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COMMUNICATIONS IN PARTIAL DIFFERENTIAL EQUATIONS
2020, VOL. 45, NO. 10, 1335–1380
https://doi.org/10.1080/03605302.2020.1774895

Inverse random source problems for time-harmonic acoustic and elastic waves

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ABSTRACT
This paper concerns the random source problems for the time-harmonic acoustic and elastic wave equations in two and three dimensions. The goal is to determine the compactly supported external force from the radiated wave field measured in a domain away from the source region. The source is assumed to be a microlocally isotropic generalized Gaussian random function such that its covariance operator is a classical pseudo-differential operator. Given such a distributional source, the direct problem is shown to have a unique solution by using an integral equation approach and the Sobolev embedding theorem. For the inverse problem, we demonstrate that the amplitude of the scattering field averaged over the frequency band, obtained from a single realization of the random source, determines uniquely the principle symbol of the covariance operator. The analysis employs asymptotic expansions of the Green functions and microlocal analysis of the Fourier integral operators associated with the Helmholtz and Navier equations.

ARTICLE HISTORY
Received 12 October 2018
Accepted 15 May 2020

KEYWORDS
Elastic wave equation; Gaussian random function; Helmholtz equation; inverse source problem; uniqueness

2010 MATHEMATICS SUBJECT CLASSIFICATION
78A46; 65C30

1. Introduction

The inverse source scattering in waves, as an important and active research subject in inverse scattering theory, are to determine the unknown sources that generate prescribed radiated wave patterns [1]. It has been considered as a basic mathematical tool for the solution of many medical imaging modalities [2], such as magnetoencephalography (MEG), electroencephalography (EEG), and electroneurography (ENG). These imaging modalities are noninvasive neurophysiological techniques that measure the electric or magnetic fields generated by neuronal activity of the brain. The spatial distributions of the measured fields are analyzed to localize the sources of the activity within the brain to provide information about both the structure and function of the brain [3–5]. The inverse source scattering problem has also attracted much research in the community of antenna design and synthesis [6]. A variety of antenna-embedding materials or substrates, including non-magnetic dielectrics, magneto-dielectrics, and double negative meta-materials are of great interest.

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Driven by these significant applications, the inverse source scattering problems have continuously received much attention and have been extensively studied by many researchers. There are a lot of available mathematical and numerical results, especially for the acoustic waves or the Helmholtz equation [7–12]. In general, the inverse source problem does not have a unique solution due to the existence of non-radiating sources [13–15]. Some additional constraint or information is needed in order to obtain a unique solution, such as to seek the minimum energy solution which represents the pseudo-inverse solution for the inverse problem. For electromagnetic waves, Ammari et al. showed uniqueness and presented an inversion scheme in [3] to reconstruct dipole sources based on a low-frequency asymptotic analysis of the time-harmonic Maxwell equations. In [16], Albanese and Monk discussed uniqueness and non-uniqueness of the inverse source problems for Maxwell's equations. Computationally, a more serious issue is the lack of stability, i.e., a small variation in the measured data may lead to a huge error in the reconstruction. Recently, it has been realized that the use of multi-frequency data can overcome the difficulties of non-uniqueness and instability which are encountered at a single frequency. In [17], Bao et al. initialized the mathematical study on the stability of the inverse source problem for the Helmholtz equation by using multi-frequency data. Since then, the increasing stability has become an interesting research topic in the study of inverse source problems [18–20]. We refer to [21] for a topic review on solving general inverse scattering problems with multi-frequencies.

Recently, the elastic wave scattering problems have received ever increasing attention for their important applications in many scientific areas such as geophysics and seismology [22–26]. However, the inverse source problem is much less studied for the elastic waves. The elastic wave equation is challenging due to the coexistence of compressional and shear waves that have different wavenumbers. Consequently, the Green tensor of the Navier equation has a more complicated expression than the Green function of the Helmholtz equation does. A more sophisticated analysis is required.

In many applications the source and hence the radiating field may inherently be considered random [27]. Therefore, their governing equations are stochastic differential equations. Although the deterministic counterparts have been well studied, little is known for the stochastic inverse problems due to randomness and uncertainties. A uniqueness result may be found in [28] for an inverse random source problem. It was shown that the auto-correlation function of the random source was uniquely determined by the auto-correlation function of the radiated field. Recently, effective mathematical models and efficient computational methods have been developed in [29–34] for inverse random source scattering problems, where the stochastic wave equations are considered and the random sources are assumed to be driven by additive white noise. In stochastic setting the inverse problems are often formulated to determine the statistical properties such as the mean and variance. The methods mentioned are based on observations of the correlations in the scattering data. By the strong law of large numbers, the correlations have to be approximated by taking fairly large number of realizations of the measurement. We refer to [35] for statistical inversion theory on general random inverse problems.

This paper takes another perspective to randomness following earlier work in [36–38]. We assume that all the data is produced by a single realization of a random source. If the data is exceptional noisy or corrupted, the recovery of this single source realization may be infeasible. However, it may be possible to recover some statistical parameters of the source if observations of the radiating field are available at multiple wavelengths.

Here, we develop a unified theory on both of the direct and inverse scattering problems for the time-harmonic acoustic and elastic wave equations. The source is assumed to be a generalized Gaussian random function which is supported in a bounded domain $D \subset \mathbb{R}^d$, $d = 2$ or 3. In addition, we assume that the covariance of the random source is described by a pseudo-differential operator with the principle symbol given by

$$\phi(x)|\xi|^{-m}, \quad m \in \left[d, d+\frac{1}{2}\right),$$

where ϕ is a smooth non-negative function supported on D and is called the micro-correlation strength of the source. The parameter m indicates how irregular realizations such a random process has. What is more, the micro-correlation strength identifies where this most irregular behavior is strongest and where it is more dampened. This large class of random fields includes stochastic processes like the fractional Brownian motion and Markov field [38].

When $m \in [d, d+\frac{1}{2})$, we can only ensure that the source belongs to a Sobolev space with negative smoothness index almost surely. Hence, the direct scattering problem requires a careful analysis since the source is non-smooth. In this work, we establish the well-posedness of the direct scattering problems for both wave equations with such rough sources in Theorems 3.3 and 4.3, respectively.

The inverse scattering problem aims at reconstructing the micro-correlation strength of the source ϕ from the scattered field measured in a bounded domain U where $\bar{U} \cap \bar{D} = \emptyset$. For a single realization of the random source, we measure the amplitude of the scattering field averaged over the frequency band in a bounded and simply connected domain U, i.e., for some large $Q > 0$, our data is given by

$$\int_1^Q \kappa^s |u(x,\kappa)|^2 \mathrm{d}\kappa \quad \text{or} \quad \int_1^Q \omega^s |\boldsymbol{u}(x,\omega)|^2 \mathrm{d}\omega, \quad x \in U, \tag{1.1}$$

where $s \in \mathbb{R}^+$ depends on m, $\kappa > 0$ and $\omega > 0$ are the wavenumber and the angular frequency, u and $\boldsymbol{u}$ represents the pressure of the acoustic wave equation and the displacement of the elastic wave equation, respectively.

Combining harmonic and microlocal analysis, we show that: for acoustic waves, the micro-correlation strength function ϕ can be recovered given data in (1.1); For elastic waves, note that the source is a vector, if the components of the random source are independent and the principle symbol of the pseudo-differential operator of each component coincides, thus, the micro-correlation strength function ϕ can be determined uniquely by these measurements.

This work is motivated by [36, 38], where an inverse problem was considered for the two-dimensional random Schrödinger equation. The potential function in the Schrödinger equation was assumed to be a Gaussian random function with a pseudo-differential operator describing its covariance. It was shown that the principle symbol of the covariance operator can be determined uniquely by the backscattered field, generated by a single realization of the random potential and a point source as the incident field. A closely related problem can be found in [37]. The authors considered the uniqueness for an inverse acoustic scattering problem in a half-space with an impedance boundary condition, where the impedance coefficient was assumed to be a Gaussian random function whose covariance operator is a pseudo-differential operator.

The paper is organized as follows. In Section 2, we introduce some commonly used Sobolev spaces, give a precise mathematical description of the generalized Gaussian random function, and present several lemmas on rough fields and random variables. Section 3 is devoted to the study of the acoustic wave equation in the two- and three-dimensional cases. The well-posedness of the direct problems are examined. The uniqueness of the inverse problem is achieved. Section 4 addresses the two- and three-dimensional elastic wave equations. Analogous results are obtained. The direct problem is shown to have a unique solution and the inverse problem is proved to have the uniqueness to recover the principle symbol of the covariance operator for the random source. This paper is concluded with some general remarks in Section 5.

2. Preliminaries

In this section, we introduce some necessary notation such as Sobolev spaces and generalized Gaussian random functions which are used throughout the paper.

2.1. Sobolev spaces

Let $\mathbb{R}^d$ be the d-dimensional space, where $d = 2$ or 3. Denote by $C_0^\infty(\mathbb{R}^d)$ the set of smooth functions with compact support and by $\mathcal{D}'(\mathbb{R}^d)$ the set of generalized (distributional) functions. For $1 < p < \infty, s \in \mathbb{R},$ the Sobolev space $H^{s,p}(\mathbb{R}^d)$ is defined by

$$H^{s,p}(\mathbb{R}^d) = \{h = (I - \Delta)^{-\frac{s}{2}} g : g \in L^p(\mathbb{R}^d)\},$$

which has the norm

$$||h||_{H^{s,p}(\mathbb{R}^d)} = ||(I - \Delta)^{\frac{s}{2}} h||_{L^p(\mathbb{R}^d)}.$$

With the definition of Sobolev spaces in the whole space, we can define the Sobolev spaces $H^{s,p}(V)$ for any Lipschitz domain $V \subset \mathbb{R}^d$ as the restrictions to V of the elements in $H^{s,p}(\mathbb{R}^d)$. The norm is defined by

$$||h||_{H^{s,p}(V)} = \inf\{||g||_{H^{s,p}(\mathbb{R}^d)} : g|_V = h\}.$$

According to [39], for $s \in \mathbb{R}$ and $1 < p < \infty,$ we can define $H_0^{s,p}(V)$ as the space of all distributions $h \in H^{s,p}(\mathbb{R}^d)$ such that $\mathrm{supp} h \subset \bar{V}$ and the the norm is defined by

$$||h||_{H_0^{s,p}(V)} = ||h||_{H^{s,p}(\mathbb{R}^d)}.$$

It is known that $C_0^\infty(V)$ is dense in $H_0^{s,p}(V)$ for any $1 < p < \infty, s \in \mathbb{R};$ $C_0^\infty(V)$ is dense in $H^{s,p}(V)$ for any $1 < p < \infty, s \leq 0;$ $C^\infty(\bar{V})$ is dense in $H^{s,p}(V)$ for any $1 < p < \infty,$ $s \in \mathbb{R}.$ Additionally, by [39, Propositions 2.4 and 2.9], for any $s \in \mathbb{R}, p, q \in (1, \infty)$ satisfying $\frac{1}{p} + \frac{1}{q} = 1,$ we have

$$H_0^{-s,q}(V) = (H^{s,p}(V))' \quad \text{and} \quad H^{-s,q}(V) = (H_0^{s,p}(V))',$$

where the prime denotes the dual space.

2.2. Generalized gaussian random functions

In this subsection, we provide a precise mathematical description of the generalized Gaussian random function. Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a complete probability space. The function q is said to be a generalized Gaussian random function if $q : \Omega \to \mathcal{D}'(\mathbb{R}^d)$ is a measurable map such that for every $\hat{\omega} \in \Omega$, the mapping $\hat{\omega} \in \Omega \mapsto \langle q(\hat{\omega}), \psi \rangle$ is a Gaussian random variable for all $\psi \in C_0^\infty(\mathbb{R}^d)$. The expectation and the covariance of the generalized Gaussian random function q can be defined by

$$\mathbb{E}q : \psi \in C_0^\infty(\mathbb{R}^d) \mapsto \mathbb{E}\langle q, \psi \rangle \in \mathbb{R},$$
$$\mathrm{Cov}q : (\psi_1, \psi_2) \in C_0^\infty(\mathbb{R}^d)^2 \mapsto \mathrm{Cov}(\langle q, \psi_1 \rangle, \langle q, \psi_2 \rangle) \in \mathbb{R},$$

where $\mathbb{E}\langle q, \psi \rangle$ denotes the expectation of $\langle q, \psi \rangle$ and

$$\mathrm{Cov}(\langle q, \psi_1 \rangle, \langle q, \psi_2 \rangle) = \mathbb{E}((\langle q, \psi_1 \rangle - \mathbb{E}\langle q, \psi_1 \rangle)(\langle q, \psi_2 \rangle - \mathbb{E}\langle q, \psi_2 \rangle))$$

denotes the covariance of $\langle q, \psi_1 \rangle$ and $\langle q, \psi_2 \rangle$. The covariance operator $C_q : C_0^\infty(\mathbb{R}^d) \to \mathcal{D}'(\mathbb{R}^d)$ is defined by

$$\langle C_q \psi_1, \psi_2 \rangle = \mathrm{Cov}(\langle q, \psi_1 \rangle, \langle q, \psi_2 \rangle) = \mathbb{E}(\langle q - \mathbb{E}q, \psi_1 \rangle \langle q - \mathbb{E}q, \psi_2 \rangle). \tag{2.1}$$

Let $k_q(x, y)$ be the Schwartz kernel of the covariance operator C_q. We also call $k_q(x, y)$ the covariance function of q. Thus, (2.1) means that

$$k_q(x, y) = \mathbb{E}((q(x) - \mathbb{E}q(x))(q(y) - \mathbb{E}q(y))) \tag{2.2}$$

in the sense of generalized functions.

In this paper, we assume that each component of the external source is a generalized, microlocally isotropic Gaussian random function. For this end, let $D \subset \mathbb{R}^d$ be a bounded and simply connected domain. We introduce the following definition.

Definition 2.1. A generalized Gaussian random function q on $\mathbb{R}^d$ is called microlocally isotropic of order m in D, if the realizations of q are almost surely supported in the domain D and its covariance operator C_{q} is a classical pseudo-differential operator having the principal symbol $\phi(x)|\xi|^{-m}$, where $\phi \in C_0^\infty(\mathbb{R}^d), \mathrm{supp}\phi \subset D$ and $\phi(x) \geq 0$ for all $x \in \mathbb{R}^d$.

We refer to [36, 38] for examples, such as fractional Brownian motion and Markov field, and numerical illustrations of random fields and covariance structures, which satisfy the above definition. Recall that the parameter m indirectly indicates the Sobolev smoothness of the realizations of the field as we will see in the next lemma and the microlocal strength identifies the amplitude of these oscillations. In particular, we are interested in the case $m \in [d, d + \frac{1}{2})$, which corresponds to rough fields. Now we introduce three lemmas and give an assumption which will be used in subsequent analysis.

Lemma 2.2. *Let f be a generalized and microlocally isotropic Gaussian random function of order m in D. If $m = d$, then $f \in H^{-\varepsilon,p}(D)$ almost surely for all $\varepsilon > 0, 1 < p < \infty$. If $m \in \left(d, d+\frac{1}{2}\right)$, then $f \in C^{\alpha}(D)$ almost surely for all $\alpha \in \left(0, \frac{m-d}{2}\right)$.*

Lemma 2.3. *Let X and Y be two zero-mean random variables such that the pair (X, Y) is a Gaussian random vector. Then we have*

$$\mathbb{E}((X^2 - \mathbb{E}X^2)(Y^2 - \mathbb{E}Y^2)) = 2(\mathbb{E}XY)^2.$$

Lemma 2.4. *Let X_t, $t \geq 0$ be a real valued stochastic process with a continuous path of zero mean, i.e., $\mathbb{E}X_t = 0$. Assume that for some constants $c > 0$ and $\beta > 0$ such that the condition*

$$|\mathbb{E}(X_t X_{t+r})| \leq c(1+r)^{-\beta} \tag{2.3}$$

holds for all $t, r \geq 0$. Then

$$\lim_{Q\to\infty} \frac{1}{Q}\int_1^Q X_t \mathrm{d}t = 0$$

almost surely.

Lemma 2.2 is a direct consequence of Theorem 2 in [38]. Lemma 2.3 is shown in [36] as Lemma 4.2. The ergodic result of Lemma 2.4 is an immediate corollary of [40, p. 94]. To establish the main results, we need the following assumption.

Assumption A. The external source f is assumed to have a compact support $D \subset \mathbb{R}^d$. Let $U \subset \mathbb{R}^d \setminus \bar{D}$ be the measurement domain of the wave field. We assume that D and U are two bounded and simply connected domains and there is a positive distance between D and U.

3. Acoustic waves

This section addresses the direct and inverse source scattering problems for the Helmholtz equation in two- and three-dimensional space. The external source is assumed to be a generalized Gaussian random function whose covariance operator is a classical pseudo-differential operator. The direct problem is shown to have a unique solution. For the inverse problem, we show that the principle symbol of the covariance operator can be determined uniquely by the scattered field obtained from a single realization of the random source.

3.1. The direct scattering problem

Consider the Helmholtz equation in a homogeneous medium

$$\Delta u + \kappa^2 u = f \quad \text{in } \mathbb{R}^d, \tag{3.1}$$

where $\kappa > 0$ is the wavenumber, u is the wave field, and f is a generalized Gaussian random function. Note that u is a random field since f is a random function. To ensure the uniqueness of the solution for (3.1), the usual Sommerfeld radiation condition is imposed

$$\lim_{r\to\infty} r^{\frac{d-1}{2}}(\partial_r u - \mathrm{i}\kappa u) = 0, \quad r = |x|, \tag{3.2}$$

uniformly for all directions $\hat{x} = x/|x|$. In addition, the external source function f satisfies the following assumption.

Assumption B. The generalized Gaussian random field f is microlocally isotropic of order m in D, where $m \in [d, d+\frac{1}{2})$. The principle symbol of its covariance operator C_f is $\phi(x)|\xi|^{-m}$ with $\phi \in C_0^\infty(D)$ and $\phi \geq 0$. Moreover, the mean value of f is zero, i.e., $\mathbb{E}(f) = 0$.

Recall that by Lemma 2.2, the random source $f(\hat{\omega})$ belongs with probability one to the Sobolev space $H^{-\varepsilon,p}(D)$ for all $\varepsilon > 0, 1 < p < \infty$. Hence it suffices to show that the direct scattering problem is well-posed when f is a deterministic non-smooth function in $H^{-\varepsilon,p}(D)$.

