

# Stability for the inverse source problems in elastic and electromagnetic waves



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## ABSTRACT

This paper concerns the inverse source problems for the time-harmonic elastic and electromagnetic wave equations. The goal is to determine the external force and the electric current density from boundary measurements of the radiated wave field, respectively. The problems are challenging due to the ill-posedness and complex model systems. Uniqueness and stability are established for both of the inverse source problems. Based on either continuous or discrete multi-frequency data, a unified increasing stability theory is developed. The stability estimates consist of two parts: the Lipschitz type data discrepancy and the high frequency tail of the source functions. As the upper bound of frequencies increases, the latter decreases and thus becomes negligible. The increasing stability results reveal that ill-posedness of the inverse problems can be overcome by using multi-frequency data. The method is based on integral equations and analytical continuation, and requires the Dirichlet data only. The analysis employs asymptotic expansions of Green's tensors and the transparent boundary conditions by using the Dirichlet-to-Neumann maps. In addition, for the first time, the stability is established on the inverse source problems for both the Navier and Maxwell equations.

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## R É S U M É

Le présent article concerne les problèmes inverses de sources pour les équations d'ondes élastiques et électromagnétiques en régime harmonique. L'objectif est de déterminer respectivement la force externe et la densité de courant électrique à partir de mesures des ondes sur le bord. Ces problèmes inverses sont difficiles en raison de leur caractère mal-posé et de la complexité des modèles associés. L'unicité et la stabilité de solutions sont établies pour les deux problèmes inverses de source. Une approche globale de stabilité croissante a été développée pour des données continues et discrètes. Ces estimations de stabilité contiennent un terme de type Lipschitz et un reste contrôlé par la plus haute fréquence de l'onde. Quand cette dernière augmente le reste devient négligeable. Ces estimations de stabilité révèlent que le caractère mal-posé des problèmes inverses étudiés peut être contourné en

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utilisant des données multi-fréquentielle. La méthode utilisée, est basée sur les équations intégrales et la continuation unique, et nécessite uniquement des données de type Dirichlet. L'analyse utilise des développements asymptotiques des tenseurs de Green, et des conditions au bord de type transparente en utilisant l'opérateur Dirichlet-Neumann. Enfin, pour la première fois des estimations de stabilité sont établies pour les problèmes inverses de source associés aux équations de Navier et de Maxwell.

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## 1. Introduction

The inverse source problems in waves arise in many scientific and industrial areas such as antenna design and synthesis, biomedical imaging, and photo-acoustic tomography [7]. For instance, in medical imaging, such as magnetoencephalography (MEG), the imaging modality is a non-invasive neurophysiological technique that measures the electric or magnetic fields generated by neuronal activity of the brain [4,27,47]. The spatial distributions of the measured fields are analyzed to localize the sources of the activity within the brain to provide information about both the structure and function of the brain. The inverse source problems are also considered as a basic mathematical tool for solving many imaging problems including reflection tomography, diffusion-based optical tomography, lidar imaging for chemical and biological threat detection, and fluorescence microscopy [31].

Motivated by these significant applications, the inverse source problems, as an important research subject in inverse scattering theory, have continuously attracted much attention by many researchers [2,4,8,9,11–13, 23,36,37,56]. Consequently, a great deal of mathematical and numerical results are available, especially for the acoustic waves or the Helmholtz equations. In general, it is known that there is no uniqueness for the inverse source problem at a fixed frequency due to the existence of non-radiating sources [18,24,28]. Therefore, additional information is required for the source in order to obtain a unique solution, such as to seek the minimum energy solution [43]. From the computational point of view, a more challenging issue is the lack of stability. A small variation of the data might lead to a huge error in the reconstruction. Recently, it has been realized that the use of multi-frequency data is an effective approach to overcome the difficulties of non-uniqueness and instability which are encountered at a single frequency. In [15], Bao et al. initialized the mathematical study on the stability of the inverse source problem for the Helmholtz equation by using multi-frequency data. The increasing stability was further studied in [20], [39] for the inverse source problem of the three-dimensional Helmholtz equation. Based on the Huygens principle, the method assumes a special form of the source function, and requires both the Dirichlet and Neumann boundary data. A different approach was developed in [39] to obtain the same increasing stability result for both the two- and three-dimensional Helmholtz equation. The method removes the assumption on the source function and requires the Dirichlet data only. An attempt was made in [40] to extend the stability result to the inverse random source of the one-dimensional stochastic Helmholtz equation. We refer to [1,16,26,57] for the study of the inverse source problems by using multiple frequency information. A topical review can be found in [14] on the inverse source problems as well as other inverse scattering problems by using multiple frequencies to overcome the ill-posedness and gain increased stability. We also refer to [33] on the increasing stability of determining potentials for the Schrödinger equation. Related results can be found in [6,30,32] on the increasing stability in the solution of the Cauchy problem for the acoustic and electromagnetic wave equations.

Although a lot of work has been done on the inverse source problem for acoustic waves, little is known on the inverse source problems for elastic and electromagnetic waves, especially their stability. This work initializes the mathematical study and provides the first stability results of the inverse source problems for elastic and electromagnetic waves. Our objective is to develop a unified stability theory of the inverse source problems for elastic and electromagnetic waves. It significantly extends the previous approaches for

the Helmholtz equations to handle the more complicated Navier and Maxwell equations. Especially, more delicate studies are needed for sophisticated Green's tensors of these two wave equations. The results shed light on the stability analysis of the more challenging inverse medium and obstacle scattering problems [14]. In addition, they motivate further study of the time-domain inverse problem where all frequencies are available in order to gain better stability [17]. It should also be pointed out that the general case is widely open for the inverse source problems on these vector wave equations in inhomogeneous media. General references on elastic and electromagnetic wave scattering problems may be found in [3,10,21,35,38,44,46,53] and [22,25,29,48,51,52,55], respectively.

For electromagnetic waves, Ammari et al. [4] showed uniqueness and stability, and presented an inversion scheme to reconstruct dipole sources based on a low-frequency asymptotic analysis of the time-harmonic Maxwell equations. In [2], Albanese and Monk discussed uniqueness and non-uniqueness of the inverse source problems for Maxwell's equations. A monograph can be found in [51] on general inverse problems for Maxwell's equations. We refer to [34,41,42,55] for solving inverse source problems on hyperbolic systems by using Carleman estimates, and to [49,50] for inverse problems which are related to Maxwell's equations. To the best of our knowledge, there is no stability result for the inverse source problem of Maxwell's equations in a general setting. The questions are completely open regarding uniqueness and stability for the inverse source problem of the elastic wave equation. In this paper, we develop new techniques and establish a unified increasing stability theory in the inverse source scattering problems for both elastic and electromagnetic waves, where the wave propagation is governed by the two- or three-dimensional Navier equation and the three dimensional Maxwell equations, respectively.

For elastic waves, the inverse source problem is to determine the external force that produces the measured displacement. We show a uniqueness result and demonstrate that the increasing stability can be achieved by using the Dirichlet boundary data only at multiple frequencies. For electromagnetic waves, the inverse source problem is to reconstruct the electric current density from the tangential trace of the electric field. First we discuss the uniqueness of the problem and distinguish the detectable radiating sources from the non-radiating sources. Then we prove that the increasing stability can be obtained to reconstruct the radiating electric current densities from the boundary measurement at multiple frequencies. For each wave, we give the stability estimates for both the continuous frequency data and the discrete frequency data. The estimates consist of two parts: the first part is the Lipschitz type of data discrepancy and the second part is the high frequency tail of the source function. The former is analyzed via the Green tensor. The latter is estimated by the analytical continuation, which decreases as the frequency of the data increases. The results reveal that the ill-posedness of the inverse problems can be overcome and the inverse problems are stable when multi-frequency data is used. In our analysis, the main ingredients are to use the transparent boundary conditions and Green's tensors for the wave equations. The transparent boundary condition establishes the relation between the Dirichlet data and the Neumann data. The Neumann data can not only be represented in terms of the Dirichlet data, but also be computed once the Dirichlet data is available in practice.

Throughout, we assume that the source of either the external force or the electric current density has a compact support  $\Omega \subset \mathbb{R}^d$ ,  $d = 2$  or  $3$ . Let  $\hat{R} > 0$  be a sufficiently large constant such that  $\bar{\Omega} \subset B_{\hat{R}} = \{\mathbf{x} = (x_1, \dots, x_d)^\top \in \mathbb{R}^d : |\mathbf{x}| < \hat{R}\}$ . Let  $R > \hat{R}$  be a constant such that  $B_{\hat{R}} \subset B_R = \{\mathbf{x} \in \mathbb{R}^d : |\mathbf{x}| < R\}$ . Denote by  $\Gamma_R = \{\mathbf{x} \in \mathbb{R}^d : |\mathbf{x}| = R\}$  the boundary of  $B_R$  where the measurement of the wave field is taken. Let  $U_R = (-R, R)^d$  be a rectangular box in  $\mathbb{R}^d$ . Clearly we have  $\Omega \subset B_{\hat{R}} \subset B_R \subset U_R$ . The problem geometry is shown in Fig. 1.

The paper is organized as follows. In Section 2, we show the increasing stability of the inverse source problem for elastic waves. Section 3 is devoted to the inverse source problem for electromagnetic waves. The uniqueness and non-uniqueness are discussed and the increasing stability is obtained. In both sections, the analysis is carried for the continuous frequency data, followed by the discussion for the discrete frequency data. The paper is concluded with some general remarks in Section 4. To make the paper easily accessible,

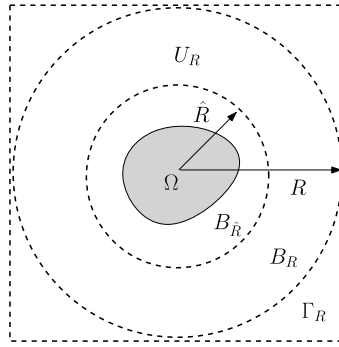


Fig. 1. Problem geometry of the inverse source scattering.

some necessary notations and useful results are provided in the appendices on the differential operators, Helmholtz decomposition, and Sobolev spaces.

## 2. Elastic waves

This section addresses the inverse source problem for elastic waves. The uniqueness and increasing stability are established to reconstruct the external force from the boundary measurement of the displacement at multiple frequencies.

### 2.1. Problem formulation

Consider the time-harmonic Navier equation in a homogeneous medium:

$$\mu \Delta \mathbf{u} + (\lambda + \mu) \nabla \nabla \cdot \mathbf{u} + \omega^2 \mathbf{u} = \mathbf{f} \quad \text{in } \mathbb{R}^d, \tag{2.1}$$

where  $\omega > 0$  is the angular frequency,  $\lambda$  and  $\mu$  are the Lamé constants satisfying  $\mu > 0$  and  $\lambda + \mu > 0$ ,  $\mathbf{u} \in \mathbb{C}^d$  is the displacement field, and  $\mathbf{f} \in \mathbb{C}^d$  accounts for the external force which is assumed to have a compact support  $\Omega \subset \mathbb{R}^d$ .

An appropriate radiation condition is needed to complete the definition of the scattering problem since it is imposed in the open domain. As discussed in Appendix B, the displacement  $\mathbf{u}$  can be decomposed into the compressional part  $\mathbf{u}_p$  and the shear part  $\mathbf{u}_s$ :

$$\mathbf{u} = \mathbf{u}_p + \mathbf{u}_s \quad \text{in } \mathbb{R}^d \setminus \bar{\Omega}.$$

The Kupradze–Sommerfeld radiation condition requires that  $\mathbf{u}_p$  and  $\mathbf{u}_s$  satisfy the Sommerfeld radiation condition:

$$\lim_{r \rightarrow \infty} r^{\frac{d-1}{2}} (\partial_r \mathbf{u}_p - i\kappa_p \mathbf{u}_p) = 0, \quad \lim_{r \rightarrow \infty} r^{\frac{d-1}{2}} (\partial_r \mathbf{u}_s - i\kappa_s \mathbf{u}_s) = 0, \quad r = |\mathbf{x}|, \tag{2.2}$$

where  $\kappa_p, \kappa_s$  are the compressional and shear wavenumbers, given by

$$\kappa_p = \frac{\omega}{(\lambda + 2\mu)^{1/2}} = c_p \omega, \quad \kappa_s = \frac{\omega}{\mu^{1/2}} = c_s \omega,$$

where

$$c_p = (\lambda + 2\mu)^{-1/2}, \quad c_s = \mu^{-1/2}. \tag{2.3}$$

Note that  $c_p, c_s$  are independent of  $\omega$  and  $c_p < c_s$ .

Given  $\mathbf{f} \in L^2(\Omega)^d$ , it is known that the scattering problem (2.1)–(2.2) has a unique solution (cf. [12]):

$$\mathbf{u}(\mathbf{x}, \omega) = \int_{\Omega} \mathbf{G}_N(\mathbf{x}, \mathbf{y}; \omega) \cdot \mathbf{f}(\mathbf{y}) d\mathbf{y}, \tag{2.4}$$

where  $\mathbf{G}_N(\mathbf{x}, \mathbf{y}; \omega) \in \mathbb{C}^{d \times d}$  is Green’s tensor for the Navier equation (2.1) and the dot is the matrix-vector multiplication. Explicitly, we have

$$\mathbf{G}_N(\mathbf{x}, \mathbf{y}; \omega) = \frac{1}{\mu} g_d(\mathbf{x}, \mathbf{y}; \kappa_s) \mathbf{I}_d + \frac{1}{\omega^2} \nabla_{\mathbf{x}} \nabla_{\mathbf{x}}^{\top} (g_d(\mathbf{x}, \mathbf{y}; \kappa_s) - g_d(\mathbf{x}, \mathbf{y}; \kappa_p)), \tag{2.5}$$

where  $\mathbf{I}_d$  is the  $d \times d$  identity matrix,

$$g_2(\mathbf{x}, \mathbf{y}; \kappa) = \frac{i}{4} H_0^{(1)}(\kappa |\mathbf{x} - \mathbf{y}|) \quad \text{and} \quad g_3(\mathbf{x}, \mathbf{y}; \kappa) = \frac{1}{4\pi} \frac{e^{i\kappa |\mathbf{x} - \mathbf{y}|}}{|\mathbf{x} - \mathbf{y}|} \tag{2.6}$$

are the fundamental solutions for the two- and three-dimensional Helmholtz equation, respectively, and  $H_0^{(1)}$  is the Hankel function of the first kind with order zero.

Define a boundary operator

$$D\mathbf{u} = \mu \partial_{\nu} \mathbf{u} + (\lambda + \mu) (\nabla \cdot \mathbf{u}) \boldsymbol{\nu} \quad \text{on } \Gamma_R, \tag{2.7}$$

where  $\boldsymbol{\nu}$  is the unit normal vector on  $\Gamma_R$ . It is shown in [19,38] that there exists a Dirichlet-to-Neumann (DtN) operator  $T_N$  such that

$$D\mathbf{u} = T_N \mathbf{u} \quad \text{on } \Gamma_R, \tag{2.8}$$

which is the transparent boundary condition for the scattering problem of the Navier equation.

**Problem 2.1** (Continuous frequency data for elastic waves). *Let the external force  $\mathbf{f}$  be a complex function with the compact support  $\Omega$ . The inverse source problem is to determine  $\mathbf{f}$  from the displacement  $\mathbf{u}(\mathbf{x}, \omega)$ ,  $\mathbf{x} \in \Gamma_R, \omega \in (0, K)$ , where  $K > 1$  is a constant.*

**Remark 2.2.** The boundary data does not have to be measured on the sphere  $\Gamma_R$ . In fact, it can be measured on any Lipschitz continuous boundary  $\Gamma$  which encloses the compact support of  $\mathbf{f}$ , e.g., take  $\Gamma = \partial\Omega$ . When  $\mathbf{u}$  is available on  $\Gamma$ , we may consider the following boundary value problem:

$$\begin{cases} \mu \Delta \mathbf{u} + (\lambda + \mu) \nabla \nabla \cdot \mathbf{u} + \omega^2 \mathbf{u} = 0 & \text{in } B_R \setminus \bar{\Omega}, \\ \mathbf{u} = \mathbf{u} & \text{on } \Gamma, \\ D\mathbf{u} = T_N \mathbf{u} & \text{on } \Gamma_R. \end{cases} \tag{2.9}$$

It can be shown that the problem (2.9) has a unique solution  $\mathbf{u}$  in  $B_R \setminus \bar{\Omega}$  [38]. Therefore, the Dirichlet data  $\mathbf{u}$  is immediately available on  $\Gamma_R$  once the problem (2.9) is solved, and then the Neumann data  $T_N \mathbf{u}$  can be computed on  $\Gamma_R$  by using (2.8).

### 2.2. Uniqueness

This section is concerned with uniqueness of the inverse problem. Introduce two auxiliary functions:

$$\mathbf{u}_p^{\text{inc}}(\mathbf{x}) = \mathbf{p} e^{-i\kappa_p \mathbf{x} \cdot \mathbf{d}} \quad \text{and} \quad \mathbf{u}_s^{\text{inc}}(\mathbf{x}) = \mathbf{q} e^{-i\kappa_s \mathbf{x} \cdot \mathbf{d}}, \tag{2.10}$$

where  $\mathbf{d} \in \mathbb{S}^{d-1}$  is the unit propagation direction vector and  $\mathbf{p}, \mathbf{q} \in \mathbb{S}^{d-1}$  are unit polarization vectors. These unit vectors may be chosen as follows:

- (i) For  $d = 2$ ,  $\mathbf{d}(\theta) = (\cos \theta, \sin \theta)^\top$ ,  $\mathbf{p}(\theta)$  and  $\mathbf{q}(\theta)$  satisfy  $\mathbf{p}(\theta) = \mathbf{d}(\theta)$  and  $\mathbf{q}(\theta) \cdot \mathbf{d}(\theta) = 0$  for all  $\theta \in [0, 2\pi]$ .
- (ii) For  $d = 3$ ,  $\mathbf{d}(\theta, \varphi) = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)^\top$ ,  $\mathbf{p}(\theta, \varphi)$  and  $\mathbf{q}(\theta, \varphi)$  satisfy  $\mathbf{p}(\theta, \varphi) = \mathbf{d}(\theta, \varphi)$  and  $\mathbf{q}(\theta, \varphi) \cdot \mathbf{d}(\theta, \varphi) = 0$  for all  $\theta \in [0, \pi], \varphi \in [0, 2\pi]$ .

In fact,  $\mathbf{u}_p^{\text{inc}}$  and  $\mathbf{u}_s^{\text{inc}}$  are known as the compressional and shear plane waves. It is easy to verify that they satisfy the homogeneous Navier equation:

$$\mu \Delta \mathbf{u} + (\lambda + \mu) \nabla \nabla \cdot \mathbf{u} + \omega^2 \mathbf{u} = 0 \quad \text{in } \mathbb{R}^d. \tag{2.11}$$

**Theorem 2.3.** *Let  $I \subset \mathbb{R}^+$  be an open interval. Then the external force  $\mathbf{f}$  can be uniquely determined by the multiple-frequency data  $\{\mathbf{u}(\mathbf{x}, \omega) : \mathbf{x} \in \Gamma_R, \omega \in I\}$ .*

**Proof.** We prove the two dimensional case in details, and then briefly present the proof for the three dimensional case since the steps are similar. Let  $\mathbf{u}(\mathbf{x}, \omega) = 0$  for  $\mathbf{x} \in \Gamma_R$  and  $\omega \in I$ . It suffices to show that  $\mathbf{f} = 0$ .

(i) Consider  $d = 2$ . Let  $\boldsymbol{\xi}_p = \kappa_p \mathbf{d}$ . The compressional plane wave in (2.10) can be written as  $\mathbf{u}_p^{\text{inc}}(\mathbf{x}) = \mathbf{p} e^{-i \boldsymbol{\xi}_p \cdot \mathbf{x}}$ . Multiplying the both sides of (2.1) by  $\mathbf{u}_p^{\text{inc}}(\mathbf{x})$ , using the integration by parts over  $B_R$ , and noting (2.11), we obtain

$$\int_{B_R} (\mathbf{p} e^{-i \boldsymbol{\xi}_p \cdot \mathbf{x}}) \cdot \mathbf{f}(\mathbf{x}) d\mathbf{x} = \int_{\Gamma_R} (\mathbf{u}_p^{\text{inc}}(\mathbf{x}) \cdot T_N \mathbf{u}(\mathbf{x}, \omega) + \mathbf{u}(\mathbf{x}, \omega) \cdot D \mathbf{u}_p^{\text{inc}}(\mathbf{x})) d\gamma(\mathbf{x}), \tag{2.12}$$

which means

$$\mathbf{p} \cdot \hat{\mathbf{f}}(\boldsymbol{\xi}_p) = 0, \quad \forall \omega \in I.$$

Since  $\hat{\mathbf{f}}(\kappa_p \mathbf{d}) = \hat{\mathbf{f}}(c_p \omega \mathbf{d})$  is an analytic function with respect to  $\omega \in \mathbb{C}$ , for each fixed  $\mathbf{d}$  and  $\mathbf{p}$ , we have  $\mathbf{p} \cdot \hat{\mathbf{f}}(\boldsymbol{\xi}_p) = 0$  for all  $\kappa_p \in (0, +\infty)$ .

Let  $\boldsymbol{\xi}_s = -\kappa_s \mathbf{d}$  with  $|\boldsymbol{\xi}_s| = \kappa_s \in (0, \infty)$ . The shear plane wave in (2.10) can be written as  $\mathbf{u}_s^{\text{inc}}(\mathbf{x}) = \mathbf{q} e^{-i \boldsymbol{\xi}_s \cdot \mathbf{x}}$ . Multiplying  $\mathbf{u}_s^{\text{inc}}$  on both sides of (2.1), using the integration by parts, we may similarly get  $\mathbf{q} \cdot \hat{\mathbf{f}}(\boldsymbol{\xi}_s) = 0$  for all  $\kappa_s \in (0, +\infty)$ . Hence, for each  $\kappa > 0$ , we have both  $\mathbf{p} \cdot \hat{\mathbf{f}}(\kappa \mathbf{d}) = 0$  and  $\mathbf{q} \cdot \hat{\mathbf{f}}(\kappa \mathbf{d}) = 0$ . Let  $\mathbf{p}(\theta) = (\cos \theta, \sin \theta)^\top$  and take  $\mathbf{q}(\theta) = (-\sin \theta, \cos \theta)^\top$ . Then  $\mathbf{p}(\theta) \cdot \mathbf{q}(\theta) = 0$  and they form an orthonormal basis in  $\mathbb{R}^2$  for any  $\theta \in [0, 2\pi]$ . Hence we have from the Pythagorean theorem that

$$|\hat{\mathbf{f}}(\kappa \mathbf{d})|^2 = |\mathbf{p} \cdot \hat{\mathbf{f}}(\kappa \mathbf{d})|^2 + |\mathbf{q} \cdot \hat{\mathbf{f}}(\kappa \mathbf{d})|^2 = 0$$

for each  $\kappa > 0$  and  $\mathbf{d} \in \mathbb{S}$ , which means  $\hat{\mathbf{f}} = 0$  and then  $\mathbf{f} = 0$ .

(ii) Consider  $d = 3$ . Repeating similar steps, we get both  $\mathbf{p} \cdot \hat{\mathbf{f}}(\kappa \mathbf{d}) = 0$  and  $\mathbf{q} \cdot \hat{\mathbf{f}}(\kappa \mathbf{d}) = 0$ . Let  $\mathbf{p}(\theta, \varphi) = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)^\top$ . We choose  $\mathbf{q}_1(\theta, \varphi) = (\cos \theta \cos \varphi, \cos \theta \sin \varphi, -\sin \theta)^\top$  and  $\mathbf{q}_2(\theta, \varphi) = \mathbf{p}(\theta, \varphi) \times \mathbf{q}_1(\theta, \varphi) = (-\sin \varphi, \cos \varphi, 0)^\top$  for the shear plane wave. It is easy to verify that  $\mathbf{p}, \mathbf{q}_1, \mathbf{q}_2$  are mutually orthogonal and thus form an orthonormal basis in  $\mathbb{R}^3$  for any  $\theta \in [0, \pi], \varphi \in [0, 2\pi]$ . Using the Pythagorean theorem yields

$$|\hat{\mathbf{f}}(\kappa \mathbf{d})|^2 = |\mathbf{p} \cdot \hat{\mathbf{f}}(\kappa \mathbf{d})|^2 + |\mathbf{q}_1 \cdot \hat{\mathbf{f}}(\kappa \mathbf{d})|^2 + |\mathbf{q}_2 \cdot \hat{\mathbf{f}}(\kappa \mathbf{d})|^2 = 0$$

for each  $\kappa > 0$  and  $\mathbf{d} \in \mathbb{S}^2$ , which means  $\hat{\mathbf{f}} = 0$  and then  $\mathbf{f} = 0$ .  $\square$

### 2.3. Stability with continuous frequency data

This section discusses the stability from the data with frequency ranging over a finite interval. Given the Dirichlet data  $\mathbf{u}$  on  $\Gamma_R$ ,  $D\mathbf{u}$  can be viewed as the Neumann data. It follows from (2.8) that the Neumann data can be computed via the DtN operator  $T_N$  once the Dirichlet data is available on  $\Gamma_R$ . Hence we may just define a boundary measurement in terms of the Dirichlet data only:

$$\|\mathbf{u}(\cdot, \omega)\|_{\Gamma_R}^2 = \int_{\Gamma_R} (|T_N \mathbf{u}(\mathbf{x}, \omega)|^2 + \omega^2 |\mathbf{u}(\mathbf{x}, \omega)|^2) \, d\gamma(\mathbf{x}).$$

Denote a functional space:

$$\mathbb{F}_M(B_R) = \{\mathbf{f} \in H^{m+1}(B_R)^d : \|\mathbf{f}\|_{H^{m+1}(B_R)^d} \leq M, \text{ supp } \mathbf{f} = \Omega\},$$

where  $m \geq d$  is an integer and  $M > 1$  is a constant. Hereafter, the notation “ $a \lesssim b$ ” stands for  $a \leq Cb$ , where  $C > 0$  is a generic constant independent of  $m, \omega, K, M$ , but may change step by step in the proofs.

The following stability estimate is the main result for Problem 2.1.

**Theorem 2.4.** *Let  $\mathbf{u}$  be the solution of the scattering problem (2.1)–(2.2) corresponding to the source  $\mathbf{f} \in \mathbb{F}_M(B_R)$ . Then*

$$\|\mathbf{f}\|_{L^2(B_R)^d}^2 \lesssim \epsilon_1^2 + \frac{M^2}{\left(\frac{K^{\frac{2}{3}} |\ln \epsilon_1|^{\frac{1}{4}}}{(R+1)(6m-6d+3)^3}\right)^{2m-2d+1}}, \tag{2.13}$$

where

$$\epsilon_1 = \left( \int_0^K \omega^{d-1} \|\mathbf{u}(\cdot, \omega)\|_{\Gamma_R}^2 \, d\omega \right)^{\frac{1}{2}}.$$

**Remark 2.5.** The stability estimate (2.13) consists of two parts: the data discrepancy and the high frequency tail. The former is of the Lipschitz type. The latter decreases as  $K$  increases which makes the problem have an almost Lipschitz stability. The result reveals that the problem becomes more stable when higher frequency data is used.

**Lemma 2.6.** *Let  $\mathbf{u}$  be the solution of the scattering problem (2.1)–(2.2) corresponding to the source  $\mathbf{f} \in L^2(B_R)^d$ . Then*

$$\|\mathbf{f}\|_{L^2(B_R)^d}^2 \lesssim \int_0^\infty \omega^{d-1} \|\mathbf{u}(\cdot, \omega)\|_{\Gamma_R}^2 \, d\omega.$$

**Proof.** Again, we prove the two dimensional case in details, and then briefly present the proof for the three dimensional case.

