previous result.
4. **Iteration**: repeat Step 3 until a prescribed highest frequency ω_J is reached.

5. **Numerical experiments.** We present two examples to show the effectiveness of the proposed method. The scattering data is obtained from solving the direct problem by using the finite element method with the perfectly matched layer (PML) technique, which is implemented via FreeFem++ [16]. The research on the PML technique has undergone a tremendous development since Berenger proposed a PML for solving the Maxwell equations [4]. The basic idea of the PML technique is to surround the domain of interest by a layer of finite thickness fictitious material which absorbs all the waves coming from inside the computational domain. When the waves reach the outer boundary of the PML region, their values are so small that the homogeneous Dirichlet boundary conditions can be imposed. However, the PML technique is much less studied for the elastic wave scattering problems, especially for the rigorous convergence analysis [7, 8, 19]. In contrast, the transparent boundary condition (TBC) is mathematically exact. It helps to reduce the scattering problem equivalently from an open domain into a boundary value problem in a bounded domain, which makes the analysis feasible. The finite element solution is interpolated uniformly on Γ_R. To test the stability, we add noise to the data:
$$\boldsymbol{u}^\delta(\boldsymbol{x}_k) = \boldsymbol{u}(\boldsymbol{x}_k)(1 + \delta\,\mathrm{rand}), \quad k = 1, \ldots, K,$$
where rand are uniformly distributed random numbers in $[-1, 1]$ and δ is the noise level, $\boldsymbol{x}_k$ are the data points. In our experiments, we pick 100 uniformly distributed points $\boldsymbol{x}_k$ on Γ_R, i.e., $K = 100$. We take $\lambda = 2, \mu = 1$, $R = 1$. The radius of the

initial $R_0 = 0.5$. The noise level $\delta = 5\%$. The step size in (40) is $\tau = 0.005/k_i$ where $k_i = [\omega_i]$. The incident field is taken as a plane compressional wave.

Example 1. Consider a bean-shaped obstacle:

$$\boldsymbol{r}(\theta,\varphi) = (r_1(\theta,\varphi), r_2(\theta,\varphi), r_3(\theta,\varphi))^\top,\ \theta \in [0,\pi],\ \varphi \in [0,2\pi],$$

where

$$\begin{aligned}
r_1(\theta,\varphi) &= 0.75\left((1-0.05\cos(\pi\cos\theta))\sin\theta\cos\varphi\right)^{1/2},\\
r_2(\theta,\varphi) &= 0.75\left((1-0.005\cos(\pi\cos\theta))\sin\theta\sin\varphi + 0.35\cos(\pi\cos\theta)\right)^{1/2},\\
r_3(\theta,\varphi) &= 0.75\cos\theta.
\end{aligned}$$

The exact surface is plotted in Figure 1(a). This obstacle is non-convex and is usually difficult to reconstruct the concave part of the obstacle. The obstacle is illuminated by the compressional wave sent from a single direction $\boldsymbol{d} = (0,1,0)^\top$; the frequency ranges from $\omega_{\min} = 1$ to $\omega_{\max} = 5$ with increment 1 at each continuation step, i.e., $\omega_i = i+1, i = 0,\ldots,4$; for any fixed frequency, repeat $L = 100$ times with previous result as starting points. The step size for the decent method is $0.005/\omega_i$. The number of recovered coefficients is $6(\omega_i+2)^2$ for corresponding frequency. Figure 1(b) shows the initial guess which is the ball with radius $R_0 = 0.5$; Figure 1(c) shows the final reconstructed surface; Figures 1(d)–(f) show the cross section of the exact surface along the plane $x_1 = 0, x_2 = 0, x_3 = 0$, respectively; Figures 1(g)–(i) show the corresponding cross section for the reconstructed surface along the plane $x_1 = 0, x_2 = 0, x_3 = 0$, respectively. As is seen, the algorithm effectively reconstructs the bean-shaped obstacle.

Example 2. Consider a cushion-shaped obstacle:

$$\boldsymbol{r}(\theta,\varphi) = r(\theta,\varphi)(\sin(\theta)\cos(\varphi), \sin(\theta)\sin(\varphi), \cos(\theta))^\top,\ \theta \in [0,\pi],\ \varphi \in [0,2\pi],$$

where

$$r(\theta,\varphi) = (0.75 + 0.45(\cos(2\varphi)-1)(\cos(4\theta)-1))^{1/2}.$$

Figure 2(a) shows the exact surface. This example is much more complex than the bean-shaped obstacle due to its multiple concave parts. Multiple incident directions are needed in order to obtain a good result. In this example, the obstacle is illuminated by the compressional wave from 6 directions, which are the unit vectors pointing to the origin from the face centers of the cube. The multiple frequencies are the same as the first example, i.e., the frequency ranges from $\omega_{\min} = 1$ to $\omega_{\max} = 5$ with $\omega_i = i+1, i = 0,\ldots,4$. For each fixed frequency and incident direction, repeat $L = 50$ times with previous result as starting points. The step size for the decent method is $0.005/\omega_i$ and number of recovered coefficients is $6(\omega_i+2)^2$ for corresponding frequency. Figure 2(b) shows the initial guess ball with radius $R_0 = 0.5$; Figure 2(c) shows the final reconstructed surface; Figure 2(d)–(f) show the cross section of the exact surface along the plane $x_1 = 0, x_2 = 0, x_3 = 0$, respectively; while Figure 2(g)–(i) show the corresponding cross section for the reconstructed surface along the plane $x_1 = 0, x_2 = 0, x_3 = 0$, respectively. It is clear to note that the algorithm can also reconstruct effectively the more complex cushion-shaped obstacle.