First, we show some regularity results of the fundamental solution. These results play an important role in the proof of the well-posedness. Let $\Phi_d(x,y,\kappa)$ be the fundamental solution for the two- and three-dimensional Helmholtz equation. Explicitly, we have

$$\Phi_2(x,y,\kappa) = \frac{\mathrm{i}}{4} H_0^{(1)}(\kappa|x-y|), \quad \Phi_3(x,y,\kappa) = \frac{1}{4\pi}\frac{e^{\mathrm{i}\kappa|x-y|}}{|x-y|}, \tag{3.3}$$

where $H_0^{(1)}$ is the Hankel function of the first kind with order zero. We shall study the asymptotic properties of the fundamental solutions and their derivatives when x is close to y. For the two-dimensional case, we recall that

$$H_n^{(1)}(t) = J_n(t) + \mathrm{i}Y_n(t), \tag{3.4}$$

where J_n and Y_n are the Bessel functions of the first and second kind with order n, respectively. They admit the following expansions

$$\begin{aligned} J_n(t) &= \sum_{p=0}^{\infty} \frac{(-1)^p}{p!(n+p)!}\left(\frac{t}{2}\right)^{n+2p}, \\ Y_n(t) &= \frac{2}{\pi}\left\{\ln\frac{t}{2} + \gamma\right\} J_n(t) - \frac{1}{\pi}\sum_{p=0}^{n-1}\frac{(n-1-p)!}{p!}\left(\frac{2}{t}\right)^{n-2p} \\ &\quad - \frac{1}{\pi}\sum_{p=0}^{\infty}\frac{(-1)^p}{p!(n+p)!}\left(\frac{t}{2}\right)^{n+2p}\{\psi(p+n)+\psi(p)\}, \end{aligned} \tag{3.5}$$

where $\gamma := \lim_{p\to\infty}\left\{\sum_{j=1}^{p} j^{-1} - \ln p\right\}$ denotes the Euler constant, $\psi(0) = 0$, $\psi(p) = \sum_{j=1}^{p} j^{-1}$, and the finite sum in (3.5) is set to be zero for $n = 0$. Using (3.4)–(3.5), we may verify that

$$H_0^{(1)}(t) = \frac{2\mathrm{i}}{\pi}\ln\frac{t}{2} + \left(1 + \frac{2\mathrm{i}}{\pi}\gamma\right) + O\left(t^2\ln\frac{t}{2}\right), \tag{3.6}$$

$$H_1^{(1)}(t) = -\frac{2\mathrm{i}}{\pi}\frac{1}{t} + \frac{\mathrm{i}}{\pi} t \ln\frac{t}{2} + \left(1 + \frac{2\mathrm{i}}{\pi}\gamma - \frac{\mathrm{i}}{\pi}\right)\frac{t}{2} + O\left(t^3 \ln\frac{t}{2}\right). \tag{3.7}$$

Using the recurrence relations for the Hankel function of the first kind (see [41, Eq. (5.6.3)])

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[t^{-n} H_n^{(1)}(t)\right] = -t^{-n} H_{n+1}^{(1)}(t), \tag{3.8}$$

we may show from (3.6) to (3.7) that

$$\begin{aligned} \Phi_2(x, y, \kappa) &= \frac{\mathrm{i}}{4} H_0^{(1)}(\kappa|x-y|) \\ &= -\frac{1}{2\pi}\ln\frac{|x-y|}{2} + \left(\frac{\mathrm{i}}{4} - \frac{\gamma}{2\pi}\right) + O\left(|x-y|^2 \ln\frac{|x-y|}{2}\right), \end{aligned} \tag{3.9}$$

$$\begin{aligned} \partial_{y_i}\Phi_2(x, y, \kappa) &= -\frac{\kappa\mathrm{i}}{4}(y_i - x_i)\frac{H_1^{(1)}(\kappa|x-y|)}{|x-y|} \\ &= -\frac{1}{2\pi}\frac{y_i - x_i}{|x-y|^2} + O((y_i - x_i)\ln\frac{|x-y|}{2}). \end{aligned} \tag{3.10}$$

For the three-dimensional case, a simple calculation yields that

$$\Phi_3(x, y, \kappa) = \frac{e^{\mathrm{i}\kappa|x-y|}}{4\pi|x-y|}, \tag{3.11}$$

$$\partial_{y_i}\Phi_3(x, y, \kappa) = \frac{(y_i - x_i)}{4\pi|x-y|^3} e^{\mathrm{i}\kappa|x-y|}(\mathrm{i}\kappa|x-y| - 1). \tag{3.12}$$

Lemma 3.1. *Given any $x \in \mathbb{R}^d$, we have $\Phi_2(x, \cdot, \kappa) \in L^2_{\mathrm{loc}}(\mathbb{R}^2) \cap H^{1,p}_{\mathrm{loc}}(\mathbb{R}^2)$ for any $p \in (1, 2)$ and $\Phi_3(x, \cdot, \kappa) \in L^2_{\mathrm{loc}}(\mathbb{R}^3) \cap H^{1,p}_{\mathrm{loc}}(\mathbb{R}^3)$ for any $p \in \left(1, \frac{3}{2}\right)$.*

Proof. For any fixed $x \in \mathbb{R}^d$, let $V \subset \mathbb{R}^d$ be a bounded domain containing x. Denote $\rho := \sup_{y\in V}|x-y|$, then we have $V \subset B_\rho(x)$.

For $d=2$, by (3.9) and (3.10), it suffices to show that

$$\ln\frac{|x-y|}{2} \in L^2(V), \quad \frac{y_i - x_i}{|x-y|^2} \in L^p(V), \quad \forall\ p \in (1, 2).$$

A direct calculation yields

$$\int_V \left|\ln\frac{|x-y|}{2}\right|^2 \mathrm{d}y \le \int_{B_\rho(x)} \left|\ln\frac{|x-y|}{2}\right|^2 \mathrm{d}y \lesssim \int_0^\rho r\left|\ln\frac{r}{2}\right|^2 \mathrm{d}r < \infty$$

and

$$\int_V \left|\frac{y_i - x_i}{|x-y|^2}\right|^p \mathrm{d}y \le \int_{B_\rho(x)} \frac{1}{|x-y|^p}\mathrm{d}y \lesssim \int_0^\rho r^{1-p}\mathrm{d}r < \infty, \quad \forall\ p \in (1, 2).$$

Hereafter, the notation $a \lesssim b$ means $a \le Cb$, where $C > 0$ is a generic constant which may change step by step in the proofs. Thus, we conclude that $\Phi_2(x, \cdot, \kappa) \in L^2_{\rm loc}(\mathbb{R}^2) \cap H^{1,p}_{\rm loc}(\mathbb{R}^2)$ for any $p \in (1, 2)$.

For $d = 3$, from (3.11) and (3.12), it suffices to prove that

$$\frac{e^{\mathrm{i}\kappa|x-y|}}{|x-y|} \in L^2(V), \quad e^{\mathrm{i}\kappa|x-y|}\frac{y_i - x_i}{|x-y|^3} \in L^p(V) \quad \forall\, p \in \left(1, \frac{3}{2}\right).$$

Similarly, we may have from a simple calculation that

$$\int_V \left|\frac{e^{\mathrm{i}\kappa|x-y|}}{|x-y|}\right|^2 \mathrm{d}y \le \int_{B_\rho(x)} \frac{1}{|x-y|^2}\mathrm{d}y \lesssim \int_0^\rho 1\mathrm{d}r < \infty$$

and

$$\int_V \left|e^{\mathrm{i}\kappa|x-y|}\frac{y_i - x_i}{|x-y|^3}\right|^p \mathrm{d}y \le \int_{B_\rho(x)} \frac{1}{|x-y|^{2p}}\mathrm{d}y \lesssim \int_0^\rho r^{2-2p}\mathrm{d}r < \infty \quad \forall\, p \in \left(1, \frac{3}{2}\right),$$

which show that $\Phi_3(x, \cdot, \kappa) \in L^2_{\rm loc}(\mathbb{R}^3) \cap H^{1,p}_{\rm loc}(\mathbb{R}^3)$ for any $p \in \left(1, \frac{3}{2}\right)$. □

Let V and G be any two bounded domains in $\mathbb{R}^d$. By Lemma 3.1 and the Sobolev embedding theorem, we obtain that $\Phi_d(x, \cdot, \kappa) \in H^s(V)$ where $s \in (0, 1)$ for $d = 2$ and $s \in \left(0, \frac{1}{2}\right)$ for $d = 3$. Hence, given $g \in H_0^{-s}(V)$, we can define the operator H_κ in the dual sense by

$$H_\kappa g(x) = \int_V \Phi_d(x, y, \kappa)g(y)\mathrm{d}y, \quad x \in G.$$

Following the similar arguments in [42, Theorem 8.2], we may show the following regularity of the operator H_κ. The proof is omitted here for brevity.

Lemma 3.2. *The operator $H_\kappa : H_0^{-s}(V) \to H^s(G)$ is bounded for $s \in (0, 1)$ in two dimensions or for $s \in \left(0, \frac{1}{2}\right)$ in three dimensions.*

Theorem 3.3. *For some fixed $s \in \left(0, 1 - \frac{d}{6}\right)$ and $\tilde{p} \in \left(1, \frac{d}{d-1}\right)$, assume $0 < \varepsilon < s, 1 < p < \min\left(\frac{\tilde{p}d}{d+\tilde{p}(\varepsilon-1)}, \frac{2d}{d+2(\varepsilon-s)}\right)$ and $\frac{1}{p} + \frac{1}{p'} = 1$, then the scattering problem (3.1)–(3.2) with the source $f \in H_0^{-\varepsilon,p'}(D)$ attains a unique solution $u \in H^{\varepsilon,p}_{\rm loc}(\mathbb{R}^d)$, which can be represented by*

$$u(x, \kappa) = -\int_D \Phi_d(x, y, \kappa)f(y)\mathrm{d}y. \tag{3.13}$$

Proof. It is clear that the scattering problem (3.1)–(3.2) with $f = 0$ only has the zero solution. Hence the uniqueness follows. Now we focus on the existence. Noting $s \in \left(0, 1 - \frac{d}{6}\right)$, we have from a simple calculation that $1 < \frac{2d}{d+2(1-s)} < \frac{3}{2}$. Since $1 < p < \frac{\tilde{p}d}{d+\tilde{p}(\varepsilon-1)}$, a direct computation shows that $1\tilde{p} - \frac{1}{d} < \frac{1}{p} - \frac{\varepsilon}{d}$. By Lemma 3.1 and the Sobolev embedding theorem, we obtain that $\Phi_d(x, \cdot, \kappa) \in H^{1,\tilde{p}}_{\rm loc}(\mathbb{R}^d) \subset H^{\varepsilon,p}_{\rm loc}(\mathbb{R}^d)$. Since

$\Delta u+\kappa^2 u=f\in H_0^{-\varepsilon,p'}(D)$, we have in the sense of distribution that

$$\int_{B_r}(\Delta u(y)+\kappa^2 u(y))\Phi_d(x,y,\kappa)\mathrm{d}y=\int_{B_r}\Phi_d(x,y,\kappa)f(y)\mathrm{d}y. \tag{3.14}$$

Here $B_r=\{y\in\mathbb{R}^d:|y|\le r\}$, where $r>0$ is sufficiently large such that $\bar{D}\subset B_r$. Denote by S_{A} the operator which acts on u on the left-hand side of (3.14). For $\varphi\in C^\infty(\mathbb{R}^d)$, we have

$$\begin{aligned}(S_{\mathrm{A}}\varphi)(x):&=\int_{B_r}(\Delta\varphi(y)+\kappa^2\varphi(y))\Phi_d(x,y,\kappa)\mathrm{d}y\\&=\int_{B_r\setminus B_\delta(x)}(\Delta\varphi(y)+\kappa^2\varphi(y))\Phi_d(x,y,\kappa)\mathrm{d}y+\int_{B_\delta(x)}(\Delta\varphi(y)+\kappa^2\varphi(y))\Phi_d(x,y,\kappa)\mathrm{d}y\\&=\int_{B_r\setminus B_\delta(x)}(\Delta\varphi(y)\Phi_d(x,y,\kappa)-\varphi(y)\Delta\Phi_d(x,y,\kappa))\mathrm{d}y+\int_{B_\delta(x)}(\Delta\varphi(y)+\kappa^2\varphi(y))\Phi_d(x,y,\kappa)\mathrm{d}y\\&=\int_{B_\delta(x)}(\Delta\varphi(y)+\kappa^2\varphi(y))\Phi_d(x,y,\kappa)\mathrm{d}y+\int_{\partial B_r}\left(\frac{\partial\varphi(y)}{\partial\nu(y)}\Phi_d(x,y,\kappa)-\varphi(y)\frac{\partial\Phi_d(x,y,\kappa)}{\partial\nu(y)}\right)\mathrm{d}s(y)\\&\quad+\int_{\partial B_\delta(x)}\left(\frac{\partial\varphi(y)}{\partial\nu(y)}\Phi_d(x,y,\kappa)-\varphi(y)\frac{\partial\Phi_d(x,y,\kappa)}{\partial\nu(y)}\right)\mathrm{d}s(y),\end{aligned}$$

where $\delta>0$ is a sufficiently small number, and $\nu(y)$ denotes the unit normal which directs to the exterior of B_r for $y\in\partial B_r$ and directs to the interior of $B_\delta(x)$ for $y\in\partial B_\delta(x)$. Using the mean value theorem, we get

$$\lim_{\delta\to 0}\int_{\partial B_\delta(x)}\left(\frac{\partial\varphi(y)}{\partial\nu(y)}\Phi_d(x,y,\kappa)-\varphi(y)\frac{\partial\Phi_d(x,y,\kappa)}{\partial\nu(y)}\right)\mathrm{d}s(y)=-\varphi(x)$$

and

$$\lim_{\delta\to 0}\int_{B_\delta(x)}(\Delta\varphi(y)+\kappa^2\varphi(y))\Phi_d(x,y,\kappa)\mathrm{d}y=0.$$

Combining the above equations gives that

$$(S_{\mathrm{A}}\varphi)(x)=-\varphi(x)+\int_{\partial B_r}\left(\frac{\partial\varphi(y)}{\partial\nu(y)}\Phi_d(x,y,\kappa)-\varphi(y)\frac{\partial\Phi_d(x,y,\kappa)}{\partial\nu(y)}\right)\mathrm{d}s(y),$$

which implies

$$(S_{\mathrm{A}}u)(x)=-u(x)+\int_{\partial B_r}\left(\frac{\partial u(y)}{\partial\nu(y)}\Phi_d(x,y,\kappa)-u(y)\frac{\partial\Phi_d(x,y,\kappa)}{\partial\nu(y)}\right)\mathrm{d}s(y).$$

Since both u and Φ_d satisfy the Sommerfeld radiation condition, we have

$$\lim_{r\to\infty}\int_{\partial B_r}\left(\frac{\partial u(y)}{\partial\nu(y)}\Phi_d(x,y,\kappa)-u(y)\frac{\partial\Phi_d(x,y,\kappa)}{\partial\nu(y)}\right)\mathrm{d}s(y)=0.$$

Therefore

$$u(x,\kappa)=-\int_D\Phi_d(x,y,\kappa)f(y)\mathrm{d}y=-H_\kappa f(x).$$

Next is to show that $u \in H^{\varepsilon,p}_{\mathrm{loc}}(\mathbb{R}^d)$. From Lemma 3.2, we have that the operator $H_\kappa : H_0^{-s}(D) \to H^s_{\mathrm{loc}}(\mathbb{R}^d)$ for $s \in \left(0, 1-\frac{d}{6}\right)$ is bounded. The assumption $1 < p < \frac{2d}{d+2(\varepsilon-s)}$ implies that $\frac{1}{2}+\frac{\varepsilon-s}{d} < \frac{1}{p} < 1$ which yields $\frac{1}{2}-\frac{s}{d} < \frac{1}{p}-\frac{\varepsilon}{d}$. Thus, since $0<\varepsilon<s$, the Sobolev embedding theorem implies that $H^s(D)$ is embedded into $H^{\varepsilon,p}(D)$ and $H_0^{-\varepsilon,p'}(D)$ is embedded into $H_0^{-s}(D)$. Thus, the operator $H_\kappa : H_0^{-\varepsilon,p'}(D) \to H^{\varepsilon,p}_{\mathrm{loc}}(\mathbb{R}^d)$ is bounded, which completes the proof. □

3.2. Asymptotics of the correlation data

The three technical results proven in this section lay out the foundations for the proofs of our main results. Notice carefully that the main results for the cases $d=2$ and $d=3$ will require separate proofs. However, here we formulate general auxiliary lemmas that make the deviations of the main proofs moderate. First, in Lemmas 3.4 and 3.5 we study the general behavior of the asymptotics of the correlation data. At the end of the section, we prove Lemma 3.6, which will be used in the recovery of the local strength ϕ.

Recall that our strategy is to utilize the ergodicity of the weighted solution process $u(x,\kappa)$ averaged in (1.1) with respect to κ and to apply Lemma 2.4. To do this, we need to show that condition (2.3) is satisfied and, therefore, we study the asymptotic correlations in $u(x,\kappa)$.

To build the general theory, we study the asymptotics of integral

$$I(x,\kappa_1,\kappa_2) := \frac{1}{\kappa_1^{l_1}\kappa_2^{l_2}} \int_{\mathbb{R}^{2d}} e^{\mathrm{i}(c_1\kappa_1|x-y|-c_2\kappa_2|x-z|)} K(x,y,z)\mathbb{E}(q(y)q(z))\mathrm{d}y\mathrm{d}z \tag{3.15}$$

where

$$K(x,y,z) := \frac{(x_1-y_1)^{m_1}\cdots(x_d-y_d)^{m_d}(x_1-z_1)^{n_1}\cdots(x_d-z_d)^{n_d}}{|x-y|^{p_1}|x-z|^{p_2}}.$$

Here q stands for a generalized Gaussian random function satisfying Assumption B, and l_1, l_2, c_1, c_2, $m_1, \dots, n_d$, p_1, p_2 are nonnegative constants. Similarly, we define the integral

$$\tilde{I}(x,\kappa_1,\kappa_2) := \frac{1}{\kappa_1^{l_1}\kappa_2^{l_2}} \int_{\mathbb{R}^{2d}} e^{\mathrm{i}(c_1\kappa_1|x-y|+c_2\kappa_2|x-z|)} K(x,y,z)\mathbb{E}(q(y)q(z))\mathrm{d}y\mathrm{d}z. \tag{3.16}$$

Below, in the two next sections we show that the integrals $I(x,\kappa_1,\kappa_2)$ and $\tilde{I}(x,\kappa_1,\kappa_2)$ approximate the correlation data $\mathbb{E}(u(x,\kappa_1)\overline{u(x,\kappa_2)})$ and $\mathbb{E}(u(x,\kappa_1)u(x,\kappa_2))$, respectively. In fact, in the three-dimensional case, these quantities coincide.

