

# Computation of interior elastic transmission eigenvalues using a conforming finite element and the secant method



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## ABSTRACT

The interior elastic transmission eigenvalue problem, arising from the inverse scattering theory of non-homogeneous elastic media, is nonlinear, non-self-adjoint and of fourth order. This paper proposes a numerical method to compute real elastic transmission eigenvalues. To avoid treating the non-self-adjoint operator, an auxiliary nonlinear function is constructed. The values of the function are generalized eigenvalues of a series of self-adjoint fourth order problems. The roots of the function are the transmission eigenvalues. The self-adjoint fourth order problems are computed using the  $H^2$ -conforming Argyris element. The secant method is employed to search the roots of the nonlinear function. The convergence of the proposed method is proved.

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## 1. Introduction

Inverse elastic scattering theory became an active research area recently (see, e.g., [1] and the references therein). In this paper, we consider the computation of the interior elastic transmission eigenvalue problem (ETE) arising from the inverse scattering theory of non-homogeneous elastic media. Similar to the acoustic and electromagnetic transmission eigenvalue problems [2], the ETE plays a critical role in the qualitative reconstruction methods for inhomogeneous media. The ETE is a new nonlinear non-self-adjoint eigenvalue problem and its theory is far from complete [3].

Numerical methods for the acoustic transmission eigenvalues have been developed by many researchers since 2010 (see, e.g., [4–18]). There exist some papers on the electromagnetic transmission eigenvalue problems [19–21]. It is non-trivial to develop finite element methods for the transmission eigenvalue problems since they are nonlinear, non-self-adjoint and of high order [22]. There are a few papers on the computation of the ETE. In [23], the authors proposed a discontinuous Galerkin method based on a mixed formulation. Recently, Chang et al. implemented an efficient quadratic Jacobi–Davidson method combining with partial locking or partial deflation techniques to compute many positive eigenvalues [24]. We also refer the readers to the two recent papers appeared on arXiv [25,26].

The goal of this paper is to develop an effective numerical method to compute real transmission eigenvalues. Real transmission eigenvalues can be reconstructed from the scattered waves and used to estimate material property of the elastic body (see, e.g., [27]). It is shown in [3] that there exists a countable set of real elastic transmission eigenvalues. The problem of the existence of complex elastic transmission eigenvalues is largely open. Unlike the Laplacian eigenvalue problem or the biharmonic eigenvalue problem, the transmission eigenvalue problem is nonlinear and non-self-adjoint. To overcome these difficulties, we reformulate the ETE as a problem to seek the root of a nonlinear function. Specifically,

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following the idea of [5] for the acoustic transmission eigenvalue problem, the ETE is first written as a nonlinear fourth order eigenvalue problem. Then a nonlinear function, whose roots are the real elastic transmission eigenvalues, is constructed. The values of the nonlinear function are generalized eigenvalues of some self-adjoint coercive fourth order problems, which can be treated using the classical  $H^2$ -conforming finite elements. Finally, the secant method is used to compute the roots of the nonlinear function.

The rest of the paper is organized as follows. In Section 2, we introduce the elastic transmission eigenvalue problem and derive a quadratic eigenvalue problem based on a fourth order partial differential equation. To avoid treating the nonlinearity and non-self-adjointness directly, the ETE is decomposed into a nonlinear function and a series of linear self-adjoint fourth order eigenvalue problems. The values of the nonlinear function are generalized eigenvalues of the fourth order problems. The roots of the nonlinear function are transmission eigenvalues. In Section 3, the  $H^2$ -conforming Argyris element for the fourth order problems is presented and the convergence is proved. The secant method is used in Section 4 to compute roots of the nonlinear function and the error estimate is obtained. Some preliminary numerical examples are presented in Section 5.

## 2. The elastic transmission eigenvalue problem

Let  $\mathbf{x} = (x, y)^\top \in \mathbb{R}^2$  and  $D \subset \mathbb{R}^2$  be a bounded Lipschitz polygon. The elastic wave equation is

$$\nabla \cdot \sigma(\mathbf{u}) + \omega^2 \rho \mathbf{u} = \mathbf{0} \quad \text{in } D \subset \mathbb{R}^2, \quad (1)$$

where  $\mathbf{u}(\mathbf{x}) = (u_1(\mathbf{x}), u_2(\mathbf{x}))^\top$  is the displacement vector of the wave field,  $\omega > 0$  is the angular frequency,  $\rho(\mathbf{x})$  is the mass density, and  $\sigma(\mathbf{u})$  is the stress tensor given by the generalized Hooke law

$$\sigma(\mathbf{u}) = 2\mu \varepsilon(\mathbf{u}) + \lambda \text{tr}(\varepsilon(\mathbf{u}))\mathbf{I},$$

where  $\mathbf{I} \in \mathbb{R}^{2 \times 2}$  is the identity matrix and  $\mu, \lambda$  are the Lamé parameters satisfying  $\mu > 0, \lambda + \mu > 0$ . The strain tensor  $\varepsilon(\mathbf{u})$  is defined as

$$\varepsilon(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^\top),$$

where  $\nabla \mathbf{u}$  is the displacement gradient tensor

$$\nabla \mathbf{u} = \begin{bmatrix} \partial_x u_1 & \partial_y u_1 \\ \partial_x u_2 & \partial_y u_2 \end{bmatrix}.$$

Explicitly, we have

$$\sigma(\mathbf{u}) = \begin{bmatrix} (\lambda + 2\mu)\partial_x u_1 + \lambda\partial_y u_2 & \mu(\partial_y u_1 + \partial_x u_2) \\ \mu(\partial_x u_2 + \partial_y u_1) & \lambda\partial_x u_1 + (\lambda + 2\mu)\partial_y u_2 \end{bmatrix}. \quad (2)$$

Given  $\mathbf{u}, \mathbf{v} \in H_0^1(D)^2$ , it follows from the integration by parts that

$$\begin{aligned} (\sigma(\mathbf{u}), \nabla \mathbf{v}) &= \int_D \sigma(\mathbf{u}) : \nabla \mathbf{v} \, d\mathbf{x} \\ &= \int_D (2\mu \varepsilon(\mathbf{u}) : \varepsilon(\mathbf{v}) + \lambda(\nabla \cdot \mathbf{u})(\nabla \cdot \mathbf{v})) \, d\mathbf{x}, \end{aligned} \quad (3)$$

where  $A : B = \text{tr}(AB^\top)$  is the Frobenius inner product of square matrices  $A$  and  $B$ . We recall the first Korn inequality [28, Corollary 11.2.25]: there exists a positive constant  $C$  such that

$$\|\varepsilon(\mathbf{u})\|_{L^2} \geq C \|\mathbf{u}\|_{H^1} \quad \text{for all } \mathbf{u} \in H_0^1(D)^2.$$

Let  $\rho_0, \rho_1 \in L^2(D)$  be the mass density of the background medium and the mass density of  $D$ , respectively. In this paper, we consider the case when  $\rho_1 > \rho_0$  such that

$$p \leq \rho_0(\mathbf{x}) \leq P, \quad p_* \leq \rho_1(\mathbf{x}) \leq P_*, \quad \mathbf{x} \in D, \quad (4)$$

where  $p, p_*, P, P_*$  are positive constants and  $p_* > P$ .