6. Conclusion.

In this paper, we study the direct and inverse obstacle scattering problems for elastic waves in three dimensions. An exact transparent boundary condition is developed. The direct problem is shown to have a unique weak solution. The domain derivative is derived for the total displacement. A frequency

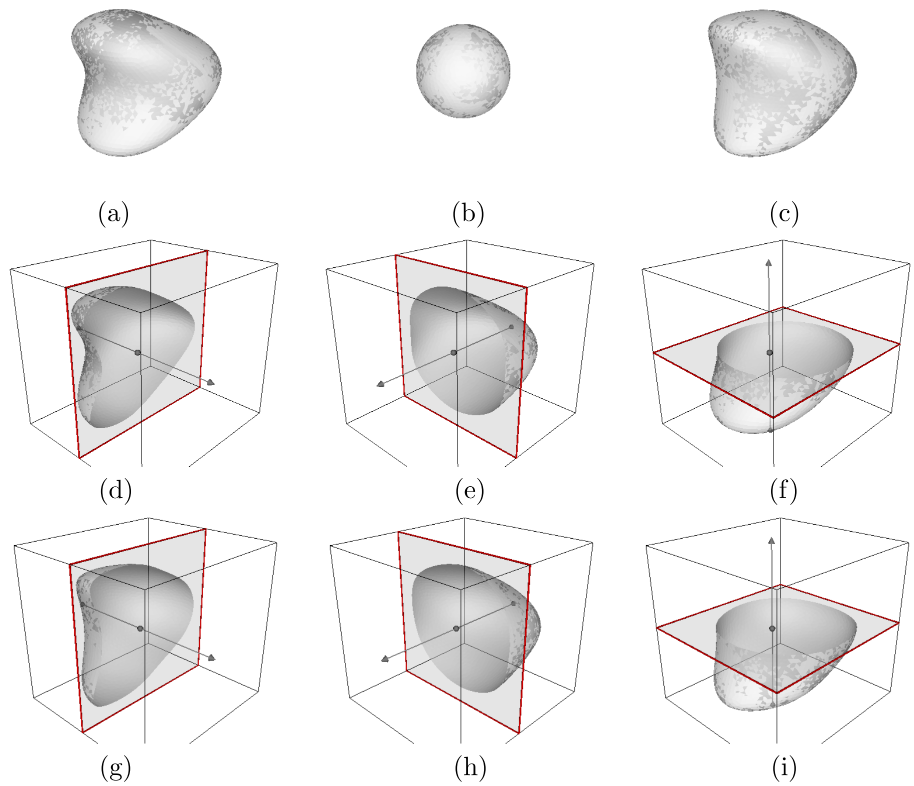

FIGURE 1. Example 1: A bean-shaped obstacle. (a) the exact surface; (b) the initial guess; (c) the reconstructed surface; (d)–(f) the corresponding cross section of the exact surface along plane $x_1 = 0, x_2 = 0, x_3 = 0$, respectively; (g)–(i) the corresponding cross section of the reconstructed surface along plane $x_1 = 0, x_2 = 0, x_3 = 0$, respectively.

continuation method is proposed to solve the inverse problem. Numerical examples are presented to demonstrate the effectiveness of the proposed method. The results show that the method is stable and accurate to reconstruct surfaces with noise. Future work includes the surfaces of different boundary conditions and multiple obstacles. We hope to be able to address these issues and report the progress elsewhere in the future.

Appendix A. **Spherical harmonics.** The spherical coordinates (r, θ, φ) are related to the Cartesian coordinates $\boldsymbol{x} = (x_1, x_2, x_3)$ by $x_1 = r\sin\theta\cos\varphi, x_2 = r\sin\theta\sin\varphi, x_3 = r\cos\theta$. The local orthonormal basis is

$$\begin{aligned}
\boldsymbol{e}_r &= (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta),\\
\boldsymbol{e}_\theta &= (\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta),\\
\boldsymbol{e}_\varphi &= (-\sin\varphi, \cos\varphi, 0),
\end{aligned}$$

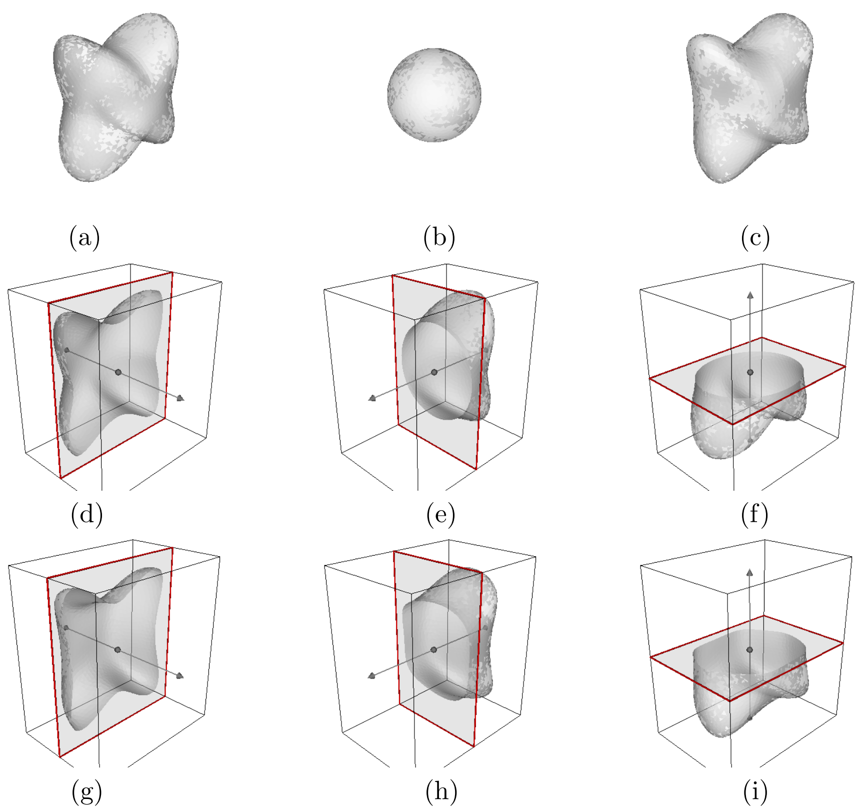

FIGURE 2. Example 2: A cushion-shaped obstacle. (a) the exact surface; (b) the initial guess; (c) the reconstructed surface; (d)–(f) the corresponding cross section of the exact surface along the plane $x_1 = 0, x_2 = 0, x_3 = 0$, respectively; (d)–(f) the corresponding cross section of the reconstructed surface along the plane $x_1 = 0, x_2 = 0, x_3 = 0$, respectively.

where θ and φ are the Euler angles. Note that $\boldsymbol{e}_r$ is also the unit outward normal vector on Γ_R.