Lemma 3.4. *For $\kappa_1,\kappa_2 \ge 1$, the estimates*

$$|I(x,\kappa_1,\kappa_2)| \le c_n(\kappa_1+\kappa_2)^{-(m+2\min(l_1,l_2))}(1+|\kappa_1-\kappa_2|)^{-n}, \tag{3.17}$$

$$|\mathbb{E}(\tilde{u}(x,\kappa_1)\tilde{u}(x,\kappa_2))| \le c_n(\kappa_1+\kappa_2)^{-n}(1+|\kappa_1-\kappa_2|)^{-m} \tag{3.18}$$

holds uniformly for $x \in U$, where $n \in \mathbb{N}$ is arbitrary.

Proof. To estimate the integral $I(x, \kappa_1, \kappa_2)$, we introduce the multiple coordinate transformation that allows to use the microlocal methods in our analysis. Noting $\mathbb{E}q = 0$ and (2.2), we conclude that the correlation function $\mathbb{E}(q(y)q(z))$ is the Schwartz kernel of a pseudo-differential operator C_q with a classical symbol $\sigma(y, \xi) \in S^{-m}_{1,0}(\mathbb{R}^d \times \mathbb{R}^d)$ which is defined by

$$S^{-m}_{1,0}(\mathbb{R}^d \times \mathbb{R}^d) := \{a(x, \xi) \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d) : |\partial^\alpha_\xi \partial^\beta_x a(x, \xi)| \le C_{\alpha,\beta}(1 + |\xi|)^{-m-|\alpha|}\}.$$

Here α, β are multiple indices, $|\alpha|$ denotes the sum of its component. The principle symbol of C_q is $\sigma^p(y, \xi) = \phi(y)|\xi|^{-m}$. The support of $\mathbb{E}(q(y)q(z))$ is contained in $D \times D$. We can write $\mathbb{E}(q(y)q(z))$ in terms of its symbol by

$$\mathbb{E}(q(y)q(z)) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{\mathrm{i}(y-z)\cdot\xi} \sigma(y, \xi) \mathrm{d}\xi. \tag{3.19}$$

To establish a uniform estimate with respect to the variable x, we extend the covariance function into the space $\mathbb{R}^{2d} \times \mathbb{R}^d$, and define $B_1(y, z, x) = \mathbb{E}(q(y)q(z))\theta(x)$ where $\theta(x) \in C^\infty_0(\mathbb{R}^d)$ equals to one in the domain U and has its support outside of the domain $\bar{D}$. Thus, we have

$$B_1(y, z, x) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{\mathrm{i}(y-z)\cdot\xi} c_1(y, x, \xi) \mathrm{d}\xi,$$

where $c_1(y, x, \xi) = \sigma(y, \xi)\theta(x) \in S^{-m}_{1,0}(\mathbb{R}^{2d} \times \mathbb{R}^d)$ with a principle symbol

$$c^p_1(y, x, \xi) = \phi(y)|\xi|^{-m}\theta(x).$$

To proceed the analysis, let us briefly revisit the conormal distributions of Hörmander type [43]. If $X \subset \mathbb{R}^d$ is an open set and $S \subset X$ is a smooth submanifold of X, we denote by $I(X; S)$ the distributions in $\mathcal{D}'(X)$ that are smooth in $X \setminus S$ and have a conormal singularity at S. In consequence, by (3.19), the correlation function $\mathbb{E}(q(y)q(z))$ is a conormal distribution in $\mathbb{R}^{2d}$ of Hörmander type having conormal singularity on the surface $S_1 = \{(y, z) \in \mathbb{R}^{2d} : y - z = 0\}$. Moreover, let $I_{\mathrm{comp}}(X; S)$ be the set of distributions supported in a compact subset of X. Let $X_1 \subset \mathbb{R}^{3d}$ be an open set containing $D \times D \times \mathrm{supp}(\theta)$ so that $B_1 \in I_{\mathrm{comp}}(X_1; S_1 \cap X_1)$.

Define the first coordinate transformation $\eta : \mathbb{R}^{3d} \to \mathbb{R}^{3d}$ by

$$(v, w, x) = \eta(y, z, x) = (y - z, y + z, x). \tag{3.20}$$

Substituting the coordinate transformation (3.20) into $B_1(y, z, x)$ gives

$$B_2(v, w, x) = B_1(\eta^{-1}(v, w, x)) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{\mathrm{i}v\cdot\xi} c_1\left(\frac{v + w}{2}, x, \xi\right) \mathrm{d}\xi,$$

which means that $B_2 \in I(\mathbb{R}^{3d}, S_2)$ where $S_2 := \{(v, w, x) : v = 0\}$. Actually, $B_2 \in I_{\mathrm{comp}}(X_2, X_2 \cap S_2)$ where $X_2 := \eta(\mathbf{D})$. To find out how the symbol transforms in the change of coordinates, we need to represent $c_1\left(\frac{v+w}{2}, x, \xi\right)$ with a symbol that does not depend on v. Using the representation theorem of conormal distribution [43, Lemma 18.2.1]), we obtain

$$B_2(v,w,x) = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{\mathrm{i}v\cdot\xi} c_2(w,x,\xi)\mathrm{d}\xi,$$

where $c_2(w,x,\xi)$ has the asymptotic expansion

$$c_2(w,x,\xi) \sim \sum_{l=0}^{\infty} \frac{\langle -\mathrm{i}D_v, D_\xi\rangle^l}{l!} c_1\left(\frac{v+w}{2}, x, \xi\right)\bigg|_{v=0} \in S^{-m}_{1,0}(\mathbb{R}^{2d}\times\mathbb{R}^d),$$

where D is defined by $D_j := -\mathrm{i}\partial_j$. In particular, the principle symbol of $c_2(w,x,\xi)$ is

$$c_2^p(w,x,\xi) = \phi\left(\frac{v+w}{2}\right)|\xi|^{-m}\theta(x)\bigg|_{v=0}.$$

We consider the phase of $I(x,\kappa_1,\kappa_2)$. A simple calculation shows that

$$\begin{aligned} c_1\kappa_1|x-y| - c_2\kappa_2|x-z| = (c_1\kappa_1 + c_2\kappa_2)\frac{|x-y|-|x-z|}{2} \\ + (c_1\kappa_1 - c_2\kappa_2)\frac{|x-y|+|x-z|}{2}. \end{aligned} \tag{3.21}$$

In the second set of coordinates, let $\frac{|x-y|\pm|x-z|}{2}$ play the role of two coordinates. We will do this change in two steps. First, for the two-dimensional case where $d=2$, we define $\tau_1 : \mathbb{R}^6 \to \mathbb{R}^6$ by

$$\tau_1(y,z,x) = (E_1,E_2,x),$$

where $E_1 = (t_1,s_1)$ and $E_2 = (t_2,s_2)$ with

$$\begin{aligned} t_1 &= \frac{1}{2}|x-y|, \quad s_1 = \frac{1}{2}|x-y|\arcsin\left(\frac{y_1-x_1}{|x-y|}\right), \\ t_2 &= \frac{1}{2}|x-z|, \quad s_2 = \frac{1}{2}|x-z|\arcsin\left(\frac{z_1-x_1}{|x-z|}\right). \end{aligned}$$

For the three-dimensional case where $d=3$, we define $\tau_1 : \mathbb{R}^9 \to \mathbb{R}^9$ by

$$\tau_1(y,z,x) = (E_1,E_2,x),$$

where $E_1 = (t_1,s_1,r_1)$ and $E_2 = (t_2,s_2,r_2)$ with

$$\begin{aligned} t_1 &= \frac{1}{2}|x-y|, \quad s_1 = \frac{1}{2}\arccos\left(\frac{y_3-x_3}{|x-y|}\right), \quad r_1 = \frac{1}{2}|x-y|\arctan\left(\frac{y_2-x_2}{y_1-x_1}\right), \\ t_2 &= \frac{1}{2}|x-z|, \quad s_2 = \frac{1}{2}\arccos\left(\frac{z_3-x_3}{|x-z|}\right), \quad r_2 = \frac{1}{2}|x-z|\arctan\left(\frac{z_2-x_2}{z_1-x_1}\right). \end{aligned}$$

Second, we define $\tau_2 : \mathbb{R}^{3d} \to \mathbb{R}^{3d}$ by

$$(g,h,x) = \tau_2(E_1,E_2,x) = (E_1-E_2, E_1+E_2, x).$$

Thus, combining the definitions of τ_1, τ_2 and (3.21), we have

$$c_1\kappa_1|x-y| - c_2\kappa_2|x-z| = (c_1\kappa_1 + c_2\kappa_2)g\cdot e_1 + (c_1\kappa_1 - c_2\kappa_2)h\cdot e_1,$$

where $e_1 = (1,0)$ for $d=2$, and $e_1 = (1,0,0)$ for $d=3$. Now we denote $\tau = \tau_2\circ\tau_1 : \mathbb{R}^{3d} \to \mathbb{R}^{3d}$ with $\tau(y,z,x) = (g,h,x)$. We consider the transformation $\rho = \eta\circ\tau^{-1} : \mathbb{R}^{3d} \to \mathbb{R}^{3d}$

with $\rho(g,h,x)=(v,w,x)$. Let us decompose the coordinate transform ρ into two parts $\rho=(\rho_1,\rho_2)$, the $\mathbb{R}^d$-valued function $\rho_1(g,h,x)=v$ and the $\mathbb{R}^{2d}$-valued function $\rho_2(g,h,x)=(w,x)$. The Jacobian J_ρ corresponding to the decomposing of the variables is given by

$$J_\rho=\begin{bmatrix}\rho'_{11} & \rho'_{12}\\ \rho'_{21} & \rho'_{22}\end{bmatrix}=\begin{bmatrix}J_g\rho_1 J_{(h,x)}\rho_1\\ J_g\rho_2 J_{(h,x)}\rho_2\end{bmatrix}.$$

By the definition of ρ, it is easy to see that $v=0$ if $g=0$. Hence we have $\rho_1(0,h,x)=0$ which implies $\rho'_{12}(0,h,x)=0$.

Next we consider the pull-back distribution $B_3=B_2\circ\rho$. It follow from [43, Theorem 18.2.9] that we get a representation for B_3:

$$B_3(g,h,x)=(2\pi)^{-d}\int_{\mathbb{R}^d}e^{\mathrm{i}g\cdot\xi}c_3(h,x,\xi)\mathrm{d}\xi, \tag{3.22}$$

where $c_3(h,x,\xi)\in S^{-m}_{1,0}(\mathbb{R}^{2d}\times\mathbb{R}^d)$ is a symbol satisfying

$$c_3(h,x,\xi)=c_2(\rho_2(g,h,x),((\rho'_{11}(g,h,x))^{-1})^T\xi)\times|\det\rho'_{11}(g,h,x)|^{-1}|_{g=0}+r(h,x,\xi),$$

where $r(h,x,\xi)\in S^{-m-1}_{1,0}(\mathbb{R}^{2d}\times\mathbb{R}^d)$. The principle symbol of $c_3(h,x,\xi)$ is given by

$$c_3^p(h,x,\xi)=\phi(y(g,h,x))|((\rho'_{11}(g,h,x))^{-1})^T\xi|^{-m}\theta(x)\times|\det\rho'_{11}(g,h,x)|^{-1}|_{g=0}.$$

Let $X_3:=\tau(X_1)$ and $S_3:=\{(g,h,x):g=0\}$, we have $B_3\in I_{\mathrm{comp}}(X_3,X_3\cap S_3)$. So we can write $I(x,\kappa_1,\kappa_2)$ in the following form

$$I(x,\kappa_1,\kappa_2)=\frac{1}{\kappa_1^{l_1}\kappa_2^{l_2}}\int_{\mathbb{R}^{2d}}e^{\mathrm{i}\left[(c_1\kappa_1+c_2\kappa_2)g\cdot e_1+(c_1\kappa_1-c_2\kappa_2)h\cdot e_1\right]}B_3(g,h,x)H(g,h,x)\mathrm{d}g\mathrm{d}h,$$

where

$$H(g,h,x)=K(x,y,z)\det((\tau^{-1})'(g,h,x)). \tag{3.23}$$

Here $y=y(g,h,x)$ and $z=z(g,h,x)$. Since H is smooth in X_3 in all variables and $I(\mathbb{R}^{3d},S_3)$ is closed under multiplication with a smooth function, we conclude that $B_3(g,h,x)H(g,h,x)\in I(\mathbb{R}^{3d},S_3)$. Multiplying (3.22) by H, we arrive at

$$B_3(g,h,x)H(g,h,x)=(2\pi)^{-d}\int_{\mathbb{R}^d}e^{\mathrm{i}g\cdot\xi}c_4(h,x,\xi)\mathrm{d}\xi, \tag{3.24}$$

where $c_4(h,x,\xi)$ has the asymptotic expansion

$$c_4(h,x,\xi)\sim\sum_{l=0}^{\infty}\frac{\langle -\mathrm{i}D_g,D_\xi\rangle^l}{l!}(c_3(h,x,\xi)H(g,h,x))\Big|_{g=0}.$$

In particular, the principle symbol of $c_4(h,x,\xi)$ is given by

$$c_4^p(h,x,\xi)=\phi(y(g,h,x))|((\rho'_{11}(g,h,x))^{-1})^T\xi|^{-m}\theta(x)|\det\rho'_{11}(g,h,x)|^{-1}H(g,h,x)|_{g=0}. \tag{3.25}$$

Combining (3.24) and the Fourier inversion rule, we obtain

$$B_3(g,h,x)H(g,h,x) = (\mathcal{F}^{-1}c_4)(h,x,g). \tag{3.26}$$

Substituting (3.26) into $I(x,\kappa_1,\kappa_2)$ gives

$$\begin{aligned} I(x,\kappa_1,\kappa_2) &= \frac{1}{\kappa_1^{l_1}\kappa_2^{l_2}}\int_{\mathbb{R}^{2d}} e^{\mathrm{i}\left[(c_1\kappa_1+c_2\kappa_2)g\cdot e_1+(c_1\kappa_1-c_2\kappa_2)h\cdot e_1\right]}(\mathcal{F}^{-1}c_4)(h,x,g)\mathrm{d}g\mathrm{d}h \\ &= \frac{1}{\kappa_1^{l_1}\kappa_2^{l_2}}\int_{\mathbb{R}^{d}} e^{\mathrm{i}(c_1\kappa_1-c_2\kappa_2)h\cdot e_1}c_4(h,x,-(c_1\kappa_1+c_2\kappa_2)e_1)\mathrm{d}h \\ &= \frac{1}{\kappa_1^{l_1}\kappa_2^{l_2}}\frac{1}{\mathrm{i}(c_1\kappa_1-c_2\kappa_2)}\int_{\mathbb{R}^{d}} c_4(h,x,-(c_1\kappa_1+c_2\kappa_2)e_1)\mathrm{d}e^{\mathrm{i}(c_1\kappa_1-c_2\kappa_2)h_1}\cdots\mathrm{d}h_d \\ &= -\frac{1}{\kappa_1^{l_1}\kappa_2^{l_2}}\frac{1}{\mathrm{i}(c_1\kappa_1-c_2\kappa_2)}\int_{\mathbb{R}^{d}} e^{\mathrm{i}(c_1\kappa_1-c_2\kappa_2)h_1}\partial_{h_1}c_4(h,x,-(c_1\kappa_1+c_2\kappa_2)e_1)\mathrm{d}h \\ &= (-1)^n\frac{1}{\kappa_1^{l_1}\kappa_2^{l_2}}\frac{1}{(\mathrm{i}(c_1\kappa_1-c_2\kappa_2))^n}\int_{\mathbb{R}^{d}} e^{\mathrm{i}(c_1\kappa_1-c_2\kappa_2)h_1}\partial_{h_1}^n c_4(h,x,-(c_1\kappa_1+c_2\kappa_2)e_1)\mathrm{d}h, \end{aligned} \tag{3.27}$$

where we use the integrations by parts n times and the fact that $c_4(h,x,\xi)$ is C^∞ smooth and compactly supported in the $(g,\ h,\ x)$ variables. Since $c_4(h,x,\xi)\in S^{-m}_{1,0}(\mathbb{R}^{2d}\times\mathbb{R}^d)$, we have $|\partial_{h_1}^n c_4(h,x,\xi)|\le c_n(1+|\xi|)^{-m}$ for all positive integer n, where c_n is independent of $(h,x)\in\mathbb{R}^{2d}$. Therefore

$$\begin{aligned} |I(x,\kappa_1,\kappa_2)| &\lesssim \frac{1}{\kappa_1^{l_1}\kappa_2^{l_2}}\frac{1}{(1+|c_1\kappa_1-c_2\kappa_2|)^n}\frac{1}{(1+|c_1\kappa_1+c_2\kappa_2|)^m} \\ &\lesssim \frac{1}{(\kappa_1\kappa_2)^{\min(l_1,l_2)}}\frac{1}{(1+|c_1\kappa_1-c_2\kappa_2|)^n}\frac{1}{(c_1\kappa_1+c_2\kappa_2)^m}, \end{aligned} \tag{3.28}$$

where we use the fact that $\kappa_1\ge 1,\kappa_2\ge 1$. We need to consider the cases where $|c_1\kappa_1-c_2\kappa_2|\ge(c_1\kappa_1+c_2\kappa_2)/2$ and $|c_1\kappa_1-c_2\kappa_2|\le(c_1\kappa_1+c_2\kappa_2)/2$. If $|c_1\kappa_1-c_2\kappa_2|\le(c_1\kappa_1+c_2\kappa_2)/2$, a simple calculation shows that $\kappa_1\kappa_2\ge 3(c_1\kappa_1+c_2\kappa_2)^2/(16c_1c_2)$ which implies

$$|I(x,\kappa_1,\kappa_2)|\lesssim\frac{1}{(1+|c_1\kappa_1-c_2\kappa_2|)^n}\frac{1}{(c_1\kappa_1+c_2\kappa_2)^{m+2\min(l_1,l_2)}}.$$

If $|c_1\kappa_1-c_2\kappa_2|\ge(c_1\kappa_1+c_2\kappa_2)/2$, we have

$$|I(x,\kappa_1,\kappa_2)|\lesssim\frac{1}{(1+|c_1\kappa_1-c_2\kappa_2|)^{n-2\min(l_1,l_2)}}\frac{1}{(c_1\kappa_1+c_2\kappa_2)^{m+2\min(l_1,l_2)}}.$$

Noting that the positive integer n is arbitrary, we conclude

$$\begin{aligned} |I(x,\kappa_1,\kappa_2)| &\lesssim\frac{1}{(1+|c_1\kappa_1-c_2\kappa_2|)^n}\frac{1}{(c_1\kappa_1+c_2\kappa_2)^{m+2\min(l_1,l_2)}} \\ &\lesssim(1+|\kappa_1-\kappa_2|)^{-n}(\kappa_1+\kappa_2)^{-(m+2\min(l_1,l_2))}, \end{aligned}$$

where we use the facts $c_1\kappa_1+c_2\kappa_2\le\min(c_1,c_2)(\kappa_1+\kappa_2)$ and $|c_1\kappa_1-c_2\kappa_2|\le c|\kappa_1-\kappa_2|$ for some constant c. So the estimate (3.17) holds.

