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Elastic scattering from rough surfaces in three dimensions ☆

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Abstract

Consider the elastic scattering of a plane or point incident wave by an unbounded and rigid rough surface. The angular spectrum representation (ASR) for the time-harmonic Navier equation is derived in three dimensions. The ASR is utilized as a radiation condition to the elastic rough surface scattering problem. The uniqueness is proved through a Rellich-type identity for surfaces given by uniformly Lipschitz functions. In the case of flat surfaces with local perturbations, an equivalent variational formulation is deduced in a truncated bounded domain and the existence of solutions are shown for general incoming waves. The main ingredient of the proof is the radiating behavior of the Green tensor to the first boundary value problem of the Navier equation in a half-space.

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1. Introduction

Rough surface scattering problems have important applications in diverse scientific areas such as remote sensing, geophysics, outdoor sound propagation, radar techniques. Significant progress has been made by Chandler-Wilde and his co-authors concerning the mathematical analysis and the numerical approximation of the acoustic scattering problems modeled by the Helmholtz equation. We refer to [10,11,14,15,41] and [8,12] for the integral equation method and the variational approach, respectively, in both two- and three-dimensional settings. In the work of Durán, Muga, and Nédélec [20], the radiation condition and well-posedness in the absence of acoustic surface waves were discussed under the non-absorbing boundary condition in a locally perturbed half plane. The electromagnetic scattering problems were studied in [35] when the medium is lossy and also in [28,36] in the more challenging case of a penetrable dielectric layer.

This paper concerns the mathematical analysis of the time-harmonic elastic scattering from unbounded rigid surfaces in three dimensions. The relevant phenomena for the elastic wave propagation can be found in geophysics and seismology (see e.g., [1,2] and the references cited therein). In linear elasticity, the existence and uniqueness of solution were first studied by Arens in [3–5] for $C^{1,\alpha}$ surfaces via the boundary integral equation method in two dimensions. The results generalize the solvability of the rough surface scattering problems discussed in [11,15,41] on acoustic waves to elastic waves. Moreover, an upward propagating radiation condition (UPRC) was proposed in [4] based on the elastic Green tensor of the Dirichlet boundary value problem for the Navier equation in a half-space. It is known that the classical Kupradze radiation condition (e.g. [18]) is not appropriate for unbounded rough surfaces. The variational approach was proposed in [22,25] to handle well-posedness of the scattering problems in periodic structures by using the Rayleigh expansion condition (REC) and in [23,24] for general rigid rough surfaces by using the angular spectrum representation (ASR). The early study may also be found in [9] for with less rigorous arguments. However, most of these works are devoted to two-dimensional elastic scattering problems and little has been done in three dimensions.

The goal of this paper is threefold. First, we present a mathematical formulation of the elastic rough surface scattering problems in three dimensions. In particular, we derive the upward angular spectrum representation (UASR) and the Green tensor to the first boundary value problem of the Navier equation in the half space. To the best of our knowledge, the UASR and the Green tensor have not been rigorously investigated in the mathematical literature. The UASR for the Navier equation can be used as a formal outgoing radiation condition in rough surface scattering problems (see [12] in the acoustic case). It leads to an equivalent Dirichlet-to-Neumann (DtN) map, which can be used as a transparent boundary condition (TBC) to truncate the unbounded domain in the vertical direction. Next, we prove the uniqueness of weak solutions if the rigid surface is the graph of a uniformly Lipschitz continuous function. Analogous to that in the two-dimensional case [23], our uniqueness proof is essentially based on a Rellich-type identity in an unbounded strip. However, the calculations of some key integral identities (see e.g., (4.1)) are much more involved than the two-dimensional problem. Finally, as an application of the half-space radiation condition and Green tensor, we show the existence of solutions to locally perturbed scattering problems. Unlike the Helmholtz or Maxwell equations (see e.g., [7,32,33,35,40]), an essential difficulty in elasticity arises from the lack of a series solution of the Navier equation satisfying the Dirichlet boundary condition on the ground plane. We refer to Remarks 5.5 and 5.7 for a detailed comparison of the well-posedness results presented in this paper and those in acoustic and electromagnetic waves. The local perturbation argument can significantly simplify the analysis for general rough surfaces, since one can derive an equivalent variational formulation in a bounded domain in which the Fredhlom alternative can be applied. Some open questions will be described in this respect in Section 6. A possible future work is to investigate the well-posedness of general (non-periodic) rough surface scattering problems.

It should be pointed out that elastic surface waves, which exponentially decay in the vertical direction, satisfy the newly established radiation condition (2.13) in a weighted Sobolev space (see e.g., [24] in two dimensions) rather than in the usual H^1-space as considered in this paper. Hence, our uniqueness result (see Theorem 4.4) does not give rise to the absence of surface waves caused by a rigid scattering interface. In fact, the horizontally decaying behavior of solutions in H^1 (see Theorem 4.4) excludes the presence of elastic surface waves. An interesting problem is to analyze the absence of elastic surface waves by proving well-posedness in weighted Sobolev spaces, if the rigid rough surface is the graph of a function. For flat surfaces with local perturbations, the well-posedness and the solution form (see Theorems 5.4 and 5.6) are not valid under the traction-free boundary condition due to the presence of surface waves in the far-field expansion. We refer to [21] for the two-dimensional Green tensor with a free flat boundary and the corresponding well-posedness result in a locally perturbed half-plane. The limiting absorption principle was justified in [19] for a free boundary in a locally perturbed half space. It is worthy to mention that our arguments for rigid flat surfaces with local perturbations depend on the asymptotic behavior of the half-space Green tensor which is different from the case of free boundaries (see Theorems 5.4 and 5.6).

The rest of this paper is organized as follows. In Section 2, we formulate the three-dimensional rough surface problems and introduce the upward and downward angular spectrum representations. The downward and upward Dirichlet-to-Neumann maps will be defined and analyzed in Section 3. Section 4 is devoted to the uniqueness of the solutions for general rough surface scattering problems, while Section 5 is devoted to the existence of the solutions for locally perturbed scattering problems. The paper is concluded with some general remarks and open questions in Section 6.

2. Problem formulation

In this section, we present the mathematical formulation of the three-dimensional elastic wave scattering by unbounded rigid rough surfaces. Let $D \subset \mathbb{R}^3$ be an unbounded connected open set such that, for some constants $f_- < f_+$,

$$U_{f_+} \subset D \subset U_{f_-}, \quad U_b := \{x = (x', x_3) : x_3 > b\}, \quad x' := (x_1, x_2).$$

For $b > f_+$, let $\Gamma_b = \{x \in \mathbb{R}^3 : x_3 = b\}$ and $S_b = D \backslash \overline{U}_b$. We assume that $\Gamma := \partial D$ is an unbounded rough surface, which is Lipschitz continuous but not necessarily the graph of some function. The space D is supposed to be filled with a homogeneous and isotropic elastic medium with unit mass density.

Let $u^{\rm in}$ be a time-harmonic elastic wave which is incident on the rough surface from above. Let $\omega > 0$ be the angular frequency of the incident wave. Denote by λ, μ the Lamé constants characterizing the medium above Γ and satisfying $\mu > 0, \lambda + 2\mu/3 > 0$. The incident wave field $u^{\rm in}$ is allowed to be a general elastic plane wave field of the following form

$$u^{\rm in}(x) = c_{\rm p} u_{\rm p}^{\rm in}(x) + c_{\rm s,1} u_{\rm s,1}^{\rm in}(x) + c_{\rm s,2} u_{\rm s,2}^{\rm in}(x), \quad c_{\rm p}, c_{\rm s,j} \in \mathbb{C},\ j = 1, 2, \tag{2.1}$$

where $u_{\rm p}^{\rm in}$ is the compressional plane wave field

$$u_{\rm p}^{\rm in}(x) = de^{{\rm i}\kappa_{\rm p} x\cdot d}, \quad d := d(\theta,\varphi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, -\cos\theta)^\top \tag{2.2}$$

and $u_{{\rm s},j}^{\rm in}$ are the shear plane wave fields

$$u_{{\rm s},j}^{\rm in}(x) = d_j^\perp e^{{\rm i}\kappa_{\rm s} x\cdot d}, \quad j=1,2. \tag{2.3}$$

Here $\theta \in [0,\pi/2), \varphi \in [0,2\pi)$ are the incident angles, $d_j^\perp$ are unit vectors satisfying $d_j^\perp \cdot d = 0$, and

$$\kappa_{\rm p} = \omega/\sqrt{\lambda+2\mu}, \quad \kappa_{\rm s} = \omega/\sqrt{\mu}$$

are the compressional and shear wavenumbers, respectively. It is clear to note that $u_{\rm p}^{\rm in}$ is a longitudinal wave and $u_{{\rm s},j}^{\rm in}$, $j=1,2$ are transversal waves. It can be verified that the incident field $u^{\rm in}$ satisfies the three-dimensional time-harmonic Navier equation:

$$\mu\Delta u^{\rm in} + (\lambda+\mu)\nabla\nabla\cdot u^{\rm in} + \omega^2 u^{\rm in} = 0 \quad \text{in } \mathbb{R}^3. \tag{2.4}$$

In this paper, we assume that the elastic medium beneath the rough surface is impenetrable and rigid. Hence the total field satisfies the homogeneous Dirichlet boundary condition

$$u = 0 \quad \text{on } \Gamma.$$

The displacement of the scattered field $u^{\rm sc} := u - u^{\rm in}$ satisfies the following boundary value problem:

$$\mu\Delta u^{\rm sc} + (\lambda+\mu)\nabla\nabla\cdot u^{\rm sc} + \omega^2 u^{\rm sc} = 0 \quad \text{in } D, \qquad u^{\rm sc} = -u^{\rm in} \quad \text{on} \quad \Gamma. \tag{2.5}$$

We may also consider a spherical point source incidence given by the Green tensor of the Navier equation in $\mathbb{R}^3$, i.e.,

$$u^{\rm in}(x) = {\rm G}(x,y), \quad x \in D\backslash\{y\}, \quad y \in D, \tag{2.6}$$

where

$${\rm G}(x,y) = \frac{1}{\mu} g_{\rm s}(x,y){\rm I} + \frac{1}{\omega^2}\nabla_y\nabla_y^\top (g_{\rm s}(x,y) - g_{\rm p}(x,y)). \tag{2.7}$$

Here I is the identity matrix and

$$g_{\rm p}(x,y) = \frac{1}{4\pi}\frac{e^{{\rm i}\kappa_{\rm p}|x-y|}}{|x-y|}, \quad g_{\rm s}(x,y) = \frac{1}{4\pi}\frac{e^{{\rm i}\kappa_{\rm s}|x-y|}}{|x-y|} \tag{2.8}$$

are the fundamental solutions of the three-dimensional Helmholtz equation with the compressional and shear wave numbers, respectively. The incident field (2.6) satisfies the three-dimensional Navier equation

$$\mu\Delta u^{\rm in} + (\lambda+\mu)\nabla\nabla\cdot u^{\rm in} + \omega^2 u^{\rm in} = \delta(x-y){\rm I}, \quad x \in \mathbb{R}^3 \setminus \{y\}.$$

Since the domain D is unbounded, a radiation condition needs to be imposed at infinity to ensure the well-posedness of the boundary value problem (2.5). Following [23], we propose a radiation condition based on the upward angular spectrum representation (UASR) for solutions of the scalar Helmholtz equation [12].

We begin with the decomposition of the scattered field into a sum of its compressional and shear parts

$$u^{\mathrm{sc}} = \frac{1}{\mathrm{i}}(\nabla\varphi + \nabla\times\psi), \quad \nabla\cdot\psi = 0, \tag{2.9}$$

where the scalar function φ and the vector function ψ satisfy the homogeneous Helmholtz equations

$$\Delta\varphi + \kappa_{\mathrm{p}}^2\varphi = 0, \quad \Delta\psi + \kappa_{\mathrm{s}}^2\psi = 0 \quad \text{in } D.$$

Denote by $\hat{v}$ the Fourier transform of v in $\mathbb{R}^2$, i.e.,

$$\hat{v}(\xi) = \mathcal{F}v(\xi) := \frac{1}{2\pi}\int_{\mathbb{R}^2} v(x')e^{-\mathrm{i}x'\cdot\xi}\mathrm{d}x', \quad \xi = (\xi_1, \xi_2) \in \mathbb{R}^2.$$

Taking the Fourier transform of (2.9) and assuming that φ, ψ satisfy the UASR for the Helmholtz equations in U_b, we obtain

$$\begin{aligned}\varphi(x', x_3) &= \frac{1}{2\pi}\int_{\mathbb{R}^2} \hat{\varphi}(\xi, b)e^{\mathrm{i}\beta(\xi)(x_3-b)}e^{\mathrm{i}\xi\cdot x'}\mathrm{d}\xi,\\ \psi(x', x_3) &= \frac{1}{2\pi}\int_{\mathbb{R}^2} \hat{\psi}(\xi, b)e^{\mathrm{i}\gamma(\xi)(x_3-b)}e^{\mathrm{i}\xi\cdot x'}\mathrm{d}\xi,\end{aligned} \tag{2.10}$$

where

$$\beta(\xi) := \begin{cases}(\kappa_{\mathrm{p}}^2 - |\xi|^2)^{1/2}, & |\xi| < \kappa_{\mathrm{p}},\\ \mathrm{i}(|\xi|^2 - \kappa_{\mathrm{p}}^2)^{1/2}, & |\xi| > \kappa_{\mathrm{p}},\end{cases}$$

and

$$\gamma(\xi) := \begin{cases}(\kappa_{\mathrm{s}}^2 - |\xi|^2)^{1/2}, & |\xi| < \kappa_{\mathrm{s}},\\ \mathrm{i}(|\xi|^2 - \kappa_{\mathrm{s}}^2)^{1/2}, & |\xi| > \kappa_{\mathrm{s}}.\end{cases}$$

Denote

$$A_{\mathrm{p}}(\xi) = \hat{\varphi}(\xi, b), \quad \tilde{\boldsymbol{A}}_{\mathrm{s}}(\xi) = \hat{\psi}(\xi, b).$$

Substituting (2.10) into (2.9), we obtain

$$u^{\mathrm{sc}}(x)=\frac{1}{2\pi}\int\limits_{\mathbb{R}^2}\Big[A_{\mathrm{p}}(\xi)\,(\xi,\beta)^\top e^{\mathrm{i}\beta(x_3-b)}+\boldsymbol{A}_{\mathrm{s}}(\xi)\,e^{\mathrm{i}\gamma(x_3-b)}\Big]e^{\mathrm{i}\xi\cdot x'}\mathrm{d}\xi, \tag{2.11}$$

where $\boldsymbol{A}_{\mathrm{s}}=(A_{\mathrm{s}}^{(1)},A_{\mathrm{s}}^{(2)},A_{\mathrm{s}}^{(3)})^\top(\xi):=(\xi,\gamma)^\top\times\tilde{\boldsymbol{A}}_{\mathrm{s}}(\xi)$. It follows from (2.11) and the orthogonality $(\xi,\gamma)\cdot\boldsymbol{A}_{\mathrm{s}}^\top=0$ that

$$\begin{bmatrix}\hat{u}^{\mathrm{sc}}(\xi,b)\\0\end{bmatrix}=\begin{bmatrix}\xi_1&1&0&0\\\xi_2&0&1&0\\\beta&0&0&1\\0&\xi_1&\xi_2&\gamma\end{bmatrix}\begin{bmatrix}A_{\mathrm{p}}(\xi)\\\boldsymbol{A}_{\mathrm{s}}^\top(\xi)\end{bmatrix}:=\tilde{\mathbb{D}}(\xi)\,\boldsymbol{A}(\xi),$$

which gives

$$\boldsymbol{A}(\xi)=\begin{bmatrix}A_{\mathrm{p}}\\\boldsymbol{A}_{\mathrm{s}}^\top\end{bmatrix}(\xi)=\tilde{\mathbb{D}}^{-1}(\xi)\begin{bmatrix}\hat{u}^{sc}(\xi,b)\\0\end{bmatrix}=\mathbb{D}(\xi)\,\hat{u}^{\mathrm{sc}}(\xi,b). \tag{2.12}$$

Here $\mathbb{D}$ is a 4×3 matrix given by

$$\mathbb{D}(\xi)=\frac{1}{\beta\gamma+|\xi|^2}\begin{bmatrix}\xi_1&\xi_2&\gamma\\\beta\gamma+\xi_2^2&-\xi_1\xi_2&-\xi_1\gamma\\-\xi_1\xi_2&\beta\gamma+\xi^2&-\xi_2\gamma\\-\xi_1\beta&-\xi_2\beta&|\xi|^2\end{bmatrix}.$$