The elastic transmission eigenvalue problem is to find  $\omega^2$  such that there exists a non-trivial solution  $(\mathbf{w}, \mathbf{v})$  satisfying

$$\nabla \cdot \sigma(\mathbf{w}) + \omega^2 \rho_0 \mathbf{w} = \mathbf{0} \quad \text{in } D, \quad (5a)$$

$$\nabla \cdot \sigma(\mathbf{v}) + \omega^2 \rho_1 \mathbf{v} = \mathbf{0} \quad \text{in } D, \quad (5b)$$

$$\mathbf{w} = \mathbf{v} \quad \text{on } \Gamma, \quad (5c)$$

$$\sigma(\mathbf{w})\mathbf{v} = \sigma(\mathbf{v})\mathbf{v} \quad \text{on } \Gamma, \quad (5d)$$

where  $\sigma(\mathbf{w})\mathbf{v}$  denotes the matrix multiplication of the stress tensor  $\sigma(\mathbf{w})$  and the unit outward normal  $\mathbf{v}$  to  $\Gamma := \partial D$ . Assume that  $p_* \geq 1 \geq P$ . It is shown in [3] that the set of elastic transmission eigenvalues is discrete, with infinity being the only possible accumulation point.

Since  $\omega$  is the angular frequency in the elastic waves, the goal of this paper is to develop an effective numerical method to compute real transmission eigenvalues which are physically meaningful. To this end, we first rewrite (5) as a fourth order problem. Define the Sobolev space

$$V = \{\boldsymbol{\phi} \in H^2(D)^2 : \boldsymbol{\phi} = \mathbf{0} \text{ and } \sigma(\boldsymbol{\phi})\mathbf{v} = \mathbf{0} \text{ on } \Gamma\}. \tag{6}$$

Subtracting (5b) from (5a) and rearranging terms, one obtains

$$\nabla \cdot \sigma(\mathbf{w} - \mathbf{v}) + \omega^2 \rho_0(\mathbf{w} - \mathbf{v}) = \omega^2(\rho_1 - \rho_0)\mathbf{v}.$$

Applying (5b) once again, one gets

$$(\nabla \cdot \sigma + \omega^2 \rho_1)(\rho_1 - \rho_0)^{-1} (\nabla \cdot \sigma + \omega^2 \rho_0)(\mathbf{w} - \mathbf{v}) = \mathbf{0}.$$

Let  $\mathbf{u} = \mathbf{w} - \mathbf{v}$ . From (5c) and (5d), one has that

$$\mathbf{u} = \mathbf{0} \quad \text{and} \quad \sigma(\mathbf{u})\mathbf{v} = \mathbf{0} \quad \text{on } \Gamma.$$

Consequently, the transmission eigenvalue problem can be formulated as follows. Find  $\omega^2$  and  $\mathbf{u} \neq \mathbf{0}$  such that

$$(\nabla \cdot \sigma + \omega^2 \rho_1)(\rho_1 - \rho_0)^{-1} (\nabla \cdot \sigma + \omega^2 \rho_0)\mathbf{u} = \mathbf{0}. \tag{7}$$

The weak formulation of (7) is to find  $\omega^2 \in \mathbb{C}$  and  $\mathbf{0} \neq \mathbf{u} \in V$  such that

$$((\rho_1 - \rho_0)^{-1} (\nabla \cdot \sigma + \omega^2 \rho_0)\mathbf{u}, (\nabla \cdot \sigma + \bar{\omega}^2 \rho_1)\boldsymbol{\varphi}) = \mathbf{0} \quad \text{for all } \boldsymbol{\varphi} \in V, \tag{8}$$

where  $\bar{\omega}$  denotes the complex conjugate of  $\omega$ . Let  $\tau = \omega^2$ . We define two sesquilinear forms on  $V \times V$

$$\begin{aligned} \mathcal{A}_\tau(\boldsymbol{\phi}, \boldsymbol{\varphi}) &= ((\rho_1 - \rho_0)^{-1} (\nabla \cdot \sigma + \tau \rho_0)\boldsymbol{\phi}, (\nabla \cdot \sigma + \bar{\tau} \rho_0)\boldsymbol{\varphi}) + \tau^2(\rho_0 \boldsymbol{\phi}, \boldsymbol{\varphi}), \\ \mathcal{B}(\boldsymbol{\phi}, \boldsymbol{\varphi}) &= (\sigma(\boldsymbol{\phi}), \nabla \boldsymbol{\varphi}). \end{aligned}$$

It is clear that  $\mathcal{A}_\tau$  is symmetric. Due to (3),  $\mathcal{B}$  is also symmetric.

The variational problem for (8) can be written as to find  $\tau \in \mathbb{R}$  and  $\mathbf{0} \neq \mathbf{u} \in V$  such that

$$\mathcal{A}_\tau(\mathbf{u}, \boldsymbol{\varphi}) = \tau \mathcal{B}(\mathbf{u}, \boldsymbol{\varphi}) \quad \text{for all } \boldsymbol{\varphi} \in V. \tag{9}$$

This is a nonlinear problem since  $\tau$  appears on both sides of the equation. For a fixed  $\tau$ , we consider an associated generalized eigenvalue problem of finding  $\gamma := \gamma(\tau)$  such that

$$\mathcal{A}_\tau(\mathbf{u}, \boldsymbol{\varphi}) = \gamma \mathcal{B}(\mathbf{u}, \boldsymbol{\varphi}) \quad \text{for all } \boldsymbol{\varphi} \in V. \tag{10}$$

Then  $\tau$  is a transmission eigenvalue if  $\tau$  is a root of the nonlinear function

$$f(\tau) := \gamma(\tau) - \tau. \tag{11}$$

The following lemma can be verified in a straightforward manner.

**Lemma 2.1.** *A value  $\tau > 0$  is a transmission eigenvalue satisfying (9) if and only if  $f(\tau) = 0$  and  $\gamma$  satisfies (10).*

In the rest of this section, we analyze the generalized eigenvalue problem (10). Denote by  $\|\cdot\|$  the  $L^2$ -norm. It is shown in [29] that there exists  $\beta > 0$  such that

$$\|\nabla \cdot \sigma(\boldsymbol{\phi})\|^2 + \|\boldsymbol{\phi}\|^2 \geq \beta \|\boldsymbol{\phi}\|_{H^2(D)^2}^2 \quad \text{for } \boldsymbol{\phi} \in V.$$

The following lemma is useful in the subsequent analysis. The proof can be carried out following the proof of Lemma 3.1 in [3] for domains with smooth boundaries and thus omitted here.

**Lemma 2.2.** *Let  $\rho_0, \rho_1$  be smooth enough and assume that  $p_* \geq 1 \geq P$ . Then  $\mathcal{A}_\tau$  is a coercive sesquilinear form on  $V \times V$ , i.e., there exists a constant  $\alpha > 0$  such that*

$$\mathcal{A}_\tau(\boldsymbol{\phi}, \boldsymbol{\phi}) \geq \alpha \|\boldsymbol{\phi}\|^2 \quad \text{for all } \boldsymbol{\phi} \in V.$$

The source problem associated with (10) is to find  $\mathbf{u} \in V$  such that, for  $\mathbf{f} \in H^1(D)^2$ ,

$$\mathcal{A}_\tau(\mathbf{u}, \boldsymbol{\phi}) = (\sigma(\mathbf{f}), \nabla \boldsymbol{\phi}) \quad \text{for all } \boldsymbol{\phi} \in V. \tag{12}$$

Due to Lemma 2.2, the following theorem is a consequence of the Lax–Milgram Lemma.