(i) Consider  $d = 2$ . Let  $\boldsymbol{\xi}_p = \kappa_p \mathbf{d}$  with  $|\boldsymbol{\xi}_p| = \kappa_p \in (0, \infty)$ . We have from (2.12) that

$$\int_{B_R} (\mathbf{p} e^{-i\boldsymbol{\xi}_p \cdot \mathbf{x}}) \cdot \mathbf{f}(\mathbf{x}) \, d\mathbf{x} = \int_{\Gamma_R} (\mathbf{u}_p^{\text{inc}}(\mathbf{x}) \cdot T_N \mathbf{u}(\mathbf{x}, \omega) + \mathbf{u}(\mathbf{x}, \omega) \cdot D\mathbf{u}_p^{\text{inc}}(\mathbf{x})) \, d\gamma(\mathbf{x}).$$

A simple calculation yields that

$$D\mathbf{u}_p^{\text{inc}}(\mathbf{x}) = -i\kappa_p(\mu(\mathbf{p} \cdot \boldsymbol{\nu})\mathbf{p} + (\lambda + \mu)\boldsymbol{\nu}) e^{-i\boldsymbol{\xi}_p \cdot \mathbf{x}},$$

which gives

$$|D\mathbf{u}_p^{\text{inc}}(\mathbf{x})| \lesssim \kappa_p.$$

Noting  $\text{supp } \mathbf{f} \subset B_R$ , we get

$$\int_{\tilde{B}_R} (\mathbf{p}e^{-i\boldsymbol{\xi}_p \cdot \mathbf{x}}) \cdot \mathbf{f}(\mathbf{x})d\mathbf{x} = \mathbf{p} \cdot \int_{\mathbb{R}^2} \mathbf{f}(\mathbf{x})e^{-i\boldsymbol{\xi}_p \cdot \mathbf{x}}d\mathbf{x} = \mathbf{p} \cdot \hat{\mathbf{f}}(\boldsymbol{\xi}_p).$$

Combining the above estimates and using the Cauchy–Schwarz inequality yields

$$|\mathbf{p} \cdot \hat{\mathbf{f}}(\boldsymbol{\xi}_p)|^2 \lesssim \int_{\Gamma_R} (|T_N \mathbf{u}(\mathbf{x}, \omega)|^2 + \kappa_p^2 |\mathbf{u}(\mathbf{x}, \omega)|^2) d\gamma(\mathbf{x}).$$

Hence

$$\int_{\mathbb{R}^2} |\mathbf{p} \cdot \hat{\mathbf{f}}(\boldsymbol{\xi}_p)|^2 d\boldsymbol{\xi}_p \lesssim \int_{\mathbb{R}^2} \int_{\Gamma_R} (|T_N \mathbf{u}(\mathbf{x}, \omega)|^2 + \kappa_p^2 |\mathbf{u}(\mathbf{x}, \omega)|^2) d\gamma(\mathbf{x})d\boldsymbol{\xi}_p.$$

Using the polar coordinates, we have

$$\begin{aligned} \int_{\mathbb{R}^2} |\mathbf{p} \cdot \hat{\mathbf{f}}(\boldsymbol{\xi}_p)|^2 d\boldsymbol{\xi}_p &\lesssim \int_0^{2\pi} d\theta \int_0^\infty \kappa_p \int_{\Gamma_R} (|T_N \mathbf{u}(\mathbf{x}, \omega)|^2 + \kappa_p^2 |\mathbf{u}(\mathbf{x}, \omega)|^2) d\gamma(\mathbf{x})d\kappa_p \\ &\leq 2\pi \int_0^\infty \kappa_p \int_{\Gamma_R} (|T_N \mathbf{u}(\mathbf{x}, \omega)|^2 + \kappa_p^2 |\mathbf{u}(\mathbf{x}, \omega)|^2) d\gamma(\mathbf{x})d\kappa_p \\ &\lesssim \int_0^\infty \omega \int_{\Gamma_R} (|T_N \mathbf{u}(\mathbf{x}, \omega)|^2 + \omega^2 |\mathbf{u}(\mathbf{x}, \omega)|^2) d\gamma(\mathbf{x})d\omega \\ &= \int_0^\infty \omega \|\mathbf{u}(\cdot, \omega)\|_{\Gamma_R}^2 d\omega. \end{aligned} \tag{2.14}$$

Let  $\boldsymbol{\xi}_s = \kappa_s \mathbf{d}$  with  $|\boldsymbol{\xi}_s| = \kappa_s \in (0, \infty)$ . The shear plane wave in (2.10) can be written as  $\mathbf{u}_s^{\text{inc}}(\mathbf{x}) = \mathbf{q}e^{-i\boldsymbol{\xi}_s \cdot \mathbf{x}}$ . Multiplying  $\mathbf{u}_s^{\text{inc}}$  on both sides of (2.1), using the integration by parts, and noting (2.11), we may similarly get

$$\begin{aligned} \int_{\mathbb{R}^2} |\mathbf{q} \cdot \hat{\mathbf{f}}(\boldsymbol{\xi}_s)|^2 d\boldsymbol{\xi}_s &= \int_{\mathbb{R}^2} \left| \int_{\mathbb{R}^2} (\mathbf{q}e^{-i\boldsymbol{\xi}_s \cdot \mathbf{x}}) \cdot \mathbf{f}(\mathbf{x})d\mathbf{x} \right|^2 d\boldsymbol{\xi}_s \\ &\lesssim \int_0^\infty \kappa_s \int_{\Gamma_R} (|T_N \mathbf{u}(\mathbf{x}, \omega)|^2 + \kappa_s^2 |\mathbf{u}(\mathbf{x}, \omega)|^2) d\gamma(\mathbf{x})d\kappa_s \end{aligned}$$



$$\begin{aligned}
 &\lesssim \int_0^\infty \omega \int_{\Gamma_R} (|T_N \mathbf{u}(\mathbf{x}, \omega)|^2 + \omega^2 |\mathbf{u}(\mathbf{x}, \omega)|^2) d\gamma(\mathbf{x}) d\omega \\
 &= \int_0^\infty \omega \|\mathbf{u}(\cdot, \omega)\|_{\Gamma_R}^2 d\omega.
 \end{aligned} \tag{2.15}$$

Using the polar coordinates, we deduce that

$$\begin{aligned}
 \int_{\mathbb{R}^2} |\mathbf{q} \cdot \hat{\mathbf{f}}(\boldsymbol{\xi}_s)|^2 d\boldsymbol{\xi}_s &= \int_0^{2\pi} d\theta \int_0^\infty \kappa_s |\mathbf{q}(\theta) \cdot \hat{\mathbf{f}}(\kappa_s \mathbf{p})|^2 d\kappa_s \\
 &= \int_0^{2\pi} d\theta \int_0^\infty \kappa_p |\mathbf{q}(\theta) \cdot \hat{\mathbf{f}}(\kappa_p \mathbf{p})|^2 d\kappa_p = \int_{\mathbb{R}^2} |\mathbf{q} \cdot \hat{\mathbf{f}}(\boldsymbol{\xi}_p)|^2 d\boldsymbol{\xi}_p.
 \end{aligned} \tag{2.16}$$

Let  $\mathbf{p}(\theta) = (\cos \theta, \sin \theta)^\top$  and take  $\mathbf{q}(\theta) = (-\sin \theta, \cos \theta)^\top$ . Then  $\mathbf{p}(\theta) \cdot \mathbf{q}(\theta) = 0$  and they form an orthonormal basis in  $\mathbb{R}^2$  for any  $\theta \in [0, 2\pi]$ . Hence we have from the Pythagorean theorem that

$$|\hat{\mathbf{f}}(\boldsymbol{\xi}_p)|^2 = |\mathbf{p} \cdot \hat{\mathbf{f}}(\boldsymbol{\xi}_p)|^2 + |\mathbf{q} \cdot \hat{\mathbf{f}}(\boldsymbol{\xi}_p)|^2. \tag{2.17}$$

Noting  $\text{supp } \mathbf{f} \subset B_R$  again, we obtain from the Parseval theorem and (2.14)–(2.17) that

$$\begin{aligned}
 \|\mathbf{f}\|_{L^2(B_R)^2}^2 &= \|\mathbf{f}\|_{L^2(\mathbb{R}^2)^2}^2 = \|\hat{\mathbf{f}}\|_{L^2(\mathbb{R}^2)^2}^2 = \int_{\mathbb{R}^2} |\hat{\mathbf{f}}(\boldsymbol{\xi}_p)|^2 d\boldsymbol{\xi}_p \\
 &= \int_{\mathbb{R}^2} |\mathbf{p} \cdot \hat{\mathbf{f}}(\boldsymbol{\xi}_p)|^2 d\boldsymbol{\xi}_p + \int_{\mathbb{R}^2} |\mathbf{q} \cdot \hat{\mathbf{f}}(\boldsymbol{\xi}_p)|^2 d\boldsymbol{\xi}_p \\
 &= \int_{\mathbb{R}^2} |\mathbf{p} \cdot \hat{\mathbf{f}}(\boldsymbol{\xi}_p)|^2 d\boldsymbol{\xi}_p + \int_{\mathbb{R}^2} |\mathbf{q} \cdot \hat{\mathbf{f}}(\boldsymbol{\xi}_s)|^2 d\boldsymbol{\xi}_s \\
 &\lesssim \int_0^\infty \omega \|\mathbf{u}(\cdot, \omega)\|_{\Gamma_R}^2 d\omega,
 \end{aligned}$$

which proves the lemma for the two-dimensional case.

(ii) Consider  $d = 3$ . Repeating similar steps and using the spherical coordinates, we get

$$\begin{aligned}
 \int_{\mathbb{R}^3} |\mathbf{p} \cdot \hat{\mathbf{f}}(\boldsymbol{\xi}_p)|^2 d\boldsymbol{\xi}_p &= \int_{\mathbb{R}^3} \left| \int_{\mathbb{R}^3} (\mathbf{p} e^{-i\boldsymbol{\xi}_p \cdot \mathbf{x}}) \cdot \mathbf{f}(\mathbf{x}) d\mathbf{x} \right|^2 d\boldsymbol{\xi}_p \\
 &\lesssim \int_0^{2\pi} d\theta \int_0^\pi \sin \varphi d\varphi \int_0^\infty \kappa_p^2 \int_{\Gamma_R} (|T_N \mathbf{u}(\mathbf{x}, \omega)|^2 + \kappa_p^2 |\mathbf{u}(\mathbf{x}, \omega)|^2) d\gamma(\mathbf{x}) d\kappa_p \\
 &\leq 2\pi^2 \int_0^\infty \kappa_p^2 \int_{\Gamma_R} (|T_N \mathbf{u}(\mathbf{x}, \omega)|^2 + \kappa_p^2 |\mathbf{u}(\mathbf{x}, \omega)|^2) d\gamma(\mathbf{x}) d\kappa_p \\
 &\lesssim \int_0^\infty \omega^2 \int_{\Gamma_R} (|T_N \mathbf{u}(\mathbf{x}, \omega)|^2 + \omega^2 |\mathbf{u}(\mathbf{x}, \omega)|^2) d\gamma(\mathbf{x}) d\omega
 \end{aligned}$$

$$= \int_0^\infty \omega^2 \|\mathbf{u}(\cdot, \omega)\|_{\Gamma_R}^2 d\omega. \tag{2.18}$$

Let  $\mathbf{p}(\theta, \varphi) = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)^\top$ . We choose  $\mathbf{q}_1(\theta, \varphi) = (\cos \theta \cos \varphi, \cos \theta \sin \varphi, -\sin \theta)^\top$  and  $\mathbf{q}_2(\theta, \varphi) = \mathbf{p}(\theta, \varphi) \times \mathbf{q}_1(\theta, \varphi) = (-\sin \varphi, \cos \varphi, 0)^\top$  for the shear plane wave. It is easy to verify that  $\{\mathbf{p}, \mathbf{q}_1, \mathbf{q}_2\}$  are mutually orthogonal and thus form an orthonormal basis in  $\mathbb{R}^3$  for any  $\theta \in [0, \pi], \varphi \in [0, 2\pi]$ . Using the Pythagorean theorem yields

$$|\hat{\mathbf{f}}(\boldsymbol{\xi}_p)|^2 = |\mathbf{p} \cdot \hat{\mathbf{f}}(\boldsymbol{\xi}_p)|^2 + |\mathbf{q}_1 \cdot \hat{\mathbf{f}}(\boldsymbol{\xi}_p)|^2 + |\mathbf{q}_2 \cdot \hat{\mathbf{f}}(\boldsymbol{\xi}_p)|^2. \tag{2.19}$$

Following similar arguments as those in (2.15)–(2.17), we get from (2.18)–(2.19) that

$$\begin{aligned} \|\mathbf{f}\|_{L^2(B_R)^3}^2 &= \|\mathbf{f}\|_{L^2(\mathbb{R}^3)^3}^2 = \int_{\mathbb{R}^3} |\hat{\mathbf{f}}(\boldsymbol{\xi}_p)|^2 d\boldsymbol{\xi}_p \\ &\lesssim \int_0^\infty \omega^2 \|\mathbf{u}(\cdot, \omega)\|_{\Gamma_R}^2 d\omega, \end{aligned}$$

which completes the proof.  $\square$

For  $d = 2$ , let

$$I_1(s) = \int_0^s \omega^3 \int_{\Gamma_R} \left| \int_{\Omega} \mathbf{G}_N(\mathbf{x}, \mathbf{y}; \omega) \cdot \mathbf{f}(\mathbf{y}) d\mathbf{y} \right|^2 d\gamma(\mathbf{x}) d\omega, \tag{2.20}$$

$$I_2(s) = \int_0^s \omega \int_{\Gamma_R} \left| \int_{\Omega} D_{\mathbf{x}}(\mathbf{G}_N(\mathbf{x}, \mathbf{y}; \omega) \cdot \mathbf{f}(\mathbf{y})) d\mathbf{y} \right|^2 d\gamma(\mathbf{x}) d\omega. \tag{2.21}$$

For  $d = 3$ , let

$$I_1(s) = \int_0^s \omega^4 \int_{\Gamma_R} \left| \int_{\Omega} \mathbf{G}_N(\mathbf{x}, \mathbf{y}; \omega) \cdot \mathbf{f}(\mathbf{y}) d\mathbf{y} \right|^2 d\gamma(\mathbf{x}) d\omega, \tag{2.22}$$

$$I_2(s) = \int_0^s \omega^2 \int_{\Gamma_R} \left| \int_{\Omega} D_{\mathbf{x}}(\mathbf{G}_N(\mathbf{x}, \mathbf{y}; \omega) \cdot \mathbf{f}(\mathbf{y})) d\mathbf{y} \right|^2 d\gamma(\mathbf{x}) d\omega. \tag{2.23}$$

Denote a sector

$$\mathcal{V} = \{z \in \mathbb{C} : -\frac{\pi}{4} < \arg z < \frac{\pi}{4}\}.$$

The integrands in (2.20)–(2.23) are analytic functions of the angular frequency  $\omega$ . The integrals with respect to  $\omega$  can be taken over any path joining points 0 and  $s$  in  $\mathcal{V}$ . Thus  $I_1(s)$  and  $I_2(s)$  are analytic functions of  $s = s_1 + is_2 \in \mathcal{V}, s_1, s_2 \in \mathbb{R}$ .

**Lemma 2.7.** *Let  $\mathbf{f} \in H^3(B_R)^d$ . For any  $s = s_1 + is_2 \in \mathcal{V}$ , the following estimates hold:*

(i) *when  $d = 3$ ,*

$$|I_1(s)| \lesssim |s|^5 e^{4c_s R|s|} \|\mathbf{f}\|_{H^2(B_R)^3}^2, \tag{2.24}$$

$$|I_2(s)| \lesssim |s|^3 e^{4c_s R|s|} \|\mathbf{f}\|_{H^3(B_R)^3}^2; \tag{2.25}$$

(ii) *when  $d = 2$ ,*

$$|I_1(s)| \lesssim |s|^3 (|s|^{\frac{3}{2}} + |s|^{\frac{1}{2}} + 1)^2 e^{4c_s R|s|} \|\mathbf{f}\|_{H^2(B_R)^2}^2, \tag{2.26}$$

$$|I_2(s)| \lesssim |s| (|s|^{\frac{3}{2}} + |s|^{\frac{1}{2}} + 1)^2 e^{4c_s R|s|} \|\mathbf{f}\|_{H^3(B_R)^2}^2. \tag{2.27}$$

**Proof.** We first prove the three-dimensional case and then show the corresponding two-dimensional case.

(i) Consider  $d = 3$ . Recalling (2.5), we split the Green tensor  $\mathbf{G}_N(\mathbf{x}, \mathbf{y})$  into two parts:

$$\mathbf{G}_N(\mathbf{x}, \mathbf{y}; \omega) = \mathbf{G}_1(\mathbf{x}, \mathbf{y}; \omega) + \mathbf{G}_2(\mathbf{x}, \mathbf{y}; \omega),$$

where

$$\begin{aligned} \mathbf{G}_1(\mathbf{x}, \mathbf{y}; \omega) &= \frac{1}{4\pi\mu} \frac{e^{i\kappa_s|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|} \mathbf{I}_3, \\ \mathbf{G}_2(\mathbf{x}, \mathbf{y}; \omega) &= \frac{1}{4\pi\omega^2} \nabla_{\mathbf{x}} \nabla_{\mathbf{x}}^{\top} \left( \frac{e^{i\kappa_s|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|} - \frac{e^{i\kappa_p|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|} \right). \end{aligned}$$

Let  $\omega = st, t \in (0, 1)$ . Noting (2.3), we have from a simple calculation that

$$|I_1(s)| \lesssim I_{1,1}(s) + I_{1,2}(s),$$

where

$$\begin{aligned} I_{1,1}(s) &= \int_0^1 |s|^5 t^4 \int_{\Gamma_R} \left| \int_{\Omega} \mathbf{G}_1(\mathbf{x}, \mathbf{y}; st) \cdot \mathbf{f}(\mathbf{y}) d\mathbf{y} \right|^2 d\gamma(\mathbf{x}) dt \\ &\lesssim \int_0^1 |s|^5 t^4 \int_{\Gamma_R} \left| \int_{\Omega} \frac{e^{i\kappa_s st|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|} \mathbf{I}_3 \cdot \mathbf{f}(\mathbf{y}) d\mathbf{y} \right|^2 d\gamma(\mathbf{x}) dt \end{aligned}$$

and

$$\begin{aligned} I_{1,2}(s) &= \int_0^1 |s|^5 t^4 \int_{\Gamma_R} \left| \int_{\Omega} \mathbf{G}_2(\mathbf{x}, \mathbf{y}; st) \cdot \mathbf{f}(\mathbf{y}) d\mathbf{y} \right|^2 d\gamma(\mathbf{x}) dt \\ &\lesssim \int_0^1 |s|^5 t^4 \int_{\Gamma_R} \left| \int_{\Omega} \frac{1}{(st)^2} \nabla_{\mathbf{y}} \nabla_{\mathbf{y}}^{\top} \left( \frac{e^{i\kappa_s st|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|} - \frac{e^{i\kappa_p st|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|} \right) \cdot \mathbf{f}(\mathbf{y}) d\mathbf{y} \right|^2 d\gamma(\mathbf{x}) dt. \end{aligned}$$

Here we have used

$$\nabla_{\mathbf{x}} \nabla_{\mathbf{x}}^{\top} \left( \frac{e^{i\kappa_s|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|} - \frac{e^{i\kappa_p|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|} \right) = \nabla_{\mathbf{y}} \nabla_{\mathbf{y}}^{\top} \left( \frac{e^{i\kappa_s|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|} - \frac{e^{i\kappa_p|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|} \right).$$

First we estimate  $I_{1,1}(s)$ . Noting that  $\text{supp } \mathbf{f} = \Omega \subset B_R$  and

$$|e^{ic_s st|\mathbf{x}-\mathbf{y}|}| \leq e^{2c_s R|s|}, \quad \forall \mathbf{x} \in \Gamma_R, \mathbf{y} \in \Omega,$$

we have from the Cauchy–Schwarz inequality that

$$\begin{aligned} |I_{1,1}(s)| &\lesssim \int_0^1 |s|^5 t^4 \int_{\Gamma_R} \left| \int_{\Omega} \frac{e^{2c_s R|s|}}{|\mathbf{x}-\mathbf{y}|} |\mathbf{f}(\mathbf{y})| d\mathbf{y} \right|^2 d\gamma(\mathbf{x}) dt \\ &\lesssim \int_0^1 |s|^5 t^4 \int_{\Gamma_R} \int_{B_R} |\mathbf{f}(\mathbf{y})|^2 d\mathbf{y} \int_{\Omega} \frac{e^{4c_s R|s|}}{|\mathbf{x}-\mathbf{y}|^2} d\mathbf{y} d\gamma(\mathbf{x}) dt \\ &\lesssim |s|^5 e^{4c_s R|s|} \|\mathbf{f}\|_{L^2(B_R)}^2. \end{aligned} \tag{2.28}$$

Next we estimate  $I_{1,2}(s)$ . For any  $c \in \mathbb{R}$ , considering the following power series

$$\frac{e^{icst|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|} = \frac{1}{|\mathbf{x}-\mathbf{y}|} + i(cst) - \frac{(cst)^2}{2!} |\mathbf{x}-\mathbf{y}| - \frac{i(cst)^3}{3!} |\mathbf{x}-\mathbf{y}|^2 + \frac{(cst)^4}{4!} |\mathbf{x}-\mathbf{y}|^3 + \dots,$$

we obtain

$$\begin{aligned} \frac{1}{(st)^2} \nabla_{\mathbf{y}} \nabla_{\mathbf{y}}^{\top} \left( \frac{e^{ic_s st|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|} - \frac{e^{ic_p st|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|} \right) &= -\frac{1}{2} (c_s^2 - c_p^2) \nabla_{\mathbf{y}} \nabla_{\mathbf{y}}^{\top} |\mathbf{x}-\mathbf{y}| \\ &\quad - \frac{i(st)}{3!} (c_s^3 - c_p^3) \nabla_{\mathbf{y}} \nabla_{\mathbf{y}}^{\top} |\mathbf{x}-\mathbf{y}|^2 + \frac{(st)^2}{4!} (c_s^4 - c_p^4) \nabla_{\mathbf{y}} \nabla_{\mathbf{y}}^{\top} |\mathbf{x}-\mathbf{y}|^3 + \dots \end{aligned} \tag{2.29}$$

Substituting (2.29) into  $I_{1,2}(s)$  and using the integration by parts, we have

$$\begin{aligned} I_{1,2}(s) &= \int_0^1 |s|^5 t^4 \int_{\Gamma_R} \left| \int_{\Omega} \nabla_{\mathbf{y}} \nabla_{\mathbf{y}}^{\top} \left( \frac{1}{2} (c_s^2 - c_p^2) |\mathbf{x}-\mathbf{y}| + \dots \right) \cdot \mathbf{f}(\mathbf{y}) d\mathbf{y} \right|^2 d\gamma(\mathbf{x}) dt \\ &= \int_0^1 |s|^5 t^4 \int_{\Gamma_R} \left| \int_{\Omega} \left( \frac{1}{2} (c_s^2 - c_p^2) |\mathbf{x}-\mathbf{y}| + \dots \right) \nabla_{\mathbf{y}} \nabla_{\mathbf{y}} \cdot \mathbf{f}(\mathbf{y}) d\mathbf{y} \right|^2 d\gamma(\mathbf{x}) dt. \end{aligned}$$

Noting  $c_p < c_s$ , we have

$$\begin{aligned} &\left| \frac{1}{2} (c_s^2 - c_p^2) |\mathbf{x}-\mathbf{y}| + \frac{i(st)}{3!} (c_s^3 - c_p^3) |\mathbf{x}-\mathbf{y}|^2 - \frac{(st)^2}{4!} (c_p^4 - c_s^4) |\mathbf{x}-\mathbf{y}|^3 + \dots \right| \\ &\leq \frac{1}{2} c_s^2 |\mathbf{x}-\mathbf{y}| + \frac{|s|}{3!} c_s^3 |\mathbf{x}-\mathbf{y}|^2 + \frac{|s|^2}{4!} c_s^4 |\mathbf{x}-\mathbf{y}|^3 + \dots \\ &\leq c_s^2 |\mathbf{x}-\mathbf{y}| \left( \frac{1}{2} + \frac{c_s |s| |\mathbf{x}-\mathbf{y}|}{3!} + \frac{c_s^2 |s|^2 |\mathbf{x}-\mathbf{y}|^2}{4!} + \dots \right) \\ &\leq 2c_s^2 R e^{c_s |s| |\mathbf{x}-\mathbf{y}|} \lesssim e^{2c_s R|s|}. \end{aligned} \tag{2.30}$$

Using (2.30) and the Cauchy–Schwarz inequality gives

$$\begin{aligned}
 |I_{1,2}(s)| &\lesssim \int_0^1 |s|^5 t^4 \int_{\Gamma_R} \left( \int_{B_R} |\nabla_{\mathbf{y}} \nabla_{\mathbf{y}} \cdot \mathbf{f}(\mathbf{y})|^2 d\mathbf{y} \right) \left( \int_{\Omega} e^{4c_s R|s|} d\mathbf{y} \right) d\gamma(\mathbf{x}) dt \\
 &\lesssim |s|^5 e^{4c_s R|s|} \|\mathbf{f}\|_{H^2(B_R)}^2.
 \end{aligned} \tag{2.31}$$

Combining (2.28) and (2.31) proves (2.24).

For  $I_2(s)$ , we have from (2.7), (2.29), and the integration by parts that

$$|I_2(s)| \lesssim I_{2,1}(s) + I_{2,2}(s),$$

where

$$\begin{aligned}
 I_{2,1}(s) &= \int_0^1 |s|^3 t^2 \int_{\Gamma_R} \left| \int_{\Omega} \nabla_{\mathbf{x}} (\mathbf{G}_N(\mathbf{x}, \mathbf{y}) \cdot \mathbf{f}(\mathbf{y})) \cdot \boldsymbol{\nu}(\mathbf{x}) d\mathbf{y} \right|^2 d\gamma(\mathbf{x}) dt \\
 &= \int_0^1 |s|^3 t^2 \int_{\Gamma_R} \left| \int_{\Omega} \mathbf{G}_N(\mathbf{x}, \mathbf{y}) \cdot (\nabla_{\mathbf{y}} \mathbf{f}(\mathbf{y}) \cdot \boldsymbol{\nu}(\mathbf{x})) d\mathbf{y} \right|^2 d\gamma(\mathbf{x}) dt
 \end{aligned}$$

and

$$\begin{aligned}
 I_{2,2}(s) &= \int_0^1 |s|^3 t^2 \int_{\Gamma_R} \left| \int_{\Omega} \nabla_{\mathbf{x}} \cdot (\mathbf{G}_N(\mathbf{x}, \mathbf{y}) \cdot \mathbf{f}(\mathbf{y})) \boldsymbol{\nu}(\mathbf{x}) d\mathbf{y} \right|^2 d\gamma(\mathbf{x}) dt \\
 &= \int_0^1 |s|^3 t^2 \int_{\Gamma_R} \left| \int_{\Omega} \mathbf{G}_N(\mathbf{x}, \mathbf{y}) \cdot (\nabla_{\mathbf{y}} \cdot \mathbf{f}(\mathbf{y})) \boldsymbol{\nu}(\mathbf{x}) d\mathbf{y} \right|^2 d\gamma(\mathbf{x}) dt \\
 &\lesssim \int_0^1 |s|^3 t^2 \int_{\Gamma_R} \left| \int_{\Omega} \frac{e^{i\kappa_s |\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|} (\nabla_{\mathbf{y}} \cdot \mathbf{f}(\mathbf{y})) \boldsymbol{\nu}(\mathbf{x}) d\mathbf{y} \right|^2 d\gamma(\mathbf{x}) dt \\
 &\quad + \int_0^1 |s|^3 t^2 \int_{\Gamma_R} \left| \int_{\Omega} \left( \frac{1}{2} (c_s^2 - c_p^2) |\mathbf{x}-\mathbf{y}| + \frac{i(st)}{3!} (c_s^3 - c_p^3) |\mathbf{x}-\mathbf{y}|^2 \right. \right. \\
 &\quad \left. \left. - \frac{(st)^2}{4!} (c_p^4 - c_s^4) |\mathbf{x}-\mathbf{y}|^3 + \dots \right) \nabla_{\mathbf{y}} \cdot (\nabla_{\mathbf{y}} \nabla_{\mathbf{y}} \cdot \mathbf{f}(\mathbf{y})) \boldsymbol{\nu}(\mathbf{x}) d\mathbf{y} \right|^2 d\gamma(\mathbf{x}) dt.
 \end{aligned}$$

Following the similar steps for  $I_{1,1}(s)$  and  $I_{1,2}(s)$ , we may estimate  $I_{2,1}(s)$  and  $I_{2,2}(s)$ , respectively, and prove the inequality (2.25).

(ii) Consider  $d = 2$ . Similarly, let  $\omega = st, t \in (0, 1)$  and

$$\mathbf{G}_N(\mathbf{x}, \mathbf{y}; \omega) = \mathbf{G}_1(\mathbf{x}, \mathbf{y}; \omega) + \mathbf{G}_2(\mathbf{x}, \mathbf{y}; \omega)$$

where

$$\begin{aligned}
 \mathbf{G}_1(\mathbf{x}, \mathbf{y}; \omega) &= \frac{i}{4\mu} H_0^{(1)}(\kappa_s |\mathbf{x}-\mathbf{y}|) \mathbf{I}_2, \\
 \mathbf{G}_2(\mathbf{x}, \mathbf{y}; \omega) &= \frac{i}{4\omega^2} \nabla_{\mathbf{x}} \nabla_{\mathbf{x}}^{\top} \left( H_0^{(1)}(\kappa_s |\mathbf{x}-\mathbf{y}|) - H_0^{(1)}(\kappa_p |\mathbf{x}-\mathbf{y}|) \right).
 \end{aligned}$$

Noting (2.3), we get

$$|I_1(s)| \lesssim I_{1,1}(s) + I_{1,2}(s),$$

where

$$\begin{aligned} I_{1,1}(s) &= \int_0^1 |s|^4 t^3 \int_{\Gamma_R} \left| \int_{\Omega} \mathbf{G}_1(\mathbf{x}, \mathbf{y}; st) \cdot \mathbf{f}(\mathbf{y}) d\mathbf{y} \right|^2 d\gamma(\mathbf{x}) dt \\ &\lesssim \int_0^1 |s|^4 t^3 \int_{\Gamma_R} \left| \int_{\Omega} H_0^{(1)}(c_s st |\mathbf{x} - \mathbf{y}|) \mathbf{I}_2 \cdot \mathbf{f}(\mathbf{y}) d\mathbf{y} \right|^2 d\gamma(\mathbf{x}) dt \end{aligned}$$

and

$$\begin{aligned} I_{1,2}(s) &= \int_0^1 |s|^4 t^3 \int_{\Gamma_R} \left| \int_{\Omega} \mathbf{G}_2(\mathbf{x}, \mathbf{y}; st) \cdot \mathbf{f}(\mathbf{y}) d\mathbf{y} \right|^2 d\gamma(\mathbf{x}) dt \\ &\lesssim \int_0^1 |s|^4 t^3 \int_{\Gamma_R} \left| \int_{\Omega} \frac{1}{(st)^2} \nabla_{\mathbf{y}} \nabla_{\mathbf{y}}^{\top} (H_0^{(1)}(c_s st |\mathbf{x} - \mathbf{y}|) - H_0^{(1)}(c_p st |\mathbf{x} - \mathbf{y}|)) \cdot \mathbf{f}(\mathbf{y}) d\mathbf{y} \right|^2 d\gamma(\mathbf{x}) dt. \end{aligned}$$

Here we have used

$$\nabla_{\mathbf{x}} \nabla_{\mathbf{x}}^{\top} (H_0^{(1)}(\kappa_s |\mathbf{x} - \mathbf{y}|) - H_0^{(1)}(\kappa_p |\mathbf{x} - \mathbf{y}|)) = \nabla_{\mathbf{y}} \nabla_{\mathbf{y}}^{\top} (H_0^{(1)}(\kappa_s |\mathbf{x} - \mathbf{y}|) - H_0^{(1)}(\kappa_p |\mathbf{x} - \mathbf{y}|)).$$

First we estimate  $I_{1,1}(s)$ . Recall  $H_0^{(1)}(z) = J_0(z) + iY_0(z)$  and the expansions of  $J_0, Y_0$  in [54]:

$$J_0(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{4^k} \frac{z^{2k}}{(k!)^2}, \quad Y_0(z) = \frac{2}{\pi} \left( \ln\left(\frac{z}{2}\right) + c_0 \right) J_0(z) + \frac{2}{\pi} \sum_{k=1}^{\infty} (-1)^{k+1} H_k \frac{z^{2k}}{4^k (k!)^2},$$

where  $c_0 = 0.5772\dots$  is the Euler–Mascheroni constant and  $H_k$  is a harmonic number defined by

$$H_k = 1 + \frac{1}{2} + \dots + \frac{1}{k}.$$

It is easy to verify

$$4^k (k!)^2 \geq (2k)!, \tag{2.32}$$

which gives

$$\begin{aligned} |J_0(c_s st |\mathbf{x} - \mathbf{y}|)| &= \left| c_s^2 \sum_{k=0}^{\infty} (-1)^k \frac{(c_s st)^{2k-2} |\mathbf{x} - \mathbf{y}|^{2k}}{4^k (k!)^2} \right| \\ &\leq c_s^2 |\mathbf{x} - \mathbf{y}|^2 \sum_{k=0}^{\infty} \frac{(c_s |s| |\mathbf{x} - \mathbf{y}|)^{2k}}{(2k)!} \leq c_s^2 |\mathbf{x} - \mathbf{y}|^2 e^{c_s |s| |\mathbf{x} - \mathbf{y}|} \lesssim e^{2c_s R |s|}. \end{aligned}$$