Observe that $\tilde{I}$ given in (3.16) is analogous to I where we replace κ_2 with $-\kappa_2$. Since the proof of (3.28) allows κ_2 to be negative, we may show that

$$\begin{aligned}|\tilde{I}(x,\kappa_1,\kappa_2)| &\lesssim \frac{1}{\kappa_1^{l_1}\kappa_2^{l_2}}\frac{1}{|c_1\kappa_1+c_2\kappa_2|^n}\frac{1}{(1+|c_1\kappa_1-c_2\kappa_2|)^m}\\ &\lesssim (\kappa_1+\kappa_2)^{-n}(1+|\kappa_1-\kappa_2|)^{-m},\end{aligned}$$

which shows the estimate (3.18) and completes the proof. □

To derive the linear relationship between the scattering data and the function in the principle symbol, it is required to compute the order of $\mathbb{E}(|\tilde{u}(x,\kappa)|^2)$ in terms of κ. To this end, we study the asymptotic of $I(x,\kappa,\kappa)$ for large κ.

Lemma 3.5. *For $\kappa_1=\kappa_2=\kappa$, the following asymptotic holds*

$$I(x,\kappa,\kappa)=R_d(x,\kappa)\kappa^{-(l_1+l_2+m)}+O(\kappa^{-(l_1+l_2+m+1)}),$$

where $R_d(x,\kappa)$ is given by

$$R_d(x,\kappa)=C_d\int_{\mathbb{R}^d}e^{\mathrm{i}(c_1-c_2)|x-y|\kappa}\frac{(x_1-y_1)^{m_1+n_1}\cdots(x_d-y_d)^{m_d+n_d}}{|x-y|^{p_1+p_2}}\phi(y)\mathrm{d}y$$

with

$$C_2=\frac{1}{4}c_m,\quad C_3=\frac{1}{8}c_m,\quad c_m=\left(\frac{2}{c_1+c_2}\right)^m.$$

Proof. Setting $\kappa_1=\kappa_2=\kappa$ in (3.27) gives

$$I(x,\kappa,\kappa)=\frac{1}{\kappa^{l_1+l_2}}\int_{\mathbb{R}^d}e^{\mathrm{i}(c_1-c_2)\kappa h\cdot e_1}c_4(h,x,-(c_1+c_2)\kappa e_1)\mathrm{d}h.$$

The symbol $c_4(h,x,\xi)\in S_{1,0}^{-m}(\mathbb{R}^{2d}\times\mathbb{R}^d)$ can be decomposed into

$$c_4(h,x,\xi)=c_4^p(h,x,\xi)+r(h,x,\xi),$$

where $c_4^p(h,x,\xi)\in S_{1,0}^{-m}(\mathbb{R}^{2d}\times\mathbb{R}^d)$ is the principal symbol which is given by (3.25) and $r(h,x,\xi)\in S_{1,0}^{-m-1}(\mathbb{R}^{2d}\times\mathbb{R}^d)$ is the lower order remainder terms which is smooth and compactly supported in (g, h, x)-variables. Thus, we have

$$\begin{aligned}I(x,\kappa,\kappa)&=\frac{1}{\kappa^{l_1+l_2}}\int_{\mathbb{R}^d}e^{\mathrm{i}(c_1-c_2)\kappa h\cdot e_1}(c_4^p(h,x,-(c_1+c_2)\kappa e_1)+r(h,x,-(c_1+c_2)\kappa e_1))\mathrm{d}h\\ &=\frac{1}{\kappa^{l_1+l_2}}\int_{\mathbb{R}^d}e^{\mathrm{i}(c_1-c_2)\kappa h\cdot e_1}c_4^p(h,x,-(c_1+c_2)\kappa e_1)\mathrm{d}h+O(\kappa^{-(l_1+l_2+m+1)}).\end{aligned}\tag{3.29}$$

By (3.25),

$$\begin{aligned}c_4^p(h,x,-(c_1+c_2)\kappa e_1)&=\phi(y(g,h,x))((c_1+c_2)|((\rho_{11}'(g,h,x))^{-1})^\top e_1|\kappa)^{-m}\theta(x)\\ &\quad\times|\det\rho_{11}'(g,h,x)|^{-1}H(g,h,x)|_{g=0}.\end{aligned}\tag{3.30}$$

Letting $a = (c_1 + c_2)^2 |((\rho'_{11}(g, h, x))^{-1})^\top e_1|^2 \neq 0$, we substitute (3.30) into formula (3.29) and obtain

$$I(x, \kappa, \kappa) = R_d(x, \kappa)\kappa^{-(l_1+l_2+m)} + O(\kappa^{-(l_1+l_2+m+1)}),$$

where

$$R_d(x, \kappa) = \theta(x) \int_{\mathbb{R}^d} e^{\mathrm{i}(c_1-c_2)\kappa h \cdot e_1} \frac{\phi(y(0, h, x))H(0, h, x)}{a^{\frac{m}{2}} |\mathrm{det}\rho'_{11}(0, h, x)|} \mathrm{d}h. \tag{3.31}$$

Next we need to compute a. Noting that $a = (c_1 + c_2)^2 |((\rho'_{11}(g, h, x))^{-1})^\top e_1|^2$, we compute $(\rho'_{11}(g, h, x))^{-1}$ first.

In two dimensions, we have from the definition of ρ_1 that

$$\rho'_{11}(g, h, x) = \partial_g v = \begin{bmatrix} \partial_{g_1} v_1 & \partial_{g_2} v_1 \\ \partial_{g_1} v_2 & \partial_{g_2} v_2 \end{bmatrix}.$$

A straightforward manipulation using the definition of ρ_1 shows that

$$v_1 = (h_1 + g_1) \sin\left(\frac{h_2 + g_2}{h_1 + g_1}\right) - (h_1 - g_1) \sin\left(\frac{h_2 - g_2}{h_1 - g_1}\right),$$
$$v_2 = (h_1 + g_1) \cos\left(\frac{h_2 + g_2}{h_1 + g_1}\right) - (h_1 - g_1) \cos\left(\frac{h_2 - g_2}{h_1 - g_1}\right).$$

Hence, a direct derivation leads to that

$$\rho'_{11}(g, h, x)|_{g=0} = 2\begin{bmatrix} \sin\tilde{\alpha} - \tilde{\alpha}\cos\tilde{\alpha} & \cos\tilde{\alpha} \\ \cos\tilde{\alpha} + \tilde{\alpha}\sin\tilde{\alpha} & -\sin\tilde{\alpha} \end{bmatrix},$$

which implies that

$$\left(\rho'_{11}(g, h, x)|_{g=0}\right)^{-1} = \frac{1}{2}\begin{bmatrix} \sin\tilde{\alpha} & \cos\tilde{\alpha} \\ \cos\tilde{\alpha} + \tilde{\alpha}\sin\tilde{\alpha} & -\sin\tilde{\alpha} + \tilde{\alpha}\cos\tilde{\alpha} \end{bmatrix},$$
$$\left(\left(\rho'_{11}(g, h, x)|_{g=0}\right)^{-1}\right)^T e_1 = \frac{1}{2}(\sin\tilde{\alpha}, \cos\tilde{\alpha})^\top.$$

where $\tilde{\alpha} = h_2/h_1$. Thus we obtain $a = (c_1 + c_2)^2/4$ and $|\mathrm{det}\rho'_{11}(0, h, x)|^{-1} = 1/4$. Next we focus on the computation of $H(0, h, x)$. From (3.23) we have $H(g, h, x) = K(x, y, z)\mathrm{det}((\tau^{-1})'(g, h, x))$, thus we compute $\mathrm{det}\tau'(0, h, x)$ first. Recalling that $\tau : \mathbb{R}^6 \to \mathbb{R}^6$ is given by $\tau(y, z, x) = (g, h, x)$, we have

$$(\tau^{-1}(g, h, x))' = \begin{bmatrix} \partial_g y & \partial_h y & \partial_x y \\ \partial_g z & \partial_h z & \partial_x z \\ \partial_g x & \partial_h x & \partial_x x \end{bmatrix}, \quad \tau(y, z, x)' = \begin{bmatrix} \partial_y g & \partial_z g & \partial_x g \\ \partial_y h & \partial_z h & \partial_x h \\ \partial_y x & \partial_z x & \partial_x x \end{bmatrix}.$$

Now we calculate $\partial_y g$. Noting $g = (t_1 - t_2, s_1 - s_2)$ and denoting $\alpha = \arcsin\left(\frac{y_1 - x_1}{|x-y|}\right)$, we obtain

$$\frac{\partial g_1}{\partial y_1} = \frac{\partial t_1}{\partial y_1} = \frac{1}{2}\frac{y_1 - x_1}{|x-y|} = \frac{1}{2}\sin\alpha, \qquad \frac{\partial g_1}{\partial y_2} = \frac{\partial t_1}{\partial y_2} = \frac{1}{2}\frac{y_2 - x_2}{|x-y|} = \frac{1}{2}\cos\alpha,$$

$$\frac{\partial g_2}{\partial y_1} = \frac{\partial s_1}{\partial y_1} = \frac{1}{2}\left[\frac{y_1 - x_1}{|x-y|}\arcsin\left(\frac{y_1 - x_1}{|x-y|}\right) + \left(1 - \left(\frac{y_1 - x_1}{|x-y|}\right)^2\right)^{\frac{1}{2}}\right] = \frac{1}{2}(\alpha\sin\alpha + \cos\alpha),$$

$$\frac{\partial g_2}{\partial y_2} = \frac{\partial s_1}{\partial y_2} = \frac{1}{2}\left[\frac{y_2 - x_2}{|x-y|}\arcsin\left(\frac{y_1 - x_1}{|x-y|}\right) - \left(1 - \left(\frac{y_1 - x_1}{|x-y|}\right)^2\right)^{-\frac{1}{2}}\frac{(y_1 - x_1)(y_2 - x_2)}{|x-y|^2}\right]$$
$$= \frac{1}{2}(\alpha\cos\alpha - \sin\alpha).$$

Thus

$$\partial_y g = \frac{1}{2}\begin{bmatrix} \sin\alpha & \cos\alpha \\ \alpha\sin\alpha + \cos\alpha & \alpha\cos\alpha - \sin\alpha \end{bmatrix}.$$

When $g=0$, we have $y=z$. A similar calculation yields that $\partial_z g = -\partial_y g, \partial_y h = \partial_y g, \partial_z h = \partial_y g$. Obviously, $\partial_y x = 0, \partial_z x = 0, \partial_x x = I$. Hence

$$\det(\tau(y,z,x)')$$
$$= \frac{1}{2^4}\begin{vmatrix} \sin\alpha & \cos\alpha & -\sin\alpha & -\cos\alpha \\ \alpha\sin\alpha + \cos\alpha & \alpha\cos\alpha - \sin\alpha & -\alpha\sin\alpha - \cos\alpha & -\alpha\cos\alpha + \sin\alpha \\ \sin\alpha & \cos\alpha & \sin\alpha & \cos\alpha \\ \alpha\sin\alpha + \cos\alpha & \alpha\cos\alpha - \sin\alpha & \alpha\sin\alpha + \cos\alpha & \alpha\cos\alpha - \sin\alpha \end{vmatrix}.$$

A direct calculation shows that $\det(\tau(y,z,x)') = 1/4$ which implies $\det(\tau^{-1}(0,h,x)') = 4$. Substituting these results into (3.31) gives

$$R_2(x,\kappa) = c_m\int_{\mathbb{R}^2} e^{\mathrm{i}(c_1-c_2)|x-y|\kappa}\frac{(x_1-y_1)^{m_1+n_1}(x_2-y_2)^{m_2+n_2}}{|x-y|^{p_1+p_2}}\phi(y)\mathrm{d}h,$$

where $(y,z,x) = \tau(g,h,x)$. Noting $|\det(\partial_y h)| = \frac{1}{4}$, we have

$$R_2(x,\kappa) = C_2\int_{\mathbb{R}^2} e^{\mathrm{i}(c_1-c_2)|x-y|\kappa}\frac{(x_1-y_1)^{m_1+n_1}(x_2-y_2)^{m_2+n_2}}{|x-y|^{p_1+p_2}}\phi(y)\mathrm{d}y.$$

In three dimensions, by the definition of ρ_1, we have

$$\rho'_{11}(g,h,x) = \partial_g v = \begin{bmatrix} \partial_{g_1}v_1 & \partial_{g_2}v_1 & \partial_{g_3}v_1 \\ \partial_{g_1}v_2 & \partial_{g_2}v_2 & \partial_{g_3}v_2 \\ \partial_{g_1}v_3 & \partial_{g_2}v_3 & \partial_{g_3}v_3 \end{bmatrix}.$$

It follows from the definition of ρ_1 that

$$v_1 = (h_1+g_1)\sin(h_2+g_2)\cos\left(\frac{h_3+g_3}{h_1+g_1}\right) - (h_1-g_1)\sin(h_2-g_2)\cos\left(\frac{h_3-g_3}{h_1-g_1}\right),$$
$$v_2 = (h_1+g_1)\sin(h_2+g_2)\sin\left(\frac{h_3+g_3}{h_1+g_1}\right) - (h_1-g_1)\sin(h_2-g_2)\sin\left(\frac{h_3-g_3}{h_1-g_1}\right),$$
$$v_3 = (h_1+g_1)\cos(h_2+g_2) - (h_1-g_1)\cos(h_2-g_2).$$

According to the above formulations, a direct derivation shows that

$$\rho_{11}'(g,h,x)|_{g=0}=2\begin{bmatrix}\sin\tilde{\alpha}\cos\tilde{\beta}+\tilde{\beta}\sin\tilde{\alpha}\sin\tilde{\beta} & h_1\cos\tilde{\alpha}\cos\tilde{\beta} & -\sin\tilde{\alpha}\sin\tilde{\beta}\\ \sin\tilde{\alpha}\sin\tilde{\beta}-\tilde{\beta}\sin\tilde{\alpha}\cos\tilde{\beta} & h_1\cos\tilde{\alpha}\sin\tilde{\beta} & \sin\tilde{\alpha}\cos\tilde{\beta}\\ \cos\tilde{\alpha} & -h_1\sin\tilde{\alpha} & 0\end{bmatrix},$$

which gives

$$\begin{aligned}&\left(\rho_{11}'(g,h,x)|_{g=0}\right)^{-1}\\ &=\frac{1}{2h_1\sin\tilde{\alpha}}\begin{bmatrix}h_1\sin^2\tilde{\alpha}\cos\tilde{\beta} & h_1\sin^2\tilde{\alpha}\sin\tilde{\beta} & h_1\sin\tilde{\alpha}\cos\tilde{\alpha}\\ \sin\tilde{\alpha}\cos\tilde{\alpha}\cos\tilde{\beta} & \sin\tilde{\alpha}\cos\tilde{\alpha}\sin\tilde{\beta} & -\sin^2\tilde{\alpha}\\ -h_1\sin\tilde{\beta}+h_1\tilde{\beta}\sin^2\tilde{\alpha}\cos\tilde{\beta} & h_1\cos\tilde{\beta}+h_1\tilde{\beta}\sin^2\tilde{\alpha}\sin\tilde{\beta} & h_1\tilde{\beta}\sin\tilde{\alpha}\cos\tilde{\alpha}\end{bmatrix},\end{aligned}$$

where $\tilde{\alpha}=h_2$ and $\tilde{\beta}=h_3/h_1$. So we conclude that

$$\left(\left(\rho_{11}'(0,h,x)\right)^{-1}\right)^{\top}e_1=\frac{1}{2}\left(\sin\tilde{\alpha}\cos\tilde{\beta},\ \sin\tilde{\alpha}\sin\tilde{\beta},\ \cos\tilde{\alpha}\right)^{\top}.$$

Thus, we obtain $a=(c_1+c_2)^2/4$ and $|\det\rho_{11}'(0,h,x)|=8h_1\sin\tilde{\alpha}=8\sqrt{(y_1-x_1)^2+(y_2-x_2)^2}$.

Next we focus on the computation of $H(0,h,x)$, which requires to compute $\det(\tau^{-1}(0,h,x)')$ first. Recalling that $\tau^{-1}:\mathbb{R}^9\to\mathbb{R}^9$ is given by $\tau^{-1}(g,h,x)=(y,z,x)$, we have

$$(\tau^{-1})'(g,h,x)=\begin{bmatrix}\partial_g y & \partial_h y & \partial_x y\\ \partial_g z & \partial_h z & \partial_x z\\ \partial_g x & \partial_h x & \partial_x x\end{bmatrix},\quad ((\tau^{-1})'(g,h,x))^{-1}=\begin{bmatrix}\partial_y g & \partial_z g & \partial_x g\\ \partial_y h & \partial_z h & \partial_x h\\ \partial_y x & \partial_z x & \partial_x x\end{bmatrix}.$$

Now we calculate $\frac{\partial g}{\partial y}$. Noting $g=(t_1-t_2,s_1-s_2,r_1-r_2)$ and denoting

$$\alpha=\arccos\left(\frac{y_3-x_3}{|x-y|}\right),\quad \beta=\arctan\left(\frac{y_2-x_2}{y_1-x_1}\right),$$

we obtain that

$$\frac{\partial g}{\partial y}=\frac{1}{2}\begin{bmatrix}\sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha\\ \dfrac{\cos\alpha\cos\beta}{|x-y|} & \dfrac{\cos\alpha\sin\beta}{|x-y|} & -\dfrac{\sin\alpha}{|x-y|}\\ \beta\sin\alpha\cos\beta-\dfrac{\sin\beta}{\sin\alpha} & \beta\sin\alpha\sin\beta+\dfrac{\cos\beta}{\sin\alpha} & \beta\cos\alpha\end{bmatrix}.$$

When $g=0$, we have $y=z$. A simple calculation yields that $\frac{\partial g}{\partial z}=-\frac{\partial g}{\partial y}$, $\frac{\partial h}{\partial y}=\frac{\partial g}{\partial y}$, $\frac{\partial h}{\partial z}=\frac{\partial g}{\partial y}$. It is clear to note that $\frac{\partial x}{\partial y}=0$, $\frac{\partial x}{\partial z}=0$, $\frac{\partial x}{\partial x}=I$. Hence

$$\det(((\tau^{-1})'(0,h,x))^{-1})=\frac{1}{8}\frac{1}{(y_1-x_1)^2+(y_2-x_2)^2},$$

which implies

$$\det((\tau^{-1})'(0,h,x)) = 8\big((y_1-x_1)^2+(y_2-x_2)^2\big).$$

Substituting these results into (3.31) gives

$$R_3(x,\kappa) = c_m\int_{\mathbb{R}^3} e^{\mathrm{i}(c_1-c_2)\kappa h\cdot e_1}\frac{(x_1-y_1)^{m_1+n_1}(x_2-y_2)^{m_2+n_2}(x_3-y_3)^{m_3+n_3}}{|x-y|^{p_1+p_2}}$$
$$\times\sqrt{(x_1-y_1)^2+(x_2-y_2)^2}\,\phi(y(0,h,x))\mathrm{d}h,$$

where $(y,z,x)=\tau^{-1}(g,h,x)$. Noting $\det\left(\frac{\partial h}{\partial y}\right)=\frac{1}{8}\frac{1}{\sqrt{(x_1-y_1)^2+(x_2-y_2)^2}}$, we arrive at

$$R_3(x,\kappa) = C_3\int_{\mathbb{R}^3} e^{\mathrm{i}(c_1-c_2)|x-y|\kappa}\frac{(x_1-y_1)^{m_1+n_1}(x_2-y_2)^{m_2+n_2}(x_3-y_3)^{m_3+n_3}}{|x-y|^{p_1+p_2}}\phi(y)\mathrm{d}y,$$

which completes the proof. □

To finish this section, we prove the following result that is utilized in the recovery of the micro-correlation strength ϕ.