**Theorem 2.3.** *There exists a unique solution  $\mathbf{u} \in V$  to (12). Furthermore, it holds that*

$$\|\mathbf{u}\|_{H^2(D)^2} \leq C \|\mathbf{f}\|_{H^1(D)^2}. \quad (13)$$

**Proof.** Due to the boundedness of  $\rho_0$  and  $\rho_1$ , for  $\phi, \varphi \in V$ , there exists some constant  $C > 0$  such that

$$\begin{aligned} & |\mathcal{A}_\tau(\phi, \varphi)| \\ &= |((\rho_1 - \rho_0)^{-1} (\nabla \cdot \sigma + \tau \rho_0) \phi, (\nabla \cdot \sigma + \bar{\tau} \rho_0) \varphi) + \tau^2 (\rho_0 \phi, \varphi)| \\ &\leq C (|\nabla \cdot \sigma(\phi)| |\nabla \cdot \sigma(\varphi)| + |\nabla \cdot \sigma(\phi)| |\varphi| + |\nabla \cdot \sigma(\varphi)| |\phi| + |\phi| |\varphi|) \\ &\leq C \|\phi\|_V \|\varphi\|_V. \end{aligned}$$

Hence  $\mathcal{A}_\tau$  is bounded. The coercivity of  $\mathcal{A}_\tau$  follows from Lemma 2.2. Let  $F$  be a linear functional on  $V$  such that

$$F(\phi) := (\sigma(\mathbf{f}), \nabla \phi),$$

for all  $\phi \in V$ . Then the Lax–Milgram Lemma implies that there exists a unique solution  $\mathbf{u}$  to the problem

$$\mathcal{A}_\tau(\mathbf{u}, \phi) = F(\phi) \quad \text{for all } \phi \in V.$$

Moreover, we have

$$\|\mathbf{u}\|_{H^2(D)^2} \leq C \|F\|_{V'},$$

where  $V'$  represents the dual space of  $V$ . Using the definition of  $\sigma(\mathbf{f})$ , we obtain that

$$\|F\|_{V'} \leq C_{\lambda, \mu} \|\mathbf{f}\|_{H^1(D)^2},$$

which shows the estimate (13) and the proof is complete.  $\square$

In the rest of the paper, we assume that the following regularity for  $\mathbf{u}$  holds

$$\|\mathbf{u}\|_{H^{2+\xi}(D)^2} \leq C \|\mathbf{f}\|_{H^1(D)^2}, \quad (14)$$

where  $\xi \geq 0$  is the elliptic regularity parameter. Note that for the biharmonic equation,  $\xi \in (\frac{1}{2}, 1]$  is determined by the angles at the corners of  $D$  and  $\xi = 1$  if  $D$  is convex [30].

It follows from Theorem 2.3 that there exists a solution operator  $T : H^1(D)^2 \rightarrow V$  such that

$$\mathbf{u} = T\mathbf{f}.$$

Clearly, the operator  $T$  is self-adjoint since  $\mathcal{A}_\tau$  is symmetric.  $T$  is also a compact operator due to the compact embedding of  $H^2(D)^2$  into  $H^1(D)^2$  (see, e.g., Theorem 1.2.1 of [22]). The generalized eigenvalue problem (10) has the following equivalent operator form

$$\eta \mathbf{u} = T\mathbf{u}, \quad \text{where } \eta = \gamma^{-1}.$$

From the classical spectral theory of compact self-adjoint operators,  $T$  has at most a countable set of real eigenvalues and zero is the only possible accumulation point. Consequently, we have the following lemma for the generalized eigenvalue problem (10).

**Lemma 2.4.** *Let  $\rho_0$  and  $\rho_1$  satisfy (4) and the conditions in Lemma 2.2 are satisfied. Then the generalized eigenvalue problem (10) has at most a countable set of positive eigenvalues and  $+\infty$  is the only possible accumulation point.*

In view of Lemma 2.1, the computation of real transmission eigenvalues can be carried out as follows:

1. Obtain an approximation  $f_h(\tau)$  of  $f(\tau)$  by computing the generalized eigenvalue  $\gamma_h(\tau)$  of (10) using the  $H^2$ -conforming Argyris element;
2. Compute the zero of  $f_h(\tau)$  using some iterative root-finding method.

### 3. The argyris element for $\gamma(\tau)$

In this section, we employ the Argyris element to compute  $\gamma(\tau)$ . The convergence for the source problem (12) is established first. Then the theory of Babuška and Osborn [31] is applied to obtain the convergence for the eigenvalue problem (10).

Let  $\mathcal{T}$  be a regular triangular mesh for  $D$  and  $K \in \mathcal{T}$  be a triangle. We employ the  $H^2$ -conforming Argyris element, which uses  $\mathcal{P}_5$ , the set of polynomials of degree up to 5 on  $K$ , to discretize (10). Note that  $\dim(\mathcal{P}_5) = 21$ . For  $\mathcal{N} = \{N_1, \dots, N_{21}\}$ , the degrees of freedom are 3 values at the vertices of  $K$ , 6 values of the first order partial derivatives at the vertices of  $K$ , 9 values of the second order derivatives at the vertices of  $K$ , and 3 values of the normal derivatives at the midpoints of three edges of  $K$  [28].

Note that the Argyris element does not belong to the affine families. This is due to the fact that normal derivatives are used as degrees of freedom. Fortunately, their interpolation properties are quite similar to those of affine families. Hence the Argyris element is referred to be almost-affine. Let  $v \in H^2(D)$  and  $I_h v$  be the interpolation of  $v$  by the Argyris element. For  $v \in H^{1+\alpha}(D)$ ,  $\alpha > 0$ , the following interpolation result holds (see, e.g., [22])

$$\|v - I_h v\|_{H^2(D)} \leq Ch^{s-1}|v|_{H^{s+1}(D)}, \tag{15}$$

where  $1 \leq s \leq \min\{5, 1 + \alpha\}$  depends on the regularity of  $v$ .

Let  $V_h \subset V$  be the Argyris finite element space associated with  $\mathcal{T}$ . The degrees of freedom of functions in  $V_h$  related to the boundary nodes need a careful treatment. Let  $e \subset \Gamma$  be an edge of a triangle  $T \subset \mathcal{T}$  with the unit outward normal  $\mathbf{v} := (v_x, v_y)^\top$  and unit tangent vector  $(t_x, t_y)^\top$ . The case when  $v_x v_y = 0$  is easy to treat. Assume that  $v_x v_y \neq 0$  and thus  $t_x t_y \neq 0$ . It is clear that

$$t_x v_x + t_y v_y = 0, \quad t_x^2 + t_y^2 = 1, \quad v_x^2 + v_y^2 = 1. \tag{16}$$

On  $e$ ,  $\mathbf{u} = (u_1, u_2)^\top = \mathbf{0}$ . Hence the tangential derivatives of  $u_1$  and  $u_2$  are also  $\mathbf{0}$ , i.e.,

$$t_x \partial_x u_1 + t_y \partial_y u_1 = 0, \quad t_x \partial_x u_2 + t_y \partial_y u_2 = 0. \tag{17}$$

The boundary condition  $\sigma(\mathbf{u})\mathbf{v} = \mathbf{0}$  implies that

$$\begin{aligned} ((\lambda + 2\mu)\partial_x u_1 + \lambda\partial_y u_2)v_x + \mu(\partial_y u_1 + \partial_x u_2)v_y &= 0, \\ \mu(\partial_x u_2 + \partial_y u_1)v_x + (\lambda\partial_x u_1 + (\lambda + 2\mu)\partial_y u_2)v_y &= 0. \end{aligned}$$