On the other hand, it can be shown that

$$\frac{H_k}{4^k (k!)^2} \lesssim \frac{1}{(2k)!}, \tag{2.33}$$

which yields

$$|Y_0(c_s st|\mathbf{x} - \mathbf{y}|)| \lesssim |J_0(c_s st|\mathbf{x} - \mathbf{y}|)| + \sum_{k=1}^{\infty} \frac{(c_s |s|t|\mathbf{x} - \mathbf{y}|)^{2k}}{(2k)!} \lesssim e^{2c_s R|s|}.$$

It follows from the Cauchy–Schwarz inequality that

$$\begin{aligned} |I_{1,1}(s)| &\lesssim \int_0^1 |s|^{4t^3} \int_{\Gamma_R} e^{4Rc_s|s|} \|\mathbf{f}\|_{H^2(B_R)^2}^2 d\gamma(\mathbf{x}) dt \\ &\lesssim |s|^4 \|\mathbf{f}\|_{L^2(B_R)^2}^2 \lesssim |s|^3 (|s|^{\frac{3}{2}} + |s|^{\frac{1}{2}} + 1)^2 e^{4c_s R|s|} \|\mathbf{f}\|_{L^2(B_R)^2}^2. \end{aligned} \tag{2.34}$$

Next we estimate  $I_{1,2}(s)$ , which requires to evaluate the integral

$$\left| \int_{\Omega} \mathbf{G}_2(\mathbf{x}, \mathbf{y}) \cdot \mathbf{f}(\mathbf{y}) d\mathbf{y} \right|^2,$$

where

$$\begin{aligned} \mathbf{G}_2(\mathbf{x}, \mathbf{y}; \omega) &= \frac{i}{4\omega^2} \nabla_{\mathbf{x}} \nabla_{\mathbf{x}}^{\top} (J_0(\kappa_s|\mathbf{x} - \mathbf{y}|) - J_0(\kappa_p|\mathbf{x} - \mathbf{y}|)) \\ &\quad - \frac{1}{4\omega^2} \nabla_{\mathbf{x}} \nabla_{\mathbf{x}}^{\top} (Y_0(\kappa_s|\mathbf{x} - \mathbf{y}|) - Y_0(\kappa_p|\mathbf{x} - \mathbf{y}|)). \end{aligned}$$

Letting  $\omega = st$  and using the expansion of  $J_0$ , we obtain

$$\begin{aligned} &\frac{1}{(st)^2} \nabla_{\mathbf{y}} \nabla_{\mathbf{y}}^{\top} (J_0(c_s st|\mathbf{x} - \mathbf{y}|) - J_0(c_p st|\mathbf{x} - \mathbf{y}|)) \\ &= c_s^2 \sum_{k=1}^{\infty} (-1)^k \frac{(c_s st)^{2k-2} \nabla_{\mathbf{y}} \nabla_{\mathbf{y}}^{\top} |\mathbf{x} - \mathbf{y}|^{2k}}{4^k (k!)^2} - c_p^2 \sum_{k=1}^{\infty} (-1)^k \frac{(c_p st)^{2k-2} \nabla_{\mathbf{y}} \nabla_{\mathbf{y}}^{\top} |\mathbf{x} - \mathbf{y}|^{2k}}{4^k (k!)^2}. \end{aligned}$$

It follows from the integration by parts that

$$\begin{aligned} &\int_{\Omega} \frac{1}{(st)^2} \left( \nabla_{\mathbf{y}} \nabla_{\mathbf{y}}^{\top} (J_0(c_s st|\mathbf{x} - \mathbf{y}|) - J_0(c_p st|\mathbf{x} - \mathbf{y}|)) \right) \cdot \mathbf{f}(\mathbf{y}) d\mathbf{y} \\ &= \int_{\Omega} \left( c_s^2 \sum_{k=1}^{\infty} (-1)^k \frac{(c_s st)^{2k-2} |\mathbf{x} - \mathbf{y}|^{2k}}{4^k (k!)^2} \right) \nabla_{\mathbf{y}} \nabla_{\mathbf{y}} \cdot \mathbf{f}(\mathbf{y}) d\mathbf{y} \\ &\quad - \int_{\Omega} \left( c_p^2 \sum_{k=1}^{\infty} (-1)^k \frac{(c_p st)^{2k-2} |\mathbf{x} - \mathbf{y}|^{2k}}{4^k (k!)^2} \right) \nabla_{\mathbf{y}} \nabla_{\mathbf{y}} \cdot \mathbf{f}(\mathbf{y}) d\mathbf{y}. \end{aligned} \tag{2.35}$$

Using the inequality (2.32), we get for any  $c > 0$  that

$$\left| c^2 \sum_{k=1}^{\infty} (-1)^k \frac{(cst)^{2k-2} |\mathbf{x} - \mathbf{y}|^{2k}}{4^k (k!)^2} \right| \leq c^2 |\mathbf{x} - \mathbf{y}|^2 \sum_{k=0}^{\infty} \frac{(c|s||\mathbf{x} - \mathbf{y}|)^{2k}}{(2k)!} \leq c^2 |\mathbf{x} - \mathbf{y}|^2 e^{c|s||\mathbf{x} - \mathbf{y}|}. \tag{2.36}$$

Combining (2.35)–(2.36) and using the Cauchy–Schwarz inequality, we obtain

$$\left| \int_{\Omega} \frac{1}{(st)^2} (\nabla_{\mathbf{y}} \nabla_{\mathbf{y}}^{\top} (J_0(c_s st|\mathbf{x} - \mathbf{y}| - J_0(c_p st|\mathbf{x} - \mathbf{y}|))) \cdot \mathbf{f}(\mathbf{y}) d\mathbf{y} \right|^2$$

$$\lesssim \left( \int_{\tilde{B}_R} |\nabla_{\mathbf{y}} \nabla_{\mathbf{y}} \cdot \mathbf{f}(\mathbf{y})|^2 d\mathbf{y} \right) \left( \int_{\Omega} c_s^4 |\mathbf{x} - \mathbf{y}|^4 e^{2c_s|s||\mathbf{x} - \mathbf{y}|} d\mathbf{y} \right) \lesssim e^{4Rc_s|s|} \|\mathbf{f}\|_{H^2(B_R)}^2. \tag{2.37}$$

Let

$$\frac{1}{(st)^2} \nabla_{\mathbf{y}} \nabla_{\mathbf{y}}^{\top} (Y_0(c_s st|\mathbf{x} - \mathbf{y}| - Y_0(c_p st|\mathbf{x} - \mathbf{y}|))) = \mathbf{A} + \mathbf{B},$$

where

$$\mathbf{A} = \frac{2}{\pi} \frac{1}{(st)^2} \nabla_{\mathbf{y}} \nabla_{\mathbf{y}}^{\top} \left[ \left( \ln\left(\frac{1}{2}c_s st|\mathbf{x} - \mathbf{y}| \right) + \gamma \right) J_0(c_s st|\mathbf{x} - \mathbf{y}|) \right]$$

$$- \frac{2}{\pi} \frac{1}{(st)^2} \nabla_{\mathbf{y}} \nabla_{\mathbf{y}}^{\top} \left[ \left( \ln\left(\frac{1}{2}c_p st|\mathbf{x} - \mathbf{y}| \right) + \gamma \right) J_0(c_p st|\mathbf{x} - \mathbf{y}|) \right],$$

$$\mathbf{B} = \frac{2}{\pi} \frac{1}{(st)^2} \nabla_{\mathbf{y}} \nabla_{\mathbf{y}}^{\top} \sum_{k=1}^{\infty} (-1)^{k+1} H_k \frac{(c_s st|\mathbf{x} - \mathbf{y}|)^{2k}}{4^k (k!)^2}$$

$$- \frac{2}{\pi} \frac{1}{(st)^2} \nabla_{\mathbf{y}} \nabla_{\mathbf{y}}^{\top} \sum_{k=1}^{\infty} (-1)^{k+1} H_k \frac{(c_p st|\mathbf{x} - \mathbf{y}|)^{2k}}{4^k (k!)^2}.$$

We consider the matrix  $\mathbf{B}$  first. Using the integration by parts yields

$$\int_{\Omega} \mathbf{B} \cdot \mathbf{f}(\mathbf{y}) d\mathbf{y} = \int_{\Omega} \frac{2}{\pi} c_s^2 |\mathbf{x} - \mathbf{y}|^2 \sum_{k=0}^{\infty} (-1)^{k+2} H_{k+1} \frac{(c_s st|\mathbf{x} - \mathbf{y}|)^{2k}}{4^{k+1} ((k+1)!)^2} \nabla_{\mathbf{y}} \nabla_{\mathbf{y}} \cdot \mathbf{f}(\mathbf{y}) d\mathbf{y}$$

$$- \int_{\Omega} \frac{2}{\pi} c_p^2 |\mathbf{x} - \mathbf{y}|^2 \sum_{k=0}^{\infty} (-1)^{k+2} H_{k+1} \frac{(c_p st|\mathbf{x} - \mathbf{y}|)^{2k}}{4^{k+1} ((k+1)!)^2} \nabla_{\mathbf{y}} \nabla_{\mathbf{y}} \cdot \mathbf{f}(\mathbf{y}) d\mathbf{y}.$$

It is easy to verify from (2.33) that

$$\frac{H_{k+1}}{4^{k+1} ((k+1)!)^2} \lesssim \frac{1}{(2k)!},$$

which gives for any  $c > 0$  that

$$\left| \sum_{k=0}^{\infty} (-1)^{k+2} H_{k+1} \frac{(cst|\mathbf{x} - \mathbf{y}|)^{2k}}{4^{k+1} ((k+1)!)^2} \right| \lesssim \sum_{k=0}^{\infty} \frac{(c|s||\mathbf{x} - \mathbf{y}|)^{2k}}{(2k)!} \lesssim e^{c|s||\mathbf{x} - \mathbf{y}|}.$$

Using the Cauchy–Schwarz inequality and noting  $c_p < c_s$ , we have

$$\left| \int_{\Omega} \mathbf{B} \cdot \mathbf{f}(\mathbf{y}) d\mathbf{y} \right| \lesssim \left( \int_{B_R} |\nabla_{\mathbf{y}} \nabla_{\mathbf{y}} \cdot \mathbf{f}|^2 d\mathbf{y} \right) \left( \int_{\Omega} |\mathbf{x} - \mathbf{y}|^4 e^{2c_s|s||\mathbf{x} - \mathbf{y}|} d\mathbf{y} \right)$$

$$\lesssim e^{4Rc_s|s|} \|\mathbf{f}\|_{H^2(B_R)}^2. \tag{2.38}$$



Now we consider the matrix  $\mathbf{A}$ . Using the identity for any two smooth functions  $l$  and  $h$ :

$$\nabla_{\mathbf{y}} \nabla_{\mathbf{y}}^{\top} l(\mathbf{y}) h(\mathbf{y}) = h(\mathbf{y}) \nabla_{\mathbf{y}} \nabla_{\mathbf{y}}^{\top} l(\mathbf{y}) + l(\mathbf{y}) \nabla_{\mathbf{y}} \nabla_{\mathbf{y}}^{\top} h(\mathbf{y}) + \nabla_{\mathbf{y}} l(\mathbf{y})^{\top} \nabla_{\mathbf{y}} h(\mathbf{y}) + \nabla_{\mathbf{y}} l(\mathbf{y})^{\top} \nabla_{\mathbf{y}} h(\mathbf{y})$$

and

$$\ln\left(\frac{1}{2}cst|\mathbf{x} - \mathbf{y}|\right) = \ln\left(\frac{1}{2}cst\right) + \ln(|\mathbf{x} - \mathbf{y}|),$$

we split  $\mathbf{A}$  into three parts:

$$\mathbf{A} = \mathbf{A}_1 + \mathbf{A}_2 + \mathbf{A}_3,$$

where

$$\begin{aligned} \mathbf{A}_1 &= \frac{2}{\pi} \frac{1}{(st)^2} \nabla_{\mathbf{y}} \nabla_{\mathbf{y}}^{\top} (\ln(|\mathbf{x} - \mathbf{y}|)) J_0(c_s st |\mathbf{x} - \mathbf{y}|) \\ &\quad - \frac{2}{\pi} \frac{1}{(st)^2} \nabla_{\mathbf{y}} \nabla_{\mathbf{y}}^{\top} (\ln(|\mathbf{x} - \mathbf{y}|)) J_0(c_p st |\mathbf{x} - \mathbf{y}|), \\ \mathbf{A}_2 &= \frac{2}{\pi} \frac{1}{(st)^2} (\nabla_{\mathbf{y}} J_0(c_s st |\mathbf{x} - \mathbf{y}|) - \nabla_{\mathbf{y}} J_0(c_p st |\mathbf{x} - \mathbf{y}|))^{\top} \cdot \nabla_{\mathbf{y}} (\ln(|\mathbf{x} - \mathbf{y}|)) \\ &\quad + \frac{2}{\pi} \frac{1}{(st)^2} (\nabla_{\mathbf{y}} J_0(c_s st |\mathbf{x} - \mathbf{y}|) - \nabla_{\mathbf{y}} J_0(c_p st |\mathbf{x} - \mathbf{y}|))^{\top} \cdot \nabla_{\mathbf{y}} (\ln(|\mathbf{x} - \mathbf{y}|)) \\ \mathbf{A}_3 &= \frac{2}{\pi} \frac{1}{(st)^2} \left( \ln\left(\frac{1}{2}c_s st |\mathbf{x} - \mathbf{y}|\right) + \gamma \right) \nabla_{\mathbf{y}} \nabla_{\mathbf{y}}^{\top} J_0(c_s st |\mathbf{x} - \mathbf{y}|) \\ &\quad - \frac{2}{\pi} \frac{1}{(st)^2} \left( \ln\left(\frac{1}{2}c_p st |\mathbf{x} - \mathbf{y}|\right) + \gamma \right) \nabla_{\mathbf{y}} \nabla_{\mathbf{y}}^{\top} J_0(c_p st |\mathbf{x} - \mathbf{y}|). \end{aligned}$$

For  $\mathbf{A}_1$ , we have from (2.32) and the expansion of  $J_0$  that

$$\left| \frac{1}{(st)^2} (J_0(c_s st |\mathbf{x} - \mathbf{y}|) - J_0(c_p st |\mathbf{x} - \mathbf{y}|)) \right| \lesssim |\mathbf{x} - \mathbf{y}|^2 e^{c_s |\mathbf{x} - \mathbf{y}|}. \quad (2.39)$$

Noting  $\ln|\mathbf{x} - \mathbf{y}|$  is analytic when  $\mathbf{x} \in \Gamma_R$ ,  $\mathbf{y} \in \Omega$ , we get from the Cauchy–Schwarz inequality that

$$\left| \int_{\Omega} \mathbf{A}_1 \cdot \mathbf{f}(\mathbf{y}) d\mathbf{y} \right|^2 \lesssim e^{4Rc_s |s|} \int_{\Omega} |\nabla_{\mathbf{y}} \nabla_{\mathbf{y}}^{\top} \ln(|\mathbf{x} - \mathbf{y}|) \cdot \mathbf{f}(\mathbf{y})|^2 d\mathbf{y} \lesssim e^{4Rc_s |s|} \|\mathbf{f}\|_{L^2(B_R)}^2. \quad (2.40)$$

For  $\mathbf{A}_2$ , using the analyticity of  $\ln|\mathbf{x} - \mathbf{y}|$  for  $\mathbf{x} \in \Gamma_R$ ,  $\mathbf{y} \in \Omega$ , the integration by parts, and the estimate of (2.39), we have

$$\left| \int_{\Omega} \mathbf{A}_2 \cdot \mathbf{f}(\mathbf{y}) d\mathbf{y} \right|^2 \lesssim e^{4Rc_s |s|} \|\mathbf{f}\|_{H^1(B_R)}^2. \quad (2.41)$$

Now we consider  $\mathbf{A}_3$ . Noting

$$\ln\left(\frac{1}{2}cst|\mathbf{x} - \mathbf{y}|\right) + c_0 = \ln\left(\frac{1}{2}c|\mathbf{x} - \mathbf{y}|\right) + c_0 + \ln(st)$$

and using the expansion of  $J_0$ :

$$\nabla_{\mathbf{y}} \nabla_{\mathbf{y}}^{\top} J_0(cst|\mathbf{x} - \mathbf{y}|) = \nabla_{\mathbf{y}} \nabla_{\mathbf{y}}^{\top} \sum_{k=1}^{\infty} \frac{(-1)^k}{4^k} \frac{(cst|\mathbf{x} - \mathbf{y}|)^{2k}}{(k!)^2},$$

we have from the integration by parts that

$$\int_{\Omega} \mathbf{A}_3 \cdot \mathbf{f}(\mathbf{y}) d\mathbf{y} = \mathbf{A}_{3,1} + \mathbf{A}_{3,2},$$

where

$$\begin{aligned} \mathbf{A}_{3,1} &= \int_{\Omega} \frac{2}{\pi} \frac{1}{(st)^2} \sum_{k=1}^{\infty} \frac{(-1)^k}{4^k} \frac{(c_s st|\mathbf{x} - \mathbf{y}|)^{2k}}{(k!)^2} \nabla_{\mathbf{y}} \nabla_{\mathbf{y}} \cdot \left[ \left( \ln\left(\frac{1}{2}c_s|\mathbf{x} - \mathbf{y}| + c_0\right) \mathbf{f}(\mathbf{y}) \right) \right] d\mathbf{y} \\ &\quad - \int_{\Omega} \frac{2}{\pi} \frac{1}{(st)^2} \sum_{k=1}^{\infty} \frac{(-1)^k}{4^k} \frac{(c_p st|\mathbf{x} - \mathbf{y}|)^{2k}}{(k!)^2} \nabla_{\mathbf{y}} \nabla_{\mathbf{y}} \cdot \left[ \left( \ln\left(\frac{1}{2}c_p|\mathbf{x} - \mathbf{y}| + c_0\right) \mathbf{f}(\mathbf{y}) \right) \right] d\mathbf{y} \end{aligned}$$

and

$$\begin{aligned} \mathbf{A}_{3,2} &= \int_{\Omega} \frac{2}{\pi} \frac{1}{(st)^2} \ln(st) J_0(c_s st|\mathbf{x} - \mathbf{y}|) \nabla_{\mathbf{y}} \nabla_{\mathbf{y}} \cdot \mathbf{f}(\mathbf{y}) d\mathbf{y} \\ &\quad - \int_{\Omega} \frac{2}{\pi} \frac{1}{(st)^2} \ln(st) J_0(c_p st|\mathbf{x} - \mathbf{y}|) \nabla_{\mathbf{y}} \nabla_{\mathbf{y}} \cdot \mathbf{f}(\mathbf{y}) d\mathbf{y}. \end{aligned}$$

Since the function  $\ln(\frac{1}{2}c|\mathbf{x} - \mathbf{y}|) + c_0$  is analytic for  $\mathbf{y} \in \Omega$ ,  $\mathbf{x} \in \Gamma_R$ , we have from (2.36) and the Cauchy–Schwarz inequality that

$$|\mathbf{A}_{3,1}|^2 \lesssim e^{4Rc_s|s|} \|\mathbf{f}\|_{H^2(B_R)^2}^2. \tag{2.42}$$

It is easy to verify that

$$\left| (|s|t)^{\frac{1}{2}} \ln(st) \right| \lesssim (|s|t)^{\frac{1}{2}} \left( |s|t + \frac{1}{(|s|t)^{\frac{1}{2}}} + \pi \right) \lesssim |s|^{\frac{3}{2}} + \pi|s|^{\frac{1}{2}} + 1.$$

Hence

$$\begin{aligned} &\left| (|s|t)^{\frac{1}{2}} \ln(st) \frac{1}{(st)^2} (J_0(c_s st|\mathbf{x} - \mathbf{y}|) - J_0(c_p st|\mathbf{x} - \mathbf{y}|)) \right| \\ &\lesssim (|s|^{\frac{3}{2}} + \pi|s|^{\frac{1}{2}} + 1) \left| c_s^2 |\mathbf{x} - \mathbf{y}|^2 \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{4^{k+1}} \frac{(c_s st|\mathbf{x} - \mathbf{y}|)^{2k}}{((k+1)!)^2} \right. \\ &\quad \left. - c_p^2 |\mathbf{x} - \mathbf{y}|^2 \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{4^{k+1}} \frac{(c_p st|\mathbf{x} - \mathbf{y}|)^{2k}}{((k+1)!)^2} \right| \\ &\lesssim c_s^2 |\mathbf{x} - \mathbf{y}|^2 (|s|^{\frac{3}{2}} + \pi|s|^{\frac{1}{2}} + 1) \left( \sum_{k=0}^{\infty} \frac{(c_s |s||\mathbf{x} - \mathbf{y}|)^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{(c_p |s||\mathbf{x} - \mathbf{y}|)^{2k}}{(2k)!} \right) \\ &\lesssim c_s^2 |\mathbf{x} - \mathbf{y}|^2 (|s|^{\frac{3}{2}} + \pi|s|^{\frac{1}{2}} + 1) e^{c_s |s||\mathbf{x} - \mathbf{y}|}. \end{aligned}$$

Multiplying  $\mathbf{A}_{3,2}$  by  $(|s|t)^{\frac{1}{2}}$  and using the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}
 |(|s|t)^{\frac{1}{2}} \mathbf{A}_{3,2}|^2 &\lesssim \left( \int_{\tilde{B}_R} |\nabla_{\mathbf{y}} \nabla_{\mathbf{y}} \cdot \mathbf{f}(\mathbf{y})|^2 d\mathbf{y} \right) \left( \int_{\Omega} c_s^4 |\mathbf{x} - \mathbf{y}|^4 (|s|^{\frac{3}{2}} + \pi |s|^{\frac{1}{2}} + 1)^2 e^{2c_s |\mathbf{x} - \mathbf{y}|} d\mathbf{y} \right) \\
 &\lesssim (|s|^{\frac{3}{2}} + \pi |s|^{\frac{1}{2}} + 1)^2 e^{2c_s |\mathbf{x} - \mathbf{y}|} \int_{B_R} |\nabla_{\mathbf{y}} \nabla_{\mathbf{y}} \cdot \mathbf{f}(\mathbf{y})|^2 d\mathbf{y} \\
 &\lesssim (|s|^{\frac{3}{2}} + \pi |s|^{\frac{1}{2}} + 1)^2 e^{4Rc_s |s|} \|\mathbf{f}\|_{H^2(B_R)}^2.
 \end{aligned} \tag{2.43}$$

Combining (2.37)–(2.43), we obtain

$$\left| (|s|t)^{\frac{1}{2}} \int_{\Omega} \mathbf{G}_2(\mathbf{x}, \mathbf{y}; st) \cdot \mathbf{f}(\mathbf{y}) d\mathbf{y} \right|^2 \lesssim (|s|^{\frac{3}{2}} + |s|^{\frac{1}{2}} + 1)^2 e^{4Rc_s |s_2|} \|\mathbf{f}\|_{H^2(B_R)}^2,$$

which implies

$$|I_{1,2}(s)| \lesssim |s|^3 (|s|^{\frac{3}{2}} + |s|^{\frac{1}{2}} + 1)^2 e^{4Rc_s |s_2|} \|\mathbf{f}\|_{H^2(B_R)}^2. \tag{2.44}$$

Combining (2.34) and (2.44) completes the proof of the inequality (2.26).

For  $I_2(s)$ , we have from (2.7) and the integration by parts that

$$\begin{aligned}
 \int_{\Omega} D_{\mathbf{x}} (\mathbf{G}_N(\mathbf{x}, \mathbf{y}; \omega) \cdot \mathbf{f}(\mathbf{y})) d\mathbf{y} &= \mu \int_{\Omega} \mathbf{G}_N(\mathbf{x}, \mathbf{y}) \cdot (\nabla_{\mathbf{y}} \mathbf{f}(\mathbf{y}) \cdot \boldsymbol{\nu}(\mathbf{x})) d\mathbf{y} \\
 &+ (\lambda + \mu) \int_{\Omega} \frac{i}{4\mu} H_0^{(1)}(\kappa_s |\mathbf{x} - \mathbf{y}|) (\nabla_{\mathbf{y}} \cdot \mathbf{f}(\mathbf{y})) \boldsymbol{\nu}(\mathbf{x}) d\mathbf{y} \\
 &+ (\lambda + \mu) \int_{\Omega} \frac{i}{4\omega^2} \left( H_0^{(1)}(\kappa_s |\mathbf{x} - \mathbf{y}|) - H_0^{(1)}(\kappa_p |\mathbf{x} - \mathbf{y}|) \right) \nabla_{\mathbf{y}} \cdot (\nabla_{\mathbf{y}} \nabla_{\mathbf{y}} \cdot \mathbf{f}(\mathbf{y})) \boldsymbol{\nu}(\mathbf{x}) d\mathbf{y}.
 \end{aligned}$$

Using the estimates for the integrals involving  $\mathbf{G}_1(\mathbf{x}, \mathbf{y})$  and  $\mathbf{G}_2(\mathbf{x}, \mathbf{y})$ , which we have obtained for  $I_1(s)$ , and the Cauchy–Schwarz inequality, we may similarly get

$$|I_2(s)| \lesssim |s| (|s|^{\frac{3}{2}} + |s|^{\frac{1}{2}} + 1)^2 e^{4Rc_s |s_2|} \|\mathbf{f}\|_{H^3(B_R)}^2,$$

which shows (2.27). The proof is now complete.  $\square$

**Lemma 2.8.** *Let  $\mathbf{f} \in \mathbb{F}_M(B_R)$ . For any  $s \geq 1$ , the following estimate holds:*

$$\int_s^\infty \omega^{d-1} \|\mathbf{u}(\cdot, \omega)\|_{\Gamma_R}^2 d\omega \lesssim s^{-(2m-2d+1)} \|\mathbf{f}\|_{H^{m+1}(B_R)}^2.$$

**Proof.** Let

$$\begin{aligned}
 &\int_s^\infty \omega^{d-1} \|\mathbf{u}(\cdot, \omega)\|_{\Gamma_R}^2 d\omega \\
 &= \int_s^\infty \omega^{d-1} \int_{\Gamma_R} (|T_N \mathbf{u}(\mathbf{x}, \omega)|^2 + \omega^2 |\mathbf{u}(\mathbf{x}, \omega)|^2) d\gamma(\mathbf{x}) d\omega = L_1 + L_2,
 \end{aligned}$$

where

$$L_1 = \int_s^\infty \omega^{d+1} \int_{\Gamma_R} |\mathbf{u}(\mathbf{x}, \omega)|^2 d\gamma(\mathbf{x}) d\omega,$$

$$L_2 = \int_s^\infty \omega^{d-1} \int_{\Gamma_R} |T_N \mathbf{u}(x, \omega)|^2 d\gamma(\mathbf{x}) d\omega.$$

(i) Consider  $d = 3$ . Using (2.4) and noting  $s \geq 1$ , we have

$$L_1 = \int_s^\infty \omega^4 \int_{\Gamma_R} |\mathbf{u}(\mathbf{x}, \omega)|^2 d\gamma(\mathbf{x}) d\omega \lesssim L_{1,1} + L_{1,2},$$

where

$$L_{1,1} = \int_s^\infty \omega^4 \int_{\Gamma_R} \left| \int_{\Omega} \frac{e^{i\kappa_s |\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|} \mathbf{I}_3 \cdot \mathbf{f}(\mathbf{y}) d\mathbf{y} \right|^2 d\gamma(\mathbf{x}) d\omega,$$

$$L_{1,2} = \int_s^\infty \int_{\Gamma_R} \left| \int_{\Omega} \nabla_{\mathbf{y}} \nabla_{\mathbf{y}}^\top \left( \frac{e^{i\kappa_s |\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|} - \frac{e^{i\kappa_p |\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|} \right) \cdot \mathbf{f}(\mathbf{y}) d\mathbf{y} \right|^2 d\gamma(\mathbf{x}) d\omega.$$

Noting  $\Omega \subset B_{\hat{R}} \subset B_R$ , using the polar coordinates  $\rho = |\mathbf{y} - \mathbf{x}|$  originated at  $\mathbf{x}$  with respect to  $\mathbf{y}$  and the integration by parts, we obtain

$$L_{1,1} = \int_s^\infty \omega^4 \int_{\Gamma_R} \left| \int_0^{2\pi} d\theta \int_0^\pi \sin \varphi d\varphi \int_{R-\hat{R}}^{R+\hat{R}} e^{i\kappa_s \rho} \mathbf{I}_3 \cdot (\mathbf{f}\rho) d\rho \right|^2 d\gamma(\mathbf{x}) d\omega$$

$$= \int_s^\infty \omega^4 \int_{\Gamma_R} \left| \int_0^{2\pi} d\theta \int_0^\pi \sin \varphi d\varphi \int_{R-\hat{R}}^{R+\hat{R}} \frac{e^{i\kappa_s \rho}}{(i\kappa_s)^m} \mathbf{I}_3 \cdot \frac{\partial^m (\mathbf{f}\rho)}{\partial \rho^m} d\rho \right|^2 d\gamma(\mathbf{x}) d\omega,$$

which gives after using  $\kappa_s = c_s \omega$  that

$$L_{1,1} \lesssim \int_s^\infty \omega^4 \int_{\Gamma_R} \left| \int_0^{2\pi} d\theta \int_0^\pi \sin \varphi d\varphi \int_{R-\hat{R}}^{R+\hat{R}} \omega^{-m} \right.$$

$$\left. \left( \left| \sum_{|\alpha|=m} \partial_{\mathbf{y}}^\alpha \mathbf{f} \right| \rho + m \left| \sum_{|\alpha|=m-1} \partial_{\mathbf{y}}^\alpha \mathbf{f} \right| \right) d\rho \right|^2 d\gamma(\mathbf{x}) d\omega$$

$$= \int_s^\infty \omega^4 \int_{\Gamma_R} \left| \int_0^{2\pi} d\theta \int_0^\pi \sin \varphi d\varphi \int_{R-\hat{R}}^{R+\hat{R}} \omega^{-m} \right.$$

$$\left. \left( \left| \sum_{|\alpha|=m} \partial_{\mathbf{y}}^\alpha \mathbf{f} \right| \frac{1}{\rho} + \left| \sum_{|\alpha|=m-1} \partial_{\mathbf{y}}^\alpha \mathbf{f} \right| \frac{m}{\rho^2} \right) \rho^2 d\rho \right|^2 d\gamma(\mathbf{x}) d\omega$$

$$\begin{aligned}
&\leq \int_s^\infty \omega^4 \int_{\Gamma_R} \left| \int_0^{2\pi} d\theta \int_0^\pi \sin \varphi d\varphi \int_{R-\hat{R}}^{R+\hat{R}} \omega^{-m} \right. \\
&\quad \left. \left( \left| \sum_{|\alpha|=m} \partial_{\mathbf{y}}^\alpha \mathbf{f} \left| \frac{1}{(R-\hat{R})} \right. + \left| \sum_{|\alpha|=m-1} \partial_{\mathbf{y}}^\alpha \mathbf{f} \left| \frac{m}{(R-\hat{R})^2} \right. \right) \rho^2 d\rho \right|^2 d\gamma(\mathbf{x}) d\omega \right. \\
&\leq \int_s^\infty \omega^4 \int_{\Gamma_R} \left| \int_0^{2\pi} d\theta \int_0^\pi \sin \varphi d\varphi \int_0^\infty \omega^{-m} \right. \\
&\quad \left. \left( \left| \sum_{|\alpha|=m} \partial_{\mathbf{y}}^\alpha \mathbf{f} \left| \frac{1}{(R-\hat{R})} \right. + \left| \sum_{|\alpha|=m-1} \partial_{\mathbf{y}}^\alpha \mathbf{f} \left| \frac{m}{(R-\hat{R})^2} \right. \right) \rho^2 d\rho \right|^2 d\gamma(\mathbf{x}) d\omega.
\end{aligned}$$

Changing back to the Cartesian coordinates with respect to  $\mathbf{y}$ , we have

$$\begin{aligned}
L_{1,1} &\leq \int_s^\infty \int_{\Gamma_R} \omega^4 \left| \int_{\Omega} \omega^{-m} \right. \\
&\quad \left. \left( \left| \sum_{|\alpha|=m} \partial_{\mathbf{y}}^\alpha \mathbf{f} \left| \frac{1}{(R-\hat{R})} \right. + \left| \sum_{|\alpha|=m-1} \partial_{\mathbf{y}}^\alpha \mathbf{f} \left| \frac{m}{(R-\hat{R})^2} \right. \right) d\mathbf{y} \right|^2 d\gamma(\mathbf{x}) d\omega \right. \\
&\lesssim m \|\mathbf{f}\|_{H^m(B_R)^3}^2 \int_s^\infty \omega^{4-2m} d\omega \\
&\lesssim \left( \frac{m}{2m-5} \right) s^{-(2m-5)} \|\mathbf{f}\|_{H^m(B_R)^3}^2 \lesssim s^{-(2m-5)} \|\mathbf{f}\|_{H^m(B_R)^3}^2,
\end{aligned} \tag{2.45}$$

where we have used the fact that  $m \geq d = 3$ .