Let $\{Y_n^m(\theta,\varphi) : n = 0, 1, 2, \dots, m = -n, \dots, n\}$ be the orthonormal sequence of spherical harmonics of order n on the unit sphere. Define rescaled spherical harmonics

$$X_n^m(\theta,\varphi) = \frac{1}{R} Y_n^m(\theta,\varphi).$$

It can be shown that $\{X_n^m(\theta,\varphi) : n = 0, 1, \dots, m = -n, \dots, n\}$ form a complete orthonormal system in $L^2(\Gamma_R)$, which is the space of square integrable functions on Γ_R.

For a smooth scalar function $u(R,\theta,\varphi)$ defined on Γ_R, let

$$\nabla_{\Gamma_R} u = \partial_\theta u\, \boldsymbol{e}_\theta + (\sin\theta)^{-1} \partial_\varphi u\, \boldsymbol{e}_\varphi$$

be the tangential gradient on Γ_R. The surface vector curl is defined by

$$\mathbf{curl}_{\Gamma_R} u = \nabla_{\Gamma_R} u \times \boldsymbol{e}_r.$$

Denote by div_{Γ_R} and curl_{Γ_R} the surface divergence and the surface scalar curl, respectively. For a smooth vector function $\boldsymbol{u}$ tangential to Γ_R, it can be represented by its coordinates in the local orthonormal basis:

$$\boldsymbol{u} = u_\theta \boldsymbol{e}_\theta + u_\varphi \boldsymbol{e}_\varphi,$$

where

$$u_\theta = \boldsymbol{u}\cdot\boldsymbol{e}_\theta, \quad u_\varphi = \boldsymbol{u}\cdot\boldsymbol{e}_\varphi.$$

The surface divergence and the surface scalar curl can be defined as

$$\begin{aligned}
\mathrm{div}_{\Gamma_R}\boldsymbol{u} &= (\sin\theta)^{-1}\left(\partial_\theta(u_\theta\sin\theta) + \partial_\varphi u_\varphi\right),\\
\mathrm{curl}_{\Gamma_R}\boldsymbol{u} &= (\sin\theta)^{-1}\left(\partial_\theta(u_\varphi\sin\theta) - \partial_\varphi u_\theta\right).
\end{aligned}$$

Define a sequence of vector spherical harmonics:

$$\begin{aligned}
\boldsymbol{T}_n^m(\theta,\varphi) &= \frac{1}{\sqrt{n(n+1)}}\nabla_{\Gamma_R}X_n^m(\theta,\varphi),\\
\boldsymbol{V}_n^m(\theta,\varphi) &= \boldsymbol{T}_n^m(\theta,\varphi)\times\boldsymbol{e}_r,\\
\boldsymbol{W}_n^m(\theta,\varphi) &= X_n^m(\theta,\varphi)\boldsymbol{e}_r,
\end{aligned}$$

where $n = 0, 1, \ldots, m = -n, \ldots, n$. Using the orthogonality of the vector spherical harmonics, we can easily show that

1. $\{(\boldsymbol{T}_n^m, \boldsymbol{V}_n^m, \boldsymbol{W}_n^m) : n = 0, 1, 2, \ldots, m = -n, \ldots, n\}$ form a complete orthonormal system in $\boldsymbol{L}^2(\Gamma_R) = L^2(\Gamma_R)^3$;
2. $\{(\boldsymbol{T}_n^m, \boldsymbol{V}_n^m) : n = 0, 1, 2, \ldots, m = -n, \ldots, n\}$ form a complete orthonormal system in $\boldsymbol{L}_{\mathrm{t}}^2(\Gamma_R) = \{\boldsymbol{w}\in\boldsymbol{L}^2(\Gamma_R), \boldsymbol{w}\cdot\boldsymbol{e}_r = 0\}$.

Appendix B. **Functional spaces.** Denote by $L^2(\Omega)$ the square integrable functions on Ω. Let $\boldsymbol{L}^2(\Omega) = L^2(\Omega)^3$ be equipped with the inner product and norm:

$$(\boldsymbol{u},\boldsymbol{v}) = \int_\Omega \boldsymbol{u}\cdot\bar{\boldsymbol{v}}\,\mathrm{d}\boldsymbol{x}, \quad \|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)} = (\boldsymbol{u},\boldsymbol{u})^{1/2}.$$

Denote by $H^1(\Omega)$ the standard Sobolev space with the norm given by

$$\|u\|_{H^1(\Omega)} = \left(\int_\Omega |u(\boldsymbol{x})|^2 + |\nabla u(\boldsymbol{x})|^2\,\mathrm{d}\boldsymbol{x}\right)^{1/2}.$$

Let $\boldsymbol{H}_{\partial D}^1(\Omega) = H_{\partial D}^1(\Omega)^3$, where $H_{\partial D}^1(\Omega) := \{u\in H^1(\Omega) : u = 0 \text{ on } \partial D\}$. Introduce the Sobolev space

$$\boldsymbol{H}(\mathrm{curl},\Omega) = \{\boldsymbol{u}\in\boldsymbol{L}^2(\Omega), \nabla\times\boldsymbol{u}\in\boldsymbol{L}^2(\Omega)\},$$

which is equipped with the norm

$$\|\boldsymbol{u}\|_{\boldsymbol{H}(\mathrm{curl},\Omega)} = \left(\|\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}^2 + \|\nabla\times\boldsymbol{u}\|_{\boldsymbol{L}^2(\Omega)}^2\right)^{1/2}.$$

Denote by $H^s(\Gamma_R)$ the standard trace functional space which is equipped with the norm

$$\|u\|_{H^s(\Gamma_R)} = \left(\sum_{n=0}^{\infty}\sum_{m=-n}^{n}(1+n(n+1))^s|u_n^m|^2\right)^{1/2},$$

where

$$u(R,\theta,\varphi) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}u_n^m X_n^m(\theta,\varphi).$$