Substituting

$$\partial_y u_1 = -\frac{t_x}{t_y} \partial_x u_1, \quad \partial_y u_2 = -\frac{t_x}{t_y} \partial_x u_2,$$

into the above equations, one has that

$$\begin{aligned} (\lambda + 2\mu)v_x \partial_x u_1 - \frac{t_x}{t_y} \lambda v_x \partial_x u_2 - \frac{t_x}{t_y} \mu v_y \partial_x u_1 + \mu v_y \partial_x u_2 &= 0, \\ \mu v_x \partial_x u_2 - \frac{t_x}{t_y} \mu v_x \partial_x u_1 + \lambda v_y \partial_x u_1 - \frac{t_x}{t_y} (\lambda + 2\mu)v_y \partial_x u_2 &= 0. \end{aligned}$$

Collecting similar terms and using the fact that  $t_x v_x + t_y v_y = 0$ , one obtains that

$$\begin{aligned} \partial_x u_1 \left[ v_x (\lambda + 2\mu) - \mu \frac{t_x}{t_y} v_y \right] + \partial_x u_2 \left[ \mu v_y - \frac{t_x}{t_y} \lambda v_x \right] &= 0, \\ \partial_x u_1 \left[ \lambda v_y - \frac{t_x}{t_y} \mu v_x \right] + \partial_x u_2 \left[ \mu v_x - (\lambda + 2\mu)v_y \frac{t_x}{t_y} \right] &= 0, \end{aligned}$$

i.e.,

$$\begin{aligned} \partial_x u_1 [t_y v_x (\lambda + 2\mu) - \mu t_x v_y] - \partial_x u_2 (\lambda + \mu)v_x t_x &= 0, \\ -\partial_x u_1 (\lambda + \mu)v_x t_x + \partial_x u_2 [\mu v_x t_y - (\lambda + 2\mu)v_y t_x] &= 0. \end{aligned}$$

The above equation can be viewed as a homogeneous linear system for  $\partial_x u_1$  and  $\partial_x u_2$ . The determinant of the coefficient matrix is

$$\begin{aligned} &\mu(\lambda + 2\mu)v_x^2(1 - t_x^2) + \mu(\lambda + 2\mu)t_x^2(1 - v_x^2) \\ &\quad + (\lambda + 2\mu)^2 v_x^2 t_x^2 + \mu^2 v_x^2 t_x^2 - (\lambda + \mu)^2 v_x^2 t_x^2 \\ &= \mu(\lambda + 2\mu)(v_x^2 + t_x^2) + v_x^2 t_x^2 ((\lambda + 2\mu)^2 + \mu^2 - (\lambda + \mu)^2 - 2\mu(\lambda + 2\mu)) \\ &= \mu(\lambda + 2\mu)(v_x^2 + t_x^2) \\ &> 0, \end{aligned}$$

where we have used (16). As a result of the above equation and (17), it holds that

$$\partial_x u_1 = \partial_y u_1 = \partial_x u_2 = \partial_y u_2 = 0,$$

i.e., all the first-order derivatives are 0.

For the Argyris element, one also need to consider the degrees of freedom related to the second order partial derivative of a boundary node. For  $u_i$ ,  $i = 1, 2$ , one has that

$$\frac{\partial^2 u_i}{\partial x^2} t_x + \frac{\partial^2 u_i}{\partial x \partial y} t_y = 0,$$

$$\frac{\partial^2 u_i}{\partial x \partial y} t_x + \frac{\partial^2 u_i}{\partial y^2} t_y = 0,$$

which implies that

$$\frac{\partial^2 u_i}{\partial x^2} = \frac{t_y^2}{t_x^2} \frac{\partial^2 u_i}{\partial y^2}, \quad \frac{\partial^2 u_i}{\partial x \partial y} = -\frac{t_y}{t_x} \frac{\partial^2 u_i}{\partial y^2}. \quad (18)$$

If a boundary node is a corner, then

$$\frac{\partial^2 u_i}{\partial x^2} = \frac{\partial^2 u_i}{\partial y^2} = \frac{\partial^2 u_i}{\partial x \partial y} = 0, \quad i = 1, 2.$$

Now we are ready to introduce the discrete problem for (12). For  $\mathbf{f} \in H^1(D)^2$ , find  $\mathbf{u}_h \in V_h$  such that

$$\mathcal{A}_\tau(\mathbf{u}_h, \boldsymbol{\phi}_h) = (\sigma(\mathbf{f}), \nabla \boldsymbol{\phi}_h) \quad \text{for all } \boldsymbol{\phi}_h \in V_h. \quad (19)$$

The existence of a unique solution  $\mathbf{u}_h$  to (19) holds as the conforming finite element is used. As a consequence, there exists a discrete solution operator  $T_h : H^1(D)^2 \rightarrow H^2(D)^2$  such that

$$\mathbf{u}_h = T_h \mathbf{f}.$$

**Theorem 3.1.** *Let  $\mathbf{u}$  and  $\mathbf{u}_h$  be the solutions of the continuous problem (12) and discrete problem (19), respectively. Then the following error estimate holds*

$$\|\mathbf{u} - \mathbf{u}_h\|_{H^1(D)^2} \leq Ch^{2\xi} \|\mathbf{f}\|_{H^1(D)^2},$$

where  $\xi = s - 1$ .

**Proof.** From C ea's Lemma, the following error estimate holds

$$\|\mathbf{u} - \mathbf{u}_h\|_{H^2(D)^2} \leq C \inf_{\mathbf{v}_h \in V_h} \|\mathbf{u} - \mathbf{v}_h\|_{H^2(D)^2},$$

for some constant  $C$ . Using (15) and (14), one has that

$$\|\mathbf{u} - \mathbf{u}_h\|_{H^2(D)^2} \leq Ch^{s-1} |\mathbf{u}|_{H^{s+1}(D)^2} = Ch^\xi |\mathbf{u}|_{H^{2+\xi}(D)^2} \leq Ch^\xi \|\mathbf{f}\|_{H^1(D)^2},$$

where  $s = 1 + \xi$ . For  $\mathbf{g} \in H_0^1(D)^2$ , let  $\boldsymbol{\phi}_g$  be the unique solution of

$$\mathcal{A}_\tau(\boldsymbol{\phi}_g, \boldsymbol{\phi}) = (\sigma(\mathbf{g}), \nabla \boldsymbol{\phi}) \quad \text{for all } \boldsymbol{\phi} \in V.$$

The rest of the proof follows the Aubin–Nitsche Lemma (see, e.g., Theorem 3.2.4 of [22]) with suitable choices of Sobolev spaces. Let  $\mathbf{e} := \mathbf{u} - \mathbf{u}_h$  and  $\mathbf{g} \in H^1(D)^2$ . Using the Galerkin orthogonality, for any  $\mathbf{v}_h \in V_h$ , we have that

$$\begin{aligned} (\sigma(\mathbf{g}), \nabla \mathbf{e}) &= \mathcal{A}_\tau(\boldsymbol{\phi}_g, \mathbf{u} - \mathbf{u}_h) \\ &= \mathcal{A}_\tau(\boldsymbol{\phi}_g - \mathbf{v}_h, \mathbf{u} - \mathbf{u}_h) \\ &\leq C \|\boldsymbol{\phi}_g - \mathbf{v}_h\|_{H^2} \|\mathbf{u} - \mathbf{u}_h\|_{H^2}, \end{aligned}$$

which yields

$$(\sigma(\mathbf{g}), \nabla \mathbf{e}) \leq C \|\mathbf{u} - \mathbf{u}_h\|_{H^2} \inf_{\mathbf{v}_h \in V_h} \|\boldsymbol{\phi}_g - \mathbf{v}_h\|_{H^2}.$$