For  $L_{1,2}$ , it follows from the integration by parts that

$$\begin{aligned}
L_{1,2} &= \int_s^\infty \int_{\Gamma_R} \left| \int_{\Omega} \left( \frac{e^{i\kappa_s |\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|} - \frac{e^{i\kappa_p |\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|} \right) \nabla_{\mathbf{y}} \nabla_{\mathbf{y}} \cdot \mathbf{f}(\mathbf{y}) d\mathbf{y} \right|^2 d\gamma(\mathbf{x}) d\omega \\
&\lesssim \int_s^\infty \int_{\Gamma_R} \left| \int_{\Omega} \frac{e^{i\kappa_s |\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|} \nabla_{\mathbf{y}} \nabla_{\mathbf{y}} \cdot \mathbf{f}(\mathbf{y}) d\mathbf{y} \right|^2 d\gamma(\mathbf{x}) d\omega \\
&\quad + \int_s^\infty \int_{\Gamma_R} \left| \int_{\Omega} \frac{e^{i\kappa_p |\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|} \nabla_{\mathbf{y}} \nabla_{\mathbf{y}} \cdot \mathbf{f}(\mathbf{y}) d\mathbf{y} \right|^2 d\gamma(\mathbf{x}) d\omega.
\end{aligned}$$

We may follow the same steps as those for (2.45) to show

$$\begin{aligned}
L_{1,2} &\leq \int_s^\infty \int_{\Gamma_R} \left| \int_{\Omega} \omega^{-(m-2)} \left( \left| \sum_{|\alpha|=m-2} \partial_{\mathbf{y}}^\alpha (\nabla_{\mathbf{y}} \nabla_{\mathbf{y}} \cdot \mathbf{f}) \left| \frac{1}{(R-\hat{R})} \right. \right. \right. \\
&\quad \left. \left. + \left| \sum_{|\alpha|=m-3} \partial_{\mathbf{y}}^\alpha (\nabla_{\mathbf{y}} \nabla_{\mathbf{y}} \cdot \mathbf{f}) \left| \frac{(m-2)}{(R-\hat{R})^2} \right. \right) d\mathbf{y} \right|^2 d\gamma(\mathbf{x}) d\omega
\end{aligned}$$

$$\begin{aligned} &\lesssim (m - 2) \|\mathbf{f}\|_{H^m(B_R)^3}^2 \int_s^\infty \omega^{4-2m} d\omega \\ &\lesssim \left(\frac{m - 2}{2m - 5}\right) s^{-(2m-5)} \|\mathbf{f}\|_{H^m(B_R)^3}^2 \lesssim s^{-(2m-5)} \|\mathbf{f}\|_{H^m(B_R)^3}^2. \end{aligned} \tag{2.46}$$

Noting  $s \geq 1$ , using (2.7) and the integration by parts, we get

$$\begin{aligned} L_2(s) &= \int_s^\infty \omega^2 \int_{\Gamma_R} \left| \int_{\Omega} D_{\mathbf{x}}(\mathbf{G}_N(\mathbf{x}, \mathbf{y}; \omega) \cdot \mathbf{f}(\mathbf{y})) d\mathbf{y} \right|^2 d\gamma(\mathbf{x}) d\omega \\ &\lesssim \int_s^\infty \omega^2 \int_{\Gamma_R} \left| \int_{\Omega} \mathbf{G}_N(\mathbf{x}, \mathbf{y}; \omega) \cdot (\nabla_{\mathbf{y}} \mathbf{f}(\mathbf{y}) \cdot \boldsymbol{\nu}(\mathbf{x})) d\mathbf{y} \right|^2 d\gamma(\mathbf{x}) d\omega \\ &\quad + \int_s^\infty \omega^2 \int_{\Gamma_R} \left| \int_{\Omega} \frac{e^{i\kappa_s |\mathbf{x} - \mathbf{y}|}}{|\mathbf{x} - \mathbf{y}|} (\nabla_{\mathbf{y}} \cdot \mathbf{f}(\mathbf{y})) \boldsymbol{\nu}(\mathbf{x}) d\mathbf{y} \right|^2 d\gamma(\mathbf{x}) d\omega \\ &\quad + \int_s^\infty \frac{1}{\omega^2} \int_{\Gamma_R} \left| \int_{\Omega} \left( \frac{e^{i\kappa_s |\mathbf{x} - \mathbf{y}|}}{|\mathbf{x} - \mathbf{y}|} - \frac{e^{i\kappa_p |\mathbf{x} - \mathbf{y}|}}{|\mathbf{x} - \mathbf{y}|} \right) \nabla_{\mathbf{y}} \cdot (\nabla_{\mathbf{y}} \nabla_{\mathbf{y}} \cdot \mathbf{f}) \boldsymbol{\nu}(\mathbf{x}) d\mathbf{y} \right|^2 d\gamma(\mathbf{x}) d\omega. \end{aligned}$$

Again, we may follow similar arguments as those for (2.45) and (2.46) to get

$$L_2 \lesssim s^{-(2m-5)} \|\mathbf{f}\|_{H^m(B_R)^3}^2. \tag{2.47}$$

Combining (2.45)–(2.47) completes the proof for the three dimensional case.

(ii) Consider  $d = 2$ . Noting  $s \geq 1$ , we have

$$L_1 = \int_s^\infty \omega^3 \int_{\Gamma_R} |\mathbf{u}(\mathbf{x}, \omega)|^2 d\gamma(\mathbf{x}) d\omega \lesssim L_{1,1} + L_{1,2},$$

where

$$\begin{aligned} L_{1,1} &= \int_s^\infty \int_{\Gamma_R} \omega^3 \left| \int_{\Omega} H_0^{(1)}(\kappa_s |\mathbf{x} - \mathbf{y}|) \mathbf{I}_2 \cdot \mathbf{f}(\mathbf{y}) d\mathbf{y} \right|^2 d\gamma(\mathbf{x}) d\omega, \\ L_{1,2} &= \int_s^\infty \int_{\Gamma_R} \frac{1}{\omega} \left| \int_{\Omega} \nabla_{\mathbf{x}} \nabla_{\mathbf{x}}^\top \left( H_0^{(1)}(\kappa_s |\mathbf{x} - \mathbf{y}|) - H_0^{(1)}(\kappa_p |\mathbf{x} - \mathbf{y}|) \right) \cdot \mathbf{f}(\mathbf{y}) d\mathbf{y} \right|^2 d\gamma(\mathbf{x}) d\omega. \end{aligned}$$

The Hankel function can be expressed by the following integral when  $t > 0$  (e.g., [54], Chapter VI):

$$H_0^1(t) = \frac{2}{i\pi} \int_1^\infty e^{it\tau} (\tau^2 - 1)^{-\frac{1}{2}} d\tau.$$

Using the polar coordinates  $\rho = |\mathbf{y} - \mathbf{x}|$  originated at  $\mathbf{x}$  with respect to  $\mathbf{y}$  and noting  $\Omega \subset \hat{R} \subset R$ , we have

$$L_{1,1} = \int_s^\infty \omega^3 \int_{\Gamma_R} \left| \int_0^{2\pi} d\theta \int_{R-\hat{R}}^{R+\hat{R}} H_0^{(1)}(\kappa_s \rho) \mathbf{I}_2 \cdot (\mathbf{f} \rho) d\rho \right|^2 d\gamma(\mathbf{x}) d\omega.$$

Let

$$W_k(t) = \frac{2}{i\pi} \int_1^\infty \frac{e^{it\tau}}{(i\tau)^k (\tau^2 - 1)^{\frac{1}{2}}} d\tau, \quad k = 1, 2, \dots \tag{2.48}$$

It is easy to verify that

$$W_0(t) = H_0^{(1)}(t) \quad \text{and} \quad \frac{dW_k(t)}{dt} = W_{k-1}(t), \quad t > 0, \quad k \in \mathbb{N}.$$

Using the integration by parts yields

$$\begin{aligned} L_{1,1} &= \int_s^\infty \omega^3 \int_{\Gamma_R} \left| \int_0^{2\pi} d\theta \int_{R-\hat{R}}^{R+\hat{R}} \frac{W_1(\kappa_s \rho)}{\kappa_s} \mathbf{I}_2 \cdot \frac{\partial(\mathbf{f} \rho)}{\partial \rho} d\rho \right|^2 d\gamma(\mathbf{x}) d\omega \\ &= \int_s^\infty \omega^3 \int_{\Gamma_R} \left| \int_0^{2\pi} d\theta \int_{R-\hat{R}}^{R+\hat{R}} \frac{W_{m+1}(\kappa_s \rho)}{\kappa_s^{m+1}} \mathbf{I}_2 \cdot \frac{\partial^{m+1}(\mathbf{f} \rho)}{\partial \rho^{m+1}} d\rho \right|^2 d\gamma(\mathbf{x}) d\omega. \end{aligned}$$

Consequently,

$$\begin{aligned} L_{1,1} &\lesssim \int_s^\infty \omega^3 \int_{\Gamma_R} \left| \int_0^{2\pi} d\theta \int_{R-\hat{R}}^{R+\hat{R}} \left| \frac{H_{m+1}(\kappa_s \rho)}{\omega^{m+1}} \right| \left| \frac{\partial^{m+1}(\mathbf{f} \rho)}{\partial \rho^{m+1}} \right| d\rho \right|^2 d\gamma(\mathbf{x}) d\omega \\ &\lesssim \int_s^\infty \omega^3 \int_{\Gamma_R} \left| \int_0^{2\pi} d\theta \int_{R-\hat{R}}^{R+\hat{R}} \left| \frac{H_{m+1}(\kappa_s \rho)}{\omega^{m+1}} \right| \right. \\ &\quad \left. \left( \left| \sum_{|\alpha|=m+1} \partial_y^\alpha \mathbf{f} \right| + \left| \sum_{|\alpha|=m} \partial_y^\alpha \mathbf{f} \left( \frac{m+1}{\rho} \right) \right| \right) \rho d\rho \right|^2 d\gamma(\mathbf{x}) d\omega \\ &\lesssim \int_s^\infty \omega^3 \int_{\Gamma_R} \left| \int_0^{2\pi} d\theta \int_{R-\hat{R}}^{R+\hat{R}} \left| \frac{H_{m+1}(\kappa_s \rho)}{\omega^{m+1}} \right| \right. \\ &\quad \left. \left( \left| \sum_{|\alpha|=m+1} \partial_y^\alpha \mathbf{f} \right| + \left| \sum_{|\alpha|=m} \partial_y^\alpha \mathbf{f} \left( \frac{m+1}{(R-\hat{R})} \right) \right| \right) \rho d\rho \right|^2 d\gamma(\mathbf{x}) d\omega. \end{aligned}$$

It is easy to note from (2.48) that there exists a constant  $C > 0$  such that  $|H_n(\kappa\rho)| \leq C$  for  $m \geq 1$ . Hence,

$$\begin{aligned} L_{1,1} &\lesssim \int_s^\infty \omega^3 \int_{\Gamma_R} \left| \int_0^{2\pi} d\theta \int_{R-\hat{R}}^{R+\hat{R}} \omega^{-(m+1)} \right. \\ &\quad \left. \left( \left| \sum_{|\alpha|=m+1} \partial_y^\alpha \mathbf{f} \right| + \left| \sum_{|\alpha|=m} \partial_y^\alpha \mathbf{f} \left( \frac{m+1}{(R-\hat{R})} \right) \right| \right) \rho d\rho \right|^2 d\gamma(\mathbf{x}) d\omega. \end{aligned}$$

Changing back to the Cartesian coordinates with respect to  $\mathbf{y}$ , we have

$$\begin{aligned}
 L_{1,1} &\lesssim \int_s^\infty \omega^3 \int_{\Gamma_R} \left| \int_{\Omega} \omega^{-(m+1)} \left( \left| \sum_{|\alpha|=m+1} \partial_{\mathbf{y}}^\alpha \mathbf{f} \right| + \left| \sum_{|\alpha|=m} \partial_{\mathbf{y}}^\alpha \mathbf{f} \left| \frac{(m+1)}{(R-\hat{R})} \right. \right) d\mathbf{y} \right|^2 d\gamma(\mathbf{x}) d\omega \\
 &\lesssim (m+1) \|\mathbf{f}\|_{H^{m+1}(B_R)^2}^2 \int_s^\infty \omega^{1-2m} d\omega \\
 &= \left( \frac{m+1}{2m-2} \right) s^{-(2m-2)} \|\mathbf{f}\|_{H^{m+1}(B_R)^2}^2 \lesssim s^{-(2m-2)} \|\mathbf{f}\|_{H^{m+1}(B_R)^2}^2.
 \end{aligned} \tag{2.49}$$

Using the integration by parts yields

$$\begin{aligned}
 L_{1,2} &= \int_s^\infty \frac{1}{\omega} \int_{\Gamma_R} \left| \int_{\Omega} \left( H_0^{(1)}(\kappa_s|\mathbf{x}-\mathbf{y}|) - H_0^{(1)}(\kappa_p|\mathbf{x}-\mathbf{y}|) \right) \nabla_{\mathbf{y}} \nabla_{\mathbf{y}} \cdot \mathbf{f}(\mathbf{y}) d\mathbf{y} \right|^2 d\gamma(\mathbf{x}) d\omega \\
 &\lesssim \int_s^\infty \frac{1}{\omega} \int_{\Gamma_R} \left| \int_{\Omega} H_0^{(1)}(\kappa_s|\mathbf{x}-\mathbf{y}|) \nabla_{\mathbf{y}} \nabla_{\mathbf{y}} \cdot \mathbf{f}(\mathbf{y}) d\mathbf{y} \right|^2 d\gamma(\mathbf{x}) d\omega \\
 &\quad + \int_s^\infty \frac{1}{\omega} \int_{\Gamma_R} \left| \int_{\Omega} H_0^{(1)}(\kappa_p|\mathbf{x}-\mathbf{y}|) \nabla_{\mathbf{y}} \nabla_{\mathbf{y}} \cdot \mathbf{f}(\mathbf{y}) d\mathbf{y} \right|^2 d\gamma(\mathbf{x}) d\omega
 \end{aligned}$$

We may follow a similar proof for (2.49) to show that

$$\begin{aligned}
 L_{1,2} &\lesssim \int_s^\infty \frac{1}{\omega} \int_{\Gamma_R} \left| \int_{\Omega} \omega^{-(m-1)} \left( \left| \sum_{|\alpha|=m-1} \partial_{\mathbf{y}}^\alpha (\nabla_{\mathbf{y}} \nabla_{\mathbf{y}} \cdot \mathbf{f}) \right| \right. \right. \\
 &\quad \left. \left. + \left| \sum_{|\alpha|=m-2} \partial_{\mathbf{y}}^\alpha (\nabla_{\mathbf{y}} \nabla_{\mathbf{y}} \cdot \mathbf{f}) \left| \frac{(m-1)}{(R-\hat{R})} \right. \right) d\mathbf{y} \right|^2 d\gamma(\mathbf{x}) d\omega \\
 &\lesssim (m-1) \|\mathbf{f}\|_{H^{m+1}(B_R)^2}^2 \int_s^\infty \omega^{1-2m} d\omega \\
 &= \left( \frac{m-1}{2m-2} \right) s^{-(2m-2)} \|\mathbf{f}\|_{H^{m+1}(B_R)^2}^2 \lesssim s^{-(2m-2)} \|\mathbf{f}\|_{H^{m+1}(B_R)^2}^2.
 \end{aligned} \tag{2.50}$$

Next is to consider  $L_2$ . Again, we use (2.7) and the integration by parts to get

$$\begin{aligned}
 L_2(s) &= \int_s^\infty \omega \int_{\Gamma_R} \left| \int_{\Omega} D_{\mathbf{x}} (\mathbf{G}_N(\mathbf{x}, \mathbf{y}) \cdot \mathbf{f}(\mathbf{y})) d\mathbf{y} \right|^2 d\gamma(\mathbf{x}) d\omega \\
 &\lesssim \int_s^\infty \omega \int_{\Gamma_R} \left| \int_{\Omega} \mathbf{G}_N(\mathbf{x}, \mathbf{y}) \cdot (\nabla_{\mathbf{y}} \mathbf{f} \cdot \boldsymbol{\nu}(\mathbf{x})) d\mathbf{y} \right|^2 d\gamma(\mathbf{x}) d\omega \\
 &\quad + \int_s^\infty \omega \int_{\Gamma_R} \left| \int_{\Omega} H_0^{(1)}(\kappa_s|\mathbf{x}-\mathbf{y}|) (\nabla_{\mathbf{y}} \cdot \mathbf{f}(\mathbf{y})) \boldsymbol{\nu}(\mathbf{x}) d\mathbf{y} \right|^2 d\gamma(\mathbf{x}) d\omega
 \end{aligned}$$



$$+ \int_s^\infty \frac{1}{\omega^3} \int_{\Gamma_R} \left| \int_{\Omega} \left( H_0^{(1)}(\kappa_s |\mathbf{x} - \mathbf{y}|) - H_0^{(1)}(\kappa_p |\mathbf{x} - \mathbf{y}|) \right) \nabla_{\mathbf{y}} \cdot (\nabla_{\mathbf{y}} \nabla_{\mathbf{y}} \cdot \mathbf{f}) \boldsymbol{\nu}(\mathbf{x}) \, d\mathbf{y} \right|^2 d\gamma(\mathbf{x}) d\omega.$$

Following similar arguments as those for (2.49) and (2.50), we have

$$L_2 \lesssim s^{-(2m-2)} \|\mathbf{f}\|_{H^{m+1}(B_R)}^2. \tag{2.51}$$

Combining (2.49)–(2.51) completes the proof for the two dimensional case.  $\square$

**Lemma 2.9.** *Let  $\mathbf{f} \in \mathbb{F}_M(B_R)$ . Then there exists a function  $\beta(s)$  satisfying*

$$\begin{cases} \beta(s) \geq \frac{1}{2}, & s \in (K, 2^{\frac{1}{4}}K), \\ \beta(s) \geq \frac{1}{\pi} \left( \left( \frac{s}{K} \right)^4 - 1 \right)^{-\frac{1}{2}}, & s \in (2^{\frac{1}{4}}K, \infty), \end{cases} \tag{2.52}$$

such that

$$|I_1(s) + I_2(s)| \lesssim M^2 e^{(4R+1)c_s s} \epsilon_1^{2\beta(s)}, \quad \forall s \in (K, \infty).$$

**Proof.** It follows from Lemma 2.7 that

$$|(I_1(s) + I_2(s)) e^{-(4R+1)c_s |s|}| \lesssim M^2, \quad \forall s \in \mathcal{V}.$$

Recalling (2.20)–(2.23), we have

$$|(I_1(s) + I_2(s)) e^{-(4R+1)c_s s}| \leq \epsilon_1^2, \quad s \in [0, K].$$

A direct application of Lemma Appendix C.2 shows that there exists a function  $\beta(s)$  satisfying (2.52) such that

$$|(I_1(s) + I_2(s)) e^{-(4R+1)c_s s}| \lesssim M^2 \epsilon_1^{2\beta}, \quad \forall s \in (K, \infty),$$

which completes the proof.  $\square$

Next we prove Theorem 2.4.

**Proof.** We can assume that  $\epsilon_1 < e^{-1}$ , otherwise the estimate is obvious. Let

$$s = \begin{cases} \frac{1}{((4R+3)c_s \pi)^{\frac{1}{3}}} K^{\frac{2}{3}} |\ln \epsilon_1|^{\frac{1}{4}}, & 2^{\frac{1}{4}}((4R+3)c_s \pi)^{\frac{1}{3}} K^{\frac{1}{3}} < |\ln \epsilon_1|^{\frac{1}{4}}, \\ K, & |\ln \epsilon_1|^{\frac{1}{4}} \leq 2^{\frac{1}{4}}((4R+3)c_s \pi)^{\frac{1}{3}} K^{\frac{1}{3}}. \end{cases}$$

If  $2^{\frac{1}{4}}((4R+3)c_s \pi)^{\frac{1}{3}} K^{\frac{1}{3}} < |\ln \epsilon_1|^{\frac{1}{4}}$ , then we have from Lemma 2.9 that

$$\begin{aligned} |I_1(s) + I_2(s)| &\lesssim M^2 e^{(4R+3)c_s s} e^{-\frac{2|\ln \epsilon_1|}{\pi} \left( \left( \frac{s}{K} \right)^4 - 1 \right)^{-\frac{1}{2}}} \\ &\lesssim M^2 e^{\frac{-(4R+3)c_s}{((4R+3)c_s \pi)^{\frac{1}{3}}} K^{\frac{2}{3}} |\ln \epsilon_1|^{\frac{1}{4}} - \frac{2|\ln \epsilon_1|}{\pi} \left( \frac{K}{s} \right)^2} \\ &= M^2 e^{-2 \left( \frac{c_s^2 (4R+3)^2}{\pi} \right)^{\frac{1}{3}} K^{\frac{2}{3}} |\ln \epsilon_1|^{\frac{1}{2}} \left( 1 - \frac{1}{2} |\ln \epsilon_1|^{-\frac{1}{4}} \right)}. \end{aligned}$$

Noting

$$\frac{1}{2} |\ln \epsilon_1|^{-\frac{1}{4}} < \frac{1}{2}, \quad \left( \frac{(4R+3)^2}{\pi} \right)^{\frac{1}{3}} > 1,$$

we have

$$|I_1(s) + I_2(s)| \lesssim M^2 e^{-(c_s K)^{\frac{2}{3}} |\ln \epsilon_1|^{\frac{1}{2}}}.$$

Using the elementary inequality

$$e^{-x} \leq \frac{(6m - 6d + 3)!}{x^{3(2m-2d+1)}}, \quad x > 0, \tag{2.53}$$

we get

$$|I_1(s) + I_2(s)| \lesssim \frac{M^2}{\left( \frac{K^2 |\ln \epsilon_1|^{\frac{3}{2}}}{(6m-6d+3)^3} \right)^{2m-2d+1}}. \tag{2.54}$$

If  $|\ln \epsilon_1|^{\frac{1}{4}} \leq 2^{\frac{1}{4}} (((4R+3)\pi)^{\frac{1}{3}} K^{\frac{1}{3}})$ , then  $s = K$ . We have from (2.20)–(2.23) that

$$|I_1(s) + I_2(s)| \leq \epsilon_1^2.$$

Here we have used the fact that

$$I_1(s) + I_2(s) = \int_0^s \omega^{d-1} \|\mathbf{u}(\cdot, \omega)\|_{\Gamma_R}^2 d\omega, \quad s > 0.$$

Hence we obtain from Lemma 2.8 and (2.54) that

$$\begin{aligned} & \int_0^\infty \omega^{d-1} \|\mathbf{u}(\cdot, \omega)\|_{\Gamma_R}^2 d\omega \\ & \leq I_1(s) + I_2(s) + \int_s^\infty \omega^{d-1} \|\mathbf{u}(\cdot, \omega)\|_{\Gamma_R}^2 d\omega \\ & \lesssim \epsilon_1^2 + \frac{M^2}{\left( \frac{K^2 |\ln \epsilon_1|^{\frac{3}{2}}}{(6m-6d+3)^3} \right)^{2m-2d+1}} + \frac{M^2}{\left( 2^{-\frac{1}{4}} ((4R+3)\pi)^{-\frac{1}{3}} K^{\frac{2}{3}} |\ln \epsilon_1|^{\frac{1}{4}} \right)^{2m-2d+1}}. \end{aligned}$$

By Lemma 2.6, we have

$$\|\mathbf{f}\|_{L^2(B_R)^d}^2 \lesssim \epsilon_1^2 + \frac{M^2}{\left( \frac{K^2 |\ln \epsilon_1|^{\frac{3}{2}}}{(6m-6d+3)^3} \right)^{2m-2d+1}} + \frac{M^2}{\left( \frac{K^{\frac{2}{3}} |\ln \epsilon_1|^{\frac{1}{4}}}{(R+1)(6m-6d+3)^3} \right)^{2m-2d+1}}.$$

Since  $K^{\frac{2}{3}} |\ln \epsilon_1|^{\frac{1}{4}} \leq K^2 |\ln \epsilon_1|^{\frac{3}{2}}$  when  $K > 1$  and  $|\ln \epsilon_1| > 1$ , we finish the proof and obtain the stability estimate (2.13).  $\square$

2.4. Stability with discrete frequency data

In this section, we discuss the stability at a discrete set of frequencies. Let us first specify the discrete frequency data. For  $\mathbf{n} \in \mathbb{Z}^d \setminus \{0\}$ , let  $n = |\mathbf{n}|$  and define two angular frequencies

$$\omega_{p,n} = \frac{n\pi}{c_p R}, \quad \omega_{s,n} = \frac{n\pi}{c_s R}.$$

The corresponding wavenumbers are

$$\kappa_{p,n} = c_p \omega_{p,n} = \frac{n\pi}{R}, \quad \kappa_{s,n} = c_s \omega_{s,n} = \frac{n\pi}{R}. \tag{2.55}$$

Recall the boundary measurement at continuous frequencies:

$$\|\mathbf{u}(\cdot, \omega)\|_{\Gamma_R}^2 = \int_{\Gamma_R} (|T_N \mathbf{u}(\mathbf{x}, \omega)|^2 + \omega^2 |\mathbf{u}(\mathbf{x}, \omega)|^2) d\gamma(\mathbf{x}).$$

Now we define the boundary measurements at discrete frequencies:

$$\begin{aligned} \|\mathbf{u}(\cdot, \omega_{p,n})\|_{\Gamma_R}^2 &= \int_{\Gamma_R} (|T_N \mathbf{u}(\mathbf{x}, \omega_{p,n})|^2 + n^2 |\mathbf{u}(\mathbf{x}, \omega_{p,n})|^2) d\gamma(\mathbf{x}), \\ \|\mathbf{u}(\cdot, \omega_{s,n})\|_{\Gamma_R}^2 &= \int_{\Gamma_R} (|T_N \mathbf{u}(\mathbf{x}, \omega_{s,n})|^2 + n^2 |\mathbf{u}(\mathbf{x}, \omega_{s,n})|^2) d\gamma(\mathbf{x}). \end{aligned}$$

Since the discrete frequency data cannot recover the Fourier coefficient of  $\mathbf{f}$  at  $\mathbf{n} = 0$ , i.e.,  $\hat{\mathbf{f}}_0 = \frac{1}{(2R)^d} \int_{U_R} \mathbf{f}(\mathbf{x}) d\mathbf{x}$  is missing, we assume that  $\hat{\mathbf{f}}_0 = 0$ . Otherwise we may replace  $\mathbf{f}(\mathbf{x})$  by  $\tilde{\mathbf{f}}(\mathbf{x}) = \mathbf{f}(\mathbf{x}) - (\int_{\Omega} \mathbf{f}(\mathbf{x}) d\mathbf{x}) \chi_{\Omega}(\mathbf{x})$ , where  $\chi$  is the characteristic function, such that  $\tilde{\mathbf{f}}$  has a compact support  $\Omega$  and  $\int_{\Omega} \tilde{\mathbf{f}}(\mathbf{x}) d\mathbf{x} = 0$ . In fact, when  $\omega = 0$ , the Navier equation (2.1) reduces to

$$\mu \Delta \mathbf{u} + (\lambda + \mu) \nabla \nabla \cdot \mathbf{u} = \mathbf{f}. \tag{2.56}$$

Integrating (2.56) on both sides on  $B_R$  and using the integration by parts, we have

$$\int_{\Gamma_R} T_N \mathbf{u}(\mathbf{x}) d\gamma = \int_{B_R} \mathbf{f}(\mathbf{x}) d\mathbf{x},$$

which implies that  $\hat{\mathbf{f}}_0$  can be indeed recovered by the data corresponding to the static Navier equation. Hence we define

$$\tilde{\mathbb{F}}_M(B_R) = \{\mathbf{f} \in \mathbb{F}_M(B_R) : \int_{\Omega} \mathbf{f}(\mathbf{x}) d\mathbf{x} = 0\}.$$

**Problem 2.10** (Discrete frequency data for elastic waves). Let  $\mathbf{f} \in \tilde{\mathbb{F}}_M(B_R)$ . The inverse source problem is to determine  $\mathbf{f}$  from the displacement  $\mathbf{u}(\mathbf{x}, \omega)$ ,  $\mathbf{x} \in \Gamma_R$ ,  $\omega \in (0, \frac{\pi}{c_p R}] \cup \cup_{n=1}^N \{\omega_{p,n}, \omega_{s,n}\}$ , where  $1 < N \in \mathbb{N}$ .

The following stability estimate is the main result for Problem 2.10.