Using (2.11)–(2.12) yields an expression of u^{sc} in U_b in terms of the Fourier transform of the Dirichlet data $u(x',b)$:

$$u^{\mathrm{sc}}(x)=\frac{1}{2\pi}\int\limits_{\mathbb{R}^2}\Big\{\frac{1}{\beta\gamma+|\xi|^2}\Big(M_{\mathrm{p}}(\xi)e^{\mathrm{i}(\xi\cdot x'+\beta(x_3-b))}+M_{\mathrm{s}}(\xi)e^{\mathrm{i}(\xi\cdot x'+\gamma(x_3-b))}\Big)\hat{u}^{\mathrm{sc}}(\xi,b)\Big\}\mathrm{d}\xi, \tag{2.13}$$

where

$$M_{\mathrm{p}}(\xi)=(\xi_1,\xi_2,\beta)\otimes(\xi_1,\xi_2,\gamma):=\begin{bmatrix}\xi_1^2&\xi_1\xi_2&\xi_1\gamma\\\xi_1\xi_2&\xi_2^2&\xi_2\gamma\\\xi_1\beta&\xi_2\beta&\beta\gamma\end{bmatrix} \tag{2.14}$$

and

$$M_{\mathrm{s}}(\xi)=\begin{bmatrix}\beta\gamma+\xi_2^2&-\xi_1\xi_2&-\gamma\xi_1\\-\xi_1\xi_2&\beta\gamma+\xi_1^2&-\gamma\xi_2\\-\xi_1\beta&-\xi_2\beta&|\xi|^2\end{bmatrix}=(\beta\gamma+|\xi|^2)\,\mathbf{I}-M_{\mathrm{p}}(\xi). \tag{2.15}$$

Define $M_{\mathrm{p}}^+:=M_{\mathrm{p}}/(\beta\gamma+|\xi|^2)$. We can rewrite (2.13) into

$$u^{\rm sc}(x) = \frac{1}{2\pi}\int\limits_{\mathbb{R}^2} \Big\{M_{\rm p}^+(\xi)e^{{\rm i}(\xi\cdot x'+\beta(x_3-b))} + \Big(\mathbf{I} - M_{\rm p}^+(\xi)\Big)e^{{\rm i}(\xi\cdot x'+\gamma(x_3-b))}\Big\}\hat{u}^{\rm sc}(\xi,b){\rm d}\xi. \quad (2.16)$$

The representation (2.13) or (2.16), which is referred to as the UASR for elastic wave fields, is the upward radiation condition. The downward ASR of $u^{\rm sc}$ in $x_3 < b$ can be similarly derived and are written as

$$u^{\rm sc}(x) = \frac{1}{2\pi}\int\limits_{\mathbb{R}^2} \Big\{\frac{1}{\beta\gamma+|\xi|^2}\Big(M_{\rm p}^{(D)}(\xi)e^{{\rm i}(\xi\cdot x'-\beta(x_3-b))} + M_{\rm s}^{(D)}(\xi)e^{{\rm i}(\xi\cdot x'-\gamma(x_3-b))}\Big)\hat{u}^{\rm sc}(\xi,b)\Big\}{\rm d}\xi$$
$$= \frac{1}{2\pi}\int\limits_{\mathbb{R}^2} \Big\{M_{\rm p}^-(\xi)e^{{\rm i}(\xi\cdot x'-\beta(x_3-b))} + \Big(\mathbf{I} - M_{\rm p}^-(\xi)\Big)e^{{\rm i}(\xi\cdot x'-\gamma(x_3-b))}\Big\}\hat{u}^{\rm sc}(\xi,b){\rm d}\xi. \quad (2.17)$$

Here $M_{\rm p}^-(\xi) := M_{\rm p}^{(D)}(\xi)/(\beta\gamma+|\xi|^2)$,

$$M_{\rm p}^{(D)}(\xi) := \begin{bmatrix} \xi_1^2 & \xi_1\xi_2 & -\xi_1\gamma \\ \xi_1\xi_2 & \xi_2^2 & -\xi_2\gamma \\ -\xi_1\beta & -\xi_2\beta & \beta\gamma \end{bmatrix},$$
$$M_{\rm s}^{(D)}(\xi) := \begin{bmatrix} \beta\gamma+\xi_2^2 & -\xi_1\xi_2 & \gamma\xi_1 \\ -\xi_1\xi_2 & \beta\gamma+\xi_1^2 & \gamma\xi_2 \\ \xi_1\beta & \xi_2\beta & |\xi|^2 \end{bmatrix}. \quad (2.18)$$

If $u^{\rm sc}$ is quasi-biperiodic on Γ_b, then the ASR of $u^{\rm sc}$ in a half space is equivalent to the Rayleigh expansion of $u^{\rm sc}$ (see [3,22,25]). We say $u^{\rm sc}$ is quasi-biperiodic with the phase-shift $\alpha = (\alpha_1,\alpha_2) \in \mathbb{R}^2$ in the variable x', if $u^{\rm sc}(x'+2\pi n, b) = e^{{\rm i}2\pi\alpha\cdot n}u^{\rm sc}(x',b)$ for all $n = (n_1,n_2) \in \mathbb{Z}^2$. Therefore, $u^{\rm sc}(x',b)$ admits the Fourier series expansion

$$u^{\rm sc}(x',b) = \sum_{n\in\mathbb{Z}^2} u_n^{\rm sc}(b)e^{{\rm i}\alpha_n\cdot x'}, \quad x'\in\mathbb{R}^2, \quad (2.19)$$

where $\alpha_n = \alpha + n$ and $u_n^{\rm sc}(b)$ is the Fourier coefficient of $u^{\rm sc}$ on Γ_b given by

$$u_n^{\rm sc}(b) = \frac{1}{2\pi}\int\limits_0^{2\pi}\int\limits_0^{2\pi} u^{\rm sc}(x',b)e^{-{\rm i}\alpha_n\cdot x'}{\rm d}x'.$$

Substituting (2.19) into (2.13) and noting that the Fourier transform of $e^{{\rm i}\alpha_n\cdot x'}$ is $2\pi\delta(\xi-\alpha_n)$, we obtain

$$u^{\rm sc}(x) = \frac{1}{2\pi}\int\limits_{\mathbb{R}^2} \Big\{\frac{1}{\beta\gamma+|\xi|^2}\Big(M_{\rm p}(\xi)e^{{\rm i}(\xi\cdot x'+\beta(x_3-b))} + M_{\rm s}(\xi)e^{{\rm i}(\xi\cdot x'+\gamma(x_3-b))}\Big)\hat{u}^{\rm sc}(\xi,b)\Big\}{\rm d}\xi$$

$$
\begin{aligned}
&= \sum_{n\in\mathbb{Z}^2} \int_{\mathbb{R}^2} \Big\{ \frac{1}{\beta\gamma + |\xi|^2} \Big(M_{\rm p}(\xi) e^{{\rm i}(\xi\cdot x' + \beta(x_3-b))} + M_{\rm s}(\xi) e^{{\rm i}(\xi\cdot x' + \gamma(x_3-b))} \Big) \delta(\xi - \alpha_n) \Big\} u_n^{\rm sc}(b) {\rm d}\xi \\
&= \sum_{n\in\mathbb{Z}^2} \frac{1}{\beta_n\gamma_n + |\alpha_n|^2} \Big(M_{{\rm p},n} e^{{\rm i}(\alpha_n\cdot x' + \beta_n(x_3-b))} + M_{{\rm s},n} e^{{\rm i}(\alpha_n\cdot x' + \gamma_n(x_3-b))} \Big) u_n^{\rm sc}(b) \\
&= \sum_{n\in\mathbb{Z}^2} \frac{(\alpha_n,\gamma_n)^\top \cdot u_n^{\rm sc}(b)}{\beta_n\gamma_n + |\alpha_n|^2} (\alpha_n,\beta_n)^\top e^{{\rm i}(\alpha_n\cdot x' + \beta_n(x_3-b))} \\
&\quad + \frac{1}{\beta_n\gamma_n + |\alpha_n|^2} \Big[(\alpha_n,\gamma_n)^\top \times \Big(u_n^{\rm sc}(b) \times (\alpha_n,\beta_n)^\top \Big) \Big] e^{{\rm i}(\alpha_n\cdot x' + \gamma_n(x_3-b))}, \qquad (2.20)
\end{aligned}
$$

where

$$
\beta_n = \beta(\alpha_n), \quad \gamma_n = \gamma(\alpha_n), \quad M_{{\rm p},n} = M_{\rm p}(\alpha_n), \quad M_{{\rm s},n} = M_{\rm s}(\alpha_n).
$$

The representation (2.20) is the upward Rayleigh expansion of $u^{\rm sc}$ in $x_3 > b$. Using the vector identities

$$
\begin{aligned}
&(\alpha_n,\gamma_n)^\top \times \Big(u_n^{\rm sc}(b) \times (\alpha_n,\beta_n)^\top \Big) \\
&= \Big((\alpha_n,\gamma_n) \cdot (\alpha_n,\beta_n)^\top \Big) u_n^{\rm sc}(b) - \Big((\alpha_n,\gamma_n)^\top \cdot u_n^{\rm sc}(b) \Big) (\alpha_n,\beta_n)^\top \\
&= (\beta_n\gamma_n + |\alpha_n|^2) u_n^{\rm sc}(b) - \Big((\alpha_n,\gamma_n)^\top \cdot u_n^{\rm sc}(b) \Big) (\alpha_n,\beta_n)^\top,
\end{aligned}
$$

we may rewrite (2.20) into

$$
u^{\rm sc}(x) = \sum_{n\in\mathbb{Z}^2} A_{{\rm p},n} (\alpha_n,\beta_n)^\top e^{{\rm i}(\alpha_n\cdot x' + \beta_n(x_3-b))} + \mathbf{A}_{{\rm s},n} e^{{\rm i}(\alpha_n\cdot x' + \gamma_n(x_3-b))}, \qquad (2.21)
$$

where

$$
A_{{\rm p},n} = \frac{(\alpha_n,\gamma_n)^\top \cdot u_n^{\rm sc}(b)}{\beta_n\gamma_n + |\alpha_n|^2} \in \mathbb{C}, \qquad \mathbf{A}_{{\rm s},n} = u_n^{\rm sc}(b) - A_{{\rm p},n}(\alpha_n,\beta_n)^\top \in \mathbb{C}^3.
$$

It is clear to note that $(\alpha_n,\gamma_n)\cdot \mathbf{A}_{{\rm s},n} = 0$ for all $n \in \mathbb{Z}^2$. The representation (2.21) is the reduction of the UASR (see (2.13) and (2.11)) to the Rayleigh expansion in quasi-periodic spaces. The equivalence of the downward radiation conditions can be justified in the same manner.

The rough surface scattering problem can be stated as follows: Given a plane incident wave field (2.1) or a point incident wave field (2.7), the scattering problem is to find the scattered field $u^{\rm sc}$ of the boundary value problem for the Navier equation (2.5) in a distributional sense, such that the upward radiation condition (2.13) is satisfied. In this work, we

(1) prove uniqueness of the solution in $H^1(S_b)^3$ for any $b > f^+$ (see Section 4.4);
(2) for locally perturbed flat surfaces, prove existence of the Kupradze radiating solution $u^{\rm sc} - u^{\rm re} \in H^1_{\rm loc}(D)^3$, where $u^{\rm re}$ denotes the reflected wave field corresponding to the unperturbed flat surface (see Section 5).

3. Dirichlet-to-Neumann map

In this section, we introduce a Dirichlet-to-Neumann (DtN) map on the artificial flat surface Γ_b for some $b > f^+$ and investigate its mapping properties.

Recall that the traction operator on a surface is defined as

$$Tu := 2\mu\partial_\nu u + \lambda(\nabla\cdot u)\nu + \mu\nu\times(\nabla\times u),$$

where $\nu = (\nu_1, \nu_2, \nu_3)$ stands for the normal vector on the surface. Given $b > f^+$, the DtN map for the rough surface scattering problem is defined as follows.

Definition 3.1. For $v \in H^{1/2}(\Gamma_b)^3$, the upward DtN map $\mathcal{T}v$ is defined as $Tu^{\rm sc}$ on Γ_b, where $u^{\rm sc}$ is the unique upward radiation solution of the homogeneous Navier equation in U_b satisfying $u^{\rm sc} = v$ on Γ_b. More explicitly, we have

$$Tu := 2\mu\partial_3 u + \lambda(\nabla\cdot u)(0,0,1)^\top + \mu(0,0,1)^\top\times(\nabla\times u), \tag{3.1}$$

where $\partial_3 u = \partial_{x_3} u$.

We mention that the upward DtN map $\mathcal{T}$ is well defined, since $u^{\rm sc}$ can be uniquely determined in U_b via the formula (2.13). Next we derive an explicit representation of the upward DtN map $\mathcal{T}$ and show some of its properties.

Applying the traction operator T given in (3.1) to (2.13) and letting $x_3 = b$, we get

$$\mathcal{F}[(Tu^{\rm sc})|_{\Gamma_b}](\xi) = {\rm i}\begin{bmatrix} 2\mu\beta\xi_1 & \mu\gamma & 0 & \mu\xi_1 \\ 2\mu\beta\xi_2 & 0 & \mu\gamma & \mu\xi_2 \\ 2\mu\beta^2+\lambda\kappa_{\rm p}^2 & 0 & 0 & 2\mu\gamma \end{bmatrix}\begin{bmatrix} A_{\rm p} \\ \boldsymbol{A}_{\rm s}^\top \end{bmatrix} =: {\rm i}G(\xi)\boldsymbol{A}(\xi). \tag{3.2}$$

Recalling $\boldsymbol{A}(\xi) = \mathbb{D}(\xi)\hat{u}^{\rm sc}(\xi, b)$ in (2.12), we have

$$\mathcal{F}[(Tu^{\rm sc})|_{\Gamma_b}](\xi) = {\rm i}G(\xi)\mathbb{D}(\xi)\hat{u}^{\rm sc}(\xi,b) = {\rm i}M(\xi)\hat{u}^{\rm sc}(\xi,b),$$

where $M(\xi) = G(\xi)\mathbb{D}(\xi) \in \mathbb{C}^{3\times 3}$ is given by

$$M(\xi) = \frac{1}{|\xi|^2+\beta\gamma} \times \begin{bmatrix} \mu[(\gamma-\beta)\xi_2^2+\kappa_{\rm s}^2\beta] & -\mu\xi_1\xi_2(\gamma-\beta) & (2\mu|\xi|^2-\omega^2+2\mu\beta\gamma)\xi_1 \\ -\mu\xi_1\xi_2(\gamma-\beta) & \mu[(\gamma-\beta)\xi_1^2+\kappa_{\rm s}^2\beta] & (2\mu|\xi|^2-\omega^2+2\mu\beta\gamma)\xi_2 \\ -(2\mu|\xi|^2-\omega^2+2\mu\beta\gamma)\xi_1 & -(2\mu|\xi|^2-\omega^2+2\mu\beta\gamma)\xi_2 & \gamma\omega^2 \end{bmatrix}. \tag{3.3}$$

Taking the inverse Fourier transform gives

$$[\mathcal{T}u^{\rm sc}(x',b)](x') = \frac{\rm i}{2\pi}\int_{\mathbb{R}^2} G(\xi)\boldsymbol{A}(\xi)e^{{\rm i}\xi\cdot x'}{\rm d}\xi = \frac{\rm i}{2\pi}\int_{\mathbb{R}^2} M(\xi)\hat{u}^{\rm sc}(\xi,b)e^{{\rm i}\xi\cdot x'}{\rm d}\xi,$$

where the matrix function M is given in (3.3). Since $v = u^{\mathrm{sc}}|_{\Gamma_b}$, we obtain the upward DtN map

$$\mathcal{T} v(x') = \frac{\mathrm{i}}{2\pi} \int_{\mathbb{R}^2} M(\xi) \hat{v}(\xi) e^{\mathrm{i}\xi \cdot x'} \mathrm{d}\xi. \tag{3.4}$$

The boundary operator $\mathcal{T}$ is non-local and it gives an equivalent representation to the upward radiation condition (2.13).

Similarly, we may show that the downward DtN map takes the form

$$\mathcal{T}^{-} v(x') = \frac{\mathrm{i}}{2\pi} \int_{\mathbb{R}^2} M^{-}(\xi) \hat{v}(\xi) e^{\mathrm{i}\xi \cdot x'} \mathrm{d}\xi, \tag{3.5}$$

where

$$M^{-}(\xi) = \frac{1}{|\xi|^2 + \beta\gamma} \times \begin{bmatrix} -\mu[(\gamma-\beta)\xi_2^2 + \kappa_s^2 \beta] & \mu\xi_1\xi_2(\gamma-\beta) & (2\mu|\xi|^2 - \omega^2 + 2\mu\beta\gamma)\xi_1 \\ \mu\xi_1\xi_2(\gamma-\beta) & -\mu[(\gamma-\beta)\xi_1^2 + \kappa_s^2\beta] & (2\mu|\xi|^2 - \omega^2 + 2\mu\beta\gamma)\xi_2 \\ -(2\mu|\xi|^2 - \omega^2 + 2\mu\beta\gamma)\xi_1 & -(2\mu|\xi|^2 - \omega^2 + 2\mu\beta\gamma)\xi_2 & \gamma\omega^2 \end{bmatrix}.$$

Comparing with the matrix M for the upward DtN (cf. (3.3)), we can easily see that the parameters $\beta(\xi)$ and $\gamma(\xi)$ are replaced by $-\beta(\xi)$ and $-\gamma(\xi)$ in the definition of $M^{-}(\xi)$, respectively.