Furthermore,

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_h\|_{H^1} &= \sup_{\mathbf{g} \in H^1(D)^2, \mathbf{g} \neq \mathbf{0}} \frac{(\mathbf{u} - \mathbf{u}_h, \mathbf{g})}{\|\mathbf{g}\|_{H^1}} \\ &\leq C \|\mathbf{u} - \mathbf{u}_h\|_{H^2} \sup_{\mathbf{g} \in H^1(D)^2, \mathbf{g} \neq \mathbf{0}} \left\{ \inf_{\mathbf{v}_h \in V_h} \frac{\|\boldsymbol{\phi}_g - \mathbf{v}_h\|}{\|\mathbf{g}\|_{H^1}} \right\}. \end{aligned}$$

Consequently, we get

$$\|\mathbf{u} - \mathbf{u}_h\|_{H^1(D)^2} \leq Ch^{2\xi} \|\mathbf{f}\|_{H^1(D)^2},$$

which completes the proof.  $\square$

Using the operators  $T$  and  $T_h$ , we can rewrite the above error estimate as

$$\|T\mathbf{f} - T_h\mathbf{f}\| \leq Ch^{2\xi} \|\mathbf{f}\|_{H^1(D)^2}.$$

Thus we have that

$$\|T - T_h\| \leq Ch^{2\xi}.$$

Now we consider the discrete eigenvalue problem: Find  $\gamma_h \in \mathbb{R}$  such that

$$\mathcal{A}_\tau(\mathbf{u}_h, \boldsymbol{\phi}_h) = \gamma_h(\sigma(\mathbf{u}_h), \nabla \boldsymbol{\phi}_h) \quad \text{for all } \boldsymbol{\phi}_h \in V_h. \tag{20}$$

The discrete eigenvalue problem is obtained by replacing  $\mathbf{f}$  with  $\gamma_h \mathbf{u}_h$  in (19). The reciprocal of the exact eigenvalue  $\gamma$  is the eigenvalue of the solution operator  $T$ . The reciprocal of  $\gamma_h$  is the eigenvalue of the finite element solution operator  $T_h$ . If  $T_h$  converges to  $T$  in norm as  $h \rightarrow 0$ , the finite element spectral approximation theory of variationally formulated eigenvalue problems by Babuška and Osborn [31] guarantees the convergence of  $\gamma_h$  to  $\gamma$ .

Since both  $\mathcal{A}_\tau$  and  $\mathcal{B}$  are symmetric,  $T$  is self-adjoint. Similarly,  $T_h$  is self-adjoint. The estimate for the eigenvalue problem follows directly from the theory of Babuška and Osborn [31] (Theorem 8.1 therein).

**Theorem 3.2.** *Let  $\gamma$  be a generalized eigenvalue of (10) with algebraic multiplicity  $m$ . Let  $\gamma_{h,1}, \dots, \gamma_{h,m}$  be the  $m$  eigenvalues of (20) approximating  $\gamma$ . Define  $\hat{\gamma}_h = \frac{1}{m} \sum_{j=1}^m \gamma_{h,j}$ . The following estimate holds*

$$|\gamma - \hat{\gamma}_h| \leq Ch^{2\xi},$$

where  $C := C(\tau) > 0$  depends on  $\tau$  but not  $h$ .

Let  $\{\boldsymbol{\phi}_i\}_{i=1}^n$  be the basis functions of the Argyris element satisfying the boundary conditions. Let  $A_{n \times n}$  be the matrix given by  $A_{ij} = \mathcal{A}_\tau(\boldsymbol{\phi}_j, \boldsymbol{\phi}_i)$  and  $B_{n \times n}$  be given by  $B_{ij} = (\sigma(\boldsymbol{\phi}_j), \nabla \boldsymbol{\phi}_i)$ . Then  $\gamma_h$ 's in (20) are the generalized eigenvalues of  $A\mathbf{x} = \gamma B\mathbf{x}$ .

#### 4. Computation of the root of $f_h(\tau)$

Now we turn to the problem of how to compute the root of the nonlinear function  $f_h(\tau)$ , the discrete version of  $f(\tau)$  defined in (11). For simplicity, we assume that  $\rho_0$  and  $\rho_1$  are constants. Consider the case when  $\gamma(\tau)$  is the first eigenvalue of (10). Similar result holds for other eigenvalues.

The continuity of  $f$  is obvious since the generalized eigenvalue  $\gamma(\tau)$  of (10) depends on  $\tau$  continuously. In fact,  $f$  is differentiable and the derivative is negative on an interval given in Theorem 4.1. We first recall the elastic eigenvalue problem which will be used later (see, e.g., [31]). Find a non-trivial eigenpair  $(\delta, \mathbf{u}) \in \mathbb{R} \times H_0^1(D)^2$  such that

$$\int_D (2\mu \boldsymbol{\epsilon}(\mathbf{u}) : \boldsymbol{\epsilon}(\mathbf{u}) + \lambda \operatorname{div} \mathbf{u} \operatorname{div} \mathbf{v}) \, d\mathbf{x} = \delta \int_D \mathbf{u} \mathbf{v} \, d\mathbf{x} \tag{21}$$

for all  $\mathbf{v} \in H_0^1(D)^2$ .

**Theorem 4.1.** *Let  $\delta_1$  be the first elastic eigenvalue. The function  $f(\tau)$  is differentiable. Furthermore,  $f(\tau)$  is a decreasing function on  $(0, \frac{\delta_1(\rho_0 + \rho_1)}{2\rho_0\rho_1})$ .*

**Proof.** Let  $\gamma_1(\tau, \rho_0, \rho_1)$  be the first generalized eigenvalue of (10). The following Rayleigh quotient holds

$$\begin{aligned} & \gamma_1(\tau, \rho_0, \rho_1) \\ &= \inf_{\mathbf{w} \in V} \frac{\mathcal{A}_\tau(\mathbf{w}, \mathbf{w})}{\mathcal{B}(\mathbf{w}, \mathbf{w})} \\ &= \inf_{\mathbf{w} \in V} \frac{\left( \frac{1}{\rho_1 - \rho_0} (\nabla \cdot \boldsymbol{\sigma} + \tau \rho_0) \mathbf{w}, (\nabla \cdot \boldsymbol{\sigma} + \tau \rho_0) \mathbf{w} \right) + \tau^2 (\rho_0 \mathbf{w}, \mathbf{w})}{(\boldsymbol{\sigma}(\mathbf{w}), \nabla \mathbf{w})} \\ &= \inf_{\mathbf{w} \in V} \frac{\left( \frac{1}{\rho_1 - \rho_0} \nabla \cdot \boldsymbol{\sigma}(\mathbf{w}), \nabla \cdot \boldsymbol{\sigma}(\mathbf{w}) \right) + 2\tau \left( \frac{\rho_0}{\rho_1 - \rho_0} \mathbf{w}, \nabla \cdot \boldsymbol{\sigma}(\mathbf{w}) \right) + \tau^2 \left( \frac{\rho_0 \rho_1}{\rho_1 - \rho_0} \mathbf{w}, \mathbf{w} \right)}{(\boldsymbol{\sigma}(\mathbf{w}), \nabla \mathbf{w})}. \end{aligned}$$