Lemma 3.6. *Let $V_1, V_2\subset\mathbb{R}^d$ be two open, bounded, and simply connected domains with positive distance. For some positive integer l and $\phi\in C_0^\infty(V_1)$, define the integral*

$$T(x) = \int_{\mathbb{R}^d}\frac{1}{|x-y|^l}\phi(y)\mathrm{d}y,\quad x\in V_2.$$

Then $T(x), x\in V_2$ uniquely determines the function ϕ.

Proof. A simple calculation yields

$$\Delta_x|x-y|^{-n} = n^2|x-y|^{-n-2},\quad n\in\mathbb{N},$$

which implies

$$\Delta_x^n T(x) = c_n\int_{V_1}\frac{1}{|x-y|^{l+2n}}\phi(y)\mathrm{d}y,$$

where c_n is a constant depending on n. Since $T(x)$ is known in an open set V_2 which has a positive distance to the support of $\phi\in C_0^\infty(\mathbb{R}^2)$, so as $\Delta_x^n T(x), n\in\mathbb{N}$ is known in the set V_2. A linear combination of $\Delta_x^n T(x)$ shows that the integral

$$\int_{V_1}\frac{1}{|x-y|^l}P\left(\frac{1}{|x-y|^2}\right)\phi(y)\mathrm{d}y \tag{3.32}$$

is known in the set V_2, where $P(t)=\sum_{j=0}^J a_j t^j$ is a polynomial of order $J\in\mathbb{N}$. In (3.32), by changing the integral variables, we deduce

$$\begin{aligned}\int_{V_1}\frac{1}{|x-y|^l}P\left(\frac{1}{|x-y|^2}\right)\phi(y)\mathrm{d}y &= \int_{r_1}^{r_2}\frac{1}{r^l}P\left(\frac{1}{r^2}\right)\int_{|y-x|=r}\phi(y)\mathrm{d}s(y)\mathrm{d}r\\ &= \int_{r_1}^{r_2}\frac{1}{r^l}P\left(\frac{1}{r^2}\right)S(x,r)\mathrm{d}r\\ &= \frac{1}{2}\int_{\frac{1}{r_2^2}}^{\frac{1}{r_1^2}}P(t)S\left(x,\frac{1}{\sqrt{t}}\right)t^{\frac{l-3}{2}}\mathrm{d}t,\end{aligned}$$

where $S(x,r)=\int_{|y-x|=r}\phi(y)\mathrm{d}s(y)$ denotes the integral of $\phi(y)$ along the circle $|y-x|=r$, $r_1=\min_{y\in V_1}|x-y|$ and $r_2=\max_{y\in V_1}|x-y|$ denote the minimum and the maximum distance between the fixed point $x\in V_2$ and the domain V_1, respectively. Due to $\phi\in C_0^\infty(\mathbb{R}^2)$, the function $S(x,\ r)$ is continuous with respect to r and is compact supported in the interval $[r_1,r_2]$. We obtain $S\left(x,\frac{1}{\sqrt{t}}\right)t^{\frac{l-3}{2}}$ is continuous in $[r_2^{-2},r_1^{-2}]$. Note that the polynomial function $P(t)$ is dense in $C([r_2^{-2},r_1^{-2}])$, thus the function $S\left(x,\frac{1}{\sqrt{t}}\right)t^{\frac{l-3}{2}}$ is uniquely determined which implies $S(x,\ r)$ is uniquely determined for all $r>0$.

Let $g(x)=e^{-\frac{|x|^2}{2}}$ for $x\in\mathbb{R}^2$, then we have

$$\begin{aligned}(g*\phi)(x)&=\int_{\mathbb{R}^2}e^{-\frac{|x-y|^2}{2}}\phi(y)\mathrm{d}y=\int_{V_1}e^{-\frac{|x-y|^2}{2}}\phi(y)\mathrm{d}y\\&=\int_{r_1}^{r_2}e^{-\frac{r^2}{2}}\int_{|y-x|=r}\phi(y)\mathrm{d}y\mathrm{d}r=\int_{r_1}^{r_2}e^{-\frac{r^2}{2}}S(x,r)\mathrm{d}r.\end{aligned}\tag{3.33}$$

Since $S(x,\ r)$ is uniquely determined for all $r>0$, we can compute the convolution $g*\phi$ by (3.33) for $x\in V_2$. Because V_2 is open and $g*\phi$ is real analytic, hence $g*\phi$ is known everywhere, and the Fourier transform $\mathcal{F}(g*\phi)$ is known everywhere. Since

$$\begin{aligned}\mathcal{F}g(\xi)&=\int_{\mathbb{R}^2}e^{-\mathrm{i}x\cdot\xi}g(x)\mathrm{d}x=\int_{\mathbb{R}^2}e^{-\left(\frac{|x|^2}{2}+\mathrm{i}x\cdot\xi\right)}\mathrm{d}x\\&=\int_{\mathbb{R}^2}e^{-\frac{1}{2}(x_1^2+2\mathrm{i}x_1\xi_1)}e^{-\frac{1}{2}(x_2^2+2\mathrm{i}x_2\xi_2)}\mathrm{d}x\\&=e^{-\frac{1}{2}|\xi|^2}\int_{\mathbb{R}}e^{-\frac{1}{2}(x_1+\mathrm{i}\xi_1)^2}\mathrm{d}x_1\int_{\mathbb{R}}e^{-\frac{1}{2}(x_2+\mathrm{i}\xi_2)^2}\mathrm{d}x_2\\&=2\pi e^{-\frac{1}{2}|\xi|^2},\end{aligned}$$

we conclude $\mathcal{F}g$ is smooth and non-zero all over $\mathbb{R}^2$. Therefore, $\mathcal{F}\phi=\mathcal{F}(g*\phi)/\mathcal{F}g$ is uniquely determined which shows that ϕ is uniquely determined. □

Remark 3.7. We point out that the proof of Lemma 3.6 requires values integral T to be available in an open set. This is the fundamental reason why observational data on a lower dimensional manifold or boundary are not sufficient in our main results, i.e., Theorems 3.10 and 3.12, with the current technique. Similar reasoning applies to the elastic wave equation.

3.3. The two-dimensional case

First we discuss the two-dimensional case and show that the function ϕ in the principle symbol can be uniquely determined by the scattered field obtained from a single realization of the random source f. Let us begin with the asymptotic of the Hankel function $H_n^{(1)}$ with a large argument. By [44, Eqs. (9.2.7)–(9.2.10)] and [44, Eqs.(5.11.4)], we have:

$$H_n^{(1)}(z) = \sqrt{\frac{1}{z}}e^{\mathrm{i}(z-(\frac{n}{2}+\frac{1}{4})\pi)}\left(\sum_{j=0}^{N} a_j^{(n)} z^{-j} + O(|z|^{-N-1})\right), \quad \text{for } |\arg z| \le \pi - \delta, \tag{3.34}$$

for large $|z|$, where δ is an arbitrarily small positive number and the coefficients $a_j^{(n)} = (-2\mathrm{i})^j\sqrt{\frac{2}{\pi}}(n,j)$ with

$$(n,j) = \frac{(4n^2-1)(4n^2-3^2)\cdots(4n^2-(2j-1)^2)}{2^{2j}j!} \quad \text{and} \quad (n,0)=1.$$

Using the first N terms in the asymptotic of $H_n^{(1)}(\kappa|z|)$, we define

$$H_{n,N}^{(1)}(\kappa|z|) := \sqrt{\frac{1}{\kappa|z|}}e^{\mathrm{i}(\kappa|z|-(\frac{n}{2}+\frac{1}{4})\pi)}\sum_{j=0}^{N} a_j^{(n)}\left(\frac{1}{\kappa|z|}\right)^j. \tag{3.35}$$

It is easy to show from (3.34) that

$$|H_n^{(1)}(\kappa|z|) - H_{n,N}^{(1)}(\kappa|z|)| \le c(\frac{1}{\kappa|z|})^{N+\frac{3}{2}}. \tag{3.36}$$

Using (3.35), we define $\tilde{u}(x,\kappa)$ as

$$\tilde{u}(x,\kappa) := -\frac{\mathrm{i}}{4}\int_{\mathbb{R}^2} H_{0,2}^{(1)}(\kappa|x-y|)f(y)\mathrm{d}y. \tag{3.37}$$

Lemma 3.8. *The random variable $u(x,\kappa) - \tilde{u}(x,\kappa)$ satisfies almost surely the condition*

$$|u(x,\kappa) - \tilde{u}(x,\kappa)| \le c\kappa^{-\frac{7}{2}}, \quad x \in U,$$

where the constant c depends only on $L^2(D)$-norm of f.

Proof. Noting Assumption A, we know that there exists a positive constant M such that $|x-y| \ge M$ holds for all $x \in U$ and $y \in D$. By (3.36), and (3.37), we have for $x \in U$ that

$$\begin{aligned}
|u(x,\kappa) - \tilde{u}(x,\kappa)| &= \left|\frac{\mathrm{i}}{4}\int_D \left[H_0^{(1)}(\kappa|x-y|) - H_{0,2}^{(1)}(\kappa|x-y|)\right]f(y)dy\right| \\
&\lesssim ||H_0^{(1)}(\kappa|x-\cdot|) - H_{0,2}^{(1)}(\kappa|x-\cdot|)||_{H^{1,p}(D)}||f||_{H_0^{-1,p'}(D)} \\
&\le c\kappa^{-\frac{7}{2}},
\end{aligned}$$

where the constant c depends only on $H_0^{-1,p}(D)$-norm of f. □

Now we are in the position to compute the covariance of $\tilde{u}(x,\kappa)$. Using (3.35) and (3.37), we have from a direct calculation that

$$\mathbb{E}(\tilde{u}(x,\kappa_1)\overline{\tilde{u}(x,\kappa_2)}) = \frac{1}{16}\sum_{j_1,j_2=0}^{2}\frac{a_{j_1}^{(0)}\overline{a_{j_2}^{(0)}}}{\kappa_1^{j_1+\frac{1}{2}}\kappa_2^{j_2+\frac{1}{2}}}\int_{\mathbb{R}^4}\frac{e^{\mathrm{i}(\kappa_1|x-y|-\kappa_2|x-z|)}}{|x-y|^{j_1+\frac{1}{2}}|x-z|^{j_2+\frac{1}{2}}}\mathbb{E}(f(y)f(z))\mathrm{d}y\mathrm{d}z. \tag{3.38}$$

Since $\mathbb{E}(\tilde{u}(x,\kappa_1)\overline{\tilde{u}(x,\kappa_2)})$ is a linear combination of I which satisfies the estimate (3.17), thus the following result is a direct consequence of Lemma 3.4.

Lemma 3.9. *For $\kappa_1, \kappa_2 \geq 1$, the estimates*

$$|\mathbb{E}(\tilde{u}(x,\kappa_1)\overline{\tilde{u}(x,\kappa_2)})| \leq c_n(\kappa_1+\kappa_2)^{-(m+1)}(1+|\kappa_1-\kappa_2|)^{-n},$$
$$|\mathbb{E}(\tilde{u}(x,\kappa_1)\tilde{u}(x,\kappa_2))| \leq c_n(\kappa_1+\kappa_2)^{-n}(1+|\kappa_1-\kappa_2|)^{-m}.$$

holds uniformly for $x \in U$, where $n \in \mathbb{N}$ is arbitrary and $c_n > 0$ is a constant depending only on n.

Now we are ready to estimate the order of $\mathbb{E}(|\tilde{u}(x,\kappa)|^2)$. Setting $\kappa_1 = \kappa_2 = \kappa$ in (3.38) and applying Lemma 3.5, we obtain

$$\mathbb{E}(|\tilde{u}(x,\kappa)|^2) = T_{\mathrm{A}}^{(2)}(x)\kappa^{-(m+1)} + O(\kappa^{-(m+2)}), \tag{3.39}$$

where

$$T_{\mathrm{A}}^{(2)}(x) = \frac{1}{32\pi}\int_{\mathbb{R}^2}\frac{1}{|x-y|}\phi(y)\mathrm{d}y. \tag{3.40}$$

We are in the position to present the main result for the time-harmonic acoustic waves.

Theorem 3.10. *Let the external source f be a microlocally isotropic Gaussian random field which satisfies Assumption B. Then for all $x \in U$, it holds almost surely that*

$$\lim_{Q\to\infty}\frac{1}{Q-1}\int_1^Q \kappa^{m+1}|u(x,\kappa)|^2\mathrm{d}\kappa = T_{\mathrm{A}}^{(2)}(x). \tag{3.41}$$

Moreover, the scattering data $T_{\mathrm{A}}^{(2)}(x), x \in U$ uniquely determines the micro-correlation strength ϕ through the linear relation (3.40).

Proof. A simple calculation shows that

$$\begin{aligned}
&\frac{1}{Q-1}\int_1^Q \kappa^{m+1}|u(x,\kappa)|^2\mathrm{d}\kappa \\
&= \frac{1}{Q-1}\int_1^Q \kappa^{m+1}|\tilde{u}(x,\kappa)+u(x,\kappa)-\tilde{u}(x,\kappa)|^2\mathrm{d}\kappa \\
&= \frac{1}{Q-1}\int_1^Q \kappa^{m+1}|\tilde{u}(x,\kappa)|^2\mathrm{d}\kappa + \frac{1}{Q-1}\int_1^Q \kappa^{m+1}|u(x,\kappa)-\tilde{u}(x,\kappa)|^2\mathrm{d}\kappa \\
&\quad + \frac{2}{Q-1}\int_1^Q \kappa^{m+1}\Re\left[\overline{\tilde{u}(x,\kappa)}(u(x,\kappa)-\tilde{u}(x,\kappa))\right]\mathrm{d}\kappa.
\end{aligned}$$

It is clear that (3.41) follows as long as we show that

$$\lim_{Q\to\infty}\frac{1}{Q-1}\int_1^Q \kappa^{m+1}|\tilde{u}(x,\kappa)|^2\mathrm{d}\kappa = T_{\mathrm{A}}^{(2)}(x), \tag{3.42}$$

$$\lim_{Q\to\infty}\frac{1}{Q-1}\int_1^Q \kappa^{m+1}|u(x,\kappa)-\tilde{u}(x,\kappa)|^2\mathrm{d}\kappa = 0, \tag{3.43}$$

$$\lim_{Q\to\infty}\frac{2}{Q-1}\int_1^Q \kappa^{m+1}\Re\left[\overline{\tilde{u}(x,\kappa)}(u(x,\kappa)-\tilde{u}(x,\kappa))\right]\mathrm{d}\kappa = 0. \tag{3.44}$$

To prove (3.42), we define $Y(x,\kappa)=\kappa^{m+1}(|\tilde{u}(x,\kappa)|^2-\mathbb{E}|\tilde{u}(x,\kappa)|^2)$. Since

$$\int_1^Q \kappa^{m+1}|\tilde{u}(x,\kappa)|^2 \mathrm{d}\kappa=\int_1^Q \kappa^{m+1}\mathbb{E}|\tilde{u}(x,\kappa)|^2 \mathrm{d}\kappa+\int_1^Q Y(x,\kappa)\mathrm{d}\kappa,$$

(3.42) holds as long as we prove

$$\lim_{Q\to\infty}\frac{1}{Q-1}\int_1^Q \kappa^{m+1}\mathbb{E}|\tilde{u}(x,\kappa)|^2 \mathrm{d}\kappa=T_{\mathrm{A}}^{(2)}(x),\qquad \lim_{Q\to\infty}\frac{1}{Q-1}\int_1^Q Y(x,\kappa)\mathrm{d}\kappa=0.$$

By (3.39), it is easy to see that

$$\frac{1}{Q-1}\int_1^Q \kappa^{m+1}\mathbb{E}|\tilde{u}(x,\kappa)|^2 \mathrm{d}\kappa=\frac{1}{Q-1}\int_1^Q \left(T_A^{(2)}(x)+O(\kappa^{-1})\right)\mathrm{d}\kappa.$$

Clearly, we have

$$\frac{1}{Q-1}\int_1^Q T_{\mathrm{A}}^{(2)}(x)\mathrm{d}\kappa=T_{\mathrm{A}}^{(2)}(x)$$

and

$$\left|\frac{1}{Q-1}\int_1^Q O(\kappa^{-1})\mathrm{d}\kappa\right|\lesssim\frac{1}{Q-1}\int_1^Q \kappa^{-1}\mathrm{d}\kappa=\frac{\ln Q}{Q-1}\to 0\quad \text{as} Q\to\infty.$$

Hence,

$$\lim_{Q\to\infty}\frac{1}{Q-1}\int_1^Q \kappa^{m+1}\mathbb{E}|\tilde{u}(x,\kappa)|^2 \mathrm{d}\kappa=T_A^{(2)}(x).$$

To prove (3.42), it suffices to show

$$\lim_{Q\to\infty}\frac{1}{Q-1}\int_1^Q Y(x,\kappa)\mathrm{d}\kappa=0.$$

By the definition of $Y(x,\kappa)$, we obtain

$$\begin{aligned}Y(x,\kappa)&=\kappa^{m+1}(|\tilde{u}(x,\kappa)|^2-\mathbb{E}|\tilde{u}(x,\kappa)|^2)\\&=\kappa^{m+1}((\Re\tilde{u}(x,\kappa))^2-\mathbb{E}(\Re\tilde{u}(x,\kappa))^2+(\Im\tilde{u}(x,\kappa))^2-\mathbb{E}(\Im\tilde{u}(x,\kappa))^2).\end{aligned}$$