Lemma 3.2. *Let $M(\xi)$ be defined in (3.3) and let $b > f^{+}$.*

(1) *Given a fixed frequency $\omega > 0$, we have $\Re(-\mathrm{i}M)(\xi) > 0$ for sufficiently large $|\xi|$.*
(2) *The DtN map $\mathcal{T}$ is a bounded operator from $H^{1/2}(\Gamma_b)^3$ to $H^{-1/2}(\Gamma_b)^3$.*

The proof of Lemma 3.2 relies on properties of the matrix M and can be carried out by following almost the same arguments as those in the quasi-periodic case [22]. The details are omitted for brevity.

4. Uniqueness

In this section, we study the uniqueness for the boundary value problem (2.5) and (2.13) if Γ is the graph of a uniformly Lipschitz continuous function f, i.e.,

$$\Gamma = \{x \in \mathbb{R}^3 : x_3 = f(x'),\ x' = (x_1, x_2) \in \mathbb{R}^2\}$$

and there exists a constant $L > 0$ such that

$$|f(x') - f(y')| \le L\,|x' - y'| \quad \forall x', y' \in \mathbb{R}^2.$$

First, we investigate the uniqueness when f is a C^2-smooth function over $\mathbb{R}^2$. Denote the unit normal vector on $\Gamma \cup \Gamma_b$ by $\nu := (\nu_1, \nu_2, \nu_3)$ pointing into the region of $x_3 > b$ on Γ_b and into the interior of D on Γ. Since we consider the uniqueness, we assume that $u^{\rm in}=0$ in this section. Thus $u=u^{\rm sc}$ is a radiation solution in S_b for any $b>f^+$. The goal is to prove that $u\equiv 0$ in D, depending on the geometry of ∂D. The proof depends on a Rellich-type identity for the Navier equation in the unbounded strip S_b. The Rellich-type identity was first used in [17] to prove uniqueness of the acoustic scattering by smooth periodic sound-soft curves and in [26] to handle periodic Lipschitz graphs. A priori estimates and explicit bounds on the solution were given in [12] for the acoustic rough surface scattering problems. We refer to [16] for more general Rellich's identities in a bounded domain.

Lemma 4.1. *If $u\in H^1(S_b)^3$ and f is a C^2-smooth function, then the following Rellich identity holds:*

$$
\begin{aligned}
&2\Re\int_{S_b}(\mu\Delta u+(\lambda+\mu)\nabla\cdot u+\omega^2 u)\cdot\partial_3\bar{u}\,{\rm d}x\\
&=\Big(-\int_{\Gamma}+\int_{\Gamma_b}\Big)\Big\{2\Re(Tu\cdot\partial_3\bar{u})-\nu_3\mathcal{E}(u,\bar{u})+\omega^2|u|^2\Big\}{\rm d}s,
\end{aligned}
$$

where the bilinear form

$$
\mathcal{E}(u,v):=2\mu\sum_{j,k=1}^{3}\partial_k u_j\partial_k v_j+\lambda(\nabla\cdot u)(\nabla\cdot v)-\mu(\nabla\times u)\cdot(\nabla\times v)\quad\forall u,v\in H^1(S_b)^3.
$$

Proof. The proof is similar as that in [23, Lemma 6]. We sketch it here. By standard elliptic regularity, we see that $u\in H^2(S_b)^3$. For $A\ge 1$, we choose a cut-off function $\chi_A(r)\in C_0^\infty(\mathbb{R}^+)$ with $r=|x|$ such that $\chi_A(r)=1$ if $r\le A$, $\chi_A(r)=0$ if $r\ge A+1$, $0\le\chi_A(r)\le 1$ if $A<r\le A+1$, and $\|\chi_A'(r)\|\le C$ for some fixed C independent of A. Multiplying both sides of (2.5) by the test function $\chi_A(r)\partial_3\bar{u}$, using the integration by parts, and letting $A\to+\infty$, we may obtain the desired identity. □

Since u satisfies the Navier equation in D, it follows from Lemma 4.1 that

$$
\int_{\Gamma}\big(2\Re(Tu\cdot\partial_3\bar{u})-\nu_3\mathcal{E}(u,\bar{u})+\omega^2|u|^2\big){\rm d}s=\int_{\Gamma_b}\big(2\Re(Tu\cdot\partial_3\bar{u})-\nu_3\mathcal{E}(u,\bar{u})+\omega^2|u|^2\big){\rm d}s.
$$

In the following lemma, we simplify the left hand side of the above identity by using the boundary condition $u=0$ on Γ and simplify the right hand side of the above identity by the radiation condition of $u=u^{\rm sc}$.

Lemma 4.2. *(i) Under the assumptions of Lemma 4.1, the following identity holds:*

$$
\int_{\Gamma}\big(2\Re(Tu\cdot\partial_3\bar{u})-\nu_3\mathcal{E}(u,\bar{u})+\omega^2|u|^2\big){\rm d}s=\int_{\Gamma}\big(\mu|\partial_\nu u|^2\nu_3+(\lambda+\mu)|\nabla\cdot u|^2\nu_3\big){\rm d}s.
$$

(ii) Let $u = u^{\mathrm{sc}}$ satisfy (2.13) in $x_3 > b$ with the parameter-dependent coefficients $A_{\mathrm{p}}(\xi)$ and $\boldsymbol{A}_{\mathrm{s}}(\xi) \in \mathbb{C}^{3\times 1}$ for $\xi \in \mathbb{R}^3$, then the following identities hold:

$$\int_{\Gamma_b} \Big\{2\Re(Tu \cdot \partial_3 \bar{u}) - \nu_3 \mathcal{E}(u, \bar{u}) + \omega^2 |u|^2\Big\} \mathrm{d}s$$
$$= 2\omega^2 \Big\{ \int_{|\xi|<\kappa_{\mathrm{p}}} \beta^2(\xi)|A_{\mathrm{p}}(\xi)|^2 \,\mathrm{d}\xi + \int_{|\xi|<\kappa_{\mathrm{s}}} \gamma^2(\xi)|\boldsymbol{A}_{\mathrm{s}}(\xi)|^2 \,\mathrm{d}\xi \Big\}, \tag{4.1}$$

$$\Im \int_{\Gamma_b} Tu \cdot \bar{u} \mathrm{d}s = \int_{|\xi|<\kappa_{\mathrm{p}}} \omega^2 \beta(\xi)|A_{\mathrm{p}}(\xi)|^2 \mathrm{d}\xi + \int_{|\xi|<\kappa_{\mathrm{s}}} \mu\gamma(\xi)|\boldsymbol{A}_{\mathrm{s}}(\xi)|^2 \mathrm{d}\xi. \tag{4.2}$$

Proof. (i) Since $u = 0$ on Γ, a direct calculation shows that on Γ (see also [22, Lemma 5]),

$$\nu \cdot \partial_3 \bar{u} \nabla \cdot u = \nu_3 |\nabla \cdot u|^2, \quad \partial_3 u = \nu_3 \partial_\nu u, \quad \partial_\nu u + \nu \times (\nabla \times u) - \nu \nabla \cdot u = 0.$$

Hence, by the definitions of the traction operator T and the bilinear form $\mathcal{E}(\cdot,\cdot)$, we get

$$Tu \cdot \partial_3 \bar{u} = \mathcal{E}(u, \bar{u}) = \nu_3 \mu |\partial_\nu u|^2 \nu_3 + (\lambda + \mu)|\nabla \cdot u|^2 \nu_3,$$

which proves the first assertion.

(ii) The proof of the second assertion depends on the upward ASR of $u = u^{\mathrm{sc}}$ and the Parseval formula.

It follows from (3.2) and the Fourier transform of Tu in terms of A_{p} and $\boldsymbol{A}_{\mathrm{s}}$ on Γ_b that $\widehat{Tu}(\xi) = \mathrm{i}G(\xi)\boldsymbol{A}(\xi)$, where $\boldsymbol{A}$ is defined in (2.12). By (2.13), the Fourier transform $\widehat{\partial_j u}$ of $\partial_j u$ on Γ_b can be represented by

$$\widehat{\partial_j u} = H_j(\xi)\, \boldsymbol{A}(\xi), \quad j = 1, 2, 3,$$

where H_j are 3-by-4 matrices defined by

$$H_1 = \mathrm{i}\begin{bmatrix} \xi_1^2 & \xi_1 & 0 & 0 \\ \xi_1\xi_2 & 0 & \xi_1 & 0 \\ \xi_1\beta & 0 & 0 & \xi_1 \end{bmatrix}, \; H_2 = \mathrm{i}\begin{bmatrix} \xi_1\xi_2 & \xi_2 & 0 & 0 \\ \xi_2^2 & 0 & \xi_2 & 0 \\ \xi_2\beta & 0 & 0 & \xi_2 \end{bmatrix}, \; H_3 = \mathrm{i}\begin{bmatrix} \beta\xi_1 & \gamma & 0 & 0 \\ \beta\xi_2 & 0 & \gamma & 0 \\ \beta^2 & 0 & 0 & \gamma \end{bmatrix}.$$

Let

$$M_1 := H_1^* G, \quad M_2 := H_1^* H_1 + H_2^* H_2 + H_3^* H_3. \tag{4.3}$$

The Fourier transforms of u, $\nabla \cdot u$ and $\nabla \times u$ on Γ_b are given respectively by

$$\hat{u}(\xi, b) = \mathbb{D}_1(\xi)\boldsymbol{A}(\xi), \quad \widehat{\nabla \cdot u} = H_4(\xi)\boldsymbol{A}(\xi), \quad \widehat{\nabla \times u} = (\xi, \gamma)^\top \times \boldsymbol{A}_{\mathrm{s}}(\xi),$$

where

$$\mathbb{D}_1(\xi)=\begin{bmatrix}\xi_1 & 1 & 0 & 0\\ \xi_2 & 0 & 1 & 0\\ \beta & 0 & 0 & 1\end{bmatrix},\quad H_4=\mathrm{i}\begin{bmatrix}\kappa_{\mathrm{p}}^2 & \xi_1\xi_2^2 & \xi_2^3 & \xi_2^2\bar{\beta}\\ \xi_1\xi_2^2 & \xi_2^2 & 0 & 0\\ \xi_2^3 & 0 & \xi_2^2 & 0\\ \xi_2^2\beta & 0 & 0 & \xi_2^2\end{bmatrix}.$$

Noting the orthogonal identity $(\xi,\gamma)\cdot \boldsymbol{A}_{\mathrm{s}}=0$, we have from a simple calculation that $|\widehat{\nabla\times u}|^2=(|\xi|^2+|\gamma|^2)|\boldsymbol{A}_{\mathrm{s}}|^2$. Define

$$M_3:=H_4^*H_4,\quad M_4:=\mathbb{D}_1^*\mathbb{D}_1,\quad M_5:=\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & |\xi|^2+|\gamma|^2 & 0 & 0\\ 0 & 0 & |\xi|^2+|\gamma|^2 & 0\\ 0 & 0 & 0 & |\xi|^2+|\gamma|^2\end{bmatrix}.$$

By the definition of M_j, $j=1,2\cdots,5$ and the Parseval formula, we obtain

$$\int_{\Gamma_b} Tu\cdot\partial_3\bar{u}\mathrm{d}s=\int_{\mathbb{R}^2} M_1(\xi)\boldsymbol{A}(\xi)\cdot\overline{\boldsymbol{A}}(\xi)\mathrm{d}\xi,$$

$$\int_{\Gamma_b}\mathcal{E}(u,\overline{u})\mathrm{d}s=\int_{\mathbb{R}^2}\Big(2\mu M_2(\xi)+\lambda M_3(\xi)-\mu M_5(\xi)\Big)\boldsymbol{A}(\xi)\cdot\overline{\boldsymbol{A}}(\xi)\mathrm{d}\xi,$$

$$\int_{\Gamma_b}|u|^2\mathrm{d}s=\int_{\mathbb{R}^2} M_4(\xi)\boldsymbol{A}(\xi)\cdot\overline{\boldsymbol{A}}(\xi)\mathrm{d}\xi.$$

Hence,

$$\int_{\Gamma_0} 2\Re(Tu\cdot\partial_3\bar{u})-\mathcal{E}(u,\bar{u})+\omega^2|u|^2\mathrm{d}s=\int_{\mathbb{R}^2}[\Re W(\xi)]\boldsymbol{A}(\xi)\cdot\overline{\boldsymbol{A}}(\xi)\mathrm{d}\xi,\tag{4.4}$$

where $W=2M_1-2\mu M_2-\lambda M_3+\mu M_5+\omega^2M_4$.

Next we calculate $\Re W$. To obtain the real part of M_1, we decompose it into the sum $J_{1,1}+J_{1,2}+J_{1,3}$, where (e.g., (4.3))

$$J_{1,1}=\begin{bmatrix}2\mu|\beta|^2(|\xi|^2+|\beta|^2)+\lambda\kappa_{\mathrm{p}}^2\bar{\beta}^2 & 0 & 0 & 0\\ 0 & \mu|\gamma|^2 & 0 & 0\\ 0 & 0 & \mu|\gamma|^2 & 0\\ 0 & 0 & 0 & \mu|\gamma|^2\end{bmatrix},$$

$$J_{1,2}=\begin{bmatrix} 0 & \mu\bar{\beta}\xi_1\gamma & \mu\bar{\beta}\xi_2\gamma & \mu\bar{\beta}|\xi|^2+2\mu\bar{\beta}^2\gamma \\ 2\mu\beta\bar{\gamma}\xi_1 & 0 & 0 & 0 \\ 2\mu\beta\bar{\gamma}\xi_2 & 0 & 0 & 0 \\ 2\mu\beta^2\bar{\gamma}+\lambda\kappa_{\rm p}^2\bar{\gamma} & 0 & 0 & 0 \end{bmatrix},$$

$$J_{1,3}=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu\xi_1\bar{\gamma} \\ 0 & 0 & 0 & \mu\xi_2\bar{\gamma} \\ 0 & 0 & 0 & \mu|\gamma|^2 \end{bmatrix}.$$

Similarly, we decompose M_2 into the sum $J_{2,1}+J_{2,2}$, where

$$J_{2,1}=\begin{bmatrix} (|\xi|^2+|\beta|^2)^2 & 0 & 0 & 0 \\ 0 & |\xi|^2+|\gamma|^2 & 0 & 0 \\ 0 & 0 & |\xi|^2+|\gamma|^2 & 0 \\ 0 & 0 & 0 & |\xi|^2+|\gamma|^2 \end{bmatrix},$$

$$J_{2,2}=\begin{bmatrix} 0 & \xi_1(|\xi|^2+\gamma\bar{\beta}) & \xi_2(|\xi|^2+\gamma\bar{\beta}) & \bar{\beta}(|\xi|^2+\gamma\bar{\beta}) \\ \xi_1(|\xi|^2+\beta\bar{\gamma}) & 0 & 0 & 0 \\ \xi_2(|\xi|^2+\beta\bar{\gamma}) & 0 & 0 & 0 \\ \beta(|\xi|^2+\beta\bar{\gamma}) & 0 & 0 & 0 \end{bmatrix},$$

and decompose M_4 into the sum $J_{4,1}+J_{4,2}$ with

$$J_{4,1}=\begin{bmatrix} |\xi|^2+|\beta|^2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},\quad J_{4,2}=\begin{bmatrix} 0 & \xi_1 & \xi_2 & \bar{\beta} \\ \xi_1 & 0 & 0 & 0 \\ \xi_2 & 0 & 0 & 0 \\ \beta & 0 & 0 & 0 \end{bmatrix}.$$

We deduce from (4.4) that

$$\Big\langle \Re W(\xi)\boldsymbol{A},\boldsymbol{A}\Big\rangle=\Big\langle Q(\xi)\boldsymbol{A},\boldsymbol{A}\Big\rangle+\Big\langle \Re\Big(2J_{1,2}-2\mu J_{2,2}+\omega^2 J_{4,2}\Big)\boldsymbol{A},\boldsymbol{A}\Big\rangle,$$

where $Q=(Q_{i,j})_{i,j=1}^4:=\Re\Big(2J_{1,1}-2\mu J_{2,1}-\lambda M_3+\mu M_5+\omega^2 J_{4,1}\Big)$. Moreover, we can obtain $\Re\Big(2J_{1,2}-2\mu J_{2,2}+\omega^2 J_{4,2}\Big)=0$, $Q_{i,j}=0$ if $i\neq j$ and

$$Q_{1,1}=\begin{cases}2\omega^2\beta^2, & |\xi|<\kappa_{\rm p},\\ 0, & |\xi|>\kappa_{\rm p},\end{cases}\quad Q_{i,i}=\begin{cases}2\omega^2\gamma^2, & |\xi|<\kappa_{\rm s},\\ 0, & |\xi|>\kappa_{\rm s},\end{cases}\quad \text{if}\quad i=2,3,4.$$

Hence,

$$\int_{\mathbb{R}^2} \Big\langle \Re W(\xi)\boldsymbol{A}, \boldsymbol{A}\Big\rangle \mathrm{d}\xi = 2\omega^2\Big(\int_{|\xi|<\kappa_1} \beta^2(\xi)|A_{\rm p}(\xi)|^2\mathrm{d}\xi + \int_{|\xi|<\kappa_2} \gamma^2(\xi)|\boldsymbol{A}_{\rm s}(\xi)|^2\mathrm{d}\xi\Big),$$

which together with (4.4) proves the relation (4.1).