Let $\boldsymbol{H}^s(\Gamma_R) = H^s(\Gamma_R)^3$ which is equipped with the normal

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^s(\Gamma_R)} = \left(\sum_{n=0}^{\infty}\sum_{m=-n}^{n}(1+n(n+1))^s|\boldsymbol{u}_n^m|^2\right)^{1/2},$$

where $\boldsymbol{u}_n^m = (u_{1n}^m, u_{2n}^m, u_{3n}^m)$ and

$$\boldsymbol{u}(R,\theta,\varphi) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} u_{1n}^m \boldsymbol{T}_n^m(\theta,\varphi) + u_{2n}^m \boldsymbol{V}_n^m(\theta,\varphi) + u_{3n}^m \boldsymbol{W}_n^m(\theta,\varphi).$$

It can be verified that $\boldsymbol{H}^{-s}(\Gamma_R)$ is the dual space of $\boldsymbol{H}^s(\Gamma_R)$ with respect to the inner product

$$\langle \boldsymbol{u}, \boldsymbol{v}\rangle_{\Gamma_R} = \int_{\Gamma_R} \boldsymbol{u}\cdot\bar{\boldsymbol{v}}\,\mathrm{d}\gamma = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} u_{1n}^m\bar{v}_{1n}^m + u_{2n}^m\bar{v}_{2n}^m + u_{3n}^m\bar{v}_{3n}^m,$$

where

$$\boldsymbol{v}(R,\theta,\varphi) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} v_{1n}^m \boldsymbol{T}_n^m(\theta,\varphi) + v_{2n}^m \boldsymbol{V}_n^m(\theta,\varphi) + v_{3n}^m \boldsymbol{W}_n^m(\theta,\varphi).$$

Introduce three tangential trace spaces:

$$\begin{aligned}
\boldsymbol{H}_{\mathrm{t}}^s(\Gamma_R) &= \{\boldsymbol{u}\in\boldsymbol{H}^s(\Gamma_R),\ \boldsymbol{u}\cdot\boldsymbol{e}_r = 0\},\\
\boldsymbol{H}^{-1/2}(\mathrm{curl},\Gamma_R) &= \{\boldsymbol{u}\in\boldsymbol{H}_{\mathrm{t}}^{-1/2}(\Gamma_R),\ \mathrm{curl}_{\Gamma_R}\boldsymbol{u}\in H^{-1/2}(\Gamma_R)\},\\
\boldsymbol{H}^{-1/2}(\mathrm{div},\Gamma_R) &= \{\boldsymbol{u}\in\boldsymbol{H}_{\mathrm{t}}^{-1/2}(\Gamma_R),\ \mathrm{div}_{\Gamma_R}\boldsymbol{u}\in H^{-1/2}(\Gamma_R)\}.
\end{aligned}$$

For any tangential field $\boldsymbol{u}\in\boldsymbol{H}_{\mathrm{t}}^s(\Gamma_R)$, it can be represented in the series expansion

$$\boldsymbol{u}(R,\theta,\varphi) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} u_{1n}^m \boldsymbol{T}_n^m(\theta,\varphi) + u_{2n}^m \boldsymbol{V}_n^m(\theta,\varphi).$$

Using the series coefficients, the norm of the space $\boldsymbol{H}_{\mathrm{t}}^s(\Gamma_R)$ can be characterized by

$$\|\boldsymbol{u}\|^2_{\boldsymbol{H}_{\mathrm{t}}^s(\Gamma_R)} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}(1+n(n+1))^s\left(|u_{1n}^m|^2 + |u_{2n}^m|^2\right);$$

the norm of the space $\boldsymbol{H}^{-1/2}(\mathrm{curl},\Gamma_R)$ can be characterized by

$$\|\boldsymbol{u}\|^2_{\boldsymbol{H}^{-1/2}(\mathrm{curl},\Gamma_R)} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\frac{1}{\sqrt{1+n(n+1)}}|u_{1n}^m|^2 + \sqrt{1+n(n+1)}|u_{2n}^m|^2;$$

the norm of the space $\boldsymbol{H}^{-1/2}(\mathrm{div},\Gamma_R)$ can be characterized by

$$\|\boldsymbol{u}\|^2_{\boldsymbol{H}^{-1/2}(\mathrm{div},\Gamma_R)} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\sqrt{1+n(n+1)}|u_{1n}^m|^2 + \frac{1}{\sqrt{1+n(n+1)}}|u_{2n}^m|^2.$$

Given a vector field $\boldsymbol{u}$ on Γ_R, denote by

$$\boldsymbol{u}_{\Gamma_R} = -\boldsymbol{e}_r\times(\boldsymbol{e}_r\times\boldsymbol{u})$$

the tangential component of $\boldsymbol{u}$ on Γ_R. Define the inner product in $\mathbb{C}^3$:

$$\langle\boldsymbol{u},\boldsymbol{v}\rangle = \boldsymbol{v}^*\boldsymbol{u},\quad \forall\,\boldsymbol{u},\boldsymbol{v}\in\mathbb{C}^3.$$

where $\boldsymbol{v}^*$ is the conjugate transpose of $\boldsymbol{v}$.

Appendix C. TBC for potential functions. It follows from the Helmholtz decomposition (7) that any solution of (6) can be written as

$$\boldsymbol{v} = \nabla\phi + \nabla\times\boldsymbol{\psi}, \quad \nabla\cdot\boldsymbol{\psi} = 0,$$

where the scalar potential function ϕ satisfies (8) and (9):

$$\begin{cases} \Delta\phi + \kappa_{\rm p}^2\phi = 0 & \text{in } \mathbb{R}^3\setminus\bar{D}, \\ \partial_r\phi - {\rm i}\kappa_{\rm p}\phi = o(r^{-1}) & \text{as } r\to\infty, \end{cases} \tag{41}$$

and the vector potential function $\boldsymbol{\psi}$ satisfies (10) and (11):

$$\begin{cases} \nabla\times(\nabla\times\boldsymbol{\psi}) - \kappa_{\rm s}^2\boldsymbol{\psi} = 0 & \text{in } \mathbb{R}^3\setminus\bar{D}, \\ (\nabla\times\boldsymbol{\psi})\times\hat{\boldsymbol{x}} - {\rm i}\kappa_{\rm s}\boldsymbol{\psi} = o(r^{-1}) & \text{as } r\to\infty, \end{cases} \tag{42}$$

where $r = |\boldsymbol{x}|$ and $\hat{\boldsymbol{x}} = \boldsymbol{x}/r$.