Therefore

$$\mathbb{E}(Y(x,\kappa_1)Y(x,\kappa_2))=I_{A,1}+I_{A,2}+I_{A,3}+I_{A,4},$$

where

$$\begin{aligned}I_{A,1}&=\kappa_1^{m+1}\kappa_2^{m+1}\mathbb{E}\big[((\Re\tilde{u}(x,\kappa_1))^2-\mathbb{E}(\Re\tilde{u}(x,\kappa_1))^2)((\Re\tilde{u}(x,\kappa_2))^2-\mathbb{E}(\Re\tilde{u}(x,\kappa_2))^2)\big],\\I_{A,2}&=\kappa_1^{m+1}\kappa_2^{m+1}\mathbb{E}\big[((\Re\tilde{u}(x,\kappa_1))^2-\mathbb{E}(\Re\tilde{u}(x,\kappa_1))^2)((\Im\tilde{u}(x,\kappa_2))^2-\mathbb{E}(\Im\tilde{u}(x,\kappa_2))^2)\big],\\I_{A,3}&=\kappa_1^{m+1}\kappa_2^{m+1}\mathbb{E}\big[((\Im\tilde{u}(x,\kappa_1))^2-\mathbb{E}(\Im\tilde{u}(x,\kappa_1))^2)((\Re\tilde{u}(x,\kappa_2))^2-\mathbb{E}(\Re\tilde{u}(x,\kappa_2))^2)\big],\\I_{A,4}&=\kappa_1^{m+1}\kappa_2^{m+1}\mathbb{E}\big[((\Im\tilde{u}(x,\kappa_1))^2-\mathbb{E}(\Im\tilde{u}(x,\kappa_1))^2)((\Im\tilde{u}(x,\kappa_2))^2-\mathbb{E}(\Im\tilde{u}(x,\kappa_2))^2)\big].\end{aligned}$$

Combing the expression of $\tilde{u}(x,\kappa)$ and the assumption $\mathbb{E}(f)=0$ gives that both $\Re\tilde{u}(x,\kappa)$ and $\Im\tilde{u}(x,\kappa)$ are zero-mean Gaussian random variables. Applying Lemmas 2.3 and 3.9 leads to

$$\begin{aligned}
I_{A,1} &= 2\kappa_1^{m+1}\kappa_2^{m+1}[\mathbb{E}(\Re\tilde{u}(x,\kappa_1)\Re\tilde{u}(x,\kappa_2))]^2 \\
&= \frac{1}{2}\kappa_1^{m+1}\kappa_2^{m+1}\left[\mathbb{E}(\Re(\tilde{u}(x,\kappa_1)\tilde{u}(x,\kappa_2)) + \Re(\tilde{u}(x,\kappa_1)\overline{\tilde{u}(x,\kappa_2)}))\right]^2 \\
&\lesssim \left[\frac{\kappa_1^{\frac{m+1}{2}}\kappa_2^{\frac{m+1}{2}}}{(\kappa_1+\kappa_2)^n(1+|\kappa_1-\kappa_2|)^m} + \frac{\kappa_1^{\frac{m+1}{2}}\kappa_2^{\frac{m+1}{2}}}{(\kappa_1+\kappa_2)^{m+1}(1+|\kappa_1-\kappa_2|)^n}\right]^2 \\
&\lesssim \left[\frac{1}{(1+|\kappa_1-\kappa_2|)^m} + \frac{1}{(1+|\kappa_1-\kappa_2|)^n}\right]^2.
\end{aligned}$$

We can obtain the same estimates for $I_{A,2}, I_{A,3}$, and $I_{A,4}$ by the similar arguments. Thus, an application of Lemma 2.4 gives that

$$\lim_{Q\to\infty}\frac{1}{Q-1}\int_1^Q Y(x,\kappa)\mathrm{d}\kappa = 0.$$

To prove (3.43), we obtain from Lemma 3.8 that

$$\begin{aligned}
&\left|\frac{1}{Q-1}\int_1^Q \kappa^{m+1}|u(x,\kappa)-\tilde{u}(x,\kappa)|^2\mathrm{d}\kappa\right| \lesssim \frac{1}{Q-1}\int_1^Q \kappa^{m+1}\kappa^{-7}\mathrm{d}\kappa \\
&= \frac{1}{Q-1}\int_1^Q \kappa^{m-6}\mathrm{d}\kappa = \frac{1}{m-5}\frac{Q^{m-5}-1}{Q-1} \to 0 \quad \text{as} Q\to\infty.
\end{aligned}$$

To prove (3.44), by the Hölder inequality, we have

$$\begin{aligned}
&\left|\frac{2}{Q-1}\int_1^Q \kappa^{m+1}\Re\left[\overline{\tilde{u}(x,\kappa)}(u(x,\kappa)-\tilde{u}(x,\kappa))\right]\mathrm{d}\kappa\right| \\
&\le \frac{2}{Q-1}\int_1^Q \kappa^{m+1}|\tilde{u}(x,\kappa)||u(x,\kappa)-\tilde{u}(x,\kappa)|\mathrm{d}\kappa \\
&\le 2\left[\frac{1}{Q-1}\int_1^Q \kappa^{m+1}|\tilde{u}(x,\kappa)|^2\mathrm{d}\kappa\right]^{\frac{1}{2}}\left[\frac{1}{Q-1}\int_1^Q \kappa^{m+1}|u(x,\kappa)-\tilde{u}(x,\kappa)|^2\mathrm{d}\kappa\right]^{\frac{1}{2}} \\
&\to 2T_{\mathrm{A}}^{(2)}(x)^{\frac{1}{2}}\cdot 0 = 0 \quad \text{as} Q\to\infty.
\end{aligned}$$

Hence, (3.42)–(3.44) hold which means that (3.41) holds. The unique determination of ϕ by the scattering data $T_{\mathrm{A}}^{(2)}(x)$ for $x\in U$ is a direct consequence of Lemma 3.6. □

3.4. The three-dimensional case

In this subsection, we show that the scattering data obtained from a single realization of the random source can determine uniquely the function ϕ in the principle symbol in the three dimensions. By (3.3) and (3.13), we have

$$u(x,\kappa) = -\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{e^{\mathrm{i}\kappa|x-y|}}{|x-y|}f(y)\mathrm{d}y,$$

which yields

$$\mathbb{E}(u(x,\kappa_1)\overline{u(x,\kappa_2)}) = \frac{1}{16\pi^2}\int_{\mathbb{R}^6} \frac{e^{\mathrm{i}(\kappa_1|x-y|-\kappa_2|x-z|)}}{|x-y||x-z|}\mathbb{E}(f(y)f(z))\mathrm{d}y\mathrm{d}z. \tag{3.45}$$

We apply directly Lemma 3.4 and obtain the estimates of $\mathbb{E}(u(x,\kappa_1)\overline{u(x,\kappa_2)})$ and $\mathbb{E}(u(x,\kappa_1)u(x,\kappa_2))$.

Lemma 3.11. *For $\kappa_1 \geq 1, \kappa_2 \geq 1$, the following estimates*

$$|\mathbb{E}(u(x,\kappa_1)\overline{u(x,\kappa_2)})| \leq c_n(\kappa_1+\kappa_2)^{-m}(1+|\kappa_1-\kappa_2|)^{-n},$$
$$|\mathbb{E}(u(x,\kappa_1)u(x,\kappa_2))| \leq c_n(\kappa_1+\kappa_2)^{-n}(1+|\kappa_1-\kappa_2|)^{-m}$$

holds uniformly for $x \in U$, where $n \in \mathbb{N}$ is arbitrary and $c_n > 0$ is a constant depending only on n.

To derive the relationship between the scattering data and the function ϕ in the principle symbol, by setting $\kappa_1 = \kappa_2 = \kappa$ in (3.45) and using Lemma 3.5, we get

$$\mathbb{E}(|u(x,\kappa)|^2) = T_{\mathrm{A}}^{(3)}(x)\kappa^{-m} + O(\kappa^{-(m+1)}), \tag{3.46}$$

where

$$T_{\mathrm{A}}^{(3)}(x) = \frac{1}{128\pi^2}\int_{\mathbb{R}^3} \frac{1}{|x-y|^2}\phi(y)\mathrm{d}y. \tag{3.47}$$

Now we are ready to present the main result for the three-dimensional case.

Theorem 3.12. *Let the external source f be a microlocally isotropic Gaussian random field which satisfies Assumption B. Then for all $x \in U$, it holds almost surely that*

$$\lim_{Q\to\infty}\frac{1}{Q-1}\int_1^Q \kappa^m|u(x,\kappa)|^2\mathrm{d}\kappa = T_{\mathrm{A}}^{(3)}(x). \tag{3.48}$$

Moreover, the scattering data $T_{\mathrm{A}}^{(3)}(x), x \in U$ uniquely determines the micro-correlation strength ϕ through the linear integral equation (3.47).

Proof. We decompose $\kappa^m|u(x,\kappa)|^2$ into two parts:

$$\kappa^m|u(x,\kappa)|^2 = \kappa^m\mathbb{E}|u(x,\kappa)|^2 + Y(x,\kappa),$$

where

$$Y(x,\kappa) := \kappa^m(|u(x,\kappa)|^2 - \mathbb{E}|u(x,\kappa)|^2).$$

Clearly,

$$\frac{1}{Q-1}\int_1^Q \kappa^m|u(x,\kappa)|^2\mathrm{d}\kappa = \frac{1}{Q-1}\int_1^Q \kappa^m\mathbb{E}|u(x,\kappa)|^2\mathrm{d}\kappa + \frac{1}{Q-1}\int_1^Q Y(x,\kappa)\mathrm{d}\kappa.$$

Hence, (3.48) holds as long as we show that

$$\lim_{Q\to\infty}\frac{1}{Q-1}\int_1^Q \kappa^m\mathbb{E}|u(x,\kappa)|^2\mathrm{d}\kappa = T_{\mathrm{A}}^{(3)}(x), \quad \lim_{Q\to\infty}\frac{1}{Q-1}\int_1^Q Y(x,\kappa)\mathrm{d}\kappa = 0. \tag{3.49}$$

The second equation in (3.49) can be obtained by a similar argument to the two-dimensional case. Using (3.46) gives

$$\lim_{Q\to\infty}\frac{1}{Q-1}\int_1^Q \kappa^m\mathbb{E}|u(x,\kappa)|^2\mathrm{d}\kappa = \lim_{Q\to\infty}\frac{1}{Q-1}\int_1^Q (T_{\mathrm{A}}^{(3)}(x)+O(\kappa^{-1}))\mathrm{d}\kappa = T_{\mathrm{A}}^{(3)}(x).$$

Hence, the first equation in (3.49) holds which implies that (3.48) holds. A direct application of Lemma 3.6 implies that ϕ is uniquely determined by the scattering data $T_{\mathrm{A}}^{(3)}(x)$ for $x\in U$. □

4. Elastic waves

This section concerns the direct and inverse source problems for the elastic wave equation in the two- and three-dimensional cases. Following the general theme for the acoustic case presented in Section 3, we discuss the well-posedness of the direct problem and show the uniqueness of the inverse problem. We prove that the direct scattering problem with a distributional source indeed has a unique solution. For the inverse problem, we assume that each component of the external source is a microlocally isotropic Gaussian random field whose covariance operator is a classical pseudo-differential operator. Moreover, the principle symbol of the covariance operator of each component is assumed to be coincided. Our main results are as follows: in either the two- or three-dimensional case, given the scattering data which is obtained from a single realization of the random source, the principle symbol of the covariance operator can be uniquely determined. The technical details differ from acoustic waves due to the different model equation and Green tensors.

4.1. The direct scattering problem

In this subsection, we introduce the model problem of the random source scattering for elastic waves, and show that the direct problem with a distributional source is well-posed.

Consider the time-harmonic Navier equation in a homogeneous medium

$$\mu\Delta\boldsymbol{u}+(\lambda+\mu)\nabla\nabla\cdot\boldsymbol{u}+\omega^2\boldsymbol{u}=\boldsymbol{f}\quad\text{in }\mathbb{R}^d,\tag{4.1}$$

where $\omega>0$ is the angular frequency, λ and μ are the Lamé constants satisfying $\mu>0$ and $\lambda+\mu>0$, the external source $\boldsymbol{f}\in\mathbb{C}^d$ is a generalized random function supported in a bounded and simply connected domain D in $\mathbb{R}^d$, and $\boldsymbol{u}\in\mathbb{C}^d$ is the displacement of the random wave field.

Since the problem is imposed in the open domain $\mathbb{R}^d$, an appropriate radiation condition is needed to complete the formulation of the scattering problem. We adopt the Kupradze–Sommerfeld radiation condition to describe the asymptotic behavior of the displacement field away from the source. According to the Helmholtz decomposition, the displacement $\boldsymbol{u}$ can be decomposed into the compressional part $\boldsymbol{u}_{\mathrm{p}}$ and the shear part $\boldsymbol{u}_{\mathrm{s}}$:

$$\boldsymbol{u}=\boldsymbol{u}_{\mathrm{p}}+\boldsymbol{u}_{\mathrm{s}}\quad\text{in }\mathbb{R}^d\setminus\bar{D}.$$

The Kupradze–Sommerfeld radiation condition requires that $\boldsymbol{u}_{\mathrm{p}}$ and $\boldsymbol{u}_{\mathrm{s}}$ satisfy the Sommerfeld radiation condition:

$$\lim_{r\to\infty} r^{\frac{d-1}{2}}(\partial_r \boldsymbol{u}_{\rm p} - {\rm i}\kappa_{\rm p}\boldsymbol{u}_{\rm p}) = 0, \quad \lim_{r\to\infty} r^{\frac{d-1}{2}}(\partial_r \boldsymbol{u}_{\rm s} - {\rm i}\kappa_{\rm s}\boldsymbol{u}_{s}) = 0, \quad r = |x|, \tag{4.2}$$

where

$$\kappa_{\rm p} = \frac{\omega}{(\lambda + 2\mu)^{1/2}} = c_{\rm p}\omega, \qquad \kappa_{\rm s} = \frac{\omega}{\mu^{1/2}} = c_{\rm s}\omega$$

are known as the compressional and shear wavenumbers with

$$c_{\rm p} = (\lambda + 2\mu)^{-1/2}, \quad c_{\rm s} = \mu^{-1/2}.$$

Note that $c_{\rm p}$ and $c_{\rm s}$ are independent of ω and $c_{\rm p} < c_{\rm s}$.

In (4.1), the external source $\boldsymbol{f}$ is a vector with components $f_i, i = 1, ..., d$. To achieve the main results, throughout this section, we assume that each component f_i satisfies the following condition.

Assumption C. Recall that $D \subset \mathbb{R}^d$ denotes a bounded and simply connected domain, f_i is assumed to be a microlocally isotropic Gaussian random field of the same order $m \in [d, d + \frac{1}{2})$ in D. Each covariance operator C_{f_i} is assumed to have the same principle symbol $\phi(x)|\xi|^{-m}$ with $\phi \in C_0^\infty(D)$ and $\phi \ge 0$. Moreover, we assume that $\mathbb{E}(f_i) = 0$ and $\mathbb{E}(f_i f_j) = 0$ if $i \ne j$ for $i, j = 1, ..., d$.

According to Lemma 2.2, if $m = d$, we have $\boldsymbol{f}(\hat\omega) \in H^{-\varepsilon,p}(D)^3$. Thus it suffices to show that the scattering problem for such a deterministic, distributional source $\boldsymbol{f} \in H^{-\varepsilon,p}(D)^3$ has a unique solution.

Introduce the Green tensor $\boldsymbol{G}(x, y, \omega) \in \mathbb{C}^{d\times d}$ for the Navier equation (4.1) which is given by

$$\boldsymbol{G}(x, y, \omega) = \frac{1}{\mu}\Phi_d(x, y, \kappa_{\rm s})\boldsymbol{I}_d + \frac{1}{\omega^2}\nabla_x\nabla_x^\top(\Phi_d(x, y, \kappa_{\rm s}) - \Phi_d(x, y, \kappa_{\rm p})), \tag{4.3}$$

where $\boldsymbol{I}_d$ is the $d \times d$ identity matrix and $\Phi_d(x, y, \kappa)$ is the fundamental solution for the d-dimensional Helmholtz equation given in (3.3). Here the notation $\nabla_x\nabla_x^\top$ is given by

$$\nabla_x\nabla_x^\top\varphi = \begin{bmatrix} \partial^2_{x_1x_1}\varphi & \partial^2_{x_1x_2}\varphi \\ \partial^2_{x_2x_1}\varphi & \partial^2_{x_2x_2}\varphi \end{bmatrix} \quad \text{if } d = 2$$

and

$$\nabla_x\nabla_x^\top\varphi = \begin{bmatrix} \partial^2_{x_1x_1}\varphi & \partial^2_{x_1x_2}\varphi & \partial^2_{x_1x_3}\varphi \\ \partial^2_{x_2x_1}\varphi & \partial^2_{x_2x_2}\varphi & \partial^2_{x_2x_3}\varphi \\ \partial^2_{x_3x_1}\varphi & \partial^2_{x_3x_2}\varphi & \partial^2_{x_3x_3}\varphi \end{bmatrix} \quad \text{if } d = 3$$

for some scalar function φ defined in $\mathbb{R}^d$. It is easily verified that the Green tensor $\boldsymbol{G}(x, y, \omega)$ is symmetric with respect to the variables x and y.