To prove the second identity (4.2), we observe that

$$\Im \int_{\Gamma_b} T u\, \bar{u}\, \mathrm{d}s = \Im \int_{\mathbb{R}^2} \langle \mathrm{i}G\boldsymbol{A}, \mathbb{D}_1\boldsymbol{A}\rangle \mathrm{d}\xi = \int_{\mathbb{R}^2} \langle (\Re \mathbb{D}_1^* G)\boldsymbol{A}, \boldsymbol{A}\rangle \mathrm{d}\xi, \tag{4.5}$$

where

$$\mathbb{D}_1^* G = \begin{bmatrix} 2\mu\beta(|\xi|^2+|\beta|^2)+\lambda\kappa_{\rm p}^2\bar{\beta} & \mu\xi_1\gamma & \mu\xi_2\gamma & \mu|\xi|^2+2\mu\bar{\beta}^2\gamma \\ 2\mu\beta\xi_1 & \mu\gamma & 0 & \mu\xi_1 \\ 2\mu\beta\xi_2 & 0 & \mu\gamma & \mu\xi_2 \\ 2\mu\beta^2+\lambda\kappa_{\rm p}^2 & 0 & 0 & 2\mu\gamma \end{bmatrix}.$$

We decompose $\mathbb{D}_1^* G$ into the sum $J_1 + J_2 + J_3$, where

$$J_1 = \begin{bmatrix} 2\mu\beta(|\xi|^2+|\beta|^2)+\lambda\kappa_{\rm p}^2\bar{\beta} & 0 & 0 & 0 \\ 0 & \mu\gamma & 0 & 0 \\ 0 & 0 & \mu\gamma & 0 \\ 0 & 0 & 0 & \mu\gamma \end{bmatrix},$$

$$J_2 = \begin{bmatrix} 0 & \mu\xi_1\gamma & \mu\xi_2\gamma & \mu|\xi|^2+2\mu\bar{\beta}\gamma \\ 2\mu\beta\xi_1 & 0 & 0 & 0 \\ 2\mu\beta\xi_2 & 0 & 0 & 0 \\ 2\mu\beta^2+\lambda\kappa_{\rm p}^2 & 0 & 0 & 0 \end{bmatrix}, \quad J_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu\xi_1 \\ 0 & 0 & 0 & \mu\xi_2 \\ 0 & 0 & 0 & \mu\gamma \end{bmatrix}.$$

It follows from straightforward calculation that $\langle \Re(J_2+J_3)\boldsymbol{A}, \boldsymbol{A}\rangle$=0 and

$$\langle \Re J_1 \boldsymbol{A}, \boldsymbol{A}\rangle = \begin{cases} \omega^2\beta|A_{\rm p}|^2 + \mu\gamma|\boldsymbol{A}_{\rm s}|^2, & |\xi|<\kappa_{\rm p}, \\ \mu\gamma|\boldsymbol{A}_{\rm s}|^2, & \kappa_{\rm p}\le|\xi|<\kappa_{\rm s}, \\ 0, & \kappa_{\rm s}<|\xi|. \end{cases}$$

Following (4.5), we deduce that

$$\Im \int_{\Gamma_b} T u\bar{u}\mathrm{d}s = \int_{|\xi|<\kappa_{\rm p}} \omega^2\beta(\xi)|A_{\rm p}(\xi)|^2\mathrm{d}\xi + \int_{|\xi|<\kappa_{\rm s}} \mu\,\gamma(\xi)|\boldsymbol{A}_{\rm s}(\xi)|^2\mathrm{d}\xi,$$

which completes the proof. □

The following lemma plays an important role in the subsequent analysis. It implies that the upward propagating modes of the compressional and shear parts must vanish when $u^{\mathrm{in}}=0$.

Lemma 4.3. *If $u^{\mathrm{in}}=0$ and the radiating solution $u^{\mathrm{sc}} \in H^1(S_b)^3$ for any $b>f_+$, then*

$$A_{\mathrm{p}}(\xi)=0 \quad \textit{for } |\xi|<\kappa_{\mathrm{p}} \quad \textit{and} \quad \mathbf{A}_{\mathrm{s}}(\xi)=0 \quad \textit{for } |\xi|<\kappa_{\mathrm{s}},$$

where $A_{\mathrm{p}}(\xi)$ and $\mathbf{A}_{\mathrm{s}}(\xi)$ are defined in (2.11).

Proof. Multiplying the Navier equation in (2.5) by the complex conjugate of u^{sc} and using Betti's formula yield

$$0=\int\limits_{S_b}\big(\mathcal{E}(u^{\mathrm{sc}},\bar{u}^{sc})-\omega^2 u^{\mathrm{sc}}\cdot\bar{u}^{\mathrm{sc}}\big)\mathrm{d}x-\int\limits_{\Gamma_b}\bar{u}^{\mathrm{sc}}\cdot Tu^{\mathrm{sc}}\mathrm{d}s.$$

Taking the imaging part of the above equation and recalling the definition of DtN operator, we obtain

$$0=\Im\int\limits_{\Gamma_b}\bar{u}^{\mathrm{sc}}\cdot Tu^{\mathrm{sc}}\mathrm{d}s=\Im\int\limits_{\Gamma_b}\bar{u}^{\mathrm{sc}}\cdot\mathcal{T}u^{\mathrm{sc}}\mathrm{d}s=0,$$

which proves the result by noting (4.2) with $u=u^{\mathrm{sc}}$. □

By Lemma 4.3, the uniqueness does not hold for general rough surfaces. In the following theorem, we investigate the uniqueness under an additional geometrical assumption of the scattering surface.

Theorem 4.4. *If Γ is the graph of a uniformly Lipschitz function and $u^{\mathrm{in}}=0$, then $u\equiv 0$ in D.*

Proof. If f is a C^2-smooth function, it follows from Lemmas 4.1–4.3 that

$$\begin{aligned}&\int\limits_{\Gamma}\big(\mu|\partial_\nu u|^2\nu_3+(\lambda+\mu)|\nabla\cdot u|^2\nu_3\big)\mathrm{d}s\\&=2\omega^2\Big\{\int\limits_{|\xi|<\kappa_{\mathrm{p}}}\beta^2(\xi)|A_{\mathrm{p}}(\xi)|^2\,\mathrm{d}\xi+\int\limits_{|\xi|<\kappa_{\mathrm{s}}}\gamma^2(\xi)|\mathbf{A}_{\mathrm{s}}(\xi)|^2\,\mathrm{d}\xi\Big\}\\&=0.\end{aligned}\tag{4.6}$$

The geometric assumption of Γ implies that

$$\nu_3(x)=\frac{1}{\sqrt{1+|\nabla_{x'}f|^2}}>C_L>0,\quad x\in\Gamma,$$

where C_L is a constant depending on L only. Hence, we get $u=\partial_\nu u=0$ on Γ. As a consequence of the unique continuation in elasticity, it holds that $u\equiv 0$ in D. This proves the uniqueness for

C^2-smooth functions. Finally, the proof can be completed by applying Nečas' approach [39, Chapter 5] of approximating a Lipschitz graph by smooth surfaces. We refer to [22] for the application of the Nečas' approximation theory to bi-periodic surfaces and [23] for rough surfaces in two dimensions in elasticity. $\square$

In the proof of Theorem 4.4, the relation (4.6) is derived based on the important identity (4.1). Combined with the identity (4.2), this identity will be used to prove the existence of solutions to the rough surface scattering problems. We remark that, for the uniqueness proof only, the relation (4.6) can be also obtained in a more straightforward way without using (4.1), which is given as follows.

Proof. Using Lemmas 4.1 and 4.2 (i), we obtain for each fixed $b > f_+$ that

$$\int_{\Gamma} \left(\mu|\partial_\nu u|^2\nu_3 + (\lambda+\mu)|\nabla\cdot u|^2\nu_3\right)\mathrm{d}s = \int_{\Gamma_b} (2\Re(Tu\cdot\partial_3\bar{u}) - \nu_3\mathcal{E}(u,\bar{u}) + \omega^2|u|^2)\mathrm{d}s. \quad (4.7)$$

It suffices to show that the right hand side of (4.7) vanishes. By Lemma 4.3, we have

$$u = u^{\mathrm{sc}} = \int_{|\xi|\geq\kappa_{\mathrm{p}}} A_{\mathrm{p}}(\xi)\,(\xi,\beta)^\top e^{\mathrm{i}\beta(x_3-b)}e^{\mathrm{i}\xi\cdot x'}\mathrm{d}\xi + \int_{|\xi|\geq\kappa_{\mathrm{s}}} \boldsymbol{A}_{\mathrm{s}}(\xi)\,e^{\mathrm{i}\gamma(x_3-b)}e^{\mathrm{i}\xi\cdot x'}\mathrm{d}\xi, \quad x_3\geq b,$$

which gives

$$\widehat{\partial_3 u}(\xi,c) = \mathrm{i}\beta(\xi)\,A_{\mathrm{p}}(\xi)\,(\xi,\beta)^\top e^{\mathrm{i}\beta(c-b)} + \mathrm{i}\gamma(\xi)\,\boldsymbol{A}_{\mathrm{s}}(\xi)\,e^{\mathrm{i}\gamma(c-b)}, \quad c>b. \quad (4.8)$$

Since the right hand side of (4.7) does not depend on the choice of b, we have for each $c > b$ that

$$\int_{\Gamma_b} \left(2\Re(Tu\cdot\partial_3\bar{u}) - \nu_3\mathcal{E}(u,\bar{u}) + \omega^2|u|^2\right)\mathrm{d}s = \int_{\Gamma_c} \left(2\Re(Tu\cdot\partial_3\bar{u}) - \nu_3\mathcal{E}(u,\bar{u}) + \omega^2|u|^2\right)\mathrm{d}s. \quad (4.9)$$

First we prove that the first term on the right hand side of the above identity vanishes as $c \to +\infty$. Using (3.4), (4.8) and Lemma 4.3, we obtain

$$\begin{aligned}
\Re\int_{\Gamma_c} Tu\cdot\partial_3\bar{u}\mathrm{d}s &= \Re\int_{\mathbb{R}^2} \widehat{Tu}\cdot\overline{\widehat{\partial_3 u}}\mathrm{d}\xi \\
&= \Im\int_{\mathbb{R}^2} M(\xi)\Big(A_{\mathrm{p}}(\xi)(\xi,\beta)^\top e^{\mathrm{i}\beta(c-b)} + \boldsymbol{A}_{\mathrm{s}}(\xi)e^{\mathrm{i}\gamma(c-b)}\Big) \\
&\quad\cdot\overline{\Big(\beta A_{\mathrm{p}}(\xi)(\xi,\beta)^\top e^{\mathrm{i}\beta(c-b)} + \gamma\boldsymbol{A}_{\mathrm{s}}(\xi)e^{\mathrm{i}\gamma(c-b)}\Big)}\mathrm{d}\xi \\
&= \Im\int_{|\xi|\geq\kappa_{\mathrm{p}}} M(\xi)\Big(A_{\mathrm{p}}(\xi)(\xi,\beta)^\top\Big)\cdot\overline{\Big(\beta A_{\mathrm{p}}(\xi)(\xi,\beta)^\top\Big)}e^{-2(c-b)\sqrt{|\xi|^2-\kappa_{\mathrm{p}}^2}}\mathrm{d}\xi
\end{aligned}$$

$$+\Im \int_{|\xi|\ge\kappa_{\rm p}} M(\xi)\Big(A_{\rm p}(\xi)(\xi,\beta)^\top\Big)\cdot\overline{\Big(\gamma \boldsymbol{A}_{\rm s}(\xi)\Big)}e^{-(c-b)\sqrt{|\xi|^2-\kappa_{\rm p}^2}}e^{-(c-b)\bar\gamma}{\rm d}\xi$$

$$+\Im \int_{|\xi|\ge\kappa_{\rm p}} M(\xi)\boldsymbol{A}_{\rm s}(\xi)\cdot\overline{\Big(\beta A_{\rm p}(\xi,\beta)^\top\Big)}e^{-(c-b)\sqrt{|\xi|^2-\kappa_{\rm p}^2}}e^{-(c-b)\bar\gamma}{\rm d}\xi$$

$$+\Im \int_{|\xi|\ge\kappa_{\rm s}} M(\xi)\boldsymbol{A}_{\rm s}(\xi)\cdot\overline{\Big(\gamma \boldsymbol{A}_{\rm s}(\xi)\Big)}e^{-2(c-b)\sqrt{|\xi|^2-\kappa_{\rm s}^2}}{\rm d}\xi, \tag{4.10}$$

where the matrix M is given by (3.3), and the dot denotes the inner product over $\mathbb{R}^2$. For each $\epsilon>0$, there exists a sufficiently small $\delta>0$, which does not depend on c, such that

$$\Im \int_{\kappa_{\rm p}\le|\xi|\le\kappa_{\rm p}+\delta} M(\xi)\Big(A_{\rm p}(\xi)(\xi,\beta)^\top\Big)\cdot\overline{\Big(\beta A_{\rm p}(\xi,\beta)^\top\Big)}e^{-2(c-b)\sqrt{|\xi|^2-\kappa_{\rm p}^2}}{\rm d}\xi<\epsilon.$$

On the other hand, we have

$$\lim_{c\to+\infty}\int_{|\xi|\ge\kappa_{\rm p}+\delta} M(\xi)\Big(A_{\rm p}(\xi)(\xi,\beta)^\top\Big)\cdot\overline{\Big(\beta A_{\rm p}(\xi,\beta)^\top\Big)}e^{-2(c-b)\sqrt{|\xi|^2-\kappa_{\rm p}^2}}{\rm d}\xi=0,$$

since it is an exponentially decaying function as $c\to+\infty$. Hence, the first term on the right hand side of (4.10) tends to zero as $c\to\infty$. Similarly, we may show that the remaining terms on the right hand side of (4.10) and those of (4.9) vanish as $c\to+\infty$. Hence we show that (4.7) vanishes due to the relation (4.9) and the arbitrariness of $c>b$. □

5. Existence

In this section, we discuss the existence of the solutions to the scattering problems where the flat surfaces are locally perturbed.

5.1. Scattering from flat surfaces

The propagation and reflection of elastic waves in a homogeneous half-space have been of significant interest in the classical seismology (see e.g., [1,2] and the references cited therein). The analytical solutions of such problems are frequently used in the literature for various purposes. In this section, we assume that $\Gamma=\Gamma_0$ (i.e., $b=0$) is a rigid flat surface. In this case, the total field consists of the incident field $u^{\rm in}$ and the reflected field $u^{\rm re}$, i.e., $u=u^{\rm in}+u^{\rm re}$, where $u^{\rm re}$ solves the boundary value problem

$$\mu\Delta u^{\rm re}+(\lambda+\mu)\nabla\nabla\cdot u^{\rm re}+\omega^2u^{\rm re}=0\quad\text{in}\quad U_0,\qquad u^{\rm re}=-u^{\rm in}\quad\text{on}\quad\Gamma_0.$$

If $u^{\rm in}$ is a compressional plane wave field of the form (2.2), then it takes the following form:

$$u^{\rm re}=u_{\rm p}^{\rm re}=-\frac{(\alpha,\gamma)^\top\cdot d}{\beta\gamma+|\alpha|^2}(\alpha,\beta)^\top e^{{\rm i}(\alpha\cdot x'+\beta x_3)}$$

$$- \frac{1}{\beta\gamma + |\alpha|^2}\Big[(\alpha,\gamma)^\top \times \Big(d \times (\alpha,\beta)^\top\Big)\Big] e^{\mathrm{i}(\alpha\cdot x' + \gamma x_3)}, \tag{5.1}$$

where

$$\alpha = \kappa_{\mathrm{p}}(\sin\theta\cos\varphi, \sin\theta\sin\varphi), \quad \beta = \sqrt{\kappa_{\mathrm{p}}^2 - |\alpha|^2}, \quad \gamma = \sqrt{\kappa_{\mathrm{s}}^2 - |\alpha|^2}.$$

For the shear incident plane wave field (2.3) with $d \cdot d_j^\perp = 0$ ($j = 1, 2$), we have

$$u^{\mathrm{re}} = u^{\mathrm{re}}_{\mathrm{s},j} = -\frac{(\alpha,\gamma)^\top \cdot d_j^\perp}{\beta\gamma + |\alpha|^2}(\alpha,\beta)^\top e^{\mathrm{i}(\alpha\cdot x' + \beta x_3)}$$
$$- \frac{1}{\beta\gamma + |\alpha|^2}\Big[(\alpha,\gamma)^\top \times \Big(d_j^\perp \times (\alpha,\beta)^\top\Big)\Big] e^{\mathrm{i}(\alpha\cdot x' + \gamma x_3)}, \tag{5.2}$$

where

$$\alpha = \kappa_{\mathrm{s}}(\sin\theta\cos\varphi, \sin\theta\sin\varphi), \quad \beta = \sqrt{\kappa_{\mathrm{p}}^2 - |\alpha|^2}, \quad \gamma = \sqrt{\kappa_{\mathrm{s}}^2 - |\alpha|^2}.$$

Thus, if u^{in} takes the general form (2.1), then it follows from the linear superposition principle that the reflected wave field is given by

$$u^{\mathrm{re}}(x) = c_{\mathrm{p}} u^{\mathrm{re}}_{\mathrm{p}}(x) + c_{\mathrm{s},1} u^{\mathrm{re}}_{\mathrm{s},1}(x) + c_{\mathrm{s},2} u^{\mathrm{re}}_{\mathrm{s},2}(x). \tag{5.3}$$

The expressions of (5.1) and (5.2) follow directly from the UPRC (2.13) with $\hat{u}^{\mathrm{re}}(\xi, 0) = -\hat{u}^{\mathrm{in}}(\xi, 0)$. They can be also obtained from the upward Rayleigh expansion (2.20) with $u_n^{\mathrm{sc}}(b) = -u_n^{\mathrm{in}}(b)$ for $n = (0, 0)$ and $u_n^{\mathrm{sc}}(b) = 0$ for $|n| \neq 0$. These analytical solutions in a half-space indicate that, in general case, a compressional (resp. shear) plane wave reflects back to the domain as a sum of both compressional and shear waves.