In this section, we introduce the TBC for the scalar potential function ϕ and the vector potential function $\boldsymbol{\psi}$, respectively. The TBCs help to reduce (41) and (42) equivalently from the open domain $\mathbb{R}^3\setminus\bar{D}$ into the bounded domain Ω.

In the exterior domain $\mathbb{R}^3\setminus\bar{B}_R$, the solution ϕ of (41) has the following Fourier expansion in the spherical coordinates:

$$\phi(r,\theta,\varphi) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\frac{h_n^{(1)}(\kappa_{\rm p}r)}{h_n^{(1)}(\kappa_{\rm p}R)}\phi_n^m X_n^m(\theta,\varphi), \tag{43}$$

where $h_n^{(1)}$ is the spherical Hankel function of the first kind with order n and the Fourier coefficient

$$\phi_n^m = \int_{\Gamma_R}\phi(R,\theta,\varphi)\bar{X}_n^m(\theta,\varphi){\rm d}\gamma.$$

We define the boundary operator $\mathscr{T}_1$ such that

$$(\mathscr{T}_1\phi)(R,\theta,\varphi) = \frac{1}{R}\sum_{n=0}^{\infty}\sum_{m=-n}^{n} z_n(\kappa_{\rm p}R)\phi_n^m X_n^m(\theta,\varphi), \tag{44}$$

where

$$z_n(t) = \frac{t h_n^{(1)'}(t)}{h_n^{(1)}(t)}$$

satisfies (cf. [28, Theorem 2.6.1])

$$-(n+1) \le {\rm Re} z_n(t) \le -1, \quad 0 < {\rm Im} z_n(t) \le t. \tag{45}$$

Evaluating the derivative of (43) with respect to r at $r = R$ and using (44), we get the transparent boundary condition for the scalar potential function ϕ:

$$\partial_r\phi = \mathscr{T}_1\phi \quad \text{on } \Gamma_R. \tag{46}$$

The following result can be easily shown from (44)–(45).

Lemma C.1. *The operator $\mathscr{T}_1$ is bounded from $H^{1/2}(\Gamma_R)$ to $H^{-1/2}(\Gamma_R)$. Moreover, it satisfies*

$${\rm Re}\langle\mathscr{T}_1 u, u\rangle_{\Gamma_R} \le 0, \quad {\rm Im}\langle\mathscr{T}_1 u, u\rangle_{\Gamma_R} \ge 0, \quad \forall u\in H^{1/2}(\Gamma_R).$$

If ${\rm Re}\langle\mathscr{T}_1 u, u\rangle_{\Gamma_R} = 0$ *or* ${\rm Im}\langle\mathscr{T}_1 u, u\rangle_{\Gamma_R} = 0$*, then $u = 0$ on Γ_R.*

Next is to derive the TBC for the vector potential function $\boldsymbol{\psi}$. Define an auxiliary function $\boldsymbol{\varphi} = (\mathrm{i}\kappa_{\mathrm{s}})^{-1}\nabla \times \boldsymbol{\psi}$. We have from (42) that

$$\nabla \times \boldsymbol{\psi} - \mathrm{i}\kappa_{\mathrm{s}}\boldsymbol{\varphi} = 0, \quad \nabla \times \boldsymbol{\varphi} + \mathrm{i}\kappa_{\mathrm{s}}\boldsymbol{\psi} = 0, \tag{47}$$

which are Maxwell's equations. Hence $\boldsymbol{\phi}$ and $\boldsymbol{\psi}$ plays the role of the electric field and the magnetic field, respectively.

Introduce the vector wave functions

$$\begin{cases} \boldsymbol{M}_n^m(r,\theta,\varphi) = \nabla \times (\boldsymbol{x} h_n^{(1)}(\kappa_{\mathrm{s}} r) X_n^m(\theta,\varphi)), \\ \boldsymbol{N}_n^m(r,\theta,\varphi) = (\mathrm{i}\kappa_{\mathrm{s}})^{-1}\nabla \times \boldsymbol{M}_n^m(r,\theta,\varphi), \end{cases} \tag{48}$$

which are the radiation solutions of (47) in $\mathbb{R}^3 \setminus \{0\}$ (cf. [26, Theorem 9.16]):

$$\nabla \times \boldsymbol{M}_n^m(r,\theta,\varphi) - \mathrm{i}\kappa_{\mathrm{s}}\boldsymbol{N}_n^m(r,\theta,\varphi) = 0, \quad \nabla \times \boldsymbol{N}_n^m(r,\theta,\varphi) + \mathrm{i}\kappa_{\mathrm{s}}\boldsymbol{M}_n^m(r,\theta,\varphi) = 0.$$

Moreover, it can be verified from (48) that they satisfy

$$\boldsymbol{M}_n^m = h_n^{(1)}(\kappa_{\mathrm{s}} r)\nabla_{\Gamma_R} X_n^m \times \boldsymbol{e}_r \tag{49}$$

and

$$\boldsymbol{N}_n^m = \frac{\sqrt{n(n+1)}}{\mathrm{i}\kappa_{\mathrm{s}} r}(h_n^{(1)}(\kappa_{\mathrm{s}} r) + \kappa_{\mathrm{s}} r h_n^{(1)'}(\kappa_{\mathrm{s}} r))\boldsymbol{T}_n^m + \frac{n(n+1)}{\mathrm{i}\kappa_{\mathrm{s}} r} h_n^{(1)}(\kappa_{\mathrm{s}} r)\boldsymbol{W}_n^m. \tag{50}$$