When  $\rho_0$  and  $\rho_1$  are constants, we have

$$\gamma_1(\tau, \rho_0, \rho_1) = \inf_{\mathbf{w} \in V} \frac{\frac{1}{\rho_1 - \rho_0} (\nabla \cdot \boldsymbol{\sigma}(\mathbf{w}), \nabla \cdot \boldsymbol{\sigma}(\mathbf{w})) + \tau^2 \frac{\rho_0 \rho_1}{\rho_1 - \rho_0} (\mathbf{w}, \mathbf{w})}{(\boldsymbol{\sigma}(\mathbf{w}), \nabla \mathbf{w})} - \frac{2\tau \rho_0}{\rho_1 - \rho_0}.$$

Note that the sesquilinear form

$$a(\mathbf{u}, \mathbf{v}) := (\boldsymbol{\sigma}(\mathbf{u}), \nabla \mathbf{v}) = 2\mu \boldsymbol{\epsilon}(\mathbf{u}) : \boldsymbol{\epsilon}(\mathbf{v}) + \lambda (\nabla \cdot \mathbf{u})(\nabla \cdot \mathbf{v})$$

is bounded, symmetric, and coercive. Hence

$$\begin{aligned} & \gamma_1(\tau, \rho_0, \rho_1) \\ &= \inf_{\mathbf{w} \in V, a(\mathbf{w}, \mathbf{w})=1} \left\{ \frac{(\nabla \cdot \boldsymbol{\sigma}(\mathbf{w}), \nabla \cdot \boldsymbol{\sigma}(\mathbf{w}))}{\rho_1 - \rho_0} + \tau^2 \frac{\rho_0 \rho_1}{\rho_1 - \rho_0} (\mathbf{w}, \mathbf{w}) \right\} - \frac{2\tau \rho_0}{\rho_1 - \rho_0}. \end{aligned}$$

Let  $\kappa := \tau^2 \frac{\rho_0 \rho_1}{\rho_1 - \rho_0}$ . We define a new function

$$s(\kappa) = \inf_{\mathbf{w} \in V, a(\mathbf{w}, \mathbf{w})=1} \left\{ \frac{1}{\rho_1 - \rho_0} \|\nabla \cdot \sigma(\mathbf{w})\|^2 + \kappa \|\mathbf{w}\|^2 \right\}.$$

For a fixed  $\kappa \in (0, \infty)$ , there exists a  $\mathbf{w}_\kappa$  such that  $\mathbf{w}_\kappa \in V$ ,  $a(\mathbf{w}_\kappa, \mathbf{w}_\kappa) = 1$ , and

$$s(\kappa) = \left\{ \frac{1}{\rho_1 - \rho_0} \|\nabla \cdot \sigma(\mathbf{w}_\kappa)\|^2 + \kappa \|\mathbf{w}_\kappa\|^2 \right\}.$$

For a small enough positive  $h$ ,

$$\begin{aligned} s(\kappa + h) - s(\kappa) &\leq \left\{ \frac{1}{\rho_1 - \rho_0} \|\nabla \cdot \sigma(\mathbf{w}_\kappa)\|^2 + (\kappa + h) \|\mathbf{w}_\kappa\|^2 \right\} \\ &\quad - \left\{ \frac{1}{\rho_1 - \rho_0} \|\nabla \cdot \sigma(\mathbf{w}_\kappa)\|^2 + \kappa \|\mathbf{w}_\kappa\|^2 \right\} \\ &= h \|\mathbf{w}_\kappa\|^2. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} s(\kappa + h) - s(\kappa) &\geq \left\{ \frac{1}{\rho_1 - \rho_0} \|\nabla \cdot \sigma(\mathbf{w}_{\kappa+h})\|^2 + (\kappa + h) \|\mathbf{w}_{\kappa+h}\|^2 \right\} \\ &\quad - \left\{ \frac{1}{\rho_1 - \rho_0} \|\nabla \cdot \sigma(\mathbf{w}_\kappa)\|^2 + \kappa \|\mathbf{w}_\kappa\|^2 \right\} \\ &= h \|\mathbf{w}_{\kappa+h}\|^2. \end{aligned}$$

Consequently,

$$\|\mathbf{w}_{\kappa+h}\|^2 \leq \frac{s(\kappa + h) - s(\kappa)}{h} \leq \|\mathbf{w}_\kappa\|^2.$$

The above inequality implies that  $\|\mathbf{w}_\kappa\|^2$  is monotonically decreasing and thus bounded. Note that  $a(\mathbf{w}_\kappa, \mathbf{w}_\kappa) = 1$ . Then the continuity of  $s$  and the compact embedding of  $V$  into  $L^2(D)^2$  implies the existences of a  $\tilde{\mathbf{w}}$  such that  $\mathbf{w}_{\kappa+h}$  converges in  $L^2(D)^2$  strongly and  $\mathbf{w}_{\kappa+h}$  converges in  $H^2(D)^2$  weakly. In addition,  $\mathbf{w}_{\kappa+h}$  satisfies

$$\left( \frac{1}{\rho_1 - \rho_0} \nabla \cdot \sigma(\mathbf{w}_{\kappa+h}), \nabla \cdot \sigma(\phi) \right) + (\kappa + h) (\mathbf{w}_{\kappa+h}, \phi) = s(\kappa + h) (\sigma(\mathbf{w}), \nabla \phi),$$

for all  $\phi \in V$ . Taking  $h \rightarrow 0$ , we obtain

$$\left( \frac{1}{\rho_1 - \rho_0} \nabla \cdot \sigma(\tilde{\mathbf{w}}), \nabla \cdot \sigma(\phi) \right) + \kappa (\tilde{\mathbf{w}}, \phi) = s(\kappa) (\sigma(\tilde{\mathbf{w}}), \nabla \phi),$$

for all  $\phi \in V$ . Thus  $\tilde{\mathbf{w}} = \mathbf{w}_\kappa$ . Consequently,

$$\|\mathbf{w}_{\kappa+h}\|^2 \rightarrow \|\mathbf{w}_\kappa\|^2, \quad h \rightarrow 0.$$

Then the derivative of  $s(\kappa)$  is  $\|\mathbf{w}_\kappa\|^2$ .