Below we derive the reflected wave field corresponding to the point source incidence field (2.7) with the source position $y \in \mathbb{R}^3_+$. In this case, the total field $u = u^{\mathrm{in}} + u^{\mathrm{re}}$ coincides with the Green tensor $\mathrm{G_H}(x, y)$ to the first boundary value problem of the Navier in a half-space, i.e., $\mathrm{G_H}(x, y)$ satisfies

$$\begin{aligned} \mu\Delta_y \mathrm{G_H}(x, y) + (\lambda + \mu)\nabla_y\nabla_y \cdot \mathrm{G_H}(x, y) + \omega^2 \mathrm{G_H}(x, y) &= -\delta(x - y)\mathrm{I} \quad && \text{in} \quad U_0, \quad x \neq y, \\ \mathrm{G_H}(x, y) &= 0 && \text{on} \quad \Gamma_0. \end{aligned}$$

Before stating the expression of $\mathrm{G_H}(x, y)$, we introduce the outgoing Kupradze radiation condition for the scattered field u^{sc} in a half-space.

Definition 5.1. An upward radiating solution to the Navier equation (2.5) with $D = U_0$ is said to satisfy the half-space Kupradze radiation condition if its compressional part φ and shear part ψ satisfy the Sommerfeld radiation condition as follows:

$$\begin{aligned} &\varphi(x) = O(r^{-1}), \quad \partial_r\varphi - \mathrm{i}\kappa_{\mathrm{p}}\varphi = o(r^{-1}), \\ &\psi(x) = O(r^{-1}), \quad \partial_r\psi - \mathrm{i}\kappa_{\mathrm{s}}\psi = o(r^{-1}), \end{aligned} \tag{5.4}$$

uniformly in all $x \in \{x \in \mathbb{R}^3 : |x| > R\} \cap U_0$ as $r := |x| \to \infty$.

In the following lemma, G is the free-space Green tensor given by (2.7) and $\tilde{x} = (x', -x_3)$ is the image point of for $x = (x', x_3) \in \mathbb{R}^3$.

Lemma 5.2. *(i) The half-space Green tensor $G_{\mathrm{H}}(\cdot, y)$ $(y_3 > 0)$ can be expressed as*

$$G_{\mathrm{H}}(x, y) = G(x, y) - G(\tilde{x}, y) + U(x, y), \quad x_3 > 0, \quad x \neq y, \tag{5.5}$$

where $U(x, y)$ is given by

$$U(x, y) = \frac{\mathrm{i}}{2\pi\omega^2} \int_{\mathbb{R}^2} \frac{1}{\beta\gamma + |\xi|^2} \Big(\widetilde{M}_{\mathrm{p}}(\xi) e^{\mathrm{i}\xi\cdot(y'-x')} e^{\mathrm{i}\beta y_3} (e^{\mathrm{i}\beta x_3} - e^{\mathrm{i}\gamma x_3}) + \widetilde{M}_{\mathrm{s}}(\xi) e^{\mathrm{i}\xi\cdot(y'-x')} e^{\mathrm{i}\gamma y_3} (e^{\mathrm{i}\beta x_3} - e^{\mathrm{i}\gamma x_3}) \Big) \mathrm{d}\xi$$

with

$$\widetilde{M}_{\mathrm{p}}(\xi) = \begin{bmatrix} \gamma\xi_1^2 & \gamma\xi_1\xi_2 & \xi_1|\xi|^2 \\ \gamma\xi_1\xi_2 & \gamma\xi_2^2 & \xi_2|\xi|^2 \\ \beta\gamma\xi_1 & \beta\gamma\xi_2 & \beta|\xi|^2 \end{bmatrix}, \quad \widetilde{M}_{\mathrm{s}}(\xi) = \begin{bmatrix} -\gamma\xi_1^2 & -\gamma\xi_2^2 & \beta\gamma\xi_1 \\ -\gamma\xi_1\xi_2 & -\gamma\xi_2^2 & \beta\gamma\xi_2 \\ \xi_1|\xi|^2 & \xi_2|\xi|^2 & -\beta|\xi|^2 \end{bmatrix}.$$

(ii) The columns of the matrix function $G_{\mathrm{H}}(x, \cdot)$ and the rows of the matrix function $G_{\mathrm{H}}(\cdot, y)$ satisfy the half-space Kupradze radiation condition.

We remark that the first two terms on the right hand side of (5.5), i.e., $\mathrm{G}(x, y) - \mathrm{G}(\tilde{x}, y)$ does not satisfy the Navier equation in $x_3 > 0$, although it vanishes on $x_3 = 0$. We refer to [4] for the expression of U in two dimensions.

Proof. Since $\mathrm{G}_{\mathrm{H}}(\cdot, \cdot)$ is symmetric, we fix $x_3 > 0$ and take y as the variable in our proof.

(i) Taking the Fourier transform of $g_{\mathrm{p}}(x, y)$ and $g_{\mathrm{s}}(x, y)$ (see (2.8)) with respect to the variable $y' \in \mathbb{R}^2$ gives

$$\hat{g}_{\mathrm{p}}(x, (\xi, y_3)) = \frac{\mathrm{i}}{2\beta} e^{\mathrm{i}\beta|x_3 - y_3|} e^{-\mathrm{i}\xi_1 x_1} e^{-\mathrm{i}\xi_2 x_2}, \quad \hat{g}_{\mathrm{s}}(x, (\xi, y_3)) = \frac{\mathrm{i}}{2\gamma} e^{\mathrm{i}\gamma|x_3 - y_3|} e^{-\mathrm{i}\xi_1 x_1} e^{-\mathrm{i}\xi_2 x_2}.$$

The Dirichlet boundary condition on $y_3 = 0$ gives the relation

$$\begin{aligned} \mathrm{U}(x, y) &= -\mathrm{G}(x, y) + \mathrm{G}(\tilde{x}, y) \\ &= -\frac{1}{\omega^2} \nabla_y \nabla_y^\top (g_{\mathrm{s}}(x, y) - g_{\mathrm{p}}(x, y)) + \frac{1}{\omega^2} \nabla_y \nabla_y^\top (g_{\mathrm{s}}(\tilde{x}, y) - g_{\mathrm{p}}(\tilde{x}, y)) \\ &= \frac{1}{\omega^2} \nabla_y \nabla_y^\top (g_{\mathrm{s}}(\tilde{x}, y) - g_{\mathrm{s}}(x, y)) - \frac{1}{\omega^2} \nabla_y \nabla_y^\top (g_{\mathrm{p}}(\tilde{x}, y) - g_{\mathrm{p}}(x, y)). \end{aligned} \tag{5.6}$$

Therefore, the Fourier transform of $\mathrm{U}(x, y)$ on $y_3 = 0$, which we denote by $\hat{\mathrm{U}}(x, \xi) := (\hat{\mathrm{U}}(x, (\xi, 0))_{ij}$, takes the form

$$\hat{\mathrm{U}}(x,\xi)=\frac{\mathrm{i}}{\omega^2}e^{-\mathrm{i}\xi_1x_1}e^{-\mathrm{i}\xi_2x_2}\Big(e^{\mathrm{i}\beta x_3}-e^{\mathrm{i}\gamma x_3}\Big)V(\xi),\qquad V(\xi):=\begin{bmatrix}0&0&\xi_1\\0&0&\xi_2\\\xi_1&\xi_2&0\end{bmatrix}.$$

Consequently, we have from the UASR (2.13) that

$$\begin{aligned}\mathrm{U}&=\frac{\mathrm{i}}{2\pi\omega^2}\int_{\mathbb{R}^2}\Big\{\frac{1}{\beta\gamma+|\xi|^2}\Big(M_{\mathrm{p}}(\xi)e^{\mathrm{i}\xi\cdot(y'-x')}e^{\mathrm{i}\beta y_3}(e^{\mathrm{i}\beta x_3}-e^{\mathrm{i}\gamma x_3})\\&\quad+M_{\mathrm{s}}(\xi)e^{\mathrm{i}\xi\cdot(y'-x')}e^{\mathrm{i}\gamma(y_3-b)}(e^{\mathrm{i}\beta x_3}-e^{\mathrm{i}\gamma x_3})\Big\}V(\xi)\mathrm{d}\xi\\&=\frac{\mathrm{i}}{2\pi\omega^2}\int_{\mathbb{R}^2}\Big\{\frac{1}{\beta\gamma+|\xi|^2}\Big(\widetilde{M}_{\mathrm{p}}(\xi)e^{\mathrm{i}\xi\cdot(y'-x')}e^{\mathrm{i}\beta y_3}(e^{\mathrm{i}\beta x_3}-e^{\mathrm{i}\gamma x_3})\\&\quad+\widetilde{M}_{\mathrm{s}}(\xi)e^{\mathrm{i}\xi\cdot(y'-x')}e^{\mathrm{i}\gamma y_3}(e^{\mathrm{i}\beta x_3}-e^{\mathrm{i}\gamma x_3})\Big\}\mathrm{d}\xi,\qquad y_3>0,\end{aligned}$$

where M_{p} and M_{s} are given respectively in (2.14) and (2.15), and

$$\widetilde{M}_{\mathrm{p}}(\xi)=M_{\mathrm{p}}(\xi)\,V(\xi)=\begin{bmatrix}\gamma\xi_1^2&\gamma\xi_1\xi_2&\xi_1|\xi|^2\\\gamma\xi_1\xi_2&\gamma\xi_2^2&\xi_2|\xi|^2\\\beta\gamma\xi_1&\beta\gamma\xi_2&\beta|\xi|^2\end{bmatrix},$$

$$\widetilde{M}_{\mathrm{s}}(\xi)=M_{\mathrm{s}}(\xi)\,V(\xi)=\begin{bmatrix}-\gamma\xi_1^2&-\gamma\xi_2^2&\beta\gamma\xi_1\\-\gamma\xi_1\xi_2&-\gamma\xi_2^2&\beta\gamma\xi_2\\\xi_1|\xi|^2&\xi_2|\xi|^2&-\beta|\xi|^2\end{bmatrix}.$$

(ii) To prove the half-space Kupradze radiation condition of $\mathrm{G_H}$, we adopt the two-dimensional arguments of Arens [6, Theorem 4.5]. Let

$$\mathrm{U_p}(x,y)=\frac{\mathrm{i}}{2\pi\omega^2}\int_{\mathbb{R}^2}\frac{1}{\beta\gamma+|\xi|^2}\Big(\widetilde{M}_{\mathrm{p}}(\xi)e^{\mathrm{i}\xi\cdot(y'-x')}e^{\mathrm{i}\beta y_3}(e^{\mathrm{i}\beta x_3}-e^{\mathrm{i}\gamma x_3})\Big)\mathrm{d}\xi,$$

$$\mathrm{U_s}(x,y)=\frac{\mathrm{i}}{2\pi\omega^2}\int_{\mathbb{R}^2}\frac{1}{\beta\gamma+|\xi|^2}\Big(\widetilde{M}_{\mathrm{s}}(\xi)e^{\mathrm{i}\xi\cdot(y'-x')}e^{\mathrm{i}\beta y_3}(e^{\mathrm{i}\gamma x_3}-e^{\mathrm{i}\gamma x_3})\Big)\mathrm{d}\xi.$$

It suffices to verify that U_α ($\alpha=\mathrm{p,s}$) satisfies the Sommerfeld radiation condition specified in Definition 5.1. Note that $(\Delta_y+k_\alpha^2)\mathrm{U}_\alpha(x,y)=0$ for $\alpha=\mathrm{p,s}$ and all $y\in\mathbb{R}^3_+\backslash\{x\}$. Since $\mathrm{U}=\mathrm{U_p}+\mathrm{U_s}$, it follows from (5.6) that

$$\mathrm{U_p}(x,y)=\frac{1}{\omega^2}\nabla_y\nabla_y^\top(g_{\mathrm{s}}(\tilde{x},y)-g_{\mathrm{s}}(x,y))-\frac{1}{\omega^2}\nabla_y\nabla_y^\top(g_{\mathrm{p}}(\tilde{x},y)-g_{\mathrm{p}}(x,y))-\mathrm{U_s}(x,y),\ y_3=0.$$

Direct calculations show that $|g_\alpha(\tilde{x},y)-g_\alpha(x,y)|\le C(1+x_3)(1+y_3)|x-y|^{-2}$ for all $x\neq y$ with $x,y\neq 0$ and $x_3,y_3\geq 0$ and $\alpha=\mathrm{p,s}$. Following the same proof as that in [6, Theorem 2.13] and applying the interior estimate, we obtain

$$w(x,y'):=\mathrm{U_p}(x,y)|_{y_3=0}\le C\,(1+|y'|)^{-2}\quad\text{for some fixed}\quad x\in\mathbb{R}^3_+.\tag{5.7}$$

Recalling the UPRC and ASR for the Helmholtz equation, we have for $y_3 > 0$ that

$$\mathrm{U_p}(x,y) = 2\int_{\Gamma_0} \frac{\partial g_{\mathrm{p}}(y,z)}{\partial z_3} w(x,z')\mathrm{d}s(z') = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{\mathrm{i}\beta(\xi)y_3 + \mathrm{i}\xi\cdot y'}\hat{w}(x,\xi)\mathrm{d}\xi.$$

We can then use the argument in [13, Section 5] and [30, Lemma 2.2 and Corollary 4.1] to conclude that the decay rate of (5.7) ensures the Sommerfeld radiating behavior of $\mathrm{U_p}$ as $|y| \to \infty$ in $y_3 > 0$. The Sommerfeld radiation condition of $\mathrm{U_s}$ can be proceeded analogously. We note that the arguments of [13,30] present the decaying behavior of the scattered field for the two-dimensional acoustic rough surface scattering problems due to a compact source term or a point source incidence and can be readily carried over to the three-dimensional case. □

5.2. Scattering from locally perturbed flat surfaces

In this section we consider the existence of weak solutions for the scattering problem (2.5) and (2.13), where Γ is a locally perturbed flat surface. Without loss of generality, we assume that Γ coincides with the ground plane $\Gamma_0 := \{x_3 = 0\}$ in $|x| > R$ for some $R > \max_{x\in\Gamma}\{x_3\}$. Hence, the domain D above Γ is a locally perturbed half space. In this case, as can be seen from the subsequent subsections, we can propose an equivalent variational formulation in a bounded domain by truncating the unbounded domain D with a transparent boundary condition and then applying the Fredholm alternative. The reduction to a bounded domain can significantly simplify the arguments for globally perturbed scattering problems, where the compact embedding of H^1 into L^2 is in general not valid any more in an unbounded domain.

In the following, we consider to cases:

(i) The perturbation lies entirely below the ground plane, i.e., $\Gamma \cap \{x_3 > 0\} = \emptyset$.
(ii) The perturbation is allowed to occur in the upper half space, i.e., $\Gamma \cap \{x_3 > 0\} \neq \emptyset$.

Note that in the literature on acoustic and electromagnetic wave propagation, Case (i) is referred to as an open cavity scattering problem, whereas Case (ii) is known as an overfilled cavity scattering problem. The above two cases will be investigated in the following two subsections separately. In particular, the existence result of Theorem 5.4 improves the well-posedness of acoustic cavity scattering problems [34], while Theorem 5.6 generalizes the two-dimensional result [23] to three dimensions. Some open questions will be discussed in Remark 5.5.