We study the asymptotic expansion of the Green's tensor $\boldsymbol{G}(x, y, \omega)$ when $|x - y|$ is close to zero. For the two-dimensional case, using (3.4)–(3.5) gives

$$H_2^{(1)}(t) = -\frac{4{\rm i}}{\pi}\frac{1}{t^2} - \frac{\rm i}{\pi} + \frac{\rm i}{4\pi}t^2 \ln\frac{t}{2} + \left(\frac{\gamma {\rm i}}{4\pi} - \frac{3{\rm i}}{16\pi} + \frac{1}{8}\right)t^2 + O\left(t^4 \ln\frac{t}{2}\right) \quad \text{as } t \to 0. \tag{4.4}$$

Recall the recurrence relations (3.8), a direct calculation shows that

$$
\begin{aligned}
\Phi_2(x,y,\kappa) &= \frac{\mathrm{i}}{4}H_0^{(1)}(\kappa|x-y|),\\
\partial_{x_i}\Phi_2(x,y,\kappa) &= -\frac{\kappa\mathrm{i}}{4}(x_i-y_i)\frac{H_1^{(1)}(\kappa|x-y|)}{|x-y|},\\
\partial^2_{x_ix_j}\Phi_2(x,y,\kappa) &= -\frac{\kappa\mathrm{i}}{4}\frac{H_1^{(1)}(\kappa|x-y|)}{|x-y|}\delta_{ij}+\frac{\kappa^2\mathrm{i}}{4}(x_i-y_i)(x_j-y_j)\frac{H_2^{(1)}(\kappa|x-y|)}{|x-y|^2},
\end{aligned}
$$

where δ_{ij} is the Kronecker delta function. Hence, by (3.7) and (4.4), we have

$$
\begin{aligned}
&\kappa_{\mathrm{s}}H_1^{(1)}(\kappa_{\mathrm{s}}|x-y|)-\kappa_{\mathrm{p}}H_1^{(1)}(\kappa_{\mathrm{p}}|x-y|) = \frac{\mathrm{i}}{\pi}|x-y|\left(\kappa_{\mathrm{s}}^2\ln\frac{\kappa_{\mathrm{s}}|x-y|}{2}-\kappa_{\mathrm{p}}^2\ln\frac{\kappa_{\mathrm{p}}|x-y|}{2}\right)\\
&\quad+\left(\frac{1}{2}+\frac{\mathrm{i}}{\pi}\gamma-\frac{\mathrm{i}}{2\pi}\right)(\kappa_{\mathrm{s}}^2-\kappa_{\mathrm{p}}^2)|x-y|+O\left(|x-y|^3\ln\frac{|x-y|}{2}\right),\\
&\kappa_{\mathrm{s}}^2H_2^{(1)}(\kappa_{\mathrm{s}}|x-y|)-\kappa_{\mathrm{p}}^2H_2^{(1)}(\kappa_{\mathrm{p}}|x-y|) = \frac{\mathrm{i}}{4\pi}\left(\kappa_{\mathrm{s}}^4\ln\frac{\kappa_{\mathrm{s}}|x-y|}{2}-\kappa_{\mathrm{p}}^4\ln\frac{\kappa_{\mathrm{p}}|x-y|}{2}\right)|x-y|^2\\
&\quad-\frac{\mathrm{i}}{\pi}(\kappa_{\mathrm{s}}^2-\kappa_{\mathrm{p}}^2)+\left(\frac{\gamma\mathrm{i}}{4\pi}-\frac{3\mathrm{i}}{16\pi}+\frac{1}{8}\right)(\kappa_{\mathrm{s}}^4-\kappa_{\mathrm{p}}^4)|x-y|^2+O\left(|x-y|^4\ln\frac{|x-y|}{2}\right),
\end{aligned}
$$

which gives

$$
\begin{aligned}
&\partial^2_{x_ix_j}\left[\Phi_2(x,y,\kappa_{\mathrm{s}})-\Phi_2(x,y,\kappa_{\mathrm{p}})\right]\\
&=-\frac{\mathrm{i}}{4}\frac{1}{|x-y|}\left[\kappa_{\mathrm{s}}H_1^{(1)}(\kappa_{\mathrm{s}}|x-y|)-\kappa_{\mathrm{p}}H_1^{(1)}(\kappa_{\mathrm{p}}|x-y|)\right]\delta_{ij}\\
&\quad+\frac{\mathrm{i}}{4}\frac{(x_i-y_i)(x_j-y_j)}{|x-y|^2}\left[\kappa_{\mathrm{s}}^2H_2^{(1)}(\kappa_{\mathrm{s}}|x-y|)-\kappa_{\mathrm{p}}^2H_2^{(1)}(\kappa_{\mathrm{p}}|x-y|)\right]\\
&=\frac{1}{4\pi}\left(\kappa_{\mathrm{s}}^2\ln\frac{\kappa_{\mathrm{s}}|x-y|}{2}-\kappa_{\mathrm{p}}^2\ln\frac{\kappa_{\mathrm{p}}|x-y|}{2}\right)\delta_{ij}+\frac{1}{4\pi}(\kappa_{\mathrm{s}}^2-\kappa_{\mathrm{p}}^2)\frac{(x_i-y_i)(x_j-y_j)}{|x-y|^2}\\
&\quad-\frac{\mathrm{i}}{8}\left(1+\frac{2\mathrm{i}}{\pi}\gamma-\frac{\mathrm{i}}{\pi}\right)(\kappa_{\mathrm{s}}^2-\kappa_{\mathrm{p}}^2)\delta_{ij}-\frac{1}{16\pi}(x_i-y_i)(x_j-y_j)\left(\kappa_{\mathrm{s}}^4\ln\frac{\kappa_{\mathrm{s}}|x-y|}{2}-\kappa_{\mathrm{p}}^4\ln\frac{\kappa_{\mathrm{p}}|x-y|}{2}\right)\\
&\quad-\frac{\mathrm{i}}{4}\left(\frac{\gamma\mathrm{i}}{4\pi}-\frac{3\mathrm{i}}{16\pi}+\frac{1}{8}\right)(\kappa_{\mathrm{s}}^4-\kappa_{\mathrm{p}}^4)(x_i-y_i)(x_j-y_j)+O\left(|x-y|^2\ln\frac{|x-y|}{2}\right).
\end{aligned}
\tag{4.5}
$$

For the three-dimensional case, it follows from direct calculations that

$$
\begin{aligned}
\partial_{x_i}\Phi_3(x,y,\kappa) &= \frac{(x_i-y_i)}{4\pi|x-y|^3}e^{\mathrm{i}\kappa|x-y|}(\mathrm{i}\kappa|x-y|-1),\\
\partial^2_{x_ix_j}\Phi_3(x,y,\kappa) &= \frac{|x-y|^2\delta_{ij}-3(x_i-y_i)(x_j-y_j)}{4\pi|x-y|^5}e^{\mathrm{i}\kappa|x-y|}(\mathrm{i}\kappa|x-y|-1)\\
&\quad-\kappa^2\frac{(x_i-y_i)(x_j-y_j)}{4\pi|x-y|^3}e^{\mathrm{i}\kappa|x-y|},
\end{aligned}
$$

which lead to

$$\begin{aligned}&\partial^2_{x_ix_j}(\Phi_3(x,y,\kappa_{\rm s})-\Phi_3(x,y,\kappa_{\rm p}))\\&=\frac{|x-y|^2\delta_{ij}-3(x_i-y_i)(x_j-y_j)}{4\pi|x-y|^5}\left(e^{{\rm i}\kappa_{\rm s}|x-y|}({\rm i}\kappa_{\rm s}|x-y|-1)-e^{{\rm i}\kappa_{\rm p}|x-y|}({\rm i}\kappa_{\rm p}|x-y|-1)\right)\\&\quad-\frac{(x_i-y_i)(x_j-y_j)}{4\pi|x-y|^3}\left(\kappa_{\rm s}^2e^{{\rm i}\kappa_{\rm s}|x-y|}-\kappa_{\rm p}^2e^{{\rm i}\kappa_{\rm p}|x-y|}\right).\end{aligned} \tag{4.6}$$

Lemma 4.1. *For some fixed $x\in\mathbb{R}^d$, $\boldsymbol{G}(x,\cdot,\omega)\in(L^2_{\rm loc}(\mathbb{R}^d)\cap H^{1,p}_{\rm loc}(\mathbb{R}^d))^{d\times d}$, where $p\in(1,2)$ for $d=2$ and $p\in\left(1,\frac{3}{2}\right)$ for $d=3$.*

Proof. For any fixed $x\in\mathbb{R}^d$, we choose a bounded domain $V\subset\mathbb{R}^d$ which contains x. Define $\rho:=\sup_{y\in V}|x-y|$, then we have $V\subset B_\rho(x)$. For the two-dimensional case, from (3.6) and (4.5), it is sufficient to show that

$$\ln\frac{|x-y|}{2}\in L^2(V),\quad \frac{y_i-x_i}{|x-y|^2}\in L^p(V),\quad \forall\ p\in(1,2),$$

which are proved in Lemma 3.1. For the three-dimensional case, it follows from the expansion of the exponential function e^t that

$$\kappa_{\rm s}^2e^{{\rm i}\kappa_{\rm s}|x-y|}-\kappa_{\rm p}^2e^{{\rm i}\kappa_{\rm p}|x-y|}=(\kappa_{\rm s}^2-\kappa_{\rm p}^2)+O(|x-y|),$$

$$e^{{\rm i}\kappa_s|x-y|}({\rm i}\kappa_s|x-y|-1)-e^{{\rm i}\kappa_p|x-y|}({\rm i}\kappa_p|x-y|-1)=\frac{1}{2}(\kappa_p^2-\kappa_s^2)|x-y|^2+O(|x-y|^3).$$

Thus, by (3.11) and (4.6), it is sufficient to prove

$$\frac{1}{|x-y|}\in L^2(V),\quad \frac{y_i-x_i}{|x-y|^3}\in L^p(V),\quad \forall\ p\in\left(1,\frac{3}{2}\right),$$

which can been similarly proved to the three-dimensional case in Lemma 3.1. □

Let V and G be any two bounded domains in $\mathbb{R}^d$. By Lemma 4.1 and the Sobolev embedding theorem, we have $\boldsymbol{G}(x,\cdot,\omega)\in(H^s(V))^{d\times d}$, where $s\in(0,1)$ for $d=2$ and $s\in\left(0,\frac{1}{2}\right)$ for $d=3$. Hence, for any given $\boldsymbol{g}\in H_0^{-s}(V)^d$, in the dual sense, we define the operator $\boldsymbol{H}_\omega$ by

$$(\boldsymbol{H}_\omega\boldsymbol{g})(x)=\int_V\boldsymbol{G}(x,y,\omega)\cdot\boldsymbol{g}(y){\rm d}y,\quad x\in G,$$

where the dot is the matrix-vector multiplication. By the similar arguments to [42, Theorem 8.2], we have the following property.

Lemma 4.2. *The operator $\boldsymbol{H}_\omega:H_0^{-s}(V)^d\to H^s(G)^d$ is bounded for $s\in(0,1),d=2$ or $s\in\left(0,\frac{1}{2}\right),d=3$.*

Theorem 4.3. *For some fixed $s\in\left(0,1-\frac{d}{6}\right)$ and $\tilde{p}\in\left(1,\frac{d}{d-1}\right)$, assume $0<\varepsilon<s,1<p<\min\left(\frac{\tilde{p}d}{d+\tilde{p}(\varepsilon-1)},\frac{2d}{d+2(\varepsilon-s)}\right)$ and $\frac{1}{p}+\frac{1}{p'}=1$, then the scattering problem (4.1)–(4.2) with the source $\boldsymbol{f}\in H_0^{-\varepsilon,p'}(D)^d$ attains a unique solution $\boldsymbol{u}\in H^{\varepsilon,p}_{\rm loc}(\mathbb{R}^d)^d$ given by*

$$\boldsymbol{u}(x,\omega) = -\int_D \boldsymbol{G}(x,y,\omega)\cdot \boldsymbol{f}(y)\mathrm{d}y. \tag{4.7}$$

Proof. The uniqueness of the scattering problem (4.1)–(4.2) is obvious. We focus on the existence. For convenience, we denote the differential operator in the Navier equation by

$$\Delta^*\boldsymbol{u} := \mu\Delta\boldsymbol{u} + (\lambda+\mu)\nabla\nabla\cdot\boldsymbol{u}.$$

Let $B_r := \{y\in\mathbb{R}^d : |y| < r\}$ be a ball which is large enough such that it contains the support of $\boldsymbol{f}$. Denote by ν the unit normal vector on the boundary ∂B_r. The generalized stress vector on ∂B_r is defined by

$$P\boldsymbol{u} := \mu\partial_\nu\boldsymbol{u} + (\lambda+\mu)(\nabla\cdot\boldsymbol{u})\nu.$$

Since $\tilde{p}\in\left(1,\frac{d}{d-1}\right)$ and $1<p<\frac{\tilde{p}d}{d+\tilde{p}(\varepsilon-1)}$, we have $\frac{1}{\tilde{p}}-\frac{1}{d}<\frac{1}{p}-\frac{\varepsilon}{d}$, by Lemma 4.1 and the Sobolev embedding theorem, we have $\boldsymbol{G}(x,\cdot,\omega)\in(H^{1,\tilde{p}}_{\mathrm{loc}}(\mathbb{R}^d))^{d\times d}\subset(H^{\varepsilon,p}_{\mathrm{loc}}(\mathbb{R}^d))^{d\times d}$. Since $\Delta^*\boldsymbol{u}+\omega^2\boldsymbol{u}=\boldsymbol{f}\in H_0^{-\varepsilon,p'}(D)^d$, in the sense of distributions, we have

$$\int_{B_r}\boldsymbol{G}(x,y,\omega)\cdot(\Delta^*\boldsymbol{u}(y)+\omega^2\boldsymbol{u}(y))\mathrm{d}y = \int_{B_r}\boldsymbol{G}(x,y,\omega)\cdot\boldsymbol{f}(y)\mathrm{d}y.$$

Define the operator acting on $\boldsymbol{u}$ in the left-hand side of the above equation by S_{E}. For $\boldsymbol{\varphi}\in C^\infty(\mathbb{R}^d)^d$, from the divergence theorem, we obtain

$$\begin{aligned}
(S_{\mathrm{E}}\boldsymbol{\varphi})(x) &= \int_{B_r}\boldsymbol{G}(x,y,\omega)\cdot(\Delta^*\boldsymbol{\varphi}(y)+\omega^2\boldsymbol{\varphi}(y))\mathrm{d}y\\
&= \int_{B_r\setminus B_\delta(x)}\boldsymbol{G}(x,y,\omega)\cdot(\Delta^*\boldsymbol{\varphi}(y)+\omega^2\boldsymbol{\varphi}(y))\mathrm{d}y + \int_{B_\delta(x)}\boldsymbol{G}(x,y,\omega)\cdot(\Delta^*\boldsymbol{\varphi}(y)+\omega^2\boldsymbol{\varphi}(y))\mathrm{d}y\\
&= \int_{B_r\setminus B_\delta(x)}(\boldsymbol{G}(x,y,\omega)\cdot\Delta^*\boldsymbol{\varphi}(y) - \Delta^*\boldsymbol{G}(x,y,\omega)\cdot\boldsymbol{\varphi}(y))\mathrm{d}y\\
&\quad + \int_{B_\delta(x)}\boldsymbol{G}(x,y,\omega)\cdot(\Delta^*\boldsymbol{\varphi}(y)+\omega^2\boldsymbol{\varphi}(y))\mathrm{d}y\\
&= \int_{B_\delta(x)}\boldsymbol{G}(x,y,\omega)\cdot(\Delta^*\boldsymbol{\varphi}(y)+\omega^2\boldsymbol{\varphi}(y))\mathrm{d}y\\
&\quad + \int_{\partial B_r\cup\partial B_\delta(x)}\big(\boldsymbol{G}(x,y,\omega)\cdot P\boldsymbol{\varphi}(y) - P\boldsymbol{G}(x,y,\omega)\cdot\boldsymbol{\varphi}(y)\big)\mathrm{d}s(y),
\end{aligned}$$

where $\delta>0$ is a sufficiently small constant. By the mean value theorem, it is easy to verify that

$$\lim_{\delta\to0}\int_{\partial B_\delta(x)}\big(\boldsymbol{G}(x,y,\omega)\cdot P\boldsymbol{\varphi}(y) - P\boldsymbol{G}(x,y,\omega)\cdot\boldsymbol{\varphi}(y)\big)\mathrm{d}s(y) = -\boldsymbol{\varphi}(x)$$

and

$$\lim_{\delta\to0}\int_{B_\delta(x)}\boldsymbol{G}(x,y,\omega)\cdot(\Delta^*\boldsymbol{\varphi}(y)+\omega^2\boldsymbol{\varphi}(y))\mathrm{d}y = 0.$$

Hence, we obtain

$$(\boldsymbol{S}_{\mathrm{E}}\boldsymbol{\varphi})(x) = -\boldsymbol{\varphi}(x) + \int_{\partial B_r} \big(\boldsymbol{G}(x,y,\omega)\cdot P\boldsymbol{\varphi}(y) - P\boldsymbol{G}(x,y,\omega)\cdot\boldsymbol{\varphi}(y)\big)\mathrm{d}s(y),$$

which implies

$$(\boldsymbol{S}_{\mathrm{E}}\boldsymbol{u})(x) = -\boldsymbol{u}(x) + \int_{\partial B_r} \big(\boldsymbol{G}(x,y,\omega)\cdot P\boldsymbol{u}(y) - P\boldsymbol{G}(x,y,\omega)\cdot\boldsymbol{u}(y)\big)\mathrm{d}s(y).$$

Since $\boldsymbol{u}(y)$ and $\boldsymbol{G}(x,y,\omega)$ satisfy the Sommerfeld radiation condition, we have

$$\lim_{r\to\infty}\int_{\partial B_r} \big(\boldsymbol{G}(x,y,\omega)\cdot \boldsymbol{P}\boldsymbol{u}(y) - \boldsymbol{P}\boldsymbol{G}(x,y,\omega)\cdot\boldsymbol{u}(y)\big)\mathrm{d}s(y) = 0.$$

Therefore

$$\boldsymbol{u}(x,\omega) = -\int_D \boldsymbol{G}(x,y,\omega)\cdot\boldsymbol{f}(y)\mathrm{d}y = -\boldsymbol{H}_\omega\boldsymbol{f}(x).$$

Next is to show that $\boldsymbol{u}\in H^{\varepsilon,p}_{\mathrm{loc}}(\mathbb{R}^d)^d$. By Lemma 4.2, we have that for $s\in\left(0,1-\frac{d}{6}\right)$, the operator $\boldsymbol{H}_\omega : H_0^{-s}(D)^d \to H^s_{\mathrm{loc}}(\mathbb{R}^d)^d$ is bounded. The assumption $1<p<\frac{2d}{d+2(\varepsilon-s)}$ implies that $\frac{1}{2}+\frac{\varepsilon-s}{d}<\frac{1}{p}<1$ which yields $\frac{1}{2}-\frac{s}{d}<\frac{1}{p}-\frac{\varepsilon}{d}$. Thus, due to $0<\varepsilon<s$, the Sobolev embedding theorem implies that $H^s(D)$ is embedded into $H^{\varepsilon,p}(D)$ and $H_0^{-\varepsilon,p'}(D)$ is embedded into $H_0^{-s}(D)$. Thus, the operator $\boldsymbol{H}_\omega : H_0^{-\varepsilon,p'}(D)^d \to H^{\varepsilon,p}_{\mathrm{loc}}(\mathbb{R}^d)^d$ is bounded, which completes the proof. □

4.2. The two-dimensional case

This subsection is devoted to study the two-dimensional case. It is required to derive a relationship between the scattering data and the principle symbol of the covariance operator of the component of $\boldsymbol{f}$. To this end, we need to express the displacement $\boldsymbol{u}(x,\omega)$ explicitly. Substituting (4.5) into (4.3) gives that $\boldsymbol{u}(x,\omega) = (u_1(x,\omega), u_2(x,\omega))^\top$ where

$$u_i(x,\omega) = u_{i1}(x,\omega) + u_{i2}(x,\omega) + u_{i3}(x,\omega) + u_{i4}(x,\omega),$$

with

$$\begin{aligned}
u_{i1}(x,\omega) &= \frac{\mathrm{i}}{4\mu}\int_D H_0^{(1)}(\kappa_{\mathrm{s}}|x-y|)f_i(y)\mathrm{d}y,\\
u_{i2}(x,\omega) &= \frac{\mathrm{i}}{4\omega^2}\int_D \Big[-\kappa_{\mathrm{s}}H_1^{(1)}(\kappa_{\mathrm{s}}|x-y|) + \kappa_{\mathrm{p}}H_1^{(1)}(\kappa_{\mathrm{p}}|x-y|)\Big]\frac{1}{|x-y|}f_i(y)\mathrm{d}y,\\
u_{i3}(x,\omega) &= \frac{\mathrm{i}}{4\omega^2}\int_D \Big[\kappa_{\mathrm{s}}^2H_2^{(1)}(\kappa_{\mathrm{s}}|x-y|) - \kappa_{\mathrm{p}}^2H_2^{(1)}(\kappa_{\mathrm{p}}|x-y|)\Big]\frac{(x_i-y_i)^2}{|x-y|^2}f_i(y)\mathrm{d}y,\\
u_{i4}(x,\omega) &= \frac{\mathrm{i}}{4\omega^2}\int_D \Big[\kappa_{\mathrm{s}}^2H_2^{(1)}(\kappa_{\mathrm{s}}|x-y|) - \kappa_{\mathrm{p}}^2H_2^{(1)}(\kappa_{\mathrm{p}}|x-y|)\Big]\frac{(x_1-y_1)(x_2-y_2)}{|x-y|^2}f_k(y)\mathrm{d}y,
\end{aligned}$$

here $i=1$, 2 and $\{i,k\}=\{1,2\}$.