Combing the above estimates, we obtain

$$\begin{aligned} f'(\tau) &= 2\tau \frac{\rho_0 \rho_1}{\rho_1 - \rho_0} \|\mathbf{w}_\kappa\|^2 - \frac{2\rho_0}{\rho_1 - \rho_0} - 1 \\ &= 2\tau \frac{\rho_0 \rho_1}{\rho_1 - \rho_0} \|\mathbf{w}_\kappa\|^2 - \frac{\rho_1 + \rho_0}{\rho_1 - \rho_0}. \end{aligned}$$

Let  $\delta_1$  be the first elastic eigenvalue. One has that

$$\|\mathbf{w}_\kappa\|^2 \leq \frac{1}{\delta_1} (\sigma(\mathbf{w}_\kappa), \nabla \mathbf{w}_\kappa) = \frac{1}{\delta_1},$$

since  $(\sigma(\mathbf{w}_\kappa), \nabla \mathbf{w}_\kappa) = 1$ . This implies that

$$f'(\tau) \leq \frac{2\tau \rho_0 \rho_1}{\delta_1 (\rho_1 - \rho_0)} - \frac{\rho_1 + \rho_0}{\rho_1 - \rho_0}. \quad (22)$$

In particular,  $f$  is decreasing, i.e.,

$$f'(\tau) \leq 0 \quad \text{if} \quad \tau < \frac{\delta_1 (\rho_0 + \rho_1)}{2\rho_0 \rho_1}.$$

It is easy to see that  $f(\tau) > 0$  if  $\tau \rightarrow 0$  and  $f(\tau) < 0$  if  $\tau \rightarrow \infty$ .  $\square$



Now we turn to the problem of approximating the root of  $f(\tau)$ . Since we only have the finite element approximation for  $f$ , the nonlinear equation to be solved is the following discrete version of (11):

$$f_h(\tau) := \gamma_h(\tau) - \tau. \tag{23}$$

It is clear that  $f_h(\tau)$  is continuous. The existence of the transmission eigenvalue, i.e., the root of  $f_h(\tau)$ , can be established if one can find an interval such that  $f_h(\tau)$  changes signs at the ends of the interval.

To this end, let  $B_r$  be the largest ball of radius  $r$  such that  $B_r \subset D$  and  $B_R$  be the smallest ball of radius  $R$  such that  $D \subset B_R$ . Let  $b_1$  and  $b_2$  be given by

$$b_1 = \max\left(\tau_1(B_R, P, p_*), \sqrt{d_1(D)/P_*}\right), \quad b_2 = \tau_1(B_r, p, P_*), \tag{24}$$

where  $\tau_1(B_R, P, p_*)$  is the first real transmission eigenvalue of  $B_R$  with  $\rho_0 = P, \rho_1 = p_*$ ,  $\tau_1(B_r, p, P_*)$  is the first real transmission eigenvalue of  $B_r$  with  $\rho_0 = p, \rho_1 = P_*$ ,  $d_1(D)$  is the first Dirichlet eigenvalue of the negative Laplacian on  $D$  (see [22]).

It is shown in [3] that the first transmission eigenvalue  $\tau$  for (9) is such that  $\tau \in (b_1, b_2)$ . It is done by showing that  $f(b_1) > 0$  and  $f(b_2) < 0$  such that  $f(\tau)$  must have a root in  $(b_1, b_2)$  due to the continuity of  $f(\tau)$ .

Using Theorem 3.2, there exists a constant  $C$  such that

$$|f_h(\tau) - f(\tau)| \leq Ch^{2\alpha}. \tag{25}$$

The following lemma is a consequence of the above discussion.

**Lemma 4.2.** *If  $h$  is small enough,  $f_h(\tau)$  has at least one root in  $(b_1, b_2)$  where  $b_1, b_2$  are defined in (24).*

**Proof.** Using (25), if  $h$  is small enough,  $f_h(b_1) > 0$  and  $f_h(b_2) < 0$  since  $f(b_1) > 0$  and  $f(b_2) < 0$ . Since  $f_h(\tau)$  is continuous, there exists at least a  $\tau_h^*$  such that  $f_h(\tau_h^*) = 0$ .  $\square$

The next lemma shows that the root of  $f_h(\tau)$  approximates that of  $f(\tau)$ .

**Lemma 4.3.** *Let  $f(\tau)$  and  $f_h(\tau)$  be two continuous functions. For a small enough  $\epsilon > 0$ , there exists some  $\eta > 0$  such that  $f'(\tau) \leq -\eta < 0$  and  $|f(\tau) - f_h(\tau)| < \epsilon$  on an interval  $[c - \epsilon/\eta, d + \epsilon/\eta]$  for some  $0 < c < d$ . If  $f(\tau^*) = 0$  for some  $\tau^* \in (c, d)$ , then there exists a  $\tau_h^*$  such that  $f_h(\tau_h^*) = 0$  and*

$$|\tau^* - \tau_h^*| < \epsilon/\eta.$$

**Proof.** For  $\epsilon > 0$  small enough, from (22), there exists  $\eta > 0$  such that

$$f'(\tau) \leq -\eta \quad \text{for } \tau \in \left(0, \frac{\delta_1(\rho_0 + \rho_1)}{2\rho_0\rho_1} - \epsilon\right). \tag{26}$$

Since  $f'(\tau) \leq -\eta < 0$ , if  $\epsilon$  is small enough, there must exist  $\tau_1$  and  $\tau_2$  such that  $f(\tau_1) > \epsilon$  and  $f(\tau_2) < -\epsilon$ . Furthermore,  $|f(\tau) - f_h(\tau)| < \epsilon$  for all  $\tau$  implies that  $f_h(\tau_1) > 0$  and  $f_h(\tau_2) < 0$ . The existence of  $\tau_h^*$  such that  $f_h(\tau_h^*) = 0$  follows immediately since  $f_h(\tau)$  is continuous.

Assume that  $|\tau^* - \tau_h^*| \geq \epsilon/\eta$ . Since  $f(\tau^*) = 0$ , we have  $f(\tau_h^*) = f'(\xi)(\tau^* - \tau_h^*)$  for  $\xi$  between  $\tau_h^*$  and  $\tau^*$ . Thus we have either  $f(\tau_h^*) > \epsilon$  or  $f(\tau_h^*) < -\epsilon$ . Both contradict the fact that  $|f_h(\tau_h^*) - f(\tau_h^*)| < \epsilon$ . This completes the proof.  $\square$

It seems straightforward to use the bisection method to compute the root of  $f_h(\tau)$  if one has  $a$  and  $b$  defined in (24) (see [5]). However, it is necessary to compute  $\tau_1(B_R, P, p_*)$  and  $\tau_1(B_r, p, P_*)$ , i.e., the first elastic transmission eigenvalue for a ball with constant mass densities, which is not simpler at all. Based on the fact that  $f_h(\tau)$  is positive close to zero and monotonically decreasing on some interval right to zero, it is suitable to use the secant method to find the root of  $f_h(\tau)$ . Let  $tol$  be the tolerance. The algorithm to compute the elastic transmission eigenvalues is as follows.

**SMETE**

- generate a regular triangular mesh  $\mathcal{T}$  for  $D$
- construct matrix  $B_h$  corresponding to  $\mathcal{B}$  in (10)
- choose  $x_1 > x_0 > 0$  small enough
- $d = \text{abs}(x_1 - x_0)$
- $\tau = x_0$  and construct the matrix  $A_{\tau,h}$
- compute the first generalized eigenvalue  $\gamma_0$  of  $A_{\tau,h}\mathbf{x} = \gamma B_h\mathbf{x}$
- $\tau = x_1$  and construct matrix  $A_{\tau,h}$
- compute the first generalized eigenvalue  $\gamma_1$  of  $A_{\tau,h}\mathbf{x} = \gamma B_h\mathbf{x}$
- while  $\delta > tol$ 
  - $\tau = x_1 - \gamma_1 \frac{x_1 - x_0}{\gamma_1 - \gamma_0}$
- construct the matrix  $A_{\tau,h}$

compute the first eigenvalue  $\gamma_\tau$  of  $A_{\tau,h}\mathbf{x} = \gamma B_h\mathbf{x}$   
 $d = \text{abs}(\gamma_\tau - \tau)$   
 $x_0 = x_1, x_1 = \tau$   
 $\gamma_0 = \gamma_1, \gamma_1 = \gamma_\tau$   
output  $\tau$

In practice, one can choose  $x_0$  and  $x_1$  be two small positive numbers, e.g.,  $x_0 = 0.1$  and  $x_1 = 0.2$ . The following theorem guarantees the converge for the proposed method.