5.2.1. Case (i): perturbation beneath the ground plane

For simplicity, we assume that Ω is connected. The problem geometry is shown in Fig. 1. If Ω is disconnected, one can apply our variational argument to each connected set of Ω. Let Λ_0 be the aperture of Ω and S be the boundary of Ω in the lower half-space. We have $\partial\Omega = \Lambda_0 \cap S$ and $D = \Omega \cup U_0 \cup \Lambda_0$. Let $\Gamma_0^c = \Gamma_0 \backslash \Lambda_0$ and $\Gamma = S \cup \Gamma_0^c$. We assume that the scattering surface Γ (especially the boundary S) is a Lipschitz continuous surface but not necessary the graph of some function.

Introduce the functional space

$$\tilde{H}^{1/2}(\Lambda_0)^3 = \{v : \tilde{v} \in H^{1/2}(\mathbb{R}^2)^3 \text{ and } \tilde{v} \text{ is the zero extension of } v \text{ from } \Lambda_0 \text{ to } \Gamma_0\}.$$

Denote by $H^{-1/2}(\Lambda_0)^3$ the dual space of $\tilde{H}^{1/2}(\Lambda_0)^3$. Define the Hilbert space

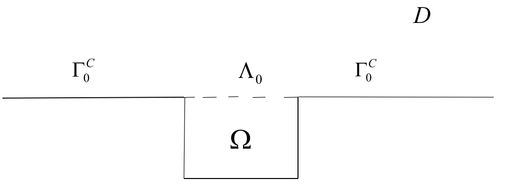

Fig. 1. The problem geometry of a local perturbation of the ground plane which lies entirely in the lower half space.

$$H^1_S(\Omega)^3 = \{u \in H^1(\Omega)^3 : u = 0 \text{ on } S,\ u|_{\Lambda_0} \in \tilde{H}^{1/2}(\Lambda_0)^3\}.$$

Consider a downward propagating pressure wave field of the form

$$u^{\rm in}_{\rm pg}(x) = \int_{\mathbb{R}^2} \frac{1}{\beta\gamma + |\xi|^2} M^{(D)}_{\rm p}(\xi)(\xi, -\beta)^\top g(\xi) e^{{\rm i}(\xi\cdot x' - \beta(x_3 - b))} {\rm d}\xi, \quad x \in S_b, \tag{5.8}$$

where $b > 0$ and g belongs to space of distributions $\mathcal{D}'(\mathbb{R}^2)$ such that ${\rm supp}(g) \subset \{|\xi| < \kappa_{\rm p}\}$. Alternatively, we may consider an incident shear wave field of the form

$$u^{\rm in}_{\rm sg}(x) = \int_{\mathbb{R}^2} \frac{1}{\beta\gamma + |\xi|^2} M^{(D)}_{\rm s}(\xi)((\xi, -\gamma) \times \boldsymbol{q}(\xi))^\top e^{{\rm i}(\xi\cdot x' - \gamma(x_3 - b))} {\rm d}\xi, \quad x \in S_b, \tag{5.9}$$

where $\boldsymbol{q} \in \mathcal{D}'(\mathbb{R}^2)^3$ is a vector distribution such that ${\rm supp}(\boldsymbol{q}) \subset \{|\xi| < \kappa_{\rm s}\}$. Here, the matrices $M^{(D)}_{\rm p}$ and $M^{(D)}_{\rm s}$ are defined in (2.18). It is easy to verify that both $u^{\rm in}_{\rm pg}(x)$ and $u^{\rm in}_{\rm sg}(x)$ satisfy the Navier equation (2.4).

Remark 5.3. We remark that the set of incident compressional (resp. shear) wave fields (5.8) (resp. (5.9)) includes the compressional (resp. shear) plane wave field (2.2) (resp. (2.3)). In fact, since the plane wave fields can be rewritten as

$$u^{\rm in}_{\rm p} = \frac{1}{{\rm i}\kappa_{\rm p}} \nabla e^{{\rm i}\kappa_{\rm p} x\cdot d},\ u^{\rm in}_{{\rm s},j}(x) = \frac{1}{{\rm i}\kappa_{\rm s}} d \times q_j e^{{\rm i}\kappa_{\rm s} x\cdot d} = q_j \nabla \times e^{{\rm i}\kappa_{\rm s} x\cdot d}, \quad j = 1, 2,$$

where q_j $(j = 1, 2)$ are unit vectors in $\mathbb{R}^3$ satisfying $q_1 \cdot q_2 = 0$ and $q_j \cdot d = 0$, it follows from the downward ASR (2.17) that $u^{\rm in}_{\rm p}$ and $u^{\rm in}_{\rm s}$ can be also formulated respectively as the representations (5.8) and (5.9) with

$$g(\xi) = \frac{1}{2\pi\kappa_{\rm p}} \widehat{e^{{\rm i}\kappa_{\rm p} x\cdot d}}(\xi)|_{\Gamma_b} = \frac{e^{{\rm i}\kappa_{\rm p} x_3 b}}{\kappa_{\rm p}} \delta(\xi - \kappa_{\rm p} d'), \quad q_j(\xi) = \frac{e^{{\rm i}\kappa_{\rm s} x_3 b}}{\kappa_{\rm s}} q_j \delta(\xi - \kappa_{\rm s} d').$$

Let $u^{\rm in}$ be an incoming wave field of the form

$$u^{\rm in}(x) = c_{\rm p} u^{\rm in}_{\rm pg}(x) + c_{\rm s} u^{\rm in}_{\rm sg}(x), \quad c_{\rm p}, c_{\rm s} \in \mathbb{C}. \tag{5.10}$$

Multiplying the complex conjugate of a test function $\phi \in H^1_S(\Omega)^3$ on both sides of the Navier equation

$$\mu\Delta u + (\lambda+\mu)\nabla\nabla\cdot u + \omega^2 u = 0 \quad \text{in } \Omega,$$

integrating over Ω and using the integration by part together with the DtN map (3.1), we deduce an equivalent variational problem: find $u \in H_S^1(\Omega)^3$ such that

$$B(u,\phi) = \int_{\Lambda_0} p\cdot\bar{\phi}\,\mathrm{d}x' \quad \forall \phi \in H_S^1(\Omega)^3, \tag{5.11}$$

where $p := Tu^{\mathrm{in}} - \mathcal{T}u^{\mathrm{in}} \in H^{-1/2}(\Lambda_0)^3$ and

$$B(u,\phi) := \int_{\Omega} \big(\mathcal{E}(u,\bar{\phi}) - \omega^2 u\cdot\bar{\phi}\big)\mathrm{d}x - \int_{\Lambda_0} \bar{\phi}\cdot\mathcal{T}\tilde{u}\,\mathrm{d}x'.$$

Note that the symbol $\tilde{f}$ stands for the zero extension of f from Λ_0 to Γ_0. In deriving (5.11), we have used the following identities on Λ_0:

$$\begin{aligned} Tu &= Tu^{\mathrm{sc}} + Tu^{\mathrm{in}} = T\tilde{u}^{\mathrm{sc}} + Tu^{\mathrm{in}} \\ &= \mathcal{T}\tilde{u}^{\mathrm{sc}} + Tu^{\mathrm{in}} = \mathcal{T}\tilde{u} - \mathcal{T}\tilde{u}^{\mathrm{in}} + Tu^{\mathrm{in}} \\ &= \mathcal{T}\tilde{u} - p. \end{aligned}$$

Moreover, using (3.4), we have an explicit form of p:

$$p(x') = \begin{cases} \displaystyle\int_{\mathbb{R}^2} \frac{\mathrm{i}}{\kappa_{\mathrm{p}}} \frac{2\omega^2\beta}{|\xi|^2+\beta\gamma}(-\xi,\gamma)^\top e^{\mathrm{i}\xi\cdot x' + \mathrm{i}\beta b} g(\xi)\mathrm{d}\xi & \text{if} \quad u^{\mathrm{in}} = u^{\mathrm{in}}_{\mathrm{pg}}, \\ \displaystyle\int_{\mathbb{R}^2} \frac{\mathrm{i}}{\kappa_{\mathrm{s}}} \frac{2\omega^2\gamma}{|\xi|^2+\beta\gamma}\boldsymbol{q}(\xi)^\top \times (\xi,-\beta)^\top e^{\mathrm{i}\xi\cdot x' + \mathrm{i}\gamma b}\mathrm{d}\xi & \text{if} \quad u^{\mathrm{in}} = u^{\mathrm{in}}_{\mathrm{sg}}. \end{cases}$$

By the trace theorem $\|u\|_{\tilde{H}^{1/2}(\Lambda_0)^3} \le C\,\|u\|_{H^1(\Omega)^3}$ for all $u \in H_S^1(\Omega)^3$ and the boundedness of the DtN map $\mathcal{T}$ (see the second assertion in Lemma 3.2), there exists a continuous linear operator $\mathcal{B}: H_S^1(\Omega)^3 \to H_S^{-1}(\Omega)^3 := (H_S^1(\Omega)^3)'$ associated with the sesquilinear form B such that

$$B(u,\phi) = (\mathcal{B}u,\phi) \quad \forall\phi \in H_S^1(\Omega)^3.$$

Hence, the variational formulation (5.11) can be rewritten as

$$\mathcal{B}u = \mathcal{F}, \tag{5.12}$$

where $\mathcal{F} \in H_S^{-1}(\Omega)^3$ is defined by the right-hand side of (5.11).

Theorem 5.4. *For incoming wave fields of the form (5.10), there always exists a solution $u \in H_S^1(\Omega)^3$ to the variational problem (5.11). Moreover, this solution can be extended from Ω to D as a solution of the scattering problem (2.5) and (2.13) in $H^1_{loc}(D)$, which can be split as*

$u = u^{\rm in} + u^{\rm re} + v^{\rm sc}$ in D. Here $u^{\rm re}$ is the reflected wave field caused by the rigid ground plane $x_3 = 0$ and $v^{\rm sc}$ satisfies the half-space Kupradze radiation condition (see Definition 5.1).

Proof. We divide the proof into two steps: the first step is to prove the well-posedness of the variational equation (5.12) and the second step is to extend the solution of (5.12) from Ω to D.

Step 1. By the Plancherel identity, we have

$$\begin{aligned}\Re \int_{\Lambda_0} \mathcal{T}\tilde{u} \cdot \bar{u}\mathrm{d}x' &= \Re \int_{\mathbb{R}^2} \mathcal{T}\tilde{u} \cdot \bar{\tilde{u}}\mathrm{d}x' = \Re \int_{\mathbb{R}^2} \widehat{\mathcal{T}\tilde{u}} \cdot \bar{\hat{\tilde{u}}}\mathrm{d}\xi \\ &= \int_{|\xi|>K} \mathrm{i}M(\xi)\hat{\tilde{u}} \cdot \bar{\hat{\tilde{u}}}\mathrm{d}\xi + \int_{|\xi|\le K} \mathrm{i}M(\xi)\hat{\tilde{u}} \cdot \bar{\hat{\tilde{u}}}\mathrm{d}\xi,\end{aligned}$$

where the matrix $M(\xi)$ defined in (3.3) and $K > 0$ is sufficiently large such that $M(\xi)$ is positive definite for all $|\xi| > K$ (see Lemma 3.2). Hence, the above identity implies that

$$-\Re \int_{\Lambda_0} \mathcal{T}\tilde{u} \cdot \bar{u}\mathrm{d}x' \ge -C \int_{|\xi|\le K} |\hat{\tilde{u}}(\xi,0)|^2 \mathrm{d}\xi \ge -C \int_{\mathbb{R}^2} |\hat{\tilde{u}}(\xi,0)|^2 \mathrm{d}\xi = -C\, \|u\|^2_{L^2(\Lambda_0)^3}.$$

Using the inequalities

$$\|u\|^2_{L^2(\Lambda_0)^3} \le \epsilon\, \|u\|^2_{H^1(\Omega)^3} + C_0(\epsilon)\|u\|^2_{L^2(\Omega)^3}, \quad \epsilon > 0$$

and

$$\int_\Omega \mathcal{E}(u,\bar{u})\mathrm{d}x + \int_\Omega |u|^2 \mathrm{d}x \ge C_1(\Omega)\|u\|^2_{H^1(\Omega)^3},$$

we obtain

$$\Re B(u,u) \ge C_2\|u\|^2_{H^1(\Omega)^3} - C_3\|u\|^2_{L^2(\Omega)^3}.$$

Since the injection of $H^1_S(\Omega)^3$ into $L^2(\Omega)^3$ is compact, the above inequality shows that the sesquilinear form B is strongly elliptic and thus the operator $\mathcal{B}$ is Fredholm with index zero. Hence, the operator equation (5.12) is solvable if its right-hand side $\mathcal{F}$ is orthogonal to all solutions $v \in H^1_S(\Omega)^3$ of the homogeneous adjoint equation $\mathcal{B}^* v = 0$. Note that such v satisfies

$$(\mathcal{B}^* v, \phi)_{L^2(\Omega)^3} = (v, \mathcal{B}\phi)_{L^2(\Omega)^3} = \overline{B(\phi, v)} = 0 \quad \forall \phi \in H^1_S(\Omega)^3. \tag{5.13}$$

Furthermore, we can extend $v \in H^1_S(\Omega)^3$ to a solution of the Navier equation (2.5) in the unbounded domain U_0 by setting

$$v(x) = \int_{\mathbb{R}^2} \Big(A_{\rm p}(\xi)(\xi, -\bar{\beta}(\xi))^\top e^{\mathrm{i}(\xi\cdot x' - \bar{\beta}x_3)} + \boldsymbol{A}_{\rm s}(\xi) e^{\mathrm{i}(\xi\cdot x' - \bar{\gamma}z)} \Big)\mathrm{d}\xi, \quad x_3 > 0,$$

where $\boldsymbol{A}_{\rm s}(\xi) \in \mathbb{C}^{3\times 3}$ satisfies the orthogonality relation $\boldsymbol{A}_{\rm s}(\xi) \cdot (\xi, -\bar{\gamma}) = 0$ and

$$\hat{v}(\xi,0)=\begin{bmatrix}\xi_1 & 1 & 0 & 0\\ \xi_2 & 0 & 1 & 0\\ -\bar{\beta} & 0 & 0 & 1\end{bmatrix}\begin{bmatrix}A_{\rm p}(\xi)\\ \boldsymbol{A}_{\rm s}^\top(\xi)\end{bmatrix},\quad \xi\in\mathbb{R}^2.$$

Analogously, we have from Lemma 4.3 and (5.13) that

$$A_{\rm p}(\xi)=0\quad\text{for } |\xi|<\kappa_{\rm p},\qquad \boldsymbol{A}_{\rm s}(\xi)=0\quad\text{for } |\xi|<\kappa_{\rm s}.$$

Hence, if the incident wave field has the form (5.8) with $\mathrm{supp}(g)\subset\{\xi:|\xi|<\kappa_{\rm p}\}$, then

$$\begin{aligned}\mathcal{F}(v)&=\int_{\mathbb{R}^2}\hat{p}\bar{\hat{v}}\mathrm{d}\xi\\ &=\int_{\mathbb{R}^2}\Big(\frac{\mathrm{i}}{\kappa_{\rm p}}\frac{2\omega^2\beta}{|\xi|^2+\beta\gamma}(-\xi,\gamma)^\top g(\xi)\Big)\cdot\Big(\bar{A}_{\rm p}(\xi)(\xi,-\beta)^\top+\bar{\boldsymbol{A}}_{\rm s}(\xi)\Big)\mathrm{d}\xi\\ &=\int_{|\xi|<\kappa_{\rm p}}\Big(\frac{\mathrm{i}}{\kappa_{\rm p}}\frac{2\omega^2\beta}{|\xi|^2+\beta\gamma}(-\xi,\gamma)^\top\Big)\cdot\Big(\bar{A}_{\rm p}(\xi)(\xi,-\beta)^\top+\bar{\boldsymbol{A}}_{\rm s}(\xi)\Big)\mathrm{d}\xi\\ &=0.\end{aligned}$$

Similarly, in the case of (5.9), we have

$$\begin{aligned}\mathcal{F}(v)&=\int_{\mathbb{R}^2}\Big(\frac{\mathrm{i}}{\kappa_{\rm s}}\frac{2\omega^2\gamma}{|\xi|^2+\beta\gamma}\mathbf{q}(\xi)^\top\times(\xi,-\beta)^\top\Big)\cdot\Big(\bar{A}_{\rm p}(\xi)(\xi,-\beta)^\top+\bar{\boldsymbol{A}}_{\rm s}(\xi)\Big)\mathrm{d}\xi\\ &=\int_{|\xi|<\kappa_{\rm s}}\Big(\frac{\mathrm{i}}{\kappa_{\rm s}}\frac{2\omega^2\gamma}{|\xi|^2+\beta\gamma}\mathbf{q}(\xi)^\top\times(\xi,-\beta)^\top\Big)\cdot\Big(\bar{A}_{\rm p}(\xi)(\xi,-\beta)^\top\Big)\mathrm{d}\xi\\ &=0.\end{aligned}$$

Therefore, the right-hand side of (5.12) is always orthogonal to each solution of (5.13). Applying the Fredholm alternative, we obtain the existence of solutions to (5.12).