To prove the main result, we need to establish the asymptotic of $\boldsymbol{u}(x,\omega)$ for $\omega\to\infty$. Recalling the definition of $H^{(1)}_{n,N}$ given in (3.35), we define $\tilde{\boldsymbol{u}}(x,\omega)=(\tilde{u}_1(x,\omega),\tilde{u}_2(x,\omega))^\top$, where

$$\tilde{u}_i(x,\omega)=\tilde{u}_{i1}(x,\omega)+\tilde{u}_{i2}(x,\omega)+\tilde{u}_{i3}(x,\omega)+\tilde{u}_{i4}(x,\omega),$$

Here

$$\begin{aligned}
\tilde{u}_{i1}(x,\omega)&=\frac{\mathrm{i}}{4\mu}\int_D H^{(1)}_{0,2}(\kappa_{\mathrm{s}}|x-y|)f_i(y)\mathrm{d}y,\\
\tilde{u}_{i2}(x,\omega)&=\frac{\mathrm{i}}{4\omega^2}\int_D\Big[-\kappa_{\mathrm{s}}H^{(1)}_{1,3}(\kappa_{\mathrm{s}}|x-y|)+\kappa_{\mathrm{p}}H^{(1)}_{1,3}(\kappa_{\mathrm{p}}|x-y|)\Big]\frac{1}{|x-y|}f_i(y)\mathrm{d}y,\\
\tilde{u}_{i3}(x,\omega)&=\frac{\mathrm{i}}{4\omega^2}\int_D\Big[\kappa_{\mathrm{s}}^2H^{(1)}_{2,4}(\kappa_{\mathrm{s}}|x-y|)-\kappa_{\mathrm{p}}^2H^{(1)}_{2,4}(\kappa_{\mathrm{p}}|x-y|)\Big]\frac{(x_i-y_i)^2}{|x-y|^2}f_i(y)\mathrm{d}y,\\
\tilde{u}_{i4}(x,\omega)&=\frac{\mathrm{i}}{4\omega^2}\int_D\Big[\kappa_{\mathrm{s}}^2H^{(1)}_{2,4}(\kappa_{\mathrm{s}}|x-y|)-\kappa_{\mathrm{p}}^2H^{(1)}_{2,4}(\kappa_{\mathrm{p}}|x-y|)\Big]\frac{(x_1-y_1)(x_2-y_2)}{|x-y|^2}f_k(y)\mathrm{d}y,
\end{aligned}$$

where $i=1,2$ and $\{i,k\}=\{1,2\}$.

Lemma 4.4. *The random variable $\boldsymbol{u}(x,\omega)-\tilde{\boldsymbol{u}}(x,\omega)$ satisfies almost surely the condition*

$$|\boldsymbol{u}(x,\omega)-\tilde{\boldsymbol{u}}(x,\omega)|\le c\omega^{-\frac{7}{2}},\quad x\in U,\quad \omega>0,$$

where the constant c depends only on $H_0^{-1,p'}(D)^2$-norm of $\boldsymbol{f}$.

Proof. By Assumption A, it is known that there exists a positive constant M such that $|x-y|\ge M$ holds for any $x\in U$ and $y\in D$. By (3.36), for $x\in U$, we have

$$\begin{aligned}
|u_{11}(x,\omega)-\tilde{u}_{11}(x,\omega)|&=\left|\frac{\mathrm{i}}{4\mu}\int_D\Big[H^{(1)}_0(\kappa_{\mathrm{s}}|x-y|)-H^{(1)}_{0,2}(\kappa_{\mathrm{s}}|x-y|)\Big]f_1(y)\mathrm{d}y\right|\\
&\lesssim||H^{(1)}_0(\kappa_s|x-\cdot|)-H^{(1)}_{0,2}(\kappa_s|x-\cdot|)||_{H^{1,p}(D)}||f_1||_{H_0^{-1,p'}(D)}\\
&\le c\omega^{-\frac{7}{2}},
\end{aligned}$$

where the constant c depends only on $H_0^{-1,p'}(D)$-norm of f_1.

Similarly, it is easy to verify that

$$|u_{ij}(x,\omega)-\tilde{u}_{ij}(x,\omega)|\le c\omega^{-\frac{7}{2}}\quad \text{for } i=1,2,\ j=1,2,3,4,$$

where the constant c depends only on $H_0^{-1,p'}(D)^2$-norm of $\boldsymbol{f}$. Therefore

$$|\boldsymbol{u}(x,\omega)-\tilde{\boldsymbol{u}}(x,\omega)|\le\sum_{i=1}^2\sum_{j=1}^4|u_{ij}(x,\omega)-\tilde{u}_{ij}(x,\omega)|\le c\omega^{-\frac{7}{2}},$$

which completes the proof. □

To derive the relationship between the scattering data and the function in the principle symbol, we need to estimate $\mathbb{E}(\boldsymbol{u}(x,\omega_1)\cdot\overline{\boldsymbol{u}(x,\omega_2)})$ for $\omega_1\ge1,\omega_2\ge1$. By

Lemma 4.4, it reduces to find the estimate of $\mathbb{E}(\tilde{\boldsymbol{u}}(x,\omega_1)\cdot\overline{\tilde{\boldsymbol{u}}(x,\omega_2)})$ for $\omega_1\geq 1,\omega_2\geq 1$. Recalling $\tilde{\boldsymbol{u}}(x,\omega)=(\tilde{u}_1(x,\omega),\tilde{u}_2(x,\omega))^\top$ and (3.35), we have

$$\begin{aligned}\mathbb{E}(\tilde{\boldsymbol{u}}(x,\omega_1)\cdot\overline{\tilde{\boldsymbol{u}}(x,\omega_2)})&=\mathbb{E}(\tilde{u}_1(x,\omega_1)\overline{\tilde{u}_1(x,\omega_2)})+\mathbb{E}(\tilde{u}_2(x,\omega_1)\overline{\tilde{u}_2(x,\omega_2)})\\&=\sum_{i,j=1}^{4}(\mathbb{E}(\tilde{u}_{1i}(x,\omega_1)\overline{\tilde{u}_{1j}(x,\omega_2)})+\mathbb{E}(\tilde{u}_{2i}(x,\omega_1)\overline{\tilde{u}_{2j}(x,\omega_2)}))\end{aligned}\tag{4.8}$$

where

$$\begin{aligned}\tilde{u}_{i1}(x,\omega)&=\frac{\mathrm{i}}{4\mu}\int_D H_{0,2}^{(1)}(\kappa_{\mathrm{s}}|x-y|)f_i(y)\mathrm{d}y\\&=\frac{\mathrm{i}}{4\mu}\int_D\kappa_{\mathrm{s}}^{-\frac{1}{2}}|x-y|^{-\frac{1}{2}}e^{\mathrm{i}\left(\kappa_{\mathrm{s}}|x-y|-\frac{1}{4}\pi\right)}\sum_{j=0}^{2}a_j^{(0)}\left(\frac{1}{\kappa_{\mathrm{s}}|x-y|}\right)^j f_i(y)\mathrm{d}y,\\ \tilde{u}_{i2}(x,\omega)&=\frac{\mathrm{i}}{4\omega^2}\int_D\left[-\kappa_{\mathrm{s}}H_{1,3}^{(1)}(\kappa_{\mathrm{s}}|x-y|)+\kappa_{\mathrm{p}}H_{1,3}^{(1)}(\kappa_{\mathrm{p}}|x-y|)\right]\frac{1}{|x-y|}f_i(y)\mathrm{d}y\\&=\frac{\mathrm{i}}{4\omega^2}\int_D-\kappa_{\mathrm{s}}^{\frac{1}{2}}|x-y|^{-\frac{3}{2}}e^{\mathrm{i}\left(\kappa_{\mathrm{s}}|x-y|-\frac{3}{4}\pi\right)}\sum_{j=0}^{3}a_j^{(1)}\left(\frac{1}{\kappa_{\mathrm{s}}|x-y|}\right)^j f_i(y)\mathrm{d}y\\&\quad+\frac{\mathrm{i}}{4\omega^2}\int_D\kappa_{\mathrm{p}}^{\frac{1}{2}}|x-y|^{-\frac{3}{2}}e^{\mathrm{i}\left(\kappa_{\mathrm{p}}|x-y|-\frac{3}{4}\pi\right)}\sum_{j=0}^{3}a_j^{(1)}\left(\frac{1}{\kappa_{\mathrm{p}}|x-y|}\right)^j f_i(y)\mathrm{d}y,\\ \tilde{u}_{i3}(x,\omega)&=\frac{\mathrm{i}}{4\omega^2}\int_D\left[\kappa_{\mathrm{s}}^2H_{2,4}^{(1)}(\kappa_{\mathrm{s}}|x-y|)-\kappa_{\mathrm{p}}^2H_{2,4}^{(1)}(\kappa_{\mathrm{p}}|x-y|)\right]\frac{(x_1-y_1)^2}{|x-y|^2}f_i(y)\mathrm{d}y\\&=\frac{\mathrm{i}}{4\omega^2}\int_D\kappa_{\mathrm{s}}^{\frac{3}{2}}|x-y|^{-\frac{5}{2}}(x_i-y_i)^2e^{\mathrm{i}\left(\kappa_{\mathrm{s}}|x-y|-\frac{5}{4}\pi\right)}\sum_{j=0}^{4}a_j^{(2)}\left(\frac{1}{\kappa_{\mathrm{s}}|x-y|}\right)^j f_i(y)\mathrm{d}y\\&\quad-\frac{\mathrm{i}}{4\omega^2}\int_D\kappa_{\mathrm{p}}^{\frac{3}{2}}|x-y|^{-\frac{5}{2}}(x_i-y_i)^2e^{\mathrm{i}\left(\kappa_{\mathrm{p}}|x-y|-\frac{5}{4}\pi\right)}\sum_{j=0}^{4}a_j^{(2)}\left(\frac{1}{\kappa_{\mathrm{p}}|x-y|}\right)^j f_i(y)\mathrm{d}y,\\ \tilde{u}_{i4}(x,\omega)&=\frac{\mathrm{i}}{4\omega^2}\int_D\left[\kappa_{\mathrm{s}}^2H_{2,4}^{(1)}(\kappa_{\mathrm{s}}|x-y|)-\kappa_{\mathrm{p}}^2H_{2,4}^{(1)}(\kappa_{\mathrm{p}}|x-y|)\right]\frac{(x_1-y_1)(x_2-y_2)}{|x-y|^2}f_k(y)\mathrm{d}y\\&=\frac{\mathrm{i}}{4\omega^2}\int_D\kappa_{\mathrm{s}}^{\frac{3}{2}}|x-y|^{-\frac{5}{2}}(x_1-y_1)(x_2-y_2)e^{\mathrm{i}\left(\kappa_{\mathrm{s}}|x-y|-\frac{5}{4}\pi\right)}\sum_{j=0}^{4}a_j^{(2)}\left(\frac{1}{\kappa_{\mathrm{s}}|x-y|}\right)^j f_k(y)\mathrm{d}y\\&\quad-\frac{\mathrm{i}}{4\omega^2}\int_D\kappa_{\mathrm{p}}^{\frac{3}{2}}|x-y|^{-\frac{5}{2}}(x_1-y_1)(x_2-y_2)e^{\mathrm{i}\left(\kappa_{\mathrm{p}}|x-y|-\frac{5}{4}\pi\right)}\sum_{j=0}^{4}a_j^{(2)}\left(\frac{1}{\kappa_{\mathrm{p}}|x-y|}\right)^j f_k(y)\mathrm{d}y.\end{aligned}$$

Using the assumption $\mathbb{E}(f_1 f_2) = 0$, we obtain

$$\mathbb{E}(\tilde{u}_{i1}(x,\omega_1)\overline{\tilde{u}_{i4}(x,\omega_2)}) = 0, \quad \mathbb{E}(\tilde{u}_{i2}(x,\omega_1)\overline{\tilde{u}_{i4}(x,\omega_2)}) = 0,$$
$$\mathbb{E}(\tilde{u}_{i3}(x,\omega_1)\overline{\tilde{u}_{i4}(x,\omega_2)}) = 0, \quad \mathbb{E}(\tilde{u}_{i4}(x,\omega_1)\overline{\tilde{u}_{i1}(x,\omega_2)}) = 0,$$
$$\mathbb{E}(\tilde{u}_{i4}(x,\omega_1)\overline{\tilde{u}_{i2}(x,\omega_2)}) = 0, \quad \mathbb{E}(\tilde{u}_{i4}(x,\omega_1)\overline{\tilde{u}_{i3}(x,\omega_2)}) = 0,$$

and

$$\begin{aligned}
&\mathbb{E}(\tilde{u}_{i1}(x,\omega_1)\overline{\tilde{u}_{i1}(x,\omega_2)}) \\
&= \frac{1}{16\mu^2}\sum_{j_1,j_2=0}^{2} \frac{a_{j_1}^{(0)}\overline{a_{j_2}^{(0)}}}{c_{\mathrm{s}}^{j_1+j_2+1}\omega_1^{j_1+\frac{1}{2}}\omega_2^{j_1+\frac{1}{2}}}\int_{\mathbb{R}^4} \frac{e^{\mathrm{i}(c_{\mathrm{s}}\omega_1|x-y|-c_{\mathrm{s}}\omega_2|x-z|)}}{|x-y|^{j_1+\frac{1}{2}}|x-z|^{j_2+\frac{1}{2}}}\mathbb{E}(f_i(y)f_i(z))\mathrm{d}y\mathrm{d}z, \\
&\mathbb{E}(\tilde{u}_{i1}(x,\omega_1)\overline{\tilde{u}_{i2}(x,\omega_2)}) = \frac{e^{\frac{\pi}{2}\mathrm{i}}}{16\mu}\sum_{j_1=0}^{2}\sum_{j_2=0}^{3}\frac{a_{j_1}^{(0)}\overline{a_{j_2}^{(1)}}}{\omega_1^{j_1+\frac{1}{2}}\omega_2^{j_2+\frac{3}{2}}} \\
&\quad\times\int_{\mathbb{R}^4}\left[-\frac{e^{\mathrm{i}(c_{\mathrm{s}}\omega_1|x-y|-c_{\mathrm{s}}\omega_2|x-z|)}}{c_{\mathrm{s}}^{j_1+j_2}}+\frac{e^{\mathrm{i}(c_{\mathrm{s}}\omega_1|x-y|-c_{\mathrm{p}}\omega_2|x-z|)}}{c_{\mathrm{s}}^{j_1+\frac{1}{2}}c_{\mathrm{p}}^{j_2-\frac{1}{2}}}\right]\frac{\mathbb{E}(f_i(y)f_i(z))}{|x-y|^{j_1+\frac{1}{2}}|x-z|^{j_2+\frac{3}{2}}}\mathrm{d}y\mathrm{d}z, \\
&\mathbb{E}(\tilde{u}_{i1}(x,\omega_1)\overline{\tilde{u}_{i3}(x,\omega_2)}) = \frac{e^{\pi\mathrm{i}}}{16\mu}\sum_{j_1=0}^{2}\sum_{j_2=0}^{4}\frac{a_{j_1}^{(0)}\overline{a_{j_2}^{(2)}}}{\omega_1^{j_1+\frac{1}{2}}\omega_2^{j_2+\frac{1}{2}}} \\
&\quad\times\int_{\mathbb{R}^4}\left[\frac{e^{\mathrm{i}(c_{\mathrm{s}}\omega_1|x-y|-c_{\mathrm{s}}\omega_2|x-z|)}}{c_{\mathrm{s}}^{j_1+j_2-1}}-\frac{e^{\mathrm{i}(c_{\mathrm{s}}\omega_1|x-y|-c_{\mathrm{p}}\omega_2|x-z|)}}{c_{\mathrm{s}}^{j_1+\frac{1}{2}}c_{\mathrm{p}}^{j_2-\frac{3}{2}}}\right]\frac{(x_i-z_i)^2\mathbb{E}(f_i(y)f_i(z))}{|x-y|^{j_1+\frac{1}{2}}|x-z|^{j_2+\frac{5}{2}}}\mathrm{d}y\mathrm{d}z, \\
&\mathbb{E}(\tilde{u}_{i2}(x,\omega_1)\overline{\tilde{u}_{i1}(x,\omega_2)}) = \frac{e^{-\frac{\pi}{2}\mathrm{i}}}{16\mu}\sum_{j_1=0}^{3}\sum_{j_2=0}^{2}\frac{a_{j_1}^{(1)}\overline{a_{j_2}^{(0)}}}{\omega_1^{j_1+\frac{3}{2}}\omega_2^{j_2+\frac{1}{2}}} \\
&\quad\times\int_{\mathbb{R}^4}\left[-\frac{e^{\mathrm{i}(c_{\mathrm{s}}\omega_1|x-y|-c_{\mathrm{s}}\omega_2|x-z|)}}{c_{\mathrm{s}}^{j_1+j_2}}+\frac{e^{\mathrm{i}(c_{\mathrm{p}}\omega_1|x-y|-c_{\mathrm{s}}\omega_2|x-z|)}}{c_{\mathrm{p}}^{j_1-\frac{1}{2}}c_{\mathrm{s}