**Theorem 2.11.** Let  $\mathbf{u}$  be the solution of the scattering problem (2.1)–(2.2) corresponding to the source  $\mathbf{f} \in \tilde{\mathbb{F}}_M(B_R)$ . Then

$$\|\mathbf{f}\|_{L^2(B_R)^d}^2 \lesssim \epsilon_2^2 + \frac{M^2}{\left(\frac{N^{\frac{5}{8}} |\ln \epsilon_3|^{\frac{1}{9}}}{(6m-3d+3)^3}\right)^{2m-d+1}}, \tag{2.57}$$

where

$$\begin{aligned} \epsilon_2 &= \left( \sum_{n=1}^N \|\mathbf{u}(\cdot, \omega_{p,n})\|_{\Gamma_R}^2 + \|\mathbf{u}(\cdot, \omega_{s,n})\|_{\Gamma_R}^2 \right)^{\frac{1}{2}}, \\ \epsilon_3 &= \sup_{\omega \in (0, \frac{\pi}{c_p R}]} \|\mathbf{u}(\cdot, \omega)\|_{\Gamma_R}. \end{aligned}$$

**Remark 2.12.** The stability estimate (2.57) for the discrete frequency data is analogous to the estimate (2.13) for the continuous frequency data. It also consists of the Lipschitz type data discrepancy and the high frequency tail of the source function. The stability increases as  $N$  increases, i.e., the inverse problem is more stable when higher frequency data is used.

The rest of this section is to prove Theorem 2.11. Similarly, we consider the auxiliary functions of compressional and shear plane waves:

$$\mathbf{u}_{p,n}^{\text{inc}} = \mathbf{p}_n e^{-i\kappa_{p,n} \mathbf{x} \cdot \hat{\mathbf{n}}} \quad \text{and} \quad \mathbf{u}_{s,n}^{\text{inc}} = \mathbf{q}_n e^{-i\kappa_{s,n} \mathbf{x} \cdot \hat{\mathbf{n}}}, \tag{2.58}$$

where  $\hat{\mathbf{n}} = \mathbf{n}/n$  represents the unit propagation direction vector and  $\mathbf{p}_n, \mathbf{q}_n$  are unit polarization vectors satisfying  $\mathbf{p}_n = \hat{\mathbf{n}}$  and  $\mathbf{q}_n \cdot \hat{\mathbf{n}} = 0$ . Substituting (2.55) into (2.58) yields

$$\mathbf{u}_{p,n}^{\text{inc}} = \mathbf{p}_n e^{-i(\frac{\pi}{R}) \mathbf{x} \cdot \mathbf{n}} \quad \text{and} \quad \mathbf{u}_{s,n}^{\text{inc}} = \mathbf{q}_n e^{-i(\frac{\pi}{R}) \mathbf{x} \cdot \mathbf{n}}.$$

It is easy to verify that  $\mathbf{u}_{p,n}^{\text{inc}}$  and  $\mathbf{u}_{s,n}^{\text{inc}}$  satisfy the homogeneous Navier equation in  $\mathbb{R}^d$ :

$$\mu \Delta \mathbf{u}_{p,n}^{\text{inc}} + (\lambda + \mu) \nabla \nabla \cdot \mathbf{u}_{p,n}^{\text{inc}} + \omega_{p,n}^2 \mathbf{u}_{p,n}^{\text{inc}} = 0 \tag{2.59}$$

and

$$\mu \Delta \mathbf{u}_{s,n}^{\text{inc}} + (\lambda + \mu) \nabla \nabla \cdot \mathbf{u}_{s,n}^{\text{inc}} + \omega_{s,n}^2 \mathbf{u}_{s,n}^{\text{inc}} = 0. \tag{2.60}$$

**Lemma 2.13.** Let  $\mathbf{u}$  be the solution of the scattering problem (2.1)–(2.2) corresponding to the source  $\mathbf{f} \in L^2(B_R)^d$ . For all  $\mathbf{n} \in \mathbb{Z}^d \setminus \{0\}$ , the Fourier coefficients of  $\mathbf{f}$  satisfy

$$|\hat{\mathbf{f}}_{\mathbf{n}}|^2 \lesssim \|\mathbf{u}(\cdot, \omega_{p,n})\|_{\Gamma_R}^2 + \|\mathbf{u}(\cdot, \omega_{s,n})\|_{\Gamma_R}^2.$$

**Proof.** (i) First consider  $d = 2$ . Multiplying the both sides of (2.1) by  $\mathbf{u}_{p,n}^{\text{inc}}(\mathbf{x})$ , using the integration by parts over  $B_R$ , and noting (2.59), we obtain

$$\int_{B_R} (\mathbf{p}_n e^{-i(\frac{\pi}{R}) \mathbf{x} \cdot \mathbf{n}}) \cdot \mathbf{f}(\mathbf{x}) d\mathbf{x} = \int_{\Gamma_R} (\mathbf{u}_{p,n}^{\text{inc}}(\mathbf{x}) \cdot T_N \mathbf{u}(\mathbf{x}, \omega_{p,n}) + \mathbf{u}(\mathbf{x}, \omega_{p,n}) \cdot D \mathbf{u}_{p,n}^{\text{inc}}(\mathbf{x})) d\gamma(\mathbf{x}).$$

A simple calculation yields that

$$D\mathbf{u}_{\mathbf{p},n}^{\text{inc}}(\mathbf{x}) = -in\left(\frac{\pi}{R}\right) (\mu(\mathbf{p}_n \cdot \boldsymbol{\nu})\mathbf{p}_n + (\lambda + \mu)\boldsymbol{\nu}) e^{-i\left(\frac{\pi}{R}\right)\mathbf{x}\cdot\mathbf{n}},$$

which gives

$$|D\mathbf{u}_{\mathbf{p}}^{\text{inc}}(\mathbf{x})| \lesssim n.$$

Noting  $\text{supp } \mathbf{f} \subset B_R \subset U_R$ , we get from Lemma [Appendix C.1](#) that

$$\frac{1}{(2R)^d} \int_{B_R} (\mathbf{p}_n e^{-i\left(\frac{\pi}{R}\right)\mathbf{x}\cdot\mathbf{n}}) \cdot \mathbf{f}(\mathbf{x}) d\mathbf{x} = \mathbf{p}_n \cdot \frac{1}{(2R)^d} \int_{U_R} \mathbf{f}(\mathbf{x}) e^{-i\left(\frac{\pi}{R}\right)\mathbf{n}\cdot\mathbf{x}} d\mathbf{x} = \mathbf{p}_n \cdot \hat{\mathbf{f}}_n.$$

Combining the above estimates and using the Cauchy–Schwarz inequality yields

$$|\mathbf{p}_n \cdot \hat{\mathbf{f}}_n|^2 \lesssim \int_{\Gamma_R} (|T_N \mathbf{u}(\mathbf{x}, \omega_{\mathbf{p},n})|^2 + n^2 |\mathbf{u}(\mathbf{x}, \omega_{\mathbf{p},n})|^2) d\gamma(\mathbf{x}).$$

Using  $\mathbf{u}_{\mathbf{s},n}^{\text{inc}}$  and [\(2.60\)](#), we may repeat the above steps and obtain similarly

$$|\mathbf{q}_n \cdot \hat{\mathbf{f}}_n|^2 \lesssim \int_{\Gamma_R} (|T_N \mathbf{u}(\mathbf{x}, \omega_{\mathbf{s},n})|^2 + n^2 |\mathbf{u}(\mathbf{x}, \omega_{\mathbf{s},n})|^2) d\gamma(\mathbf{x}).$$

It follows from the Pythagorean theorem and the above estimates that we get

$$\begin{aligned} |\hat{\mathbf{f}}_n|^2 &= |\mathbf{p}_n \cdot \hat{\mathbf{f}}_n|^2 + |\mathbf{q}_n \cdot \hat{\mathbf{f}}_n|^2 \\ &\lesssim \int_{\Gamma_R} (|T_N \mathbf{u}(\mathbf{x}, \omega_{\mathbf{p},n})|^2 + n^2 |\mathbf{u}(\mathbf{x}, \omega_{\mathbf{p},n})|^2) d\gamma(\mathbf{x}) \\ &\quad + \int_{\Gamma_R} (|T_N \mathbf{u}(\mathbf{x}, \omega_{\mathbf{s},n})|^2 + n^2 |\mathbf{u}(\mathbf{x}, \omega_{\mathbf{s},n})|^2) d\gamma(\mathbf{x}) \\ &= \|\mathbf{u}(\cdot, \omega_{\mathbf{p},n})\|_{\Gamma_R}^2 + \|\mathbf{u}(\cdot, \omega_{\mathbf{s},n})\|_{\Gamma_R}^2. \end{aligned}$$

(ii) Next is to consider  $d = 3$ . Let  $\mathbf{p}_n = \hat{\mathbf{n}}$ . We pick two unit vectors  $\mathbf{q}_{1,n}$  and  $\mathbf{q}_{2,n}$  such that  $\{\mathbf{p}_n, \mathbf{q}_{1,n}, \mathbf{q}_{2,n}\}$  are mutually orthogonal and form an orthonormal basis in  $\mathbb{R}^3$ . Thus

$$|\hat{\mathbf{f}}_n|^2 = |\mathbf{p}_n \cdot \hat{\mathbf{f}}_n|^2 + |\mathbf{q}_{1,n} \cdot \hat{\mathbf{f}}_n|^2 + |\mathbf{q}_{2,n} \cdot \hat{\mathbf{f}}_n|^2.$$

Using  $\mathbf{p}_n$  as the polarization vector for  $\mathbf{u}_{\mathbf{p},n}^{\text{inc}}$  and  $\mathbf{q}_{1,n}, \mathbf{q}_{2,n}$  as the polarization vectors for  $\mathbf{u}_{\mathbf{s},n}^{\text{inc}}$  in [\(2.58\)](#), we may follow similar arguments for  $d = 2$  and obtain

$$|\hat{\mathbf{f}}_n|^2 \lesssim \|\mathbf{u}(\cdot, \omega_{\mathbf{p},n})\|_{\Gamma_R}^2 + \|\mathbf{u}(\cdot, \omega_{\mathbf{s},n})\|_{\Gamma_R}^2,$$

which completes the proof.  $\square$

**Lemma 2.14.** *Let  $\mathbf{f} \in H^{m+1}(B_R)^d$ . For any  $N_0 \in \mathbb{N}$ , the following estimate holds:*

$$\sum_{n=N_0}^{\infty} |\hat{\mathbf{f}}_n|^2 \lesssim N_0^{-(2m-d+1)} \|\mathbf{f}\|_{H^{m+1}(B_R)^d}^2.$$

**Proof.** Let  $\mathbf{n} = (n_1, \dots, n_d)^\top$  and choose  $n_j = \max\{n_1, \dots, n_d\}$ . Then we have  $n^2 \leq dn_j^2$ , which implies that  $n_j^{-(m+1)} \leq d^{\frac{m+1}{2}} n^{-(m+1)}$ . Let  $\mathbf{f} = (f_1, \dots, f_d)^\top$ . Noting  $\text{supp } \mathbf{f} \subset B_R \subset U_R$  and using integrating by parts, we obtain

$$\left| \int_{B_R} f_1(\mathbf{x}) e^{-i(\frac{\pi}{R})\mathbf{n}\cdot\mathbf{x}} d\mathbf{x} \right|^2 \lesssim \left| \int_{B_R} n_j^{-(m+1)} e^{-i(\frac{\pi}{R})\mathbf{n}\cdot\mathbf{x}} \partial_{x_j}^{m+1} f_1(\mathbf{x}) d\mathbf{x} \right|^2 \lesssim n^{-2(m+1)} \|\mathbf{f}\|_{H^{m+1}(B_R)^d}^2.$$

Hence

$$|\hat{\mathbf{f}}_{\mathbf{n}}|^2 \lesssim \left| \int_{B_R} \mathbf{f}(\mathbf{x}) e^{-i(\frac{\pi}{R})\mathbf{n}\cdot\mathbf{x}} d\mathbf{x} \right|^2 \lesssim n^{-2(m+1)} \|\mathbf{f}\|_{H^{m+1}(B_R)^d}^2.$$

It is easy to note that there are at most  $O(n^d)$  elements in  $\{\mathbf{n} \in \mathbb{Z}^d, |\mathbf{n}| = n\}$ . Combining the above estimates yields

$$\begin{aligned} \sum_{n=N_0}^{\infty} |\hat{\mathbf{f}}_{\mathbf{n}}|^2 &\lesssim \left( \sum_{n=N_0}^{\infty} n^{d-2(m+1)} \right) \|\mathbf{f}\|_{H^{m+1}(B_R)^d}^2 \\ &\lesssim \left( \int_0^{\infty} (N_0 + t)^{d-2(m+1)} dt \right) \|\mathbf{f}\|_{H^{m+1}(B_R)^d}^2 \\ &= \frac{N_0^{-(2m-d+1)}}{(2m-d+1)} \|\mathbf{f}\|_{H^{m+1}(B_R)^d}^2 \lesssim N_0^{-(2m-d+1)} \|\mathbf{f}\|_{H^{m+1}(B_R)^d}^2, \end{aligned}$$

which completes the proof.  $\square$

**Lemma 2.15.** Let  $\mathbf{u}$  be the solution of the scattering problem (2.1)–(2.2) corresponding to the source  $\mathbf{f} \in L^2(B_R)^d$ . For any  $\kappa \in (0, \frac{\pi}{R}]$  and  $\mathbf{d} \in \mathbb{S}^{d-1}$ , the following estimate holds:

$$\left| \int_{B_R} \mathbf{f}(\mathbf{x}) e^{-i\kappa\mathbf{x}\cdot\mathbf{d}} d\mathbf{x} \right|^2 \lesssim \epsilon_3^2.$$

**Proof.** Taking the compressional plane wave  $\mathbf{u}_p^{\text{inc}}(\mathbf{x}) = \mathbf{d} e^{-i c_p (\frac{\kappa}{c_p}) \mathbf{x} \cdot \mathbf{d}}$  and using similar arguments as those in Lemma 2.13, we obtain

$$\begin{aligned} \left| \mathbf{d} \cdot \int_{B_R} \mathbf{f}(\mathbf{x}) e^{-i\kappa\mathbf{x}\cdot\mathbf{d}} d\mathbf{x} \right|^2 &= \left| \mathbf{d} \cdot \int_{B_R} \mathbf{f}(\mathbf{x}) e^{-i c_p (\frac{\kappa}{c_p}) \mathbf{x} \cdot \mathbf{d}} d\mathbf{x} \right|^2 \\ &\lesssim \int_{\Gamma_R} \left( \left| T_N \mathbf{u} \left( \mathbf{x}, \frac{\kappa}{c_p} \right) \right|^2 + \left( \frac{\kappa}{c_p} \right)^2 \left| \mathbf{u} \left( \mathbf{x}, \frac{\kappa}{c_p} \right) \right|^2 \right) d\gamma(\mathbf{x}). \end{aligned}$$

Let the shear plane wave be  $\mathbf{u}_s^{\text{inc}}(\mathbf{x}) = \mathbf{p} e^{-i c_s (\frac{\kappa}{c_s}) \mathbf{x} \cdot \mathbf{d}}$ , where  $\mathbf{p}$  is a unit vector such that  $\mathbf{d} \perp \mathbf{p}$ . We may similarly get

$$\begin{aligned} \left| \mathbf{p} \cdot \int_{B_R} \mathbf{f}(\mathbf{x}) e^{-i\kappa \mathbf{x} \cdot \mathbf{d}} d\mathbf{x} \right|^2 &= \left| \mathbf{p} \cdot \int_{B_R} \mathbf{f}(\mathbf{x}) e^{-ic_s \left(\frac{\kappa}{c_s}\right) \mathbf{x} \cdot \mathbf{d}} d\mathbf{x} \right|^2 \\ &\lesssim \int_{\Gamma_R} \left( \left| T_N \mathbf{u} \left( \mathbf{x}, \frac{\kappa}{c_s} \right) \right|^2 + \left( \frac{\kappa}{c_s} \right)^2 \left| \mathbf{u} \left( \mathbf{x}, \frac{\kappa}{c_s} \right) \right|^2 \right) d\gamma(\mathbf{x}). \end{aligned}$$

Noting  $c_p < c_s$ , we have from the Pythagorean theorem that

$$\left| \int_{B_R} \mathbf{f}(\mathbf{x}) e^{-i\kappa \mathbf{x} \cdot \mathbf{d}} d\mathbf{x} \right|^2 = \left| \mathbf{d} \cdot \int_{B_R} \mathbf{f}(\mathbf{x}) e^{-i\kappa \mathbf{x} \cdot \mathbf{d}} d\mathbf{x} \right|^2 + \left| \mathbf{p} \cdot \int_{B_R} \mathbf{f}(\mathbf{x}) e^{-i\kappa \mathbf{x} \cdot \mathbf{d}} d\mathbf{x} \right|^2 \lesssim \epsilon_3^2.$$

The proof is the same for the three-dimensional case when we take two orthonormal polarization vectors  $\mathbf{p}_1$  and  $\mathbf{p}_2$  such that  $\{\mathbf{d}, \mathbf{p}_1, \mathbf{p}_2\}$  form an orthonormal basis in  $\mathbb{R}^3$ . The details are omitted for brevity.  $\square$

**Lemma 2.16.** *Let  $\mathbf{f} \in \tilde{\mathbb{F}}_M(B_R)$ . Then there exists a function  $\beta(s)$  satisfying*

$$\begin{cases} \beta(s) \geq \frac{1}{2}, & s \in \left(\frac{\pi}{R}, 2^{\frac{1}{4}} \frac{\pi}{R}\right), \\ \beta(s) \geq \frac{1}{\pi} \left( \left(\frac{Rs}{\pi}\right)^4 - 1 \right)^{-\frac{1}{2}}, & s \in \left(2^{\frac{1}{4}} \frac{\pi}{R}, \infty\right), \end{cases} \tag{2.61}$$

such that

$$\left| \int_{B_R} \mathbf{f}(\mathbf{x}) e^{-i\left(\frac{\pi}{R}\right) \mathbf{n} \cdot \mathbf{x}} d\mathbf{x} \right|^2 \lesssim M^2 e^{2nR} \epsilon_3^{2n\beta\left(\frac{n\pi}{R}\right)}, \quad \forall \mathbf{n} \in \mathbb{Z}^d, n > 1.$$

**Proof.** We fix a propagation direction vector  $\mathbf{d} \in \mathbb{S}^{d-1}$  and consider those  $\mathbf{n} \in \mathbb{Z}^d$  which are parallel to  $\mathbf{d}$ . Define

$$I(s) = \left| \int_{B_R} \mathbf{f}(\mathbf{x}) e^{-is\mathbf{d} \cdot \mathbf{x}} d\mathbf{x} \right|^2.$$

It follows from the Cauchy–Schwarz inequality that there exists a positive constant  $C$  depending on  $R, d$  such that

$$I(s) \leq C(R, d) e^{2|s|R} M^2, \quad \forall s \in \mathcal{V},$$

which gives

$$e^{-2|s|R} I(s) \lesssim M^2, \quad \forall s \in \mathcal{V}.$$

Noting  $\int_{\Omega} \mathbf{f}(\mathbf{x}) d\mathbf{x} = 0$  and using Lemma 2.15, we have

$$e^{-2|s|R} \left| \int_{B_R} \mathbf{f}(\mathbf{x}) e^{-is\mathbf{d} \cdot \mathbf{x}} d\mathbf{x} \right|^2 \lesssim \epsilon_3^2, \quad \forall s \in \left[0, \frac{\pi}{R}\right].$$

Applying Lemma Appendix C.2 shows that there exists a function  $\beta(s)$  satisfying (2.61) such that

$$|I(s)e^{-2sR}| \lesssim M^2 \epsilon_3^{2\beta}, \quad \forall s \in (\frac{\pi}{R}, \infty),$$

which yields that

$$|I(s)| \lesssim M^2 e^{2sR} \epsilon_3^{2\beta}, \quad \forall s \in (\frac{\pi}{R}, \infty).$$

Noting that the constant  $C(R, d)$  does not depend on  $\mathbf{d}$ , we have obtained for all  $\mathbf{n} \in \mathbb{Z}^d, n > 1$  that

$$\left| \int_{\hat{B}_R} \mathbf{f}(\mathbf{x}) e^{-i(\frac{\pi}{R})\mathbf{n}\cdot\mathbf{x}} d\mathbf{x} \right|^2 = \left| \int_{\hat{B}_R} \mathbf{f}(\mathbf{x}) e^{-i(\frac{n\pi}{R})\hat{\mathbf{n}}\cdot\mathbf{x}} d\mathbf{x} \right|^2 \lesssim M^2 e^{2nR} \epsilon_3^{2n\beta(\frac{n\pi}{R})},$$

which completes the proof.  $\square$

Next we prove Theorem 2.11.

**Proof.** Applying Lemma Appendix C.1 and Lemma 2.13, we have

$$\int_{B_R} |\mathbf{f}|^2 d\mathbf{x} \lesssim \sum_{n=0}^{N_0} |\hat{\mathbf{f}}_n|^2 + \sum_{n=N_0+1}^{\infty} |\hat{\mathbf{f}}_n|^2.$$

Let

$$N_0 = \begin{cases} [N^{\frac{3}{4}} |\ln \epsilon_3|^{\frac{1}{9}}], & N^{\frac{3}{8}} < \frac{1}{2^{\frac{5}{6}} \pi^{\frac{2}{3}}} |\ln \epsilon_3|^{\frac{1}{9}}, \\ N, & N^{\frac{3}{8}} \geq \frac{1}{2^{\frac{5}{6}} \pi^{\frac{2}{3}}} |\ln \epsilon_3|^{\frac{1}{9}}. \end{cases} \tag{2.62}$$

Using Lemma 2.16 leads to

$$\begin{aligned} \left| \int_{B_R} \mathbf{f}(\mathbf{x}) e^{-i(\frac{\pi}{R})\mathbf{n}\cdot\mathbf{x}} d\mathbf{x} \right|^2 &\lesssim M^2 e^{2nR} \epsilon_3^{2n\beta} \lesssim M^2 e^{2nR} e^{2n\beta |\ln \epsilon_3|} \\ &\lesssim M^2 e^{2nR} e^{-\frac{2}{\pi}(n^4-1)^{-\frac{1}{2}} |\ln \epsilon_3|} \lesssim M^2 e^{2nR - \frac{2}{\pi} n^{-2} |\ln \epsilon_3|} \\ &\lesssim M^2 e^{-\frac{2}{\pi} n^{-2} |\ln \epsilon_3| (1-2\pi n^3 |\ln \epsilon_2|^{-1})}, \quad \forall n \in (2^{\frac{1}{4}}, \infty). \end{aligned}$$

Hence

$$\left| \int_{B_R} \mathbf{f}(\mathbf{x}) e^{-i(\frac{\pi}{R})\mathbf{n}\cdot\mathbf{x}} d\mathbf{x} \right|^2 \lesssim M^2 e^{-\frac{2}{\pi^3} N^{-2} |\ln \epsilon_3| (1-2\pi^4 N^3 |\ln \epsilon_3|^{-1})}, \quad \forall n \in (2^{\frac{1}{4}}, N_0\pi]. \tag{2.63}$$

If  $N^{\frac{3}{8}} < \frac{1}{2^{\frac{5}{6}} \pi^{\frac{2}{3}}} |\ln \epsilon_3|^{\frac{1}{9}}$ , then  $2\pi^4 N_0^3 |\ln \epsilon_3|^{-1} < \frac{1}{2}$  and

$$e^{-\frac{2}{\pi^3} \frac{|\ln \epsilon_3|}{N_0^2}} \leq e^{-\frac{2}{\pi^3} \frac{|\ln \epsilon_3|}{N^{\frac{3}{2}} |\ln \epsilon_3|^{\frac{2}{9}}}} \leq e^{-\frac{2}{\pi^3} \frac{|\ln \epsilon_3|^{\frac{7}{9}}}{N^{\frac{3}{2}}}} \leq e^{-\frac{2}{\pi^3} \frac{2^5 \pi^4 |\ln \epsilon_3|^{\frac{1}{9}} N^{\frac{9}{4}}}{N^{\frac{3}{2}}}} = e^{-64\pi |\ln \epsilon_3|^{\frac{1}{9}} N^{\frac{3}{4}}}. \tag{2.64}$$

Combining (2.63) and (2.64), we obtain

$$\begin{aligned} \left| \int_{B_R} \mathbf{f}(\mathbf{x}) e^{-i(\frac{\pi}{R})\mathbf{n}\cdot\mathbf{x}} d\mathbf{x} \right|^2 &\lesssim M^2 e^{-\frac{2}{\pi^3} N_0^{-2} |\ln \epsilon_3| (1-2\pi^4 N_0^3 |\ln \epsilon_2|^{-1})} \\ &\lesssim M^2 e^{-\frac{1}{\pi^3} N_0^{-2} |\ln \epsilon_2|} \lesssim M^2 e^{-32\pi |\ln \epsilon_3|^{\frac{1}{9}} N^{\frac{3}{4}}}, \quad \forall n \in (2^{\frac{1}{4}}, N_0\pi]. \end{aligned}$$



It follows from the elementary inequality  $e^{-x} \leq \frac{(6m-3d+1)!}{x^{3(2m-d+1)}}$  that

$$\left| \int_{B_R} \mathbf{f}(\mathbf{x}) e^{-i(\frac{\pi}{R})\mathbf{n}\cdot\mathbf{x}} d\mathbf{x} \right|^2 \lesssim \frac{M^2}{\left( \frac{|\ln \epsilon_3|^{\frac{1}{3}} N^{\frac{9}{4}}}{(6m-3d+3)^3} \right)^{2m-d+1}}, \quad n = 1, \dots, N_0.$$

Consequently,

$$\begin{aligned} \sum_{n=1}^{N_0} \left| \int_{B_R} \mathbf{f}(\mathbf{x}) e^{-i(\frac{\pi}{R})\mathbf{n}\cdot\mathbf{x}} d\mathbf{x} \right|^2 &\lesssim \frac{M^2 N_0}{\left( \frac{|\ln \epsilon_3|^{\frac{1}{3}} N^{\frac{9}{4}}}{(6m-3d+3)^3} \right)^{2m-d+1}} \\ &\lesssim \frac{M^2 N^{\frac{3}{4}} |\ln \epsilon_3|^{\frac{1}{9}}}{\left( \frac{|\ln \epsilon_3|^{\frac{1}{3}} N^{\frac{9}{4}}}{(6m-3d+3)^3} \right)^{2m-d+1}} \lesssim \frac{M^2}{\left( \frac{|\ln \epsilon_3|^{\frac{2}{9}} N^{\frac{3}{2}}}{(6m-3d+3)^3} \right)^{2m-d+1}} \lesssim \frac{M^2}{\left( \frac{|\ln \epsilon_3|^{\frac{1}{9}} N^{\frac{3}{2}}}{(6m-3d+3)^3} \right)^{2m-d+1}}. \end{aligned}$$

Here we have used that  $|\ln \epsilon_3| > 1$  when  $N^{\frac{3}{8}} < \frac{1}{2^{\frac{5}{6}} \pi^{\frac{2}{3}}} |\ln \epsilon_3|^{\frac{1}{9}}$ . If  $N^{\frac{3}{8}} < \frac{1}{2^{\frac{5}{6}} \pi^{\frac{2}{3}}} |\ln \epsilon_3|^{\frac{1}{9}}$ , we have

$$\left( \left[ |\ln \epsilon_3|^{\frac{1}{9}} N^{\frac{3}{4}} \right] + 1 \right)^{2m-d+1} \geq \left( |\ln \epsilon_3|^{\frac{1}{9}} N^{\frac{3}{4}} \right)^{2m-d+1}.$$

If  $N^{\frac{3}{8}} \geq \frac{1}{2^{\frac{5}{6}} \pi^{\frac{2}{3}}} |\ln \epsilon_3|^{\frac{1}{9}}$ , then  $N_0 = N$ . It follows from Lemma 2.13 that

$$\sum_{n=1}^{N_0} \left| \int_{B_R} \mathbf{f}(\mathbf{x}) e^{-i(\frac{\pi}{R})\mathbf{n}\cdot\mathbf{x}} d\mathbf{x} \right|^2 \lesssim \epsilon_2^2.$$

Combining the above estimates and Lemma 2.14, we obtain

$$\begin{aligned} \sum_{n=1}^{\infty} \left| \int_{B_R} \mathbf{f}(\mathbf{x}) e^{-i(\frac{\pi}{R})\mathbf{n}\cdot\mathbf{x}} d\mathbf{x} \right|^2 &\lesssim \epsilon_2^2 + \frac{M^2}{\left( \frac{|\ln \epsilon_3|^{\frac{1}{9}} N^{\frac{3}{2}}}{(6m-3d+3)^3} \right)^{2m-d+1}} \\ &+ \frac{M^2}{\left( |\ln \epsilon_3|^{\frac{1}{9}} N^{\frac{3}{4}} \right)^{2m-d+1}} + \frac{M^2 (2^{\frac{5}{6}} \pi^{\frac{2}{3}})^{2m-d+1}}{\left( |\ln \epsilon_3|^{\frac{1}{9}} N^{\frac{5}{8}} \right)^{2m-d+1}}. \end{aligned}$$

Noting that  $N^{\frac{5}{8}} \leq N^{\frac{3}{4}} \leq N^{\frac{3}{2}}$  and  $2^{\frac{5}{6}} \pi^{\frac{2}{3}} \leq (6m-3d+3)^3, \forall m \geq d$ , we complete the proof after combining the above estimates.  $\square$

### 3. Electromagnetic waves

This section concerns the inverse source problem for electromagnetic waves. Following the general theme for the elasticity case presented in Section 2, we discuss the uniqueness of the problem and then show that the increasing stability can be achieved to reconstruct the radiating electric current density from the tangential trace of the electric field at multiple frequencies. The technical details differ from elastic waves due to different model equations and Green’s tensors.