Step 2. Let $v^{\rm sc}:=u-u^{\rm in}-u^{\rm re}$ in Ω. Let $\tilde{v}^{\rm sc}$ be the zero extension of $v^{\rm sc}|_{\Lambda_0}$ onto Γ_0. Note that the sum of the incident field $u^{\rm in}$ and the reflected field $u^{\rm re}$ vanishes on Γ_0^C. We extend $v^{\rm sc}$ from Ω to D by (2.13) with $b=0$, i.e.,

$$v^{\rm sc}(x)=\frac{1}{2\pi}\int_{\mathbb{R}^2}\Big\{\frac{1}{\beta\gamma+|\xi|^2}\Big(M_{\rm p}(\xi)e^{\mathrm{i}(\xi\cdot x'+\beta x_3)}+M_{\rm s}(\xi)e^{\mathrm{i}(\xi\cdot x'+\gamma x_3)}\Big)\hat{\tilde{v}}^{\rm sc}(\xi,0)\Big\}\mathrm{d}\xi,\quad x\in U_0. \tag{5.14}$$

We claim that the scattered field $v^{\rm sc}$ defined in (5.14) can be represented as

$$v^{\rm sc}(x)=\int_{\Gamma_0}T_y\mathrm{G}_{\rm H}(x,y)v^{\rm sc}(y)\mathrm{d}s(y),\quad x\in U_0, \tag{5.15}$$

where $\mathrm{G_H}(x, y)$ is the half-space Green tensor (see (5.5)) and $T_y\mathrm{G_H}(x, y)$ represents the column-wisely action of the stress operator T to $\mathrm{G_H}(x, y)$ with respect to the variable y. Since the trace of v^{sc} on Γ_0 is compactly supported in Λ_0, by Lemma 5.2, $v^{\rm sc}$ satisfies the half-space Kupradze radiation condition, which completes the proof of the second part of Theorem 5.4.

It remains to prove (5.15). Since $v^{\rm sc}$ has compact support on Γ_0, applying the Fourier transform with respect to y' gives

$$\int_{\Gamma_0} T_y\mathrm{G_H}(x, y)v^{\rm sc}(y)\mathrm{d}s(y) = \int_{\mathbb{R}^2} \widehat{T_y\mathrm{G_H}}(x, (-\xi, 0))\hat{v}^{\rm sc}(\xi)\mathrm{d}\xi.$$

For simplicity, we denote $\widehat{T_y\mathrm{G_H}}(x, (-\xi, 0))$ by $\widehat{T_y\mathrm{G_H}}(x, -\xi)$, which will be calculated as follows. By (5.5),

$$\widehat{T_y\mathrm{G_H}}(x, -\xi) = \widehat{T_y\mathrm{G}}(x, -\xi) + \widehat{T_y\mathrm{G}}(\tilde{x}, -\xi) + \widehat{\mathrm{U}}(x, -\xi).$$

The Fourier transform of $\mathrm{G}(x, y)$ with respect to the variable y' on Γ_0 is

$$\begin{aligned}\hat{\mathrm{G}}(x, \xi, 0) &= \frac{1}{\mu}\hat{g}_{\rm p}(x, \xi, 0)\mathrm{I} \\ &+ \frac{(-\mathrm{i})^2}{\omega^2}\hat{g}_{\rm p}(x, \xi, 0)\begin{bmatrix} \xi_1^2 & \xi_1\xi_2 & \xi_1\beta \\ \xi_1\xi_2 & \xi_2^2 & \xi_2\beta \\ \xi_1\beta & \xi^2\beta & \beta^2 \end{bmatrix} - \frac{(-\mathrm{i})^2}{\omega^2}\hat{g}_{\rm s}(x, \xi, 0)\begin{bmatrix} \xi_1^2 & \xi_1\xi_2 & \xi_1\gamma \\ \xi_1\xi_2 & \xi_2^2 & \xi_2\gamma \\ \xi_1\gamma & \xi^2\gamma & \gamma^2 \end{bmatrix}.\end{aligned}$$

The expression of $\hat{\mathrm{G}}(\tilde{x}, \xi, 0)$ can be obtained analogously. For $x_3 > 0$, the functions $\mathrm{G}(x, \cdot)$ and $\mathrm{G}(\tilde{x}, \cdot)$ propagate downward and upward propagating near Γ_0, respectively. It follows from the downward and upward DtN maps that

$$\widehat{T_y\mathrm{G}}(x, \xi) = \mathrm{i}M^-(\xi)\hat{\mathrm{G}}(x, \xi, 0), \quad \widehat{T_y\mathrm{G}}(\tilde{x}, \xi) = \mathrm{i}M(\xi)\hat{\mathrm{G}}(\tilde{x}, \xi, 0), \tag{5.16}$$

where the matrices M and M^- are given by (3.3) and (3.5), respectively. Moreover, we have from (5.6) that

$$\widehat{T_y\mathrm{U}}(x, \xi) = \frac{\mathrm{i}}{2\pi\omega^2}\frac{e^{-\mathrm{i}\xi\cdot x'}}{\beta\gamma + |\xi|^2}\Big(T_{\rm p}(\xi)\widetilde{M}_{\rm p}(\xi) + T_{\rm s}(\xi)\widetilde{M}_{\rm s}(\xi)\Big)(e^{\mathrm{i}\beta x_3} - e^{\mathrm{i}\gamma x_3}), \tag{5.17}$$

where

$$T_{\rm p}(\xi) := \mathrm{i}\begin{bmatrix} \mu\beta & 0 & \mu\xi_1 \\ 0 & \mu\beta & \mu\xi_2 \\ \lambda\xi_1 & \lambda\xi_2 & (\lambda + 2\mu)\beta \end{bmatrix}, \quad T_{\rm s}(\xi) := \mathrm{i}\begin{bmatrix} \mu\gamma & 0 & \mu\xi_1 \\ 0 & \mu\gamma & \mu\xi_2 \\ \lambda\xi_1 & \lambda\xi_2 & (\lambda + 2\mu)\gamma \end{bmatrix}.$$

Combining (5.16)–(5.17), we obtain after tedious but straightforward calculations that

$$
\begin{aligned}
&\widehat{T_y\mathrm{G}}(x,-\xi)+\widehat{T_y\mathrm{G}}(\tilde{x},-\xi)+\widehat{T_y\mathrm{U}}(x,-\xi)\\
&=\mathrm{i}\,M^{-}(-\xi)\hat{\mathrm{G}}(x,-\xi,0)+\mathrm{i}\,M(-\xi)\hat{\mathrm{G}}(\tilde{x},-\xi,0)\\
&\quad+\frac{\mathrm{i}}{2\pi\omega^2}\frac{e^{\mathrm{i}\xi\cdot x'}}{\beta\gamma+|\xi|^2}\Big(T_{\mathrm{p}}(-\xi)\widetilde{M}_{\mathrm{p}}(-\xi)+T_{\mathrm{s}}(-\xi)\widetilde{M}_{\mathrm{s}}(-\xi)\Big)(e^{\mathrm{i}\beta x_3}-e^{\mathrm{i}\gamma x_3})\\
&=\frac{1}{\beta\gamma+|\xi|^2}\Big(M_{\mathrm{p}}(\xi)e^{\mathrm{i}(\xi\cdot x'+\beta(x_3-b))}+M_{\mathrm{s}}(\xi)e^{\mathrm{i}(\xi\cdot x'+\gamma(x_3-b))}\Big).
\end{aligned}
$$

Furthermore, we obtain from (5.14) that

$$
v^{\mathrm{sc}}(x)=\int_{\mathbb{R}^2}\widehat{T_y\mathrm{G_H}}(x,-\xi)\hat{v}^{\mathrm{sc}}(\xi)\mathrm{d}\xi=\int_{\Gamma_0}T_y\mathrm{G_H}(x,y)v^{\mathrm{sc}}(y)\mathrm{d}s(y),
$$

which completes the proof of (5.15). □

Remark 5.5. We make a few comments on the existence result in Theorem 5.4.

(i) If u^{in} is of the form (5.8), then the reflected wave field $u^{\mathrm{re}}_{\mathrm{pg}}$ is given by (cf. (5.1))

$$
\begin{aligned}
u^{\mathrm{re}}_{\mathrm{pg}}(x)=&-\int_{\mathbb{R}^2}\frac{(\xi,\gamma)^\top\cdot(\xi,-\beta)^\top}{(\beta\gamma+|\xi|^2)^2}M_{\mathrm{p}}(\xi)(\xi,\beta)^\top g(\xi)e^{\mathrm{i}(\xi\cdot x'+\beta(x_3-b))}\mathrm{d}\xi\\
&-\int_{\mathbb{R}^2}\frac{1}{(\beta\gamma+|\xi|^2)^2}M_{\mathrm{p}}(\xi)\Big[(\xi,\gamma)^\top\times\Big((\xi,-\beta)^\top\times(\xi,\beta)^\top\Big)\Big]g(\xi)e^{\mathrm{i}(\xi\cdot x'+\gamma(x_3-b))}\mathrm{d}\xi.
\end{aligned}
$$

If u^{in} is of the form (5.9), then the reflected wave field $u^{\mathrm{re}}_{\mathrm{sg}}$ is given by (cf. (5.2))

$$
\begin{aligned}
u^{\mathrm{re}}_{\mathrm{sg}}(x)=&-\int_{\mathbb{R}^2}\frac{(\xi,\gamma)^\top\cdot((\xi,-\beta)^\top\times\mathbf{q}(\xi))}{(\beta\gamma+|\xi|^2)^2}M_{\mathrm{s}}(\xi)(\xi,\beta)^\top e^{\mathrm{i}(\xi\cdot x'+\beta(x_3-b))}\mathrm{d}\xi\\
&-\int_{\mathbb{R}^2}\frac{1}{(\beta\gamma+|\xi|^2)^2}M_{\mathrm{s}}(\xi)\\
&\times\Big[(\xi,\gamma)^\top\times\Big(((\xi,-\beta)^\top\times\mathbf{q}(\xi))\times(\xi,\beta)^\top\Big)\Big]e^{\mathrm{i}(\xi\cdot x'+\gamma(x_3-b))}\mathrm{d}\xi.
\end{aligned}
$$

Thus, if u^{in} takes the general form (5.10), it follows from the linear superposition that the reflected wave field is given by

$$
u^{\mathrm{re}}(x)=c_{\mathrm{p}}u^{\mathrm{re}}_{\mathrm{pg}}(x)+c_{\mathrm{s}}u^{\mathrm{re}}_{\mathrm{sg}}(x).
$$

(ii) It is unclear whether the solution given by Theorem 5.4 is unique or not. By the proof of Theorem 4.4, the uniqueness is correct if the third component of the normal at the boundary S is non-negative (i.e., $\nu_3\geq 0$). Note that this condition includes interfaces given by step functions and is thus weaker than the assumption used in Section 4.4. For the Helmholtz and Maxwell equations, the well-posedness results have been established for general locally perturbed flat

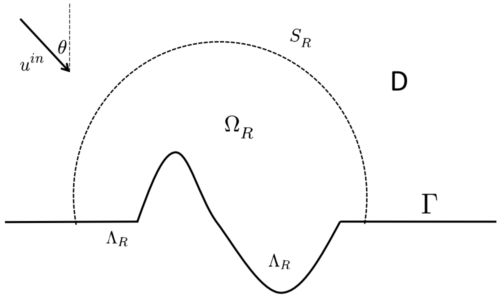

Fig. 2. Geometry of the scattering problem in a locally perturbed half plane.

surfaces which are not necessarily the graph of a function (see [32,35,40]). The arguments rely heavily on properties of the DtN maps derived from the corresponding reflection principle. However, due to the lack of a pointwise reflection principle for the first boundary value problem of the Navier equation, we are not sure whether the DtN approach can be applied to our scattering problem. Thus, we can only obtain the existence result in the general case.

(iii) The result in Theorem 5.4 improves the acoustic and electromagnetic counterparts in the following sense. First, it shows that the existence results can be verified for general incoming waves from the upper half-space even if the uniqueness is unknown. One can expect the same conclusion for acoustic and electromagnetic transmission problems. Second, the split of u^{sc} into the sum $u^{\mathrm{re}} + v^{\mathrm{sc}}$ is rigorously justified under the mild assumption that u^{sc} satisfies the UASR (2.13).

5.2.2. *Case (ii): perturbation above the ground plane*

In this subsection, we consider the scattering surface $\Gamma = \{x \in \mathbb{R}^3 : x_3 = f(x'),\ x' \in \mathbb{R}^2\}$, where f is a Lipschitz continuous function and is assumed to satisfy $f(x') = 0$ when $|x'| > R$ for some $R > 0$. This means that Γ is a local perturbation of the ground plane $x_3 = 0$. The problem geometry is shown in Fig. 2. Let $D = \{x \in \mathbb{R}^3 : x_3 > f(x'),\ x' \in \mathbb{R}^2\}$ and $\Lambda_R := \Gamma \cap \{x : |x'| \le R\}$, which contains the perturbed part of Γ. Denote by $\Omega_R = \{x \in D : |x| < R\}$ the truncated bounded domain and by $B_R^+ = \{x \in \mathbb{R}^3 : |x| < R,\ x_3 > 0\}$ the upper half-sphere. Let $S_R = \{x \in D : |x| = R\}$ and denote by ν the unit normal vector on S_R, pointing into the exterior of Ω_R. Obviously, $\partial\Omega_R = \Lambda_R \cup S_R$.

Let u^{in} be the incident elastic plane wave field (2.1). Due to the local perturbation, we assume that the scattered field $u^{\mathrm{sc}} = u^{\mathrm{re}} + v^{\mathrm{sc}}$ can be further decomposed into the sum of the reflected wave fields u^{re} and v^{sc}, where u^{re} is the reflected field of the form (5.3) solving the unperturbed scattering problem and v^{sc} satisfies the outgoing Kupradze radiation condition as defined in Definition 5.1.

Define the Sobolev space $X_R = \{v \in H^1(\Omega_R)^3 : v = 0 \text{ on } \Lambda_R\}$ and denote by X_R^{-1} the dual space of X_R. Introduce the Sobolev spaces on the open surface (see e.g., [38]):

$$H^{1/2}(S_R)^3 := \{u|_{S_R} : u \in H^{1/2}(\partial\Omega_R)^3\}, \quad \tilde{H}^{1/2}(S_R)^3 := \{u \in H^{1/2}(\partial\Omega_R)^3 : \mathrm{supp}(u) \subset S_R\}.$$

Denote by $H^{-1/2}(S_R)^3$ the dual space of $\tilde{H}^{1/2}(S_R)^3$ and by $\tilde{H}^{-1/2}(S_R)^3$ the dual space of $H^{1/2}(S_R)^3$.