In the domain $\mathbb{R}^3 \setminus \bar{B}_R$, the solution of $\boldsymbol{\psi}$ in (47) can be written in the series

$$\boldsymbol{\psi} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} \alpha_n^m \boldsymbol{N}_n^m + \beta_n^m \boldsymbol{M}_n^m, \tag{51}$$

which is uniformly convergent on any compact subsets in $\mathbb{R}^3 \setminus \bar{B}_R$. Correspondingly, the solution of $\boldsymbol{\varphi}$ in (47) is given by

$$\boldsymbol{\varphi} = (\mathrm{i}\kappa_{\mathrm{s}})^{-1}\nabla \times \boldsymbol{\psi} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} \beta_n^m \boldsymbol{N}_n^m - \alpha_n^m \boldsymbol{M}_n^m. \tag{52}$$

It follows from (49)–(50) that

$$\begin{aligned} -\boldsymbol{e}_r \times (\boldsymbol{e}_r \times \boldsymbol{M}_n^m) &= -\sqrt{n(n+1)} h_n^{(1)}(\kappa_{\mathrm{s}} r)\boldsymbol{V}_n^m, \\ -\boldsymbol{e}_r \times (\boldsymbol{e}_r \times \boldsymbol{N}_n^m) &= \frac{\sqrt{n(n+1)}}{\mathrm{i}\kappa_{\mathrm{s}} r}(h_n^{(1)}(\kappa_{\mathrm{s}} r) + \kappa_{\mathrm{s}} r h_n^{(1)'}(\kappa_{\mathrm{s}} r))\boldsymbol{T}_n^m \end{aligned}$$

and

$$\begin{aligned} \boldsymbol{e}_r \times \boldsymbol{M}_n^m &= \sqrt{n(n+1)} h_n^{(1)}(\kappa_{\mathrm{s}} r)\boldsymbol{T}_n^m, \\ \boldsymbol{e}_r \times \boldsymbol{N}_n^m &= \frac{\sqrt{n(n+1)}}{\mathrm{i}\kappa_{\mathrm{s}} r}(h_n^{(1)}(\kappa_{\mathrm{s}} r) + \kappa_{\mathrm{s}} r h_n^{(1)'}(\kappa_{\mathrm{s}} r))\boldsymbol{V}_n^m. \end{aligned}$$

Therefore, by (51), the tangential component of $\boldsymbol{\psi}$ on Γ_R is

$$\begin{aligned} \boldsymbol{\psi}_{\Gamma_R} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} &\frac{\sqrt{n(n+1)}}{\mathrm{i}\kappa_{\mathrm{s}} R}(h_n^{(1)}(\kappa_{\mathrm{s}} R) + \kappa_{\mathrm{s}} R h_n^{(1)'}(\kappa_{\mathrm{s}} R))\alpha_n^m \boldsymbol{T}_n^m \\ &+ \sqrt{n(n+1)} h_n^{(1)}(\kappa_{\mathrm{s}} R)\beta_n^m \boldsymbol{V}_n^m. \end{aligned}$$

Similarly, by (52), the tangential trace of $\boldsymbol{\varphi}$ on Γ_R is

$$\boldsymbol{\varphi}\times\boldsymbol{e}_r=\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\sqrt{n(n+1)}h_n^{(1)}(\kappa_{\rm s}R)\alpha_n^m\boldsymbol{T}_n^m - \frac{\sqrt{n(n+1)}}{{\rm i}\kappa_{\rm s}R}(h_n^{(1)}(\kappa_{\rm s}R)+\kappa_{\rm s}Rh_n^{(1)'}(\kappa_{\rm s}R))\beta_n^m\boldsymbol{V}_n^m.$$

Given any tangential component of the electric field on Γ_R with the expression

$$\boldsymbol{u}=\sum_{n=0}^{\infty}\sum_{m=-n}^{n}u_{1n}^m\boldsymbol{T}_n^m+u_{2n}^m\boldsymbol{V}_n^m,$$

where

$$u_{1n}^m=\int_{\Gamma_R}\boldsymbol{u}(R,\theta,\varphi)\cdot\bar{\boldsymbol{T}}_n^m(\theta,\varphi){\rm d}\gamma,\quad u_{2n}^m=\int_{\Gamma_R}\boldsymbol{u}(R,\theta,\varphi)\cdot\bar{\boldsymbol{V}}_n^m(\theta,\varphi){\rm d}\gamma,$$

we define

$$\mathscr{T}_2\boldsymbol{u}=\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\frac{{\rm i}\kappa_{\rm s}R}{1+z_n(\kappa_{\rm s}R)}u_{1n}^m\boldsymbol{T}_n^m+\frac{1+z_n(\kappa_{\rm s}R)}{{\rm i}\kappa_{\rm s}R}u_{2n}^m\boldsymbol{V}_n^m. \tag{53}$$

Using (53), we obtain the TBC for the vector potential $\boldsymbol{\psi}$:

$$(\nabla\times\boldsymbol{\psi})\times\boldsymbol{e}_r={\rm i}\kappa_{\rm s}\mathscr{T}_2\boldsymbol{\psi}_{\Gamma_R}\quad\text{on }\Gamma_R. \tag{54}$$

The following result can also be easily shown from (45) and (53)

Lemma C.2. *The operator $\mathscr{T}_2$ is bounded from $\boldsymbol{H}^{1/2}({\rm curl},\Gamma_R)$ to $\boldsymbol{H}^{-1/2}({\rm div},\Gamma_R)$. Moreover, it satisfies*

$${\rm Re}\langle\mathscr{T}_2\boldsymbol{u},\boldsymbol{u}\rangle_{\Gamma_R}\geq 0,\quad\forall\boldsymbol{u}\in\boldsymbol{H}^{1/2}({\rm curl},\Gamma_R).$$

If ${\rm Re}\langle\mathscr{T}_2\boldsymbol{u},\boldsymbol{u}\rangle_{\Gamma_R}=0$, then $\boldsymbol{u}=0$ on Γ_R.