### 3.1. Problem formulation

We consider the time-harmonic Maxwell equations in a homogeneous medium:

$$\nabla \times \mathbf{E} - i\kappa \mathbf{H} = 0, \quad \nabla \times \mathbf{H} + i\kappa \mathbf{E} = \mathbf{J} \quad \text{in } \mathbb{R}^3, \tag{3.1}$$

where  $\kappa > 0$  is the wavenumber,  $\mathbf{E} \in \mathbb{C}^3$  and  $\mathbf{H} \in \mathbb{C}^3$  are the electric field and the magnetic field, respectively,  $\mathbf{J} \in \mathbb{C}^3$  is the electric current density and is assumed to have a compact support  $\Omega$ . The problem geometry is the same as that for elastic waves and is shown in Fig. 1. The Silver–Müller radiation condition is required to make the direct problem well-posed:

$$\lim_{r \rightarrow \infty} ((\nabla \times \mathbf{E}) \times \mathbf{x} - i\kappa r \mathbf{E}) = 0, \quad r = |\mathbf{x}|. \tag{3.2}$$

Eliminating the magnetic field  $\mathbf{H}$  from (3.1), we obtain the decoupled Maxwell system for the electric field  $\mathbf{E}$ :

$$\nabla \times (\nabla \times \mathbf{E}) - \kappa^2 \mathbf{E} = i\kappa \mathbf{J} \quad \text{in } \mathbb{R}^3. \tag{3.3}$$

Given  $\mathbf{J} \in L^2(\Omega)^3$ , it is known that the scattering problem (3.2)–(3.3) has a unique solution (cf. [45]):

$$\mathbf{E}(\mathbf{x}, \kappa) = \int_{\Omega} \mathbf{G}_M(\mathbf{x}, \mathbf{y}; \kappa) \cdot \mathbf{J}(\mathbf{y}) d\mathbf{y}, \tag{3.4}$$

where  $\mathbf{G}_M(\mathbf{x}, \mathbf{y}; \kappa)$  is Green’s tensor for the Maxwell system (3.3). Explicitly, we have

$$\mathbf{G}_M(\mathbf{x}, \mathbf{y}; \kappa) = i\kappa g_3(\mathbf{x}, \mathbf{y}; \kappa) \mathbf{I}_3 + \frac{i}{\kappa} \nabla_{\mathbf{x}} \nabla_{\mathbf{x}}^{\top} g_3(\mathbf{x}, \mathbf{y}; \kappa), \tag{3.5}$$

where  $g_3$  is the fundamental solution of the three-dimensional Helmholtz equation and is given in (2.6).

Let  $\mathbf{E} \times \boldsymbol{\nu}$  and  $\mathbf{H} \times \boldsymbol{\nu}$  be the tangential trace of the electric field and the magnetic field, respectively. It is shown in [5] that there exists a capacity operator  $T_M$  such that

$$\mathbf{H} \times \boldsymbol{\nu} = T_M(\mathbf{E} \times \boldsymbol{\nu}) \quad \text{on } \Gamma_R, \tag{3.6}$$

which implies that  $\mathbf{H} \times \boldsymbol{\nu}$  can be computed once  $\mathbf{E} \times \boldsymbol{\nu}$  is available on  $\Gamma_R$ . The transparent boundary condition (3.6) can be equivalently written as

$$(\nabla \times \mathbf{E}) \times \boldsymbol{\nu} = i\kappa T_M(\mathbf{E} \times \boldsymbol{\nu}) \quad \text{on } \Gamma_R. \tag{3.7}$$

It follows from (3.7) that we define the following boundary measurement in terms of the tangential trace of the electric field only:

$$\|\mathbf{E}(\cdot, \kappa) \times \boldsymbol{\nu}\|_{\Gamma_R}^2 = \int_{\Gamma_R} (|T_M(\mathbf{E}(\mathbf{x}, \kappa) \times \boldsymbol{\nu})|^2 + |\mathbf{E}(\mathbf{x}, \kappa) \times \boldsymbol{\nu}|^2) d\gamma(\mathbf{x}).$$

**Problem 3.1.** Let  $\mathbf{J}$  be the electric current density with the compact support  $\Omega$ . The inverse source problem of electromagnetic waves is to determine  $\mathbf{J}$  from the tangential trace of the electric field  $\mathbf{E}(\mathbf{x}, \kappa) \times \boldsymbol{\nu}$  for  $\mathbf{x} \in \Gamma_R$ .

### 3.2. Uniqueness

In this section, we discuss the uniqueness and non-uniqueness of Problem 3.1. We begin with a simple uniqueness result.

**Theorem 3.2.** *Let  $I \subset \mathbb{R}^+$  be an open interval. If  $\nabla \cdot \mathbf{J} = 0$ , then the multiple-frequency data  $\{\mathbf{E}(\mathbf{x}, \kappa) \times \boldsymbol{\nu} : \mathbf{x} \in \Gamma_R, \omega \in I\}$  can uniquely determine  $\mathbf{J}$ .*

**Proof.** We assume  $\mathbf{E}(\mathbf{x}, \kappa) \times \boldsymbol{\nu} = 0$  on  $\Gamma_R$  for all  $\kappa \in I$ . Let  $\mathbf{E}^{\text{inc}}$  and  $\mathbf{H}^{\text{inc}}$  be the electric and magnetic plane waves. Explicitly, we have

$$\mathbf{E}^{\text{inc}} = \mathbf{p}e^{i\kappa\mathbf{x}\cdot\mathbf{d}} \quad \text{and} \quad \mathbf{H}^{\text{inc}} = \mathbf{q}e^{i\kappa\mathbf{x}\cdot\mathbf{d}}, \tag{3.8}$$

where  $\mathbf{d}(\theta, \varphi) = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)^\top$  is the unit propagation vector, and  $\mathbf{p}, \mathbf{q}$  are two unit polarization vectors and satisfy  $\mathbf{p}(\theta, \varphi) \cdot \mathbf{d}(\theta, \varphi) = 0, \mathbf{q}(\theta, \varphi) = \mathbf{p}(\theta, \varphi) \times \mathbf{d}(\theta, \varphi)$  for all  $\theta \in [0, \pi], \varphi \in [0, 2\pi]$ . It is easy to verify that  $\mathbf{E}^{\text{inc}}$  and  $\mathbf{H}^{\text{inc}}$  satisfy the homogeneous Maxwell equations in  $\mathbb{R}^3$ :

$$\nabla \times (\nabla \times \mathbf{E}^{\text{inc}}) - \kappa^2 \mathbf{E}^{\text{inc}} = 0 \tag{3.9}$$

and

$$\nabla \times (\nabla \times \mathbf{H}^{\text{inc}}) - \kappa^2 \mathbf{H}^{\text{inc}} = 0. \tag{3.10}$$

Let  $\boldsymbol{\xi} = \kappa\mathbf{d}$  with  $|\boldsymbol{\xi}| = \kappa \in I$ . We have from (3.8) that  $\mathbf{E}^{\text{inc}} = \mathbf{p}e^{-i\boldsymbol{\xi}\cdot\mathbf{x}}$  and  $\mathbf{H}^{\text{inc}} = \mathbf{q}e^{-i\boldsymbol{\xi}\cdot\mathbf{x}}$ . Multiplying the both sides of (3.3) by  $\mathbf{E}^{\text{inc}}$ , using the integration by parts over  $B_R$  and (3.9), we obtain

$$i\kappa \int_{B_R} \mathbf{p}e^{-i\boldsymbol{\xi}\cdot\mathbf{x}} \cdot \mathbf{J}(\mathbf{x})d\mathbf{x} = - \int_{\Gamma_R} (i\kappa T_M(\mathbf{E}(\mathbf{x}, \kappa) \times \boldsymbol{\nu}) \cdot \mathbf{E}^{\text{inc}} + (\mathbf{E}(\mathbf{x}, \kappa) \times \boldsymbol{\nu}) \cdot (\nabla \times \mathbf{E}^{\text{inc}})) d\gamma,$$

which means  $\mathbf{p} \cdot \hat{\mathbf{J}}(\kappa\mathbf{d}) = 0$  for all  $\omega \in I$ . Similarly, we have  $\mathbf{q} \cdot \hat{\mathbf{J}}(\kappa\mathbf{d}) = 0$  for all  $\kappa \in I$ . On the other hand, since  $\hat{\mathbf{J}}(\kappa\mathbf{d})$  is an analytic function with respect to  $\kappa \in \mathbb{C}$ , where  $\mathbb{C}$  denotes the complex number domain, we have both  $\mathbf{p} \cdot \hat{\mathbf{J}}(\kappa\mathbf{d}) = 0$  and  $\mathbf{q} \cdot \hat{\mathbf{J}}(\kappa\mathbf{d}) = 0$  for all  $\kappa > 0$ . Since  $\mathbf{p}, \mathbf{q}, \mathbf{d}$  are orthonormal vectors, they form an orthonormal basis in  $\mathbb{R}^3$ . We have from the Pythagorean theorem that

$$|\hat{\mathbf{J}}(\boldsymbol{\xi})|^2 = |\mathbf{p} \cdot \hat{\mathbf{J}}(\boldsymbol{\xi})|^2 + |\mathbf{q} \cdot \hat{\mathbf{J}}(\boldsymbol{\xi})|^2 + |\mathbf{d} \cdot \hat{\mathbf{J}}(\boldsymbol{\xi})|^2.$$

On the other hand, since  $\nabla \cdot \mathbf{J} = 0$ , we have for each  $\kappa \in I$  that

$$i\kappa\mathbf{d} \cdot \hat{\mathbf{J}}(\boldsymbol{\xi}) = i\kappa \int_{B_R} \mathbf{d}e^{i\kappa\mathbf{x}\cdot\mathbf{d}} \cdot \mathbf{J}(\mathbf{x})d\mathbf{x} = \int_{B_R} \nabla e^{i\kappa\mathbf{x}\cdot\mathbf{d}} \cdot \mathbf{J}d\mathbf{x} = \int_{B_R} e^{i\kappa\mathbf{x}\cdot\mathbf{d}} \nabla \cdot \mathbf{J}d\mathbf{x} = 0,$$

which yields that

$$|\hat{\mathbf{J}}(\boldsymbol{\xi})|^2 = |\mathbf{p} \cdot \hat{\mathbf{J}}(\boldsymbol{\xi})|^2 + |\mathbf{q} \cdot \hat{\mathbf{J}}(\boldsymbol{\xi})|^2 = 0,$$

which means  $\hat{\mathbf{J}}(\boldsymbol{\xi}) = 0$  for all  $\boldsymbol{\xi} \in \mathbb{R}^3$ , and then  $\mathbf{J}(\mathbf{x}) = 0$ .  $\square$

Next we discuss the uniqueness result much further. The goal is to distinguish the radiating and non-radiating current densities. We study a variational equation relating the unknown current density  $\mathbf{J}$  to the data  $\mathbf{E} \times \boldsymbol{\nu}$  on  $\Gamma_R$ .

Multiplying (3.3) by the complex conjugate of a test function  $\xi$  on both sides, integrating over  $B_R$ , and using the integration by parts, we obtain

$$\int_{B_R} (\nabla \times \mathbf{E} \cdot \nabla \times \bar{\xi} - \kappa^2 \mathbf{E} \cdot \bar{\xi}) \, d\mathbf{x} - \int_{\Gamma_R} [(\nabla \times \mathbf{E}) \times \boldsymbol{\nu}] \cdot \bar{\xi} \, d\gamma = i\kappa \int_{B_R} \mathbf{J} \cdot \bar{\xi} \, d\mathbf{x}. \tag{3.11}$$

Substituting (3.7) into (3.11), we obtain the variational problem: To find  $\mathbf{E} \in H(\text{curl}, B_R)$  such that

$$\begin{aligned} \int_{B_R} (\nabla \times \mathbf{E} \cdot \nabla \times \bar{\xi} - \kappa^2 \mathbf{E} \cdot \bar{\xi}) \, d\mathbf{x} - i\kappa \int_{\Gamma_R} T_M(\mathbf{E} \times \boldsymbol{\nu}) \cdot \bar{\xi}_{\Gamma_R} \, d\gamma \\ = i\kappa \int_{B_R} \mathbf{J} \cdot \bar{\xi} \, d\mathbf{x}, \quad \forall \xi \in H(\text{curl}, B_R). \end{aligned} \tag{3.12}$$

Given  $\mathbf{J} \in L^2(B_R)^3$ , the variational problem (3.12) can be shown to have a unique weak solution  $\mathbf{E} \in H(\text{curl}, B_R)$  (cf. [45,48]).

Assuming that  $\xi$  is a smooth function, we take the integration by parts one more time of (3.12) and get the identity:

$$\begin{aligned} i\kappa \int_{B_R} \mathbf{J} \cdot \bar{\xi} \, d\mathbf{x} = \int_{B_R} \mathbf{E} \cdot (\nabla \times (\nabla \times \bar{\xi}) - \kappa^2 \bar{\xi}) \, d\mathbf{x} \\ - \int_{\Gamma_R} ((\mathbf{E} \times \boldsymbol{\nu}) \cdot (\nabla \times \bar{\xi}) + i\kappa T_M(\mathbf{E} \times \boldsymbol{\nu}) \cdot \bar{\xi}_{\Gamma_R}) \, d\gamma. \end{aligned} \tag{3.13}$$

Now we choose  $\xi \in H(\text{curl}, B_R)$  to satisfy

$$\int_{B_R} (\nabla \times \xi \cdot \nabla \times \psi - \kappa^2 \xi \cdot \psi) \, d\mathbf{x} = 0, \quad \forall \psi \in C_0^\infty(B_R)^3, \tag{3.14}$$

which implies that  $\xi$  is a weak solution of the Maxwell system:

$$\nabla \times (\nabla \times \xi) - \kappa^2 \xi = 0 \quad \text{in } B_R.$$

Using this choice of  $\xi$ , we can see that (3.13) becomes

$$i\kappa \int_{B_R} \mathbf{J} \cdot \bar{\xi} \, d\mathbf{x} = - \int_{\Gamma_R} ((\mathbf{E} \times \boldsymbol{\nu}) \cdot (\nabla \times \bar{\xi}) + i\kappa T_M(\mathbf{E} \times \boldsymbol{\nu}) \cdot \bar{\xi}_{\Gamma_R}) \, d\gamma \tag{3.15}$$

for all  $\xi \in H(\text{curl}, B_R)$  satisfying (3.14).

Denote by  $\mathbb{X}(B_R)$  the closure of the set  $\{\mathbf{E} \in H(\text{curl}, B_R) : \mathbf{E} \text{ satisfies (3.14)}\}$  in the  $L^2(B_R)^3$  norm. We have the following orthogonal decomposition of  $L^2(B_R)^3$ :

$$L^2(B_R)^3 = \mathbb{X}(B_R) \oplus \mathbb{Y}(B_R).$$

It is shown in [2] that  $\mathbb{Y}(B_R)$  is an infinitely dimensional subspace of  $L^2(B_R)^3$ , which is stated in the following lemma.

**Lemma 3.3.** *Let  $\psi \in C_0^\infty(B_R)^3$ . If  $\phi = \nabla \times (\nabla \times \psi) - \kappa^2 \psi$ , then  $\phi \in \mathbb{Y}(B_R)$ .*

It follows from Lemma 3.3 that  $\mathbb{X}(B_R)$  is a proper subspace of  $L^2(B_R)^3$ . Given  $\mathbf{J} \in L^2(B_R)^3$ , only the component of  $\mathbf{J}$  in  $\mathbb{X}(B_R)$  can be determined from the data  $\mathbf{E} \times \boldsymbol{\nu}$  on  $\Gamma_R$ . Moreover, it is impossible that some other equation could be derived to determine the component of  $\mathbf{J}$  in  $\mathbb{Y}(B_R)$  from the data  $\mathbf{E} \times \boldsymbol{\nu}$  on  $\Gamma_R$ .

**Theorem 3.4.** *Suppose  $\mathbf{J} \in \mathbb{Y}(B_R)$ . Then  $\mathbf{J}$  does not produce any tangential trace of electric fields on  $\Gamma_R$  and thus cannot be identified.*

**Proof.** Since  $\mathbf{J} \in \mathbb{Y}(B_R)$ , we have from (3.15) that

$$\int_{\Gamma_R} ((\mathbf{E} \times \boldsymbol{\nu}) \cdot (\nabla \times \bar{\boldsymbol{\xi}}) + i\kappa T_M(\mathbf{E} \times \boldsymbol{\nu}) \cdot \bar{\boldsymbol{\xi}}_{\Gamma_R}) d\gamma = 0, \quad \forall \boldsymbol{\xi} \in \mathbb{X}(B_R),$$

which yields

$$\int_{\Gamma_R} (\mathbf{E} \times \boldsymbol{\nu}) \cdot (\overline{\nabla \times \boldsymbol{\xi} - i\kappa T_M^*(\boldsymbol{\xi}_{\Gamma_R})}) d\gamma = 0. \tag{3.16}$$

Here  $T_M^*$  is the adjoint operator of  $T_M$ . Let

$$\nabla \times \boldsymbol{\xi} - i\kappa T_M^*(\boldsymbol{\xi}_{\partial B_R}) = \bar{\boldsymbol{\eta}} \quad \text{on } \Gamma_R.$$

More precisely,  $\boldsymbol{\xi} \in H(\text{curl}, B_R)$  satisfies the variational problem

$$\int_{B_R} (\nabla \times \boldsymbol{\xi} \cdot \nabla \times \boldsymbol{\phi} - \kappa^2 \boldsymbol{\xi} \cdot \boldsymbol{\phi}) d\mathbf{x} + \int_{\Gamma_R} (\bar{\boldsymbol{\eta}} - i\kappa T_M^*(\boldsymbol{\xi}_{\Gamma_R})) \cdot (\boldsymbol{\phi} \times \boldsymbol{\nu}) d\gamma = 0, \quad \forall \boldsymbol{\phi} \in H(\text{curl}, B_R).$$

It is shown in [2] that there exists a unique solution  $\boldsymbol{\xi} \in \mathbb{X}(B_R)$  to the above boundary value problem for any  $\boldsymbol{\eta} \in H^{-1/2}(\text{curl}, \Gamma_R)$ , where  $\nabla_{\Gamma_R}$  is the surface gradient. Hence we have from (3.16) that

$$\int_{\Gamma_R} (\mathbf{E} \times \boldsymbol{\nu}) \cdot \boldsymbol{\eta} d\gamma = 0, \quad \forall \boldsymbol{\eta} \in H^{-1/2}(\text{curl}, \Gamma_R),$$

which yields that  $\mathbf{E} \times \boldsymbol{\nu} = 0$  on  $\Gamma_R$  and completes the proof.  $\square$

**Remark 3.5.** The electric current densities in  $\mathbb{Y}(B_R)$  are called non-radiating sources. It corresponds to find a minimum norm solution when computing the component of the source in  $\mathbb{X}(B_R)$ .

It is shown in Theorem 3.4 that  $\mathbf{J}$  cannot be determined from the tangential trace of the electric field  $\mathbf{E} \times \boldsymbol{\nu}$  on  $\Gamma_R$  if  $\mathbf{J} \in \mathbb{Y}(B_R)$ . We show in the following theorem that it is also impossible to determine  $\mathbf{J}$  from the normal component of the electric field  $\mathbf{E} \cdot \boldsymbol{\nu}$  on  $\Gamma_R$  if  $\mathbf{J} \in \mathbb{Y}(B_R)$ .

**Theorem 3.6.** *Suppose  $\mathbf{J} \in \mathbb{Y}(B_R)$ . Then  $\mathbf{J}$  does not produce any normal component of electric fields on  $\Gamma_R$ .*

**Proof.** Let  $\phi \in C^\infty(B_R)$ . Multiplying both sides of (3.3) by  $\nabla\phi$  and integrating on  $B_R$ , we have

$$\int_{B_R} (\nabla \times (\nabla \times \mathbf{E}) - \kappa^2 \mathbf{E}) \cdot \nabla\phi d\mathbf{x} = i\kappa \int_{B_R} \mathbf{J} \cdot \nabla\phi d\mathbf{x}.$$

It follows from the integration by parts that

$$\int_{\dot{B}_R} (\nabla \times (\nabla \times \mathbf{E})) \cdot \nabla \phi d\mathbf{x} = \int_{\dot{B}_R} (\nabla \times \mathbf{E}) \cdot (\nabla \times \nabla \phi) d\mathbf{x} - \int_{\Gamma_R} (\boldsymbol{\nu} \times (\nabla \times \mathbf{E})) \cdot \nabla \phi d\gamma.$$

Noting  $\nabla \times \nabla \phi = 0$  and (3.7), and using Theorem 3.4, we obtain

$$\int_{\Gamma_R} (\boldsymbol{\nu} \times (\nabla \times \mathbf{E})) \cdot \nabla \phi d\gamma = 0.$$

Combining the above equations gives

$$-\kappa^2 \int_{\dot{B}_R} \mathbf{E} \cdot \nabla \phi d\mathbf{x} = i\kappa \int_{\dot{B}_R} \mathbf{J} \cdot \nabla \phi d\mathbf{x},$$

which implies

$$i\kappa \int_{\dot{B}_R} \mathbf{E} \cdot \nabla \phi d\mathbf{x} = \int_{\dot{B}_R} \mathbf{J} \cdot \nabla \phi d\mathbf{x}. \tag{3.17}$$

We have from the integration by parts that

$$i\kappa \int_{\dot{B}_R} \mathbf{E} \cdot \nabla \phi d\mathbf{x} = -i\kappa \int_{\dot{B}_R} \nabla \cdot \mathbf{E} \phi d\mathbf{x} + i\kappa \int_{\Gamma_R} (\mathbf{E} \cdot \boldsymbol{\nu}) \phi d\gamma. \tag{3.18}$$

On the other hand, since

$$\nabla \times \mathbf{H} + i\kappa \mathbf{E} = \mathbf{J},$$

then by taking the divergence on both sides, we have

$$i\kappa \nabla \cdot \mathbf{E} = \nabla \cdot \mathbf{J}.$$

Hence

$$i\kappa \int_{\dot{B}_R} \nabla \cdot \mathbf{E} \phi d\mathbf{x} = \int_{\dot{B}_R} \nabla \cdot \mathbf{J} \phi d\mathbf{x} = - \int_{\dot{B}_R} \mathbf{J} \cdot \nabla \phi d\mathbf{x}. \tag{3.19}$$

Combining (3.17)–(3.19), we get

$$\int_{\Gamma_R} (\mathbf{E} \cdot \boldsymbol{\nu}) \phi d\mathbf{x} = 0, \quad \forall \phi \in C^\infty(B_R),$$

which implies that  $\mathbf{E} \cdot \boldsymbol{\nu}$  on  $\Gamma_R$  and completes the proof.  $\square$

The following theorem concerns the uniqueness result of Problem 3.1.

**Theorem 3.7.** *Suppose  $\mathbf{J} \in \mathbb{X}(B_R)$ , then  $\mathbf{J}$  can be uniquely determined by the data  $\mathbf{E} \times \boldsymbol{\nu}$  on  $\Gamma_R$ .*

**Proof.** It suffices to show that  $\mathbf{J} = 0$  if  $\mathbf{E} \times \boldsymbol{\nu} = 0$  on  $\Gamma_R$ . It follows from (3.15) that we have

$$\int_{B_R} \mathbf{J} \cdot \bar{\boldsymbol{\xi}} \, d\mathbf{x} = 0, \quad \forall \boldsymbol{\xi} \in \mathbb{X}(B_R).$$

Taking  $\boldsymbol{\xi} = \mathbf{J}$  yields that

$$\int_{B_R} |\mathbf{J}|^2 \, d\mathbf{x} = 0,$$

which completes the proof.  $\square$

Taking account of the uniqueness result, we revise the inverse source problem for electromagnetic waves, which is to determine  $\mathbf{J}$  in the smaller space  $\mathbb{X}(B_R)$ .

**Problem 3.8** (Continuous frequency data for electromagnetic waves). Let  $\mathbf{J} \in \mathbb{X}(B_R)$ . The inverse source problem of electromagnetic waves is to determine  $\mathbf{J}$  from the tangential trace of the electric field  $\mathbf{E}(\mathbf{x}, \kappa) \times \boldsymbol{\nu}$  for  $\mathbf{x} \in \Gamma_R, \kappa \in (0, K)$ , where  $K > 1$  is a constant.

### 3.3. Stability with continuous frequency data

Define a functional space

$$\mathbb{J}_M(B_R) = \{\mathbf{J} \in \mathbb{X}(B_R) \cap H^m(B_R)^3 : \|\mathbf{J}\|_{H^m(B_R)^3} \leq M\},$$

where  $m \geq d$  is an integer and  $M > 1$  is a constant. The following is our main result regarding the stability for Problem 3.8.

**Theorem 3.9.** Let  $\mathbf{E}$  be the solution of the scattering problem (3.2)–(3.3) corresponding to  $\mathbf{J} \in \mathbb{J}_M(B_R)$ . Then

$$\|\mathbf{J}\|_{L^2(B_R)^3}^2 \lesssim \epsilon_4^2 + \frac{M^2}{\left(\frac{K^{\frac{2}{3}} |\ln \epsilon_4|^{\frac{1}{4}}}{(R+1)(6m-15)^3}\right)^{2m-5}}, \tag{3.20}$$

where

$$\epsilon_4 = \left( \int_0^K \kappa^2 \|\mathbf{E}(\cdot, \kappa) \times \boldsymbol{\nu}\|_{\Gamma_R}^2 \, d\kappa \right)^{\frac{1}{2}}$$

**Remark 3.10.** The stability estimate (3.20) is consistent with that for elastic waves in (2.13). It also has two parts: the data discrepancy and the high frequency tail. The ill-posedness of the inverse problem decreases as  $K$  increases.

We begin with several useful lemmas.

**Lemma 3.11.** Let  $\mathbf{E}$  be the solution of (3.2)–(3.3) corresponding to the source  $\mathbf{J} \in \mathbb{X}(B_R)$ . Then

$$\|\mathbf{J}\|_{L^2(B_R)^3}^2 \lesssim \int_0^\infty \kappa^2 \|\mathbf{E}(\cdot, \kappa) \times \boldsymbol{\nu}\|_{\Gamma_R}^2 \, d\kappa.$$

**Proof.** Let  $\mathbf{E}^{\text{inc}}$  and  $\mathbf{H}^{\text{inc}}$  be the electric and magnetic plane waves:

$$\mathbf{E}^{\text{inc}}(\mathbf{x}) = \mathbf{p}e^{-i\kappa\mathbf{x}\cdot\mathbf{d}} \quad \text{and} \quad \mathbf{H}^{\text{inc}}(\mathbf{x}) = \mathbf{q}e^{-i\kappa\mathbf{x}\cdot\mathbf{d}}, \tag{3.21}$$

where  $\mathbf{d}(\theta, \varphi) = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)^\top$  is the unit propagation vector, and  $\mathbf{p}, \mathbf{q}$  are two unit polarization vectors and satisfy  $\mathbf{p}(\theta, \varphi) \cdot \mathbf{d}(\theta, \varphi) = 0, \mathbf{q}(\theta, \varphi) = \mathbf{p}(\theta, \varphi) \times \mathbf{d}(\theta, \varphi)$  for all  $\theta \in [0, \pi], \varphi \in [0, 2\pi]$ . The electric and magnetic plane waves satisfy

$$\nabla \times (\nabla \times \mathbf{E}^{\text{inc}}) - \kappa^2 \mathbf{E}^{\text{inc}} = 0 \tag{3.22}$$

and

$$\nabla \times (\nabla \times \mathbf{H}^{\text{inc}}) - \kappa^2 \mathbf{H}^{\text{inc}} = 0. \tag{3.23}$$

Let  $\boldsymbol{\xi} = \kappa\mathbf{d}$  with  $|\boldsymbol{\xi}| = \kappa \in (0, \infty)$ . We have from (3.21) that  $\mathbf{E}^{\text{inc}} = \mathbf{p}e^{-i\boldsymbol{\xi}\cdot\mathbf{x}}$  and  $\mathbf{H}^{\text{inc}} = \mathbf{q}e^{-i\boldsymbol{\xi}\cdot\mathbf{x}}$ . Multiplying the both sides of (3.3) by  $\mathbf{E}^{\text{inc}}$ , using the integration by parts over  $B_R$  and (3.22), we obtain

$$i\kappa \int_{B_R} (\mathbf{p}e^{-i\boldsymbol{\xi}\cdot\mathbf{x}}) \cdot \mathbf{J}(\mathbf{x})d\mathbf{x} = - \int_{\Gamma_R} (i\kappa T_M(\mathbf{E}(\mathbf{x}, \kappa) \times \boldsymbol{\nu}) \cdot \mathbf{E}^{\text{inc}} + (\mathbf{E}(\mathbf{x}, \kappa) \times \boldsymbol{\nu}) \cdot (\nabla \times \mathbf{E}^{\text{inc}})) d\gamma.$$

A simple calculation yields that

$$\nabla \times \mathbf{E}^{\text{inc}} = -i\kappa\mathbf{d} \times \mathbf{p}e^{-i\kappa\mathbf{x}\cdot\mathbf{d}},$$

which gives

$$|\nabla \times \mathbf{E}^{\text{inc}}| = \kappa.$$

Since  $\text{supp} \mathbf{f} = \Omega \subset B_R$ , we have

$$\int_{B_R} (\mathbf{p}e^{-i\boldsymbol{\xi}\cdot\mathbf{x}}) \cdot \mathbf{J}(\mathbf{x})d\mathbf{x} = \mathbf{p} \cdot \int_{\mathbb{R}^3} \mathbf{J}(\mathbf{x})e^{-i\boldsymbol{\xi}\cdot\mathbf{x}} d\mathbf{x} = \mathbf{p} \cdot \hat{\mathbf{J}}(\boldsymbol{\xi}).$$

Combining the above estimates yields

$$|\mathbf{p} \cdot \hat{\mathbf{J}}(\boldsymbol{\xi})|^2 \lesssim \int_{\Gamma_R} (|T_M(\mathbf{E}(\mathbf{x}, \kappa) \times \boldsymbol{\nu})|^2 + |\mathbf{E}(\mathbf{x}, \kappa) \times \boldsymbol{\nu}|^2) d\gamma(\mathbf{x}) = \|\mathbf{E}(\cdot, \kappa)\|_{\Gamma_R}^2.$$

Hence

$$\int_{\mathbb{R}^3} |\mathbf{p} \cdot \hat{\mathbf{J}}(\boldsymbol{\xi})|^2 d\boldsymbol{\xi} \lesssim \int_{\mathbb{R}^3} \|\mathbf{E}(\cdot, \kappa)\|_{\Gamma_R}^2 d\boldsymbol{\xi}.$$

Using the spherical coordinates, we get

$$\int_{\mathbb{R}^3} |\mathbf{p} \cdot \hat{\mathbf{J}}(\boldsymbol{\xi})|^2 d\boldsymbol{\xi} \lesssim \int_0^{2\pi} d\theta \int_0^\pi \sin\varphi d\varphi \int_0^\infty \kappa^2 \|\mathbf{E}(\cdot, \kappa)\|_{\Gamma_R}^2 d\kappa \lesssim \int_0^\infty \kappa^2 \|\mathbf{E}(\cdot, \kappa)\|_{\Gamma_R}^2 d\kappa.$$

Similarly we may show from (3.23) and the integration by parts that



$$\int_{\mathbb{R}^3} \left| \int_{\mathbb{R}^3} (\mathbf{q}e^{-i\xi \cdot \mathbf{x}}) \cdot \mathbf{J}(\mathbf{x}) d\mathbf{x} \right|^2 d\xi = \int_{\mathbb{R}^3} |\mathbf{q} \cdot \hat{\mathbf{J}}(\xi)|^2 d\xi \lesssim \int_0^\infty \kappa^2 \|\mathbf{E}(\cdot, \kappa)\|_{\Gamma_R}^2 d\kappa.$$