Next, we introduce the generalized stress (or traction) operator and the corresponding bilinear form

$$T_{a,b}u = (\mu + a)\partial_\nu u + b\nu\nabla\cdot u + a\nu\times(\nabla\times u),$$

$$\mathcal{E}(u,w) = (\mu + a)\sum_{j,k=1}^{3}\partial_k u_j \partial_k w_j + b(\nabla\cdot u)(\nabla\cdot w) - a(\nabla\times u)\cdot(\nabla\times w),$$

where $a, b \in \mathbb{R}$ satisfying $a + b = \lambda + \mu$. Throughout this section, we choose

$$a = \frac{\mu(\lambda+\mu)}{\lambda+3\mu}, \quad b = \frac{(\lambda+\mu)(\lambda+2\mu)}{\lambda+3\mu}.$$

The above choice of a and b yields a compact double layer operator $\mathcal{D}$ with a weakly singular kernel (see [29]) as defined below in (5.21). For simplicity we still denote $T_{a,b}$ by T_ν, which is called the pseudo stress operator [29] with the new choice of a and b. Note that the usual stress operator corresponds to $a = \mu$ and $b = \lambda$ and the Betti's formula are still valid for the new choice, i.e.,

$$\int_{\Omega_R} \big(\mathcal{E}(u,v) - \omega^2 u\cdot v\big)\mathrm{d}x - \int_{S_R} v\cdot T_\nu u\,\mathrm{d}s = 0. \tag{5.18}$$

By applying Green's formula and the half-plane Kupradze radiation condition, it is easy to derive the Green representation formula for the scattered wave field v^{sc}:

$$v^{\mathrm{sc}}(x) = \int_{S_R} T_{\nu(y)}\mathrm{G_H}(x,y)\cdot v^{\mathrm{sc}}(y) - \mathrm{G_H}(x,y)\cdot T_{\nu(y)}v^{\mathrm{sc}}(y)\mathrm{d}s(y), \quad x\in D\backslash\overline{\Omega}_R. \tag{5.19}$$

Taking the limit $x \to S_R$ in (5.19) and setting $p = T_\nu v^{\mathrm{sc}}|_{S_R} \in H^{-1/2}(S_R)^3$, we obtain

$$(\frac{1}{2}\mathcal{I} - \mathcal{D})(v^{\mathrm{sc}}|_{S_R}) + \mathcal{S}p = 0 \quad \text{on } S_R. \tag{5.20}$$

Here $\mathcal{I}$ is the identity operator, $\mathcal{D}$ and $\mathcal{S}$ are the double-layer and single-layer operators over S_R, respectively, and are defined by

$$(\mathcal{D}g)(x) = \int_{S_R} T_{\nu(y)}\mathrm{G_H}(x,y)g(y)\mathrm{d}s(y), \quad (\mathcal{S}g)(\boldsymbol{x}) = \int_{S_R}\mathrm{G_H}(x,y)g(y)\mathrm{d}s(y). \tag{5.21}$$

Combining (5.18) and (5.20) yields the variational formulation for the unknown solution pair $(u,p) \in X_R \times H^{-1/2}(S_R)^3 := X$ as follows:

$$\mathbb{B}((u,p),(\varphi,\chi)) = \begin{bmatrix} b_1((u,p),(\varphi,\chi)) \\ b_2((u,p),(\varphi,\chi)) \end{bmatrix} = \begin{bmatrix} \int_{S_R} T_\nu u_0\cdot\overline{\varphi}\mathrm{d}s \\ \int_{S_R}(\frac{1}{2}\mathcal{I} - \mathcal{D})(u_0|_{S_R})\cdot\overline{\chi}\mathrm{d}s \end{bmatrix} \tag{5.22}$$

for all $(\varphi,\chi) \in X$, where $u_0 = u^{\mathrm{in}} + u^{\mathrm{re}}$ is the reference field and

$$b_1((u,p),(\varphi,\chi)) = \int\limits_{\Omega_R} \left(\mathcal{E}(u,\overline{\varphi}) - \omega^2 u\cdot\overline{\varphi}\right)\mathrm{d}x - \int\limits_{S_R} \overline{\varphi}\cdot p\,\mathrm{d}s,$$

$$b_2((u,p),(\varphi,\chi)) = \int\limits_{S_R} \left((\frac{1}{2}\mathcal{I} - \mathcal{D})(u|_{S_R}) + \mathcal{S}p\right)\overline{\chi}\,\mathrm{d}s.$$

The Fredholm property of the sesquilinear form $\mathbb{B}$ can be proved by following almost the same lines in [31]. To prove the uniqueness, one has to assume that ω^2 is not a Dirichlet eigenvalue of the operator $-(\mu\Delta + (\lambda+\mu)\nabla(\nabla\cdot))$ over Ω_R. This assumption implies the equivalence of the variational problem (5.22) posed on Ω_R and our scattering problem in D. As a consequence of Theorem 4.4, one obtains the uniqueness. We refer to [31] for the details and only state the well-posedness results below.

Theorem 5.6. *Assume that ω^2 is not a Dirichlet eigenvalue of the operator $-(\mu\Delta + (\lambda+\mu)\nabla(\nabla\cdot))$ over Ω_R. Then, there exists a unique solution $u \in X_R$ to the variational formulation (5.22). Moreover, one may extend $v^{sc} := u - u^{in} - u^{sc}$ from Ω_R to $D\backslash\overline{\Omega}_R$ through (5.19) and the extended solution satisfies the radiation solution (5.4).*

Remark 5.7. We make some comments on the well-posedness results in Theorem 5.6.

(i) In contrast with Theorem 5.4, Theorem 5.6 is justified under the strong assumption that $u = u^{in} + u^{re} + v^{sc}$ where v^{sc} satisfies the half-plane Kupradze radiation condition; see (5.19) where this assumption was used. This automatically implies that $u - u^{in}$ fulfills the weaker radiation condition UPRC (2.13). We refer to Remark (5.5) (ii) for the reason why we cannot prove the uniqueness for non-graph scattering surfaces.

(ii) By the proof of Theorem 5.6, one can discuss the well-posedness of the elastic scattering from a trapezoidal surface, which is a non-local perturbation of flat surfaces. This requires a modified radiating assumption on $u - u^{in}$ which depends on both the incident wave and the scattering surface; see [37] for the acoustic scattering problem with a trapezoidal sound-soft curve.

Now we consider the boundary value problem in a locally perturbed half-space:

$$\mu\Delta v + (\lambda+\mu)\nabla(\nabla\cdot v) + \omega^2 v = 0 \quad \text{in } D, \qquad v = h \quad \text{on } \Gamma, \tag{5.23}$$

where $h \in (H^{1/2}(\Gamma))^3$ and v is required to satisfy the UPRC (2.13) in $x_3 > 0$. We can always find a function $h_0 \in (H^{1/2}(\Gamma_0))^3$ such that $h_0 = h$ in $\Gamma \cap \{x : |x| > R\}$ for the R specified at the beginning of this subsection. Let

$$v_0(x) = \frac{1}{2\pi}\int\limits_{\mathbb{R}^2} \left\{\frac{1}{\beta\gamma + |\xi|^2}\left(M_p(\xi)e^{i(\xi\cdot x' + \beta x_3)} + M_s(\xi)e^{i(\xi\cdot x' + \gamma x_3)}\right)\hat{h}_0(\xi)\right\}\mathrm{d}\xi, \quad x \in D.$$

Then $v_0 \in H^1(\tilde{S}_b)^3$ for any $b > 0$ with strip $\tilde{S}_b := \{x : 0 < |x_3| < b\}$ and it is an upward propagating solution to the Navier equation with the Dirichlet data $v_0 = h_0$ on $x_3 = 0$. By Sobolev extension theorem (see e.g., [27, Theorem 7.25]), v_0 can be extended to a function $v_1 \in H^1(S_b)$ from $x_3 > 0$ to D such that $v_1 \equiv v_0$ in $x_3 > 0$. Defining $w_1 = v - v_1$, we deduce that

$$\mu\Delta w_1 + (\lambda+\mu)\nabla(\nabla\cdot w_1) + \omega^2 w_1 = f_1 \quad \text{in } D, \qquad w_1 = h_1 \quad \text{on } \Gamma,$$

where $f_1 \in (H^1(\Omega_R))'$ is compactly supported in $D \cap \{x_3 < 0\}$ and $h_1 \in H^{1/2}(\Gamma)$ is compactly supported in Λ_R. Finally, by a lifting argument, the previous problem can be reduced to a homogeneous boundary value problem

$$\mu\Delta w_2 + (\lambda+\mu)\nabla(\nabla\cdot w_2) + \omega^2 w_2 = f_2 \quad \text{in } D, \qquad w_2 = 0 \quad \text{on } \Gamma,$$

with $f_2 \in (H^1(\Omega_R))'$ compactly supported in Ω_R, where $w_2 = w_1 - v_2$ for some $v_2 \in H^1(S_b)^3$ ($b > 0$) such that $v_2 \equiv h_1$ on Γ and $v_2 \equiv 0$ in $x_3 > 2R$. Choose $R > 0$ such that ω^2 is not a Dirichlet eigenvalue of the operator $-(\mu\Delta + (\lambda+\mu)\nabla(\nabla\cdot))$ over Ω_R. Then the above inhomogeneous source problem can be equivalently formulated as the variational problem:

$$\mathbb{B}((w_2, p), (\varphi, \chi)) = \begin{bmatrix} \int_{\Omega_R} f_2 \cdot \overline{\varphi} \mathrm{d}x \\ 0 \end{bmatrix}, \qquad p := T_\nu w_2|_{S_R} \in H^{-1/2}(S_R)^3, \ \forall (\varphi, \chi) \in X.$$

By the proof of Theorem 5.6, there exists a unique solution $w_2 \in H^1(\Omega_R)^3$, which can be extended to a Sommerfeld radiating solution in $D \cap \{|x| > R\}$. We summarize the solvability result as follows.

Corollary 5.8. *The boundary value problem* (5.23) *admits a unique upward propagating solution* $v = \tilde{v} + w_2 \in H^1(S_b)^3$ *for any* $b > 0$*, where* $\tilde{v}$ *satisfies the UASR (2.13) and* w_2 *satisfies the half-space Kupradze radiation condition.*

6. Concluding remarks

We have presented the mathematical formulation of time-harmonic elastic scattering from general unbounded rough surfaces in three dimensions. In particular, the ASR in a half-space is derived and properties of the DtN map are analyzed. The uniqueness is proved for the Lipschitz continuous rough surface which is given by the graph of a function. We deduce the Green tensor for the first boundary value problem of the Navier equation in a half-space. The existence of weak solution to locally perturbed scattering problem is established by applying the Fredholm alternative to an equivalent variational formulation in a truncated bounded domain.

Below we list three interesting questions for locally perturbed scattering problems which deserve to be further investigated.

- The uniqueness result for perturbations given by non-graph functions.
- Equivalent variational formulation in a bounded domain without the coupling scheme between the finite element method and the integral representation. In particular, a numerical scheme avoiding the half-space Green tensor and involving the free-space's tensor only would be desirable from the numerical point of view.
- Explicit dependence of the solution on the frequency of incidence in linear elasticity. The variational approach developed [12] leads to an explicit wavenumber dependence of solutions to the acoustic rough surface scattering problems. However, the derivation of such kind of estimates relies on the positivity of the real part of the DtN map (see [12, Lemma 3.2]), which unfortunately is not applicable to the Navier equation.

Based on the framework presented in this work, we plan to carry out the study of the elastic scattering from globally perturbed (non-periodic) rough surfaces, for example, due to an inhomogeneous elastic source term or an incoming point source incidence. This will extend at least the acoustic results of [12] and [8] in weighted and non-weighted Sobolev spaces to linear elasticity in three dimensions. In particular, the absence of elastic surface waves can be proved as a consequence of well-posedness in weighted Sobolev spaces.

References

[1] I. Abubakar, Scattering of plane elastic waves at rough surface I, Proc. Camb. Philos. Soc. 58 (1962) 136–157.
[2] J.D. Achenbach, Wave Propagation in Elastic Solids, North Holland, Amsterdam, 1973.
[3] T. Arens, The scattering of plane elastic waves by a one-dimensional periodic surface, Math. Methods Appl. Sci. 22 (1999) 55–72.
[4] T. Arens, Uniqueness for elastic wave scattering by rough surfaces, SIAM J. Math. Anal. 33 (2001) 461–476.
[5] T. Arens, Existence of solution in elastic wave scattering by unbounded rough surfaces, Math. Methods Appl. Sci. 25 (2002) 507–528.
[6] T. Arens, The Scattering of Elastic Waves by Rough Surfaces, Ph.D. thesis, Brunel University, Uxbridge, UK, 2000.
[7] G. Bao, G. Hu, T. Yin, Time-harmonic acoustic scattering from locally perturbed half-planes, SIAM J. Appl. Math. 78 (2018) 2672–2691.
[8] S.N. Chandler-Wilde, J. Elschner, Variational approach in weighted Sobolev spaces to scattering by unbounded rough surfaces, SIAM J. Math. Anal. 42 (2010) 2554–2580.
[9] A. Charalambopoulos, D. Gintides, K. Kiriaki, Radiation conditions for rough surfaces in linear elasticity, Q. J. Mech. Appl. Math. 55 (2002) 421–441.
[10] S.N. Chandler-Wilde, C. Ross, Scattering by rough surfaces: the Dirichlet problem for the Helmholtz equation in a non-locally perturbed half-plane, Math. Methods Appl. Sci. 19 (1996) 959–976.
[11] S.N. Chandler-Wilde, B. Zhang, A uniqueness result for scattering by infinite rough surfaces, SIAM J. Appl. Math. 58 (1998) 1774–1790.
[12] S.N. Chandler-Wilde, P. Monk, Existence, uniqueness, and variational methods for scattering by unbounded rough surfaces, SIAM J. Math. Anal. 37 (2005) 598–618.
[13] S.N. Chandler-Wilde, C.R. Ross, B. Zhang, Scattering by rough surfaces, in: Mathematical and Numerical Aspects of Wave Propagation, Golden, CO, 1998, SIAM, Philadelphia, PA, 1998, pp. 164–168.
[14] S.N. Chandler-Wilde, E. Heinemeyer, R. Potthast, A well-posed integral equation formulation for three dimensional rough surface scattering, Proc. R. Soc. A 462 (2006) 3683–3705.
[15] S.N. Chandler-Wilde, B. Zhang, Scattering of electromagnetic waves by rough surfaces and inhomogeneous layers, SIAM J. Math. Anal. 30 (1999) 559–583.
[16] P. Cummings, X. Feng, Sharp regularity coefficient estimates for complex-valued acoustic and elastic Helmholtz equations, Math. Models Methods Appl. Sci. 16 (2006) 139–160.
[17] A. Kirsch, Diffraction by periodic structures, in: L. Päivärinta, et al. (Eds.), Proc. Lapland Conf. Inverse Problems, Springer, Berlin, 1993, pp. 87–102.
[18] V.D. Kupradze, et al., Three-Dimensional Problems of the Mathematical Theory of Elasticity and Thermoelasticity, North-Holland, Amsterdam, 1979.
[19] Y. Dermenjian, J.C. Guillot, Scattering of elastic waves in a perturbed isotropic half space with a free boundary. The limiting absorption principle, Math. Methods Appl. Sci. 10 (1988) 87–124.
[20] M. Durán, I. Muga, J.-C. Nédélec, The Helmholtz equation in a locally perturbed half-space with non-absorbing boundary, Arch. Ration. Mech. Anal. 191 (2009) 143–172.
[21] M. Durán, I. Muga, J.-C. Nédélec, The outgoing time-harmonic elastic wave in a half-plane with free boundary, SIAM J. Appl. Math. 71 (2011) 443–464.
[22] J. Elschner, G. Hu, Scattering of plane elastic waves by three-dimensional diffraction gratings, Math. Models Methods Appl. Sci. 22 (2012) 1150019.
[23] J. Elschner, G. Hu, Elastic scattering by unbounded rough surfaces, SIAM J. Math. Anal. 44 (2012) 4101–4127.
[24] J. Elschner, G. Hu, Elastic scattering by unbounded rough surfaces: solvability in weighted Sobolev spaces, Appl. Anal. 94 (2015) 251–278.
[25] J. Elschner, G. Hu, Variational approach to scattering of plane elastic waves by diffraction gratings, Math. Methods Appl. Sci. 33 (2010) 1924–1941.

[26] J. Elschner, M. Yamamoto, An inverse problem in periodic diffractive optics: reconstruction of Lipschitz grating profiles, Appl. Anal. 81 (2002) 1307–1328.
[27] D. Gilbarg, N. Trudinger, Elliptic Partial Differential Equations of Second Order, 2nd ed., Springer, 1983.
[28] H. Haddar, A. Lechleiter, Electromagnetic wave scattering from rough penetrable layers, SIAM J. Math. Anal. 43 (2011) 2418–2443.
[29] P. Hähner, On Acoustic, Electromagnetic, and Elastic Scattering Problems in Inhomogeneous Media, Habilitationsschrift, Göttingen, 1998.
[30] G. Hu, A. Rathsfeld, Acoustic scattering from locally perturbed periodic surfaces, WIAS Preprint 2522, 2018.
[31] G. Hu, X. Yuan, Y. Zhao, Direct and inverse elastic scattering from a locally perturbed half-plane, Commun. Math. Sci. 16 (2018) 1635–1658.
[32] P. Li, Coupling of finite element and boundary integral methods for electromagnetic scattering in a two-layered medium, J. Comput. Phys. 229 (2010) 481–497.
[33] P. Li, A survey of open cavity scattering problems, J. Comput. Math. 36 (2018) 1–16.
[34] P. Li, A. Wood, A two-dimensional Helmholtz equation solution for the multiple cavity scattering problem, J. Comput. Phys. 240 (2013) 100–120.
[35] P. Li, H. Wu, W. Zheng, An overfilled cavity problem for Maxwell's equations, Math. Methods Appl. Sci. 35 (2012) 1951–1979.
[36] P. Li, G. Zheng, W. Zheng, Maxwell's equations in an unbounded structure, Math. Methods Appl. Sci. 40 (2017) 573–588.
[37] W. Lu, G. Hu, Time-harmonic acoustic scattering from a non-locally perturbed trapezoidal surface, SIAM J. Sci. Comput. 41 (2019) B522–B544.
[38] W. Mclean, Strongly Elliptic Systems and Boundary Integral Equations, Cambridge University Press, Cambridge, 2010.
[39] J. Nečas, Les Méthodes Directes en Théorie des Équations Elliptiques, Masson, 1967.
[40] A. Wood, Analysis of electromagnetic scattering from an overfilled cavity in the ground plane, J. Comput. Phys. 215 (2006) 630–641.
[41] B. Zhang, S.N. Chandler-Wilde, Acoustic scattering by an inhomogeneous layer on a rigid plate, SIAM J. Appl. Math. 58 (1998) 1931–1950.