Appendix D. **Fourier coefficients.** Recalling the Helmholtz decomposition (7):

$$\boldsymbol{v}=\nabla\phi+\nabla\times\boldsymbol{\psi},\quad\nabla\cdot\boldsymbol{\psi}=0,$$

we derive the mutual representations of the Fourier coefficients between $\boldsymbol{v}$ and $(\phi,\boldsymbol{\psi})$.

First we have from (43) that

$$\phi(r,\theta,\varphi)=\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\frac{h_n^{(1)}(\kappa_{\rm p}r)}{h_n^{(1)}(\kappa_{\rm p}R)}\phi_n^mX_n^m(\theta,\varphi). \tag{55}$$

Substituting (49)–(50) into (51) yields

$$\boldsymbol{\psi}(r,\theta,\varphi)=\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\frac{\sqrt{n(n+1)}}{{\rm i}\kappa_{\rm s}r}(h_n^{(1)}(\kappa_{\rm s}r)+\kappa_{\rm s}rh_n^{(1)'}(\kappa_{\rm s}r))\alpha_n^m\boldsymbol{T}_n^m + \sqrt{n(n+1)}h_n^{(1)}(\kappa_{\rm s}r)\beta_n^m\boldsymbol{V}_n^m+\frac{n(n+1)}{{\rm i}\kappa_{\rm s}r}h_n^{(1)}(\kappa_{\rm s}r)\alpha_n^m\boldsymbol{W}_n^m. \tag{56}$$

Given $\boldsymbol{\psi}$ on Γ_R, it has the Fourier expansion:

$$\boldsymbol{\psi}(R,\theta,\varphi)=\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\psi_{1n}^m\boldsymbol{T}_n^m(\theta,\varphi)+\psi_{2n}^m\boldsymbol{V}_n^m(\theta,\varphi)+\psi_{3n}^m\boldsymbol{W}_n^m(\theta,\varphi), \tag{57}$$

where the Fourier coefficients

$$\psi_{1n}^m = \int_{\Gamma_R} \boldsymbol{\psi}(R,\theta,\varphi)\cdot \bar{\boldsymbol{T}}_n^m(\theta,\varphi)\mathrm{d}\gamma,$$
$$\psi_{2n}^m = \int_{\Gamma_R} \boldsymbol{\psi}(R,\theta,\varphi)\cdot \bar{\boldsymbol{V}}_n^m(\theta,\varphi)\mathrm{d}\gamma,$$
$$\psi_{3n}^m = \int_{\Gamma_R} \boldsymbol{\psi}(R,\theta,\varphi)\cdot \bar{\boldsymbol{W}}_n^m(\theta,\varphi)\mathrm{d}\gamma.$$

Evaluating (56) at $r=R$ and then comparing it with (57), we get

$$\alpha_n^m = \frac{\mathrm{i}\kappa_{\rm s}R}{n(n+1)h_n^{(1)}(\kappa_{\rm s}R)}\psi_{3n}^m, \quad \beta_n^m = \frac{1}{\sqrt{n(n+1)}h_n^{(1)}(\kappa_{\rm s}R)}\psi_{2n}^m. \tag{58}$$

Plugging (58) back into (56) gives

$$\begin{aligned}\boldsymbol{\psi}(r,\theta,\varphi) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}&\left(\frac{R}{r}\right)\left(\frac{h_n^{(1)}(\kappa_{\rm s}r)+\kappa_{\rm s}rh_n^{(1)'}(\kappa_{\rm s}r)}{\sqrt{n(n+1)}h_n^{(1)}(\kappa_{\rm s}R)}\right)\psi_{3n}^m\boldsymbol{T}_n^m\\ &+\left(\frac{h_n^{(1)}(\kappa_{\rm s}r)}{h_n^{(1)}(\kappa_{\rm s}R)}\right)\psi_{2n}^m\boldsymbol{V}_n^m + \left(\frac{R}{r}\right)\left(\frac{h_n^{(1)}(\kappa_{\rm s}r)}{h_n^{(1)}(\kappa_{\rm s}R)}\right)\psi_{3n}^m\boldsymbol{W}_n^m.\end{aligned} \tag{59}$$

In the spherical coordinates, we have from (55) and (59) that

$$\begin{aligned}\nabla\phi &= \partial_r\phi\,\boldsymbol{e}_r + \frac{1}{r}\nabla_{\Gamma_R}\phi\\ &= \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left(\frac{\kappa_{\rm p}h_n^{(1)'}(\kappa_{\rm p}r)}{h_n^{(1)}(\kappa_{\rm p}R)}\right)\phi_n^m X_n^m\boldsymbol{e}_r + \left(\frac{h_n^{(1)}(\kappa_{\rm p}r)}{rh_n^{(1)}(\kappa_{\rm p}R)}\right)\phi_n^m\nabla_{\Gamma_R}X_n^m\\ &= \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left(\frac{\kappa_{\rm p}h_n^{(1)'}(\kappa_{\rm p}r)}{h_n^{(1)}(\kappa_{\rm p}R)}\right)\phi_n^m\boldsymbol{W}_n^m + \left(\frac{\sqrt{n(n+1)}h_n^{(1)}(\kappa_{\rm p}r)}{rh_n^{(1)}(\kappa_{\rm p}R)}\right)\phi_n^m\boldsymbol{T}_n^m.\end{aligned}$$

and

$$\nabla\times\boldsymbol{\psi} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\boldsymbol{I}_{1n}^m + \boldsymbol{I}_{2n}^m + \boldsymbol{I}_{3n}^m,$$