Since  $\mathbf{p}, \mathbf{q}, \mathbf{d}$  are orthonormal vectors, they form an orthonormal basis in  $\mathbb{R}^3$ . We have from the Pythagorean theorem that

$$|\hat{\mathbf{J}}(\xi)|^2 = |\mathbf{p} \cdot \hat{\mathbf{J}}(\xi)|^2 + |\mathbf{q} \cdot \hat{\mathbf{J}}(\xi)|^2 + |\mathbf{d} \cdot \hat{\mathbf{J}}(\xi)|^2.$$

On the other hand, since  $\mathbf{J}$  has a compact support  $\Omega$  contained in  $B_R$  and  $\mathbf{J} \in \mathbb{X}(B_R)$ , we obtain that  $\mathbf{J}$  is a weak solution of the Maxwell system:

$$\nabla \times (\nabla \times \mathbf{J}) - \kappa^2 \mathbf{J} = 0 \quad \text{in } B_R.$$

Multiplying the above equation by  $\mathbf{d}e^{i\kappa \mathbf{x} \cdot \mathbf{d}}$  and using integration by parts, we get

$$\int_{B_R} (\nabla \times \mathbf{J}) \cdot (\nabla \times (\mathbf{d}e^{i\kappa \mathbf{x} \cdot \mathbf{d}})) d\mathbf{x} = \kappa^2 \mathbf{d} \cdot \int_{B_R} \mathbf{J}(\mathbf{x}) e^{i\kappa \mathbf{x} \cdot \mathbf{d}} d\mathbf{x} = \kappa^2 \mathbf{d} \cdot \hat{\mathbf{J}}(\xi).$$

Noting  $\nabla \times (\mathbf{d}e^{i\kappa \mathbf{x} \cdot \mathbf{d}}) = i\kappa \mathbf{d} \times \mathbf{d}e^{i\kappa \mathbf{x} \cdot \mathbf{d}} = 0$ , we get  $\mathbf{d} \cdot \hat{\mathbf{J}}(\xi) = 0$ , which yields that

$$|\hat{\mathbf{J}}(\xi)|^2 = |\mathbf{p} \cdot \hat{\mathbf{J}}(\xi)|^2 + |\mathbf{q} \cdot \hat{\mathbf{J}}(\xi)|^2.$$

Hence, we obtain from the Parseval theorem that

$$\begin{aligned} \|\mathbf{J}\|_{L^2(B_R)^3}^2 &= \|\mathbf{J}\|_{L^2(\mathbb{R}^3)^3}^2 = \|\hat{\mathbf{J}}\|_{L^2(\mathbb{R}^3)^3}^2 = \int_{\mathbb{R}^3} |\hat{\mathbf{J}}(\xi)|^2 d\xi \\ &= \int_{\mathbb{R}^3} |\mathbf{p} \cdot \hat{\mathbf{J}}(\xi)|^2 d\xi + \int_{\mathbb{R}^3} |\mathbf{q} \cdot \hat{\mathbf{J}}(\xi)|^2 d\xi \lesssim \int_0^\infty \kappa^2 \|\mathbf{E}(\cdot, \kappa)\|_{\Gamma_R}^2 d\kappa, \end{aligned}$$

which completes the proof.  $\square$

Eliminating  $\mathbf{E}$  from (3.1), we obtain

$$\nabla \times (\nabla \times \mathbf{H}) - \kappa^2 \mathbf{H} = \nabla \times \mathbf{J} \quad \text{in } \mathbb{R}^3. \tag{3.24}$$

It is easy to verify from (3.1) that  $\nabla \cdot \mathbf{H} = 0$ . Using the identity

$$\nabla \times (\nabla \times \mathbf{H}) = -\Delta \mathbf{H} + \nabla \nabla \cdot \mathbf{H} = -\Delta \mathbf{H},$$

we get from (3.24) that  $\mathbf{H}$  satisfies the inhomogeneous Helmholtz equation:

$$\Delta \mathbf{H} + \kappa^2 \mathbf{H} = -\nabla \times \mathbf{J} \quad \text{in } \mathbb{R}^3. \tag{3.25}$$

It is known that (3.25) has a unique solution:

$$\mathbf{H}(\mathbf{x}, \kappa) = - \int_{\Omega} g_3(\mathbf{x}, \mathbf{y}) \mathbf{I}_3 \cdot \nabla \times \mathbf{J}(\mathbf{y}) d\mathbf{y}. \tag{3.26}$$

Let

$$I_1(s) = \int_0^s \kappa^2 \int_{\Gamma_R} \left| \int_{\Omega} \mathbf{G}_M(\mathbf{x}, \mathbf{y}; \kappa) \cdot \mathbf{J}(\mathbf{x}) d\mathbf{y} \times \boldsymbol{\nu}(\mathbf{x}) \right|^2 d\gamma(\mathbf{x}) d\kappa, \tag{3.27}$$

$$I_2(s) = \int_0^s \kappa^2 \int_{\Gamma_R} \left| \int_{\Omega} g_3(\mathbf{x}, \mathbf{y}; \kappa) \mathbf{I}_3 \cdot \nabla \times \mathbf{J}(\mathbf{y}) d\mathbf{y} \times \boldsymbol{\nu}(\mathbf{x}) \right|^2 d\gamma(\mathbf{x}) d\kappa. \tag{3.28}$$

Again, the integrands in (3.27)–(3.28) are analytic functions of  $\kappa$ . The integrals with respect to  $\kappa$  can be taken over any path joining points 0 and  $s$  in  $\mathcal{V}$ . Thus  $I_1(s)$  and  $I_2(s)$  are analytic functions of  $s = s_1 + is_2 \in \mathcal{V}$ ,  $s_1, s_2 \in \mathbb{R}$ .

**Lemma 3.12.** *Let  $\mathbf{J} \in H^2(B_R)^3$ . For any  $s = s_1 + is_2 \in \mathcal{V}$ , the following estimates hold:*

$$|I_1(s)| \lesssim (|s|^5 + |s|) e^{4R|s|} \|\mathbf{J}\|_{H^2(B_R)^3}^2, \tag{3.29}$$

$$|I_2(s)| \lesssim |s|^3 e^{4R|s|} \|\mathbf{J}\|_{H^1(B_R)^3}^2. \tag{3.30}$$

**Proof.** Let  $\kappa = st, t \in (0, 1)$ . Noting (3.5), we have from direct calculation that

$$|I_1(s)| \leq I_{1,1}(s) + I_{1,2}(s),$$

where

$$I_{1,1}(s) = \int_0^1 |s|^3 t^2 \int_{\Gamma_R} \left| \int_{\Omega} st g_3(\mathbf{x}, \mathbf{y}; \kappa) \mathbf{I}_3 \cdot \mathbf{J}(\mathbf{y}) d\mathbf{y} \right|^2 d\gamma(\mathbf{x}) dt,$$

$$I_{1,2}(s) = \int_0^1 |s|^3 t^2 \int_{\Gamma_R} \left| \int_{\Omega} \frac{1}{st} \nabla_{\mathbf{y}} \nabla_{\mathbf{y}}^\top g(\mathbf{x}, \mathbf{y}) \cdot \mathbf{J}(\mathbf{y}) d\mathbf{y} \right|^2 d\gamma(\mathbf{x}) dt.$$

Since  $\text{supp} \mathbf{J} = \Omega \subset B_R$  and

$$|e^{ist|\mathbf{x}-\mathbf{y}|}| \leq e^{2R|s|}, \quad \forall \mathbf{x} \in \Gamma_R, \mathbf{y} \in \Omega,$$

we get from the Cauchy–Schwarz inequality that

$$\begin{aligned} I_{1,1}(s) &\lesssim \int_0^1 |s|^5 t^4 \int_{\Gamma_R} \left| \int_{\Omega} \frac{e^{2R|s|}}{|\mathbf{x}-\mathbf{y}|} |\mathbf{J}(\mathbf{y})| d\mathbf{y} \right|^2 d\gamma(\mathbf{x}) dt \\ &\lesssim \int_0^1 |s|^5 t^4 \int_{\Gamma_R} \left( \int_{B_R} |\mathbf{J}(\mathbf{y})|^2 d\mathbf{y} \right) \left( \int_{\Omega} \frac{e^{4R|s|}}{|\mathbf{x}-\mathbf{y}|^2} d\mathbf{y} \right) d\gamma(\mathbf{x}) dt \\ &\lesssim |s|^5 e^{4R|s|} \|\mathbf{J}\|_{L^2(B_R)^3}^2. \end{aligned} \tag{3.31}$$

On the other hand, we obtain from the integration by parts that

$$I_{1,2}(s) \lesssim \int_0^1 |s|^3 t^2 \int_{\Gamma_R} \left| \int_{\Omega} \frac{1}{st} \nabla_{\mathbf{y}} \nabla_{\mathbf{y}}^\top g_3(\mathbf{x}, \mathbf{y}) \cdot \mathbf{J}(\mathbf{y}) d\mathbf{y} \right|^2 d\gamma(\mathbf{x}) dt$$

$$\begin{aligned} &\lesssim \int_0^1 |s| \int_{\Gamma_R} \left| \int_{\Omega} g_3(\mathbf{x}, \mathbf{y}) \nabla_{\mathbf{y}} \nabla_{\mathbf{y}} \cdot \mathbf{J} d\mathbf{y} \right|^2 d\gamma(\mathbf{x}) dt \\ &\lesssim |s| e^{4R|s|} \|\mathbf{J}\|_{H^2(B_R)}^2. \end{aligned} \tag{3.32}$$

Combining (3.31) and (3.32) yields (3.29).

Using the Cauchy–Schwarz inequality and (3.31), we have

$$\begin{aligned} I_2(s) &\lesssim \int_0^1 |s|^3 t^2 \int_{\Gamma_R} \left| \int_{\Omega} g_3(\mathbf{x}, \mathbf{y}; \kappa) \mathbf{I}_3 \cdot \nabla \times \mathbf{J}(\mathbf{y}) d\mathbf{y} \right|^2 d\gamma(\mathbf{x}) dt \\ &\lesssim \int_0^1 |s|^3 \left( \int_{B_R} |\nabla \times \mathbf{J}(\mathbf{y})|^2 d\mathbf{y} \right) \int_{\Gamma_R} \left( \int_{\Omega} \frac{e^{4R|s|}}{|\mathbf{x} - \mathbf{y}|^2} d\mathbf{y} \right) d\gamma(\mathbf{x}) dt \\ &\lesssim |s|^3 e^{4R|s|} \|\mathbf{J}\|_{H^1(B_R)}^2, \end{aligned}$$

which complete the proof of (3.30).  $\square$

**Lemma 3.13.** *Let  $\mathbf{J} \in \mathbb{J}_M(B_R)$ . For any  $s \geq 1$ , the following estimate holds:*

$$\int_s^\infty \kappa^2 \|\mathbf{E} \times \boldsymbol{\nu}\|_{\Gamma_R}^2 d\kappa \lesssim s^{-(2m-5)} \|\mathbf{J}\|_{H^m(B_R)}^2.$$

**Proof.** Let

$$\int_s^\infty \int_{\Gamma_R} \kappa^2 \|\mathbf{E} \times \boldsymbol{\nu}\|_{\Gamma_R}^2 d\kappa = L_1 + L_2,$$

where

$$\begin{aligned} L_1 &= \int_s^\infty \int_{\Gamma_R} \kappa^2 |\mathbf{E}(\mathbf{x}, \kappa) \times \boldsymbol{\nu}(\mathbf{x})|^2 d\gamma(\mathbf{x}) d\kappa, \\ L_2 &= \int_s^\infty \int_{\Gamma_R} \kappa^2 |\mathbf{H}(\mathbf{x}, \kappa) \times \boldsymbol{\nu}(\mathbf{x})|^2 d\gamma(\mathbf{x}) d\kappa. \end{aligned}$$

First we estimate  $L_1$ . Using (3.4) and noting  $s \geq 1$ , we obtain

$$L_1 = \int_s^\infty \int_{\Gamma_R} \kappa^2 |\mathbf{E}(\mathbf{x}, \kappa) \times \boldsymbol{\nu}(\mathbf{x})|^2 d\gamma(\mathbf{x}) d\kappa \lesssim L_{1,1} + L_{1,2},$$

where

$$L_{1,1} = \int_s^\infty \int_{\Gamma_R} \kappa^4 \left| \int_{\Omega} \frac{e^{i\kappa|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x} - \mathbf{y}|} \mathbf{I}_3 \cdot \mathbf{J}(\mathbf{y}) d\mathbf{y} \right|^2 d\gamma(\mathbf{x}) d\kappa,$$

$$L_{1,2} = \int_s^\infty \int_{\Gamma_R} \left| \int_\Omega \nabla_{\mathbf{y}} \nabla_{\mathbf{y}}^\top \frac{e^{i\kappa|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|} \cdot \mathbf{J}(\mathbf{y}) d\mathbf{y} \right|^2 d\gamma(\mathbf{x}) d\kappa.$$

Noting  $\text{supp} \mathbf{J} = \Omega \subset B_{\hat{R}} \subset B_R$ , and using integration by parts and the polar coordinates  $\rho = |\mathbf{y} - \mathbf{x}|$  originated at  $\mathbf{x}$  with respect to  $\mathbf{y}$ , we have

$$L_{1,1} = \int_s^\infty \int_{\Gamma_R} \kappa^4 \left| \int_0^{2\pi} d\theta \int_0^\pi \sin \varphi d\varphi \int_{R-\hat{R}}^{R+\hat{R}} \frac{e^{i\kappa\rho}}{(i\kappa)^m} \mathbf{I}_3 \cdot \frac{\partial^m(\mathbf{J}\rho)}{\partial \rho^m} d\rho \right|^2 d\gamma(\mathbf{x}) d\kappa.$$

Consequently,

$$\begin{aligned} L_{1,1} &\leq \int_s^\infty \int_{\Gamma_R} \kappa^4 \left| \int_0^{2\pi} d\theta \int_0^\pi \sin \varphi d\varphi \int_{R-\hat{R}}^{R+\hat{R}} \kappa^{-m} \right. \\ &\quad \left. \left( \left| \sum_{|\alpha|=m} \partial_{\mathbf{y}}^\alpha \mathbf{J} \right| \rho + m \left| \sum_{|\alpha|=m-1} \partial_{\mathbf{y}}^\alpha \mathbf{J} \right| \right) d\rho \right|^2 d\gamma(\mathbf{x}) d\kappa \\ &= \int_s^\infty \int_{\Gamma_R} \kappa^4 \left| \int_0^{2\pi} d\theta \int_0^\pi \sin \varphi d\varphi \int_{R-\hat{R}}^{R+\hat{R}} \kappa^{-m} \right. \\ &\quad \left. \left( \left| \sum_{|\alpha|=m} \partial_{\mathbf{y}}^\alpha \mathbf{J} \right| \frac{1}{\rho} + \left| \sum_{|\alpha|=m-1} \partial_{\mathbf{y}}^\alpha \mathbf{J} \right| \frac{m}{\rho^2} \right) \rho^2 d\rho \right|^2 d\gamma(\mathbf{x}) d\kappa \\ &\leq \int_s^\infty \int_{\Gamma_R} \kappa^4 \left| \int_0^{2\pi} d\theta \int_0^\pi \sin \varphi d\varphi \int_{R-\hat{R}}^{R+\hat{R}} \kappa^{-m} \right. \\ &\quad \left. \left( \left| \sum_{|\alpha|=m} \partial_{\mathbf{y}}^\alpha \mathbf{J} \right| \frac{1}{(R-\hat{R})} + \left| \sum_{|\alpha|=m-1} \partial_{\mathbf{y}}^\alpha \mathbf{J} \right| \frac{m}{(R-\hat{R})^2} \right) \rho^2 d\rho \right|^2 d\gamma(\mathbf{x}) d\kappa \\ &= \int_s^\infty \int_{\Gamma_R} \omega^4 \left| \int_0^{2\pi} d\theta \int_0^\pi \sin \varphi d\varphi \int_0^\infty \kappa^{-m} \right. \\ &\quad \left. \left( \left| \sum_{|\alpha|=m} \partial_{\mathbf{y}}^\alpha \mathbf{J} \right| \frac{1}{(R-\hat{R})} + \left| \sum_{|\alpha|=m-1} \partial_{\mathbf{y}}^\alpha \mathbf{J} \right| \frac{m}{(R-\hat{R})^2} \right) \rho^2 d\rho \right|^2 d\gamma(\mathbf{x}) d\kappa. \end{aligned}$$

Changing back to the Cartesian coordinates with respect to  $\mathbf{y}$ , we have

$$L_{1,1} \leq \int_s^\infty \int_{\Gamma_R} \kappa^4 \left| \int_\Omega \kappa^{-m} \right. \tag{3.33}$$

$$\left. \left( \left| \sum_{|\alpha|=m} \partial_{\mathbf{y}}^\alpha \mathbf{J} \right| \frac{1}{(R-\hat{R})} + \left| \sum_{|\alpha|=m-1} \partial_{\mathbf{y}}^\alpha \mathbf{J} \right| \frac{m}{(R-\hat{R})^2} \right) d\mathbf{y} \right|^2 d\gamma(\mathbf{x}) d\kappa \tag{3.34}$$

$$\lesssim m \|\mathbf{J}\|_{H^m(B_R)^3}^2 \int_s^\infty \kappa^{4-2m} d\kappa \tag{3.35}$$

$$\lesssim \left(\frac{m}{2m-5}\right) s^{-(2m-5)} \|\mathbf{J}\|_{H^m(B_R)^3}^2 \lesssim s^{-(2m-5)} \|\mathbf{J}\|_{H^m(B_R)^3}^2. \tag{3.36}$$

For  $L_{1,2}$ , it follows from the integration by parts and similar steps for (3.33) that we obtain

$$\begin{aligned} L_{1,2} &\lesssim \int_s^\infty \int_{\Gamma_R} \left| \int_\Omega \frac{e^{i\kappa|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|} \nabla_{\mathbf{y}} \nabla_{\mathbf{y}} \cdot \mathbf{J}(\mathbf{y}) d\mathbf{y} \right|^2 d\gamma(\mathbf{x}) d\kappa \\ &\lesssim \int_s^\infty \int_{\Gamma_R} \left| \int_\Omega \frac{1}{\kappa^{m-2}} \left( \left| \sum_{|\alpha|=m-2} \partial_{\mathbf{y}}^\alpha (\nabla_{\mathbf{y}} \nabla_{\mathbf{y}} \cdot \mathbf{J}) \right| \frac{1}{(R-\hat{R})} \right. \right. \\ &\quad \left. \left. + \left| \sum_{|\alpha|=m-3} \partial_{\mathbf{y}}^\alpha (\nabla_{\mathbf{y}} \nabla_{\mathbf{y}} \cdot \mathbf{J}) \right| \frac{(m-2)}{(R-\hat{R})^2} \right) d\mathbf{y} \right|^2 d\gamma(\mathbf{x}) d\kappa \\ &\lesssim (m-2) \|\mathbf{J}\|_{H^m(B_R)^3}^2 \int_s^\infty \kappa^{4-2m} d\kappa \\ &\lesssim \left(\frac{m-2}{2m-5}\right) s^{-(2m-5)} \|\mathbf{J}\|_{H^m(B_R)^3}^2 \lesssim s^{-(2m-5)} \|\mathbf{J}\|_{H^m(B_R)^3}^2. \end{aligned} \tag{3.37}$$

Following from the similar steps as those for (3.33) and (3.37), we may obtain from (3.26) that

$$L_2 \lesssim \int_s^\infty \int_{\Gamma_R} \kappa^2 \left| \int_\Omega \frac{e^{i\kappa|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|} \mathbf{I}_3 \cdot \nabla_{\mathbf{y}} \times \mathbf{J}(\mathbf{y}) d\mathbf{y} \right|^2 d\gamma(\mathbf{x}) d\kappa \lesssim s^{-(2m-5)} \|\mathbf{J}\|_{H^m(B_R)^3}^2. \tag{3.38}$$

Combining (3.33)–(3.38) completes the proof.  $\square$

**Lemma 3.14.** *Let  $\mathbf{f} \in \mathbb{J}_M(B_R)$ . Then there exists a function  $\beta(s)$  satisfying (2.52) such that*

$$|I_1(s) + I_2(s)| \lesssim M^2 e^{(4R+1)s} \epsilon_4^{2\beta(s)}, \quad \forall s \in (K, \infty).$$

**Proof.** It follows from Lemma 3.12 that

$$|(I_1(s) + I_2(s)) e^{-(4R+1)|s|}| \lesssim M^2, \quad \forall s \in \mathcal{V}.$$

Recalling (3.27)–(3.28), we have

$$|(I_1(s) + I_2(s)) e^{-(4R+1)s}| \leq \epsilon_4^2, \quad s \in [0, K].$$

Using Lemma Appendix C.2 shows that there exists a function  $\beta(s)$  satisfying (2.52) such that

$$|(I_1(s) + I_2(s)) e^{-(4R+1)s}| \lesssim M^2 \epsilon_4^{2\beta}, \quad \forall s \in (K, \infty),$$

which completes the proof.  $\square$

The proof of Theorem 3.9 is similar to that of Theorem 2.4. We sketch it here for completeness.

**Proof.** Let

$$s = \begin{cases} \frac{1}{((4R+3)\pi)^{\frac{1}{3}}} K^{\frac{2}{3}} |\ln \epsilon_4|^{\frac{1}{4}}, & 2^{\frac{1}{4}} ((4R+3)\pi)^{\frac{1}{3}} K^{\frac{1}{3}} < |\ln \epsilon_4|^{\frac{1}{4}}, \\ K, & |\ln \epsilon_4| \leq 2^{\frac{1}{4}} ((4R+3)\pi)^{\frac{1}{3}} K^{\frac{1}{3}}. \end{cases}$$

If  $2^{\frac{1}{4}}((4R + 3)\pi)^{\frac{1}{3}}K^{\frac{1}{3}} < |\ln \epsilon_4|^{\frac{1}{4}}$ , then we have from Lemma 3.14 that

$$|I_1(s) + I_2(s)| \lesssim M^2 e^{-2\left(\frac{(4R+3)^2}{\pi}\right)^{\frac{1}{3}} K^{\frac{2}{3}} |\ln \epsilon_4|^{\frac{1}{2}} \left(1 - \frac{1}{2} |\ln \epsilon_4|^{-\frac{1}{4}}\right)},$$

which gives

$$|I_1(s) + I_2(s)| \lesssim M^2 e^{-K^{\frac{2}{3}} |\ln \epsilon_4|^{\frac{1}{2}}}.$$

It follows from the elementary inequality (2.53) that we get

$$|I_1(s) + I_2(s)| \lesssim \frac{M^2}{\left(\frac{K^2 |\ln \epsilon_4|^{\frac{3}{2}}}{(6m-15)^3}\right)^{2m-5}}. \tag{3.39}$$

We have from Lemma 3.1 and (3.27)–(3.28) that

$$|I_1(s) + I_2(s)| \leq \epsilon_4^2.$$

Note that for  $s > 0$ ,

$$I_1(s) + I_2(s) = \int_0^s \kappa^2 |\mathbf{E}(\cdot, \kappa) \times \boldsymbol{\nu}|_{\Gamma_R}^2 d\kappa.$$

Hence we obtain from Lemma 3.13 and (3.39) that

$$\begin{aligned} & \int_0^\infty \kappa^2 |\mathbf{E}(\cdot, \kappa) \times \boldsymbol{\nu}|_{\Gamma_R}^2 d\kappa \\ & \leq I_1(s) + I_2(s) + \int_s^\infty \kappa^2 |\mathbf{E}(\cdot, \kappa) \times \boldsymbol{\nu}|_{\Gamma_R}^2 d\kappa \\ & \lesssim \epsilon_4^2 + \frac{M^2}{\left(\frac{K^2 |\ln \epsilon_4|^{\frac{3}{2}}}{(6m-15)^3}\right)^{2m-5}} + \frac{M^2}{\left(2^{-\frac{1}{4}}((4R + 3)\pi)^{-\frac{1}{3}}K^{\frac{2}{3}}|\ln \epsilon_4|^{\frac{1}{4}}\right)^{2m-5}}. \end{aligned}$$

By Lemma 3.11, we have

$$\|\mathbf{J}\|_{L^2(B_R)^3}^2 \lesssim \epsilon_4^2 + \frac{M^2}{\left(\frac{K^2 |\ln \epsilon_4|^{\frac{3}{2}}}{(6m-15)^3}\right)^{2m-5}} + \frac{M^2}{\left(\frac{K^{\frac{2}{3}} |\ln \epsilon_4|^{\frac{1}{4}}}{(R+1)(6m-15)^3}\right)^{2m-5}},$$

which completes proof.  $\square$

### 3.4. Stability with discrete frequency data

First we specify the discrete frequency data. For  $\mathbf{n} \in \mathbb{R}^d \setminus \{0\}$ , let  $n = |\mathbf{n}|$ , denote the wavenumber

$$\kappa_n = \frac{n\pi}{R}.$$

We define the discrete frequency boundary data:

$$\|\mathbf{E}(\cdot, \kappa_n) \times \boldsymbol{\nu}\|_{\Gamma_R}^2 = \int_{\Gamma_R} (|T_M(\mathbf{E}(\mathbf{x}, \kappa_n) \times \boldsymbol{\nu})|^2 + |\mathbf{E}(\mathbf{x}, \kappa_n) \times \boldsymbol{\nu}|^2) d\gamma(\mathbf{x}).$$

Similarly, the Fourier coefficient  $\hat{\mathbf{J}}_0$  cannot be recovered by the discrete frequency data. It is necessary to revise the functional space. Denote

$$\tilde{\mathbb{J}}_M(B_R) = \{\mathbf{J} \in \mathbb{J}_M(B_R) : \int_{\Omega} \mathbf{J}(\mathbf{x}) d\mathbf{x} = 0\}.$$

**Problem 3.15** (Discrete frequency data for electromagnetic waves). Let  $\mathbf{J} \in \tilde{\mathbb{J}}_M(B_R)$ . The inverse source problem of electromagnetic waves is to determine  $\mathbf{J}$  from the tangential trace of the electric field  $\mathbf{E}(\mathbf{x}, \kappa) \times \boldsymbol{\nu}$  for  $\mathbf{x} \in \Gamma_R, \kappa \in (0, \frac{\pi}{R}] \cup \cup_{n=1}^N \{\kappa_n\}$ , where  $1 < N \in \mathbb{N}$ .

The following stability estimate is the main result of Problem 3.15.

**Theorem 3.16.** Let  $\mathbf{E}$  be the solution of the scattering problem (3.2)–(3.3) corresponding to the source  $\mathbf{J} \in \tilde{\mathbb{J}}_M(B_R)$ . Then

$$\|\mathbf{J}\|_{L^2(B_R)^d}^2 \lesssim \epsilon_5^2 + \frac{M^2}{\left(\frac{N^{\frac{5}{8}} |\ln \epsilon_6|^{\frac{1}{9}}}{(6m-12)^3}\right)^{2m-4}}, \tag{3.40}$$

where

$$\begin{aligned} \epsilon_5 &= \left( \sum_{n \leq N} \|\mathbf{E}(\cdot, \kappa_n)\|_{\Gamma_R}^2 \right)^{\frac{1}{2}}, \\ \epsilon_6 &= \sup_{\kappa \in (0, \frac{\pi}{R}]} \|\mathbf{E}(\cdot, \kappa)\|_{\Gamma_R}. \end{aligned}$$

**Remark 3.17.** The estimate for the discrete frequency data (3.40) is also consistent with the estimate for the continuous frequency data (3.20). They are analogous to the relationship between (2.57) and (2.13) for elastic waves.

We begin with several useful lemmas.