**Theorem 4.4.** Assume that the Argyris element method is used to solve the generalized eigenvalue problem (10) on a regular triangular mesh  $\mathcal{T}$  for  $D$  with mesh size  $h$  and the secant method is employed to compute the root of  $f_h(\tau)$  with tolerance  $tol$ . Let  $\tau$  be the exact transmission eigenvalue, i.e., the root of (11), and  $\tau_h^s$  be the computed root of (23) by the secant method such that  $\tau, \tau_h^s \in \left(0, \frac{\delta_1(\rho_0 + \rho_1)}{2\rho_0\rho_1} - \epsilon\right)$ . Then

$$|\tau - \tau_h^s| \leq Ch^{2\xi}/\eta + tol. \quad (27)$$

**Proof.** Let  $\tau_h$  be the exact root of  $f_h(\tau)$ . Since  $\tau_h^s$  is computed by the secant method, one has that

$$|\tau_h - \tau_h^s| < tol.$$

Since  $\tau$  is the root of  $f(\tau)$ , we have that  $\gamma = \tau$ . Similarly,  $\gamma_h = \tau_h$ . Using Lemma 4.3 and Theorem 3.2, for  $h$  small enough, we have

$$|\tau - \tau_h| = |\gamma - \gamma_h| < Ch^{2\xi}/\eta.$$

Triangle inequality implies that

$$|\tau - \tau_h^s| \leq |\tau - \tau_h| + |\tau_h - \tau_h^s| < Ch^{2\xi}/\eta + tol. \quad \square$$

## 5. Numerical examples

In this section, we present some numerical results for three domains:

1. the unit square,
2. an L-shape domain given by

$$(0, 1) \times (0, 1) \setminus [1/2, 1] \times [0, 1/2],$$

3. a triangle whose vertices are

$$(0, 0), (0, 1), \text{ and } (1, 0).$$

Four levels of uniformly refined triangular meshes are generated. The size of the initial mesh is  $h_1 = 1/8$  and  $h_i = h_{i-1}/2$ ,  $i = 2, 3, 4$ . All examples are done using Matlab 2016a on a MacBook Pro with 16G memory and 3.3 GHz Intel Core i7 processor.

Since the Argyris element leads to many degrees of freedom, we were only able to compute four levels of meshes, i.e.,  $h \approx 1/8, 1/16, 1/32, 1/64$ . Further refinements would lead to very large matrix eigenvalue problems which take too long to solve (more than a few hours).

### 5.1. Generalized eigenvalues of (10)

We check the convergence rate of the Argyris method for the fourth order generalized eigenvalue problem (10) with fixed  $\tau = 2$ . Other parameters are chosen as follows

$$\mu = 1/16, \quad \lambda = 1/4, \quad \rho_0 = 1, \quad \rho_1 = 4. \quad (28)$$

The relative error is defined as

$$E_{i+1} = \frac{|\gamma_{i+1} - \gamma_i|}{|\gamma_i|}, \quad i = 1, 2, 3,$$

where  $\gamma_i$  is the generalized eigenvalue computed using the mesh with size  $h_i$ . Then the convergence order is defined as

$$\text{convergence order} = \log_2 \frac{E_{i+1}}{E_{i+2}}, \quad i = 1, 2. \quad (29)$$

The first eigenvalues for three domains are shown in Table 1. The lower convergence order for the L-shape domain is expected since the domain is non-convex. The convergence orders for the unit square and the triangle are much higher.

**Table 1**

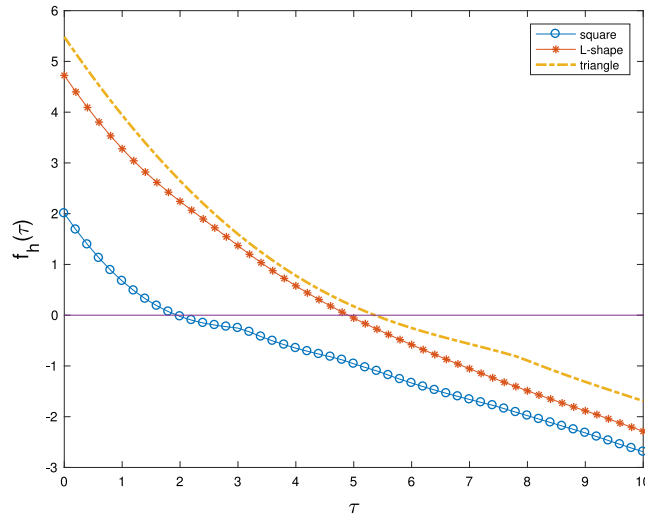
The smallest generalized eigenvalue of (10) for three domains with  $\mu = 1/16, \lambda = 1/4, \rho_0 = 1, \rho_1 = 4$ .

$h$	Unit square	Order	L-shape	Order	Triangle	Order
1/8	1.94313093		4.26827573		4.65382003	
1/16	1.94288801		4.25165430		4.64788746	
1/32	1.94288581	6.7867	4.24202856	0.7824	4.64781308	6.3157
1/64	1.94288527	2.0265	4.23602214	0.6771	4.64781232	6.5994

**Table 2**

The first (real) transmission eigenvalues computed by the secant method with  $\mu = 1/16, \lambda = 1/4, \rho_0 = 1, \rho_1 = 4$ . "NOI" denotes the number of iterations.

Unit square	NOI	L-shape	NOI	Triangle	NOI
1.94288499	7	4.90134451	6	5.36467493	7



**Fig. 1.**  $f_h(\tau)$  v.s.  $\tau$ .

5.2. Monotonicity of  $f_h(\tau)$

We study  $f_h(\tau)$  for three domains using the meshes with  $h_3 \approx 1/32$  and parameters given in (28). The first elastic eigenvalues are  $\delta_1 = 3.251402$  for the unit square,  $\delta_1 = 4.325472$  for the L-shape domain, and  $\delta_1 = 13.444678$  for the disk. According to Theorem 4.1,  $f(\tau)$  is a decreasing function on  $(0, \frac{\delta_1(\rho_0 + \rho_1)}{2\rho_0\rho_1})$ . Plugging the values for  $\delta_1, \rho_0, \rho_1$ , one has that  $f(\tau)$  is decreasing on

$$(0, 2.032126), (0, 2.703420) \text{ and } (0, 8.402924), \tag{30}$$

for the unit square, the L-shape domain and the triangle, respectively. In Fig. 1, we plot  $f_h(\tau) = \gamma_h(\tau) - \tau$  for three domains, where  $\gamma_h(\tau)$  is the smallest eigenvalue of (10). It can be seen that  $f_h(\tau)$  is decreasing on much larger intervals than those in (30) predicted by Theorem 4.1.

5.3. Transmission eigenvalues

We choose  $x_0 = 0.1$  and  $x_1 = 0.2$  in the secant method. In Table 2, we show the first (real) transmission eigenvalues computed by the secant method for three domains with mesh size  $h \approx 1/32$  and  $tol = 1.0e - 6$ . The numbers of iterations are shown as well.

**Declaration of competing interest**

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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