where

$$\begin{aligned}\boldsymbol{I}_{1n}^m &= \nabla\times\left[\left(\frac{R}{r}\right)\left(\frac{h_n^{(1)}(\kappa_{\rm s}r)+\kappa_{\rm s}rh_n^{(1)'}(\kappa_{\rm s}r)}{\sqrt{n(n+1)}h_n^{(1)}(\kappa_{\rm s}R)}\right)\psi_{3n}^m\boldsymbol{T}_n^m\right]\\ &= \frac{Rh_n^{(1)}(\kappa_{\rm s}r)}{\sqrt{n(n+1)}h_n^{(1)}(\kappa_{\rm s}R)}\left(\kappa_{\rm s}^2 - \frac{n(n+1)}{r^2}\right)\psi_{3n}^m\boldsymbol{V}_n^m,\\ \boldsymbol{I}_{2n}^m &= \nabla\times\left[\left(\frac{h_n^{(1)}(\kappa_{\rm s}r)}{h_n^{(1)}(\kappa_{\rm s}R)}\right)\psi_{2n}^m\boldsymbol{V}_n^m\right]\\ &= \left(\frac{h_n^{(1)}(\kappa_{\rm s}r)+\kappa_{\rm s}rh_n^{(1)'}(\kappa_{\rm s}r)}{rh_n^{(1)}(\kappa_{\rm s}R)}\right)\psi_{2n}^m\boldsymbol{T}_n^m + \frac{\sqrt{n(n+1)}h_n^{(1)}(\kappa_{\rm s}r)}{rh_n^{(1)}(\kappa_{\rm s}R)}\psi_{2n}^m\boldsymbol{W}_n^m,\\ \boldsymbol{I}_{3n}^m &= \nabla\times\left[\left(\frac{R}{r}\right)\left(\frac{h_n^{(1)}(\kappa_{\rm s}r)}{h_n^{(1)}(\kappa_{\rm s}R)}\right)\psi_{3n}^m\boldsymbol{W}_n^m\right]\end{aligned}$$

$$= \frac{R\sqrt{n(n+1)}h_n^{(1)}(\kappa_{\rm s}r)}{r^2 h_n^{(1)}(\kappa_{\rm s}R)}\psi_{3n}^m \boldsymbol{V}_n^m.$$

Combining the above equations, we obtain

$$\begin{aligned}
&\boldsymbol{v}(r,\theta,\varphi) = \nabla\phi(r,\theta,\varphi) + \nabla\times\boldsymbol{\psi}(r,\theta,\varphi)\\
&= \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left(\frac{\sqrt{n(n+1)}h_n^{(1)}(\kappa_{\rm p}r)}{rh_n^{(1)}(\kappa_{\rm p}R)}\phi_n^m + \frac{(h_n^{(1)}(\kappa_{\rm s}r)+\kappa_{\rm s}rh_n^{(1)'}(\kappa_{\rm s}r))}{rh_n^{(1)}(\kappa_{\rm s}R)}\psi_{2n}^m\right)\boldsymbol{T}_n^m\\
&\quad + \left(\frac{\kappa_{\rm p}h_n^{(1)'}(\kappa_{\rm p}r)}{h_n^{(1)}(\kappa_{\rm p}R)}\phi_n^m + \frac{\sqrt{n(n+1)}h_n^{(1)}(\kappa_{\rm s}r)}{rh_n^{(1)}(\kappa_{\rm s}R)}\psi_{2n}^m\right)\boldsymbol{W}_n^m\\
&\quad + \frac{\kappa_{\rm s}^2 R h_n^{(1)}(\kappa_{\rm s}r)}{\sqrt{n(n+1)}h_n^{(1)}(\kappa_{\rm s}R)}\psi_{3n}^m\boldsymbol{V}_n^m,
\end{aligned} \tag{60}$$

which gives

$$\begin{aligned}
\boldsymbol{v}(R,\theta,\varphi) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}&\frac{1}{R}\left(\sqrt{n(n+1)}\phi_n^m + (1+z_n(\kappa_{\rm s}R))\psi_{2n}^m\right)\boldsymbol{T}_n^m\\
&+\frac{\kappa_{\rm s}^2 R}{\sqrt{n(n+1)}}\psi_{3n}^m\boldsymbol{V}_n^m + \frac{1}{R}\left(z_n(\kappa_{\rm p}R)\phi_n^m + \sqrt{n(n+1)}\psi_{2n}^m\right)\boldsymbol{W}_n^m.
\end{aligned} \tag{61}$$

On the other hand, $\boldsymbol{v}$ has the Fourier expansion:

$$\boldsymbol{v}(R,\theta,\varphi) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} v_{1n}^m\boldsymbol{T}_n^m + v_{2n}^m\boldsymbol{V}_n^m + v_{3n}^m\boldsymbol{W}_n^m. \tag{62}$$

Comparing (61) with (62), we obtain

$$\begin{cases}
v_{1n}^m = \dfrac{\sqrt{n(n+1)}}{R}\phi_n^m + \dfrac{(1+z_n(\kappa_{\rm s}R))}{R}\psi_{2n}^m,\\
v_{2n}^m = \dfrac{\kappa_{\rm s}^2 R}{\sqrt{n(n+1)}}\psi_{3n}^m,\\
v_{3n}^m = \dfrac{z_n(\kappa_{\rm p}R)}{R}\phi_n^m + \dfrac{\sqrt{n(n+1)}}{R}\psi_{2n}^m,
\end{cases} \tag{63}$$

and

$$\begin{cases}
\phi_n^m = \dfrac{R(1+z_n(\kappa_{\rm s}R))}{\Lambda_n}v_{3n}^m - \dfrac{R\sqrt{n(n+1)}}{\Lambda_n}v_{1n}^m,\\
\psi_{2n}^m = \dfrac{Rz_n(\kappa_{\rm p}R)}{\Lambda_n}v_{1n}^m - \dfrac{R\sqrt{n(n+1)}}{\Lambda_n}v_{3n}^m,\\
\psi_{3n}^m = \dfrac{\sqrt{n(n+1)}}{\kappa_{\rm s}^2 R}v_{2n}^m,
\end{cases} \tag{64}$$

where

$$\Lambda_n = z_n(\kappa_{\rm p}R)(1+z_n(\kappa_{\rm s}R)) - n(n+1).$$

Noting (45), we have from a simple calculation that

$$\mathrm{Im}\Lambda_n = \mathrm{Re}z_n(\kappa_{\rm p}R)\mathrm{Im}z_n(\kappa_{\rm s}R) + (1+\mathrm{Re}z_n(\kappa_{\rm s}R))\mathrm{Im}z_n(\kappa_{\rm p}R) < 0,$$

which implies that $\Lambda_n \neq 0$ for $n = 0, 1, \ldots$.

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