**Lemma 3.18.** Let  $\mathbf{E}$  be the solution of (3.2)–(3.3) corresponding to the source  $\mathbf{J} \in \mathbb{X}(B_R)$ . Then for all  $\mathbf{n} \in \mathbb{Z}^d \setminus \{0\}$ , the Fourier coefficients of  $\mathbf{J}$  satisfy

$$|\hat{\mathbf{J}}_{\mathbf{n}}|^2 \lesssim \|\mathbf{E}(\cdot, \kappa_n)\|_{\Gamma_R}^2.$$

**Proof.** Give any  $\mathbf{n} \in \mathbb{Z}^3$ , let  $\hat{\mathbf{n}} = \mathbf{n}/n$ . Consider the following electric and magnetic plane waves:

$$\mathbf{E}^{\text{inc}}(\mathbf{x}) = \mathbf{p}e^{-i\kappa_n \mathbf{x} \cdot \hat{\mathbf{n}}} = \mathbf{p}e^{-i(\frac{\pi}{R})\mathbf{x} \cdot \mathbf{n}} \quad \text{and} \quad \mathbf{H}^{\text{inc}}(\mathbf{x}) = \mathbf{q}e^{-i\kappa_n \mathbf{x} \cdot \hat{\mathbf{n}}} = \mathbf{q}e^{-i(\frac{\pi}{R})\mathbf{x} \cdot \mathbf{n}},$$

where  $\mathbf{p}$  and  $\mathbf{q}$  are chosen such that  $\{\hat{\mathbf{n}}, \mathbf{p}, \mathbf{q}\}$  form an orthonormal basis in  $\mathbb{R}^3$ . It is easy to verify that  $\mathbf{E}^{\text{inc}}$  and  $\mathbf{H}^{\text{inc}}$  satisfy the Maxwell equations:

$$\nabla \times (\nabla \times \mathbf{E}^{\text{inc}}) - \kappa_n^2 \mathbf{E}^{\text{inc}} = 0 \tag{3.41}$$

and

$$\nabla \times (\nabla \times \mathbf{H}^{\text{inc}}) - \kappa_n^2 \mathbf{H}^{\text{inc}} = 0.$$

Multiplying the both sides of (3.3) by  $\mathbf{E}^{\text{inc}}$ , using the integration by parts over  $B_R$  and (3.41), we obtain

$$\begin{aligned} in \int_{\tilde{B}_R} (\mathbf{p}e^{-i(\frac{\pi}{R})\mathbf{n}\cdot\mathbf{x}}) \cdot \mathbf{J}(\mathbf{x})d\mathbf{x} &= - \int_{\Gamma_R} (i\kappa T_M(\mathbf{E}(\mathbf{x}, \kappa_n) \times \boldsymbol{\nu}) \cdot \mathbf{E}^{\text{inc}} \\ &\quad + (\mathbf{E}(\mathbf{x}, \kappa_n) \times \boldsymbol{\nu}) \cdot (\nabla \times \mathbf{E}^{\text{inc}}))d\gamma. \end{aligned}$$

A simple calculation yields that

$$\nabla \times \mathbf{E}^{\text{inc}} = -i\mathbf{n} \times \mathbf{p}e^{-i\kappa_n\mathbf{x}\cdot\hat{\mathbf{n}}},$$

which gives

$$|\nabla \times \mathbf{E}_n^{\text{inc}}| = n.$$

Combining the above estimates leads to

$$|\mathbf{p} \cdot \hat{\mathbf{J}}_n|^2 \lesssim \int_{\Gamma_R} (|T_M(\mathbf{E}(\mathbf{x}, \kappa_n) \times \boldsymbol{\nu})|^2 + |\mathbf{E}(\mathbf{x}, \kappa_n) \times \boldsymbol{\nu}|^2)d\gamma(\mathbf{x}) \lesssim \|\mathbf{E}(\cdot, \kappa_n)\|_{\Gamma_R}^2.$$

Similarly, we have

$$|\mathbf{q}_n \cdot \hat{\mathbf{J}}_n|^2 \lesssim \int_{\Gamma_R} (|T_M(\mathbf{E}(\mathbf{x}, \kappa_n) \times \boldsymbol{\nu})|^2 + |\mathbf{E}(\mathbf{x}, \kappa_n) \times \boldsymbol{\nu}|^2)d\gamma(\mathbf{x}) \lesssim \|\mathbf{E}(\cdot, \kappa_n)\|_{\Gamma_R}^2.$$

On the other hand, since  $\mathbf{J}$  has a compact support  $\Omega$  contained in  $B_R$  and  $\mathbf{J} \in \mathbb{X}(B_R)$ , we obtain that  $\mathbf{J}$  is a weak solution of the Maxwell system:

$$\nabla \times (\nabla \times \mathbf{J}) - \kappa_n^2 \mathbf{J} = 0 \quad \text{in } B_R.$$

Multiplying the above equation by  $\hat{\mathbf{n}}e^{-i\kappa_n\mathbf{x}\cdot\hat{\mathbf{n}}}$  and using integration by parts, we get

$$\int_{B_R} (\nabla \times \mathbf{J}) \cdot (\nabla \times (\hat{\mathbf{n}}e^{i\kappa_n\mathbf{x}\cdot\hat{\mathbf{n}}})d\mathbf{x} = \kappa_n^2 \hat{\mathbf{n}} \cdot \int_{B_R} \mathbf{J}(\mathbf{x})e^{-i(\frac{\pi}{R})\mathbf{x}\cdot\mathbf{n}}d\mathbf{x} = \kappa_n^2 \hat{\mathbf{n}} \cdot \hat{\mathbf{J}}_n.$$

Noting  $\nabla \times (\hat{\mathbf{n}}e^{-i\kappa_n\mathbf{x}\cdot\hat{\mathbf{n}}}) = -i\kappa_n \hat{\mathbf{n}} \times \hat{\mathbf{n}}e^{-i\kappa_n\mathbf{x}\cdot\hat{\mathbf{n}}} = 0$ , we get  $\hat{\mathbf{n}} \cdot \hat{\mathbf{J}}_n = 0$ , which yields from the Pythagorean theorem that

$$|\hat{\mathbf{J}}_n|^2 = |\mathbf{p} \cdot \hat{\mathbf{J}}_n|^2 + |\mathbf{q} \cdot \hat{\mathbf{J}}_n|^2 \lesssim \|\mathbf{E}(\cdot, \kappa_n)\|_{\Gamma_R}^2,$$

which completes the proof.  $\square$



**Lemma 3.19.** *Let  $\mathbf{J} \in H^m(B_R)^3$ . For any  $N_0 \in \mathbb{N}$ , the following estimate holds:*

$$\sum_{n=N_0}^{\infty} |\hat{\mathbf{J}}_{\mathbf{n}}|^2 \lesssim N_0^{-(2m-4)} \|\mathbf{J}\|_{H^m(B_R)^3}^2.$$

**Proof.** Let  $\mathbf{n} = (n_1, n_2, n_3)$  and choose  $n_j = \max\{n_1, n_2, n_3\}$ . Then we have  $n^2 \leq 3n_j^2$ , which means that  $n_j^{-2m} \leq 3^m n^{-2m}$ . Let  $\mathbf{J} = (J_1, J_2, J_3)$ . Noting  $\text{supp}\mathbf{J} \subset B_R \subset U_R$  and using integration by parts, we obtain

$$\left| \int_{B_R} J_1(\mathbf{x}) e^{-i(\frac{\pi}{R})\mathbf{n}\cdot\mathbf{x}} d\mathbf{x} \right|^2 \lesssim \left| \int_{B_R} n_j^{-m} e^{-i(\frac{\pi}{R})\mathbf{n}\cdot\mathbf{x}} \partial_{x_j}^m J_1(\mathbf{x}) d\mathbf{x} \right|^2 \lesssim n^{-2m} \|\mathbf{J}\|_{H^m(B_R)^d}^2.$$

Hence

$$|\hat{\mathbf{J}}_{\mathbf{n}}|^2 \lesssim \left| \int_{B_R} \mathbf{J}(\mathbf{x}) e^{-i(\frac{\pi}{R})\mathbf{n}\cdot\mathbf{x}} d\mathbf{x} \right|^2 \lesssim n^{-2m} \|\mathbf{J}\|_{H^m(B_R)^d}^2.$$

Noting that there are at most  $O(n^3)$  elements in  $\{\mathbf{n} \in \mathbb{Z}^3, |\mathbf{n}| = n\}$ , we get

$$\begin{aligned} \sum_{n=N_0}^{\infty} |\hat{\mathbf{J}}_{\mathbf{n}}|^2 &\lesssim \left( \sum_{n=N_0}^{\infty} n^{(3-2m)} \right) \|\mathbf{J}\|_{H^m(B_R)^d}^2 \\ &\lesssim \left( \int_0^{\infty} (N_0 + t)^{(3-2m)} dt \right) \|\mathbf{J}\|_{H^m(B_R)^d}^2 \\ &\lesssim \frac{N_0^{-(2m-4)}}{(2m-4)} \|\mathbf{J}\|_{H^m(B_R)^d}^2 \lesssim N_0^{-(2m-4)} \|\mathbf{J}\|_{H^m(B_R)^d}^2, \end{aligned}$$

which completes the proof.  $\square$

**Lemma 3.20.** *Let  $\mathbf{E}$  be the solution of (3.2)–(3.3) corresponding to the source  $\mathbf{J} \in \mathbb{X}(B_R)$ . For any  $\kappa \in (0, \frac{\pi}{R}]$  and  $\mathbf{d} \in \mathbb{S}^{d-1}$ , the following estimate holds:*

$$\left| \int_{B_R} \mathbf{J}(\mathbf{x}) e^{i\kappa\mathbf{x}\cdot\mathbf{d}} d\mathbf{x} \right|^2 \lesssim \epsilon_6^2.$$

**Proof.** Let  $\mathbf{p}, \mathbf{q} \in \mathbb{S}^{d-1}$  such that  $\mathbf{p} \cdot \mathbf{d} = 0$  and  $\mathbf{q} = \mathbf{p} \times \mathbf{d}$ . Consider the electric plane wave  $\mathbf{E}^{\text{inc}} = \mathbf{p}e^{-i\kappa\mathbf{x}\cdot\mathbf{d}}$  and magnetic plane wave  $\mathbf{H}^{\text{inc}} = \mathbf{q}e^{-i\kappa\mathbf{x}\cdot\mathbf{d}}$ . Noting  $\text{supp}\mathbf{J} \subset B_R$  and using similar arguments as those in Lemma 3.18, we get

$$\begin{aligned} |\mathbf{p} \cdot \hat{\mathbf{J}}(\kappa\mathbf{d})|^2 &= \left| \mathbf{p} \cdot \int_{B_R} \mathbf{J}(\mathbf{x}) e^{-i\kappa\mathbf{x}\cdot\mathbf{d}} d\mathbf{x} \right|^2 \\ &\lesssim \int_{\Gamma_R} (|T_M(\mathbf{E}(\mathbf{x}, \kappa) \times \boldsymbol{\nu})|^2 + |\mathbf{E}(\mathbf{x}, \kappa) \times \boldsymbol{\nu}|^2) d\gamma(\mathbf{x}) \lesssim \|\mathbf{E}(\cdot, \kappa)\|_{\Gamma_R}^2, \end{aligned}$$

and

$$\begin{aligned}
 |\mathbf{q} \cdot \hat{\mathbf{J}}(\kappa \mathbf{d})|^2 &= \left| \mathbf{q} \cdot \int_{B_R} \mathbf{J}(\mathbf{x}) e^{-i\kappa \mathbf{x} \cdot \mathbf{d}} d\mathbf{x} \right|^2 \\
 &\lesssim \int_{\Gamma_R} (|T_M(\mathbf{E}(\mathbf{x}, \kappa) \times \boldsymbol{\nu})|^2 + |\mathbf{E}(\mathbf{x}, \kappa) \times \boldsymbol{\nu}|^2) d\gamma(\mathbf{x}) \lesssim \|\mathbf{E}(\cdot, \kappa)\|_{\Gamma_R}^2.
 \end{aligned}$$

Hence we have from the Pythagorean theorem that

$$|\hat{\mathbf{J}}(\kappa \mathbf{d})|^2 = |\mathbf{p} \cdot \hat{\mathbf{J}}(\kappa \mathbf{d})|^2 + |\mathbf{q} \cdot \hat{\mathbf{J}}(\kappa \mathbf{d})|^2 \lesssim \epsilon_6^2,$$

which completes the proof.  $\square$

**Lemma 3.21.** *Let  $\mathbf{J} \in \tilde{\mathfrak{J}}_M(B_R)$ . Then there exists a function  $\beta(s)$  satisfying (2.61) such that*

$$\left| \int_{\tilde{B}_R} \mathbf{J}(\mathbf{x}) e^{-i(\frac{\pi}{R})\mathbf{n} \cdot \mathbf{x}} d\mathbf{x} \right|^2 \lesssim M^2 e^{2nR} \epsilon_6^{2n\beta(\frac{n\pi}{R})}, \quad \forall n \in (1, \infty).$$

**Proof.** We fix  $\mathbf{d} \in \mathbb{S}^{d-1}$  and consider  $\mathbf{n} \in \mathbb{Z}^3$  which parallel to  $\mathbf{d}$ . Define

$$I(s) = \left| \int_{B_R} \mathbf{J}(\mathbf{x}) e^{-is\mathbf{d} \cdot \mathbf{x}} d\mathbf{x} \right|^2.$$

It is easy to show from the Cauchy–Schwarz inequality that there exists a constant  $C$  depending on  $R, d$  such that

$$I(s) \leq C(R, d) e^{2|s|R} M^2, \quad \forall s \in \mathcal{V},$$

which gives

$$e^{-2|s|R} I(s) \lesssim M^2, \quad \forall s \in \mathcal{V}.$$

Using Lemma 3.20 yields

$$e^{-2|s|R} \left| \int_{B_R} \mathbf{J}(\mathbf{x}) e^{-is\mathbf{d} \cdot \mathbf{x}} d\mathbf{x} \right|^2 \leq \epsilon_6^2, \quad \forall s \in [0, \frac{\pi}{R}].$$

An direct application of Lemma Appendix C.2 shows that there exists a function  $\beta(s)$  satisfying (2.61) such that

$$|I(s) e^{-2sR}| \lesssim M^2 \epsilon_6^{2\beta}, \quad \forall s \in (\frac{\pi}{R}, \infty).$$

Hence

$$|I(s)| \lesssim M^2 e^{2sR} \epsilon_6^{2\beta}, \quad \forall s \in (\frac{\pi}{R}, \infty).$$

Noting that the constant  $C(R, d)$  does not depend on  $\mathbf{d}$ , we have obtained that for all  $\mathbf{n} \in \mathbb{Z}^3$  with  $n > 1$  such that

$$\left| \int_{B_R} \mathbf{J}(\mathbf{x}) e^{-i(\frac{\pi}{R})\mathbf{n}\cdot\mathbf{x}} d\mathbf{x} \right|^2 = \left| \int_{B_R} \mathbf{J}(\mathbf{x}) e^{-i(\frac{n\pi}{R})\hat{\mathbf{n}}\cdot\mathbf{x}} d\mathbf{x} \right|^2 \lesssim M^2 e^{2nR} \epsilon_6^{2n\beta(\frac{n\pi}{R})},$$

which completes the proof.  $\square$

The proof of Theorem 3.16 is similar to that for Theorem 2.11. We also briefly present it for completeness.

**Proof.** Applying Lemma Appendix C.1 and the Parseval identity, we have

$$\int_{B_R} |\mathbf{J}|^2 d\mathbf{x} \lesssim \sum_{n=0}^{N_0} |\hat{\mathbf{J}}_n|^2 + \sum_{n=N_0+1}^{\infty} |\hat{\mathbf{J}}_n|^2,$$

where  $N_0$  is given in (2.62). Using Lemma 3.21 leads to

$$\left| \int_{B_R} \mathbf{J}(\mathbf{x}) e^{-i(\frac{\pi}{R})\mathbf{n}\cdot\mathbf{x}} d\mathbf{x} \right|^2 \lesssim M^2 e^{-\frac{2}{\pi}n^{-2} |\ln \epsilon_6| (1-2\pi n^3 |\ln \epsilon_6|^{-1})}, \quad \forall n \in (2^{\frac{1}{4}}, \infty).$$

Therefore

$$\left| \int_{B_R} \mathbf{J}(\mathbf{x}) e^{-i(\frac{\pi}{R})\mathbf{n}\cdot\mathbf{x}} d\mathbf{x} \right|^2 \lesssim M^2 e^{-\frac{2}{\pi^3}N_0^{-2} |\ln \epsilon_6| (1-2\pi^4 N_0^3 |\ln \epsilon_6|^{-1})}, \quad \forall n \in (2^{\frac{1}{4}}, N_0\pi]. \tag{3.42}$$

If  $N^{\frac{3}{8}} < \frac{1}{2^{\frac{5}{6}}\pi^{\frac{2}{3}}} |\ln \epsilon_6|^{\frac{1}{9}}$ , then  $2\pi^4 N_0^3 |\ln \epsilon_6|^{-1} < \frac{1}{2}$  and

$$e^{-\frac{2}{\pi^3} \frac{|\ln \epsilon_6|}{N_0^2}} \leq e^{-64\pi |\ln \epsilon_6|^{\frac{1}{9}} N^{\frac{3}{4}}}. \tag{3.43}$$

Combining (3.42) and (3.43), we obtain

$$\left| \int_{B_R} \mathbf{J}(\mathbf{x}) e^{-i(\frac{\pi}{R})\mathbf{n}\cdot\mathbf{x}} d\mathbf{x} \right|^2 \lesssim M^2 e^{-32\pi |\ln \epsilon_6|^{\frac{1}{9}} N^{\frac{3}{4}}}, \quad \forall n \in (2^{\frac{1}{4}}, N_0\pi].$$

Using an elementary inequality of  $e^{-x}$  similar to that of (2.53), we have

$$\left| \int_{B_R} \mathbf{J}(\mathbf{x}) e^{-i(\frac{\pi}{R})\mathbf{n}\cdot\mathbf{x}} d\mathbf{x} \right|^2 \lesssim M^2 \frac{1}{\left( \frac{|\ln \epsilon_6|^{\frac{1}{9}} N^{\frac{3}{4}}}{(6m-12)^3} \right)^{2m-4}}, \quad n = 1, \dots, N_0.$$

Consequently,

$$\sum_{n=0}^{N_0} \left| \int_{B_R} \mathbf{J}(\mathbf{x}) e^{-i(\frac{\pi}{R})\mathbf{n}\cdot\mathbf{x}} d\mathbf{x} \right|^2 \lesssim \frac{M^2}{\left( \frac{|\ln \epsilon_6|^{\frac{1}{9}} N^{\frac{3}{4}}}{(6m-12)^3} \right)^{2m-4}}.$$

It follows from Lemma 3.18 that

$$\sum_{n=0}^{N_0} \left| \int_{B_R} \mathbf{J}(\mathbf{x}) e^{-i(\frac{\pi}{R})\mathbf{n}\cdot\mathbf{x}} d\mathbf{x} \right|^2 \lesssim \epsilon_5^2.$$

Combining the above estimates and Lemma 3.19, we obtain

$$\begin{aligned} \sum_{n=0}^{\infty} \left| \int_{B_R} \mathbf{J}(\mathbf{x}) e^{-i(\frac{\pi}{R})\mathbf{n}\cdot\mathbf{x}} d\mathbf{x} \right|^2 &\lesssim \epsilon_5^2 + \frac{M^2}{\left(\frac{|\ln \epsilon_6|^{\frac{1}{9}} N^{\frac{3}{2}}}{(6m-12)^3}\right)^{2m-4}} \\ &+ \frac{M^2}{\left(|\ln \epsilon_6|^{\frac{1}{9}} N^{\frac{3}{4}}\right)^{2m-4}} + \frac{M^2 (2^{\frac{5}{6}} \pi^{\frac{2}{3}})^{2m-4}}{\left(|\ln \epsilon_6|^{\frac{1}{9}} N^{\frac{5}{8}}\right)^{2m-4}}, \end{aligned}$$

which completes the proof.  $\square$

#### 4. Conclusion

We have presented a unified stability theory of the inverse source problems for both elastic and electromagnetic waves. For elastic waves, the increasing stability is achieved to reconstruct the external force. For electromagnetic waves, the increasing stability is obtained to reconstruct the radiating electric current density. The analysis requires the Dirichlet data only at multiple frequencies. The stability estimates consist of the data discrepancy and the high frequency tail. The result shows that the ill-posedness of the inverse source problems decreases as the frequency increases for the data. A possible continuation of this work is to investigate the stability with a limited aperture data, i.e., the data is only available on a part of the boundary. Since the Neumann data cannot be represented via the limited Dirichlet data by using the DtN map, a new technique must be developed, and both the Dirichlet and Neumann data are required in order to obtain the increasing stability. Another interesting direction is to study the stability in the inverse source problems for inhomogeneous media, where the analytical Green tensors are not available any more and the present method may not be directly applicable. We also point out even more challenging inverse medium and obstacle scattering problems. These nonlinear problems are largely open.

#### Appendix A. Differential operators

We list the notations for differential operators used in this paper.

First we introduce the notation in two-dimensions. Let  $\mathbf{x} = (x_1, x_2)^\top$ . Let  $u$  and  $\mathbf{u} = (u_1, u_2)^\top$  and be a scalar and vector function, respectively. We introduce the gradient and the Jacobi matrix:

$$\nabla u = (\partial_{x_1} u, \partial_{x_2} u)^\top, \quad \nabla \mathbf{u} = \begin{bmatrix} \partial_{x_1} u_1 & \partial_{x_2} u_1 \\ \partial_{x_1} u_2 & \partial_{x_2} u_2 \end{bmatrix}$$

and the scalar curl and the vector curl:

$$\text{curl} u = \partial_{x_1} u_2 - \partial_{x_2} u_1, \quad \mathbf{curl} u = (\partial_{x_2} u, -\partial_{x_1} u)^\top.$$

It is easy to verify that

$$\nabla \nabla^\top u = \begin{bmatrix} \partial_{x_1 x_1} u & \partial_{x_1 x_2} u \\ \partial_{x_2 x_1} u & \partial_{x_2 x_2} u \end{bmatrix}$$

and

$$\nabla \nabla \cdot \mathbf{u} = \begin{bmatrix} \partial_{x_1 x_1} u_1 + \partial_{x_1 x_2} u_2 \\ \partial_{x_2 x_1} u_1 + \partial_{x_2 x_2} u_2 \end{bmatrix}.$$

Next we introduce the notation in three-dimensions. Let  $\mathbf{x} = (x_1, x_2, x_3)^\top$ . Let  $u$  and  $\mathbf{u} = (u_1, u_2, u_3)^\top$  and be a scalar and vector function, respectively. We introduce the gradient, the curl, and the Jacobi matrix:

$$\nabla u = (\partial_{x_1} u, \partial_{x_2} u, \partial_{x_3} u)^\top, \quad \nabla \times \mathbf{u} = \begin{bmatrix} \partial_{x_2} u_3 - \partial_{x_3} u_2 \\ \partial_{x_3} u_1 - \partial_{x_1} u_3 \\ \partial_{x_1} u_2 - \partial_{x_2} u_1 \end{bmatrix}, \quad \nabla \mathbf{u} = \begin{bmatrix} \partial_{x_1} u_1 & \partial_{x_2} u_1 & \partial_{x_3} u_1 \\ \partial_{x_1} u_2 & \partial_{x_2} u_2 & \partial_{x_3} u_2 \\ \partial_{x_1} u_3 & \partial_{x_2} u_3 & \partial_{x_3} u_3 \end{bmatrix}.$$

It can be also verified that

$$\nabla \nabla^\top u = \begin{bmatrix} \partial_{x_1 x_1} u & \partial_{x_1 x_2} u & \partial_{x_1 x_3} u \\ \partial_{x_2 x_1} u & \partial_{x_2 x_2} u & \partial_{x_2 x_3} u \\ \partial_{x_3 x_1} u & \partial_{x_3 x_2} u & \partial_{x_3 x_3} u \end{bmatrix}$$

and

$$\nabla \nabla \cdot \mathbf{u} = \begin{bmatrix} \partial_{x_1 x_1} u_1 + \partial_{x_1 x_2} u_2 + \partial_{x_1 x_3} u_3 \\ \partial_{x_2 x_1} u_1 + \partial_{x_2 x_2} u_2 + \partial_{x_2 x_3} u_3 \\ \partial_{x_3 x_1} u_1 + \partial_{x_3 x_2} u_2 + \partial_{x_3 x_3} u_3 \end{bmatrix}.$$

**Appendix B. The Helmholtz decomposition**

We present the Helmholtz decomposition for the displacement which is used to introduce the Kupradze–Sommerfeld radiation condition in Section 2. Since the source  $\mathbf{f}$  has a compact support  $\Omega$ , the elastic wave equation (2.1) reduces to

$$\mu \Delta \mathbf{u} + (\lambda + \mu) \nabla \nabla \cdot \mathbf{u} + \omega^2 \mathbf{u} = 0 \quad \text{in } \mathbb{R}^d \setminus \bar{\Omega}. \tag{B.1}$$

First we introduce the Helmholtz decomposition in the two-dimensions. For any solution  $\mathbf{u}$  of (B.1), we let

$$\mathbf{u} = \nabla \phi + \mathbf{curl} \psi, \tag{B.2}$$

where  $\phi$  and  $\psi$  are scalar potential functions. Substituting (B.2) into (B.1) gives

$$\nabla((\lambda + 2\mu)\Delta \phi + \omega^2 \phi) + \mathbf{curl}(\mu \Delta \psi + \omega^2 \psi) = 0,$$

which is fulfilled if  $\phi$  and  $\psi$  satisfy the Helmholtz equations:

$$\Delta \phi + \kappa_p^2 \phi = 0, \quad \Delta \psi + \kappa_s^2 \psi = 0. \tag{B.3}$$

It follows from (B.2) and (B.3) that we get

$$\nabla \cdot \mathbf{u} = \Delta \phi = -\kappa_p^2 \phi, \quad \mathbf{curl} \mathbf{u} = -\Delta \psi = \kappa_s^2 \psi.$$

Using (B.2) again yields

$$\mathbf{u} = \mathbf{u}_p + \mathbf{u}_s,$$

where  $\mathbf{u}_p$  and  $\mathbf{u}_s$  are the compressional part the shear part, respectively, given by

$$\mathbf{u}_p = -\frac{1}{\kappa_p^2} \nabla \nabla \cdot \mathbf{u}, \quad \mathbf{u}_s = \frac{1}{\kappa_s^2} \mathbf{curl} \mathbf{curl} \mathbf{u}.$$

Next we introduce the Helmholtz decomposition in the three-dimensions. For any solution  $\mathbf{u}$  of (B.1), the Helmholtz decomposition reads

$$\mathbf{u} = \nabla\varphi + \nabla \times \boldsymbol{\psi}, \quad \nabla \cdot \boldsymbol{\psi} = 0, \tag{B.4}$$

where  $\varphi$  is a scalar potential function and  $\boldsymbol{\psi}$  is a vector potential function. Substituting (B.4) into (B.1) gives

$$\nabla((\lambda + 2\mu)\Delta\varphi + \omega^2\varphi) + \nabla \times (\mu\Delta\boldsymbol{\psi} + \omega^2\boldsymbol{\psi}) = 0,$$

which implies that  $\varphi$  and  $\boldsymbol{\psi}$  satisfy the Helmholtz equations:

$$\Delta\varphi + \kappa_p^2\varphi = 0, \quad \Delta\boldsymbol{\psi} + \kappa_s^2\boldsymbol{\psi} = 0. \tag{B.5}$$

Similarly, we have from (B.4) and (B.5) that

$$\mathbf{u} = \mathbf{u}_p + \mathbf{u}_s,$$

where

$$\mathbf{u}_p = -\frac{1}{\kappa_p^2}\nabla\nabla \cdot \mathbf{u}, \quad \mathbf{u}_s = \frac{1}{\kappa_s^2}\nabla \times (\nabla \times \mathbf{u}).$$

**Appendix C. Sobolev spaces**

Denote by  $L^2(B_R)$  the Hilbert space of square integrable functions. Denote by  $H^m(B_R), m \in \mathbb{N}$  the Sobolev space which consists of square integrable weak derivatives up to  $m$ th order and has the norm characterized by

$$\|u\|_{H^m(B_R)}^2 = \sum_{|\alpha| \leq m} \int_{B_R} |D^\alpha u(\mathbf{x})| d\mathbf{x}.$$

Introduce the Sobolev space

$$H(\text{curl}, B_R) = \{\mathbf{u} \in L^2(B_R)^3, \nabla \times \mathbf{u} \in L^2(B_R)^3\},$$

which is equipped with the norm

$$\|\mathbf{u}\|_{H(\text{curl}, B_R)} = \left( \|\mathbf{u}\|_{L^2(B_R)^3}^2 + \|\nabla \times \mathbf{u}\|_{L^2(B_R)^3}^2 \right)^{1/2}.$$

Let  $H^s(\Gamma_R), s \in \mathbb{R}$  be the standard trace functional space. Given  $u(\mathbf{x}) \in L^2(\Gamma_R), \mathbf{x} \in \mathbb{R}^2$ , it has the Fourier expansion

$$u(R, \theta) = \sum_{n \in \mathbb{Z}} \hat{u}_n e^{in\theta}, \quad \hat{u}_n = \frac{1}{2\pi} \int_0^{2\pi} u(R, \theta) e^{-in\theta} d\theta.$$

The  $H^s(\Gamma_R)$ -norm is characterized by

$$\|u\|_{H^s(\Gamma_R)}^2 = \sum_{n \in \mathbb{Z}} (1 + n^2)^s |\hat{u}_n|^2.$$

Given  $u(\mathbf{x}) \in L^2(\Gamma_R)$ ,  $\mathbf{x} \in \mathbb{R}^3$ , it has the Fourier expansion

$$u(R, \theta, \varphi) = \sum_{n=0}^{\infty} \sum_{m=-n}^n \hat{u}_n^m Y_n^m(\theta, \varphi), \quad \hat{u}_n^m = \int_{\Gamma_R} u(R, \theta, \varphi) \bar{Y}_n^m(\theta, \varphi) d\gamma,$$

where  $Y_n^m$  is the spherical harmonics of order  $n$ . The  $H^s(\Gamma_R)$ -norm is characterized by

$$\|u\|_{H^s(\Gamma_R)}^2 = \sum_{n=0}^{\infty} \sum_{m=-n}^m (1 + n(n+1))^s |\hat{u}_n^m|^2.$$

Define a tangential trace functional space

$$H^{-1/2}(\text{curl}, \Gamma_R) = \{\mathbf{u} \in H^{-1/2}(\Gamma_R)^3 : \mathbf{u} \cdot \boldsymbol{\nu} = 0 \text{ on } \Gamma_R, \text{curl}_{\Gamma_R} \mathbf{u} \in H^{-1/2}(\Gamma_R)\},$$

where  $\boldsymbol{\nu}$  is the unit outward normal vector on  $\Gamma_R$  and  $\text{curl}_{\Gamma_R}$  is the surface scalar curl on  $\Gamma_R$ .

Below is a classical result from the theory of Fourier analysis.

**Lemma Appendix C.1.** *Let  $U_R = (-R, R)^d \subset \mathbb{R}^d$  be a box. For  $\mathbf{f} \in L^2(U_R)^d$ , define the Fourier coefficients*

$$\hat{\mathbf{f}}_{\mathbf{n}} = \frac{1}{(2R)^d} \int_{U_R} \mathbf{f}(\mathbf{x}) e^{-i(\frac{\pi}{R})\mathbf{x} \cdot \mathbf{n}} d\mathbf{x}, \quad \mathbf{n} \in \mathbb{Z}^d.$$

Then  $\mathbf{f}$  has the Fourier series expansion

$$\mathbf{f}(\mathbf{x}) = \sum_{\mathbf{n} \in \mathbb{Z}^d} \hat{\mathbf{f}}_{\mathbf{n}} e^{i(\frac{\pi}{R})\mathbf{x} \cdot \mathbf{n}}$$

in the  $L^2$ -sense, i.e.,

$$\int_{U_R} \left| \mathbf{f}(\mathbf{x}) - \sum_{|\mathbf{n}| \leq N} \hat{\mathbf{f}}_{\mathbf{n}} e^{i(\frac{\pi}{R})\mathbf{x} \cdot \mathbf{n}} \right|^2 d\mathbf{x} \rightarrow 0, \quad N \rightarrow \infty.$$

Moreover,

$$\|\mathbf{f}\|_{L^2(U_R)}^2 = (2R)^d \sum_{\mathbf{n} \in \mathbb{Z}^d} |\hat{\mathbf{f}}_{\mathbf{n}}|^2.$$

The following lemma (cf. [20, Lemma 3.2]) gives a link between the values of an analytical function for small and large arguments.

**Lemma Appendix C.2.** *Let  $p(z)$  be analytic in the sector*

$$\mathcal{V} = \{z \in \mathbb{C} : -\frac{\pi}{4} < \arg z < \frac{\pi}{4}\}$$

and continuous in  $\bar{\mathcal{V}}$  satisfying

$$\begin{cases} |p(z)| \leq \epsilon, & z \in (0, K], \\ |p(z)| \leq M, & z \in \mathcal{V}, \\ |p(0)| = 0, & z = 0, \end{cases}$$

where  $\epsilon, K, M$  are positive constants. Then there exists a function  $\beta(z)$  satisfying

$$\begin{cases} \beta(z) \geq \frac{1}{2}, & z \in (K, 2^{\frac{1}{4}}K), \\ \beta(z) \geq \frac{1}{\pi} \left( \left( \frac{z}{K} \right)^4 - 1 \right)^{-\frac{1}{2}}, & z \in (2^{\frac{1}{4}}K, \infty), \end{cases}$$

such that

$$|p(z)| \leq M e^{\beta(z)}, \quad \forall z \in (K, \infty).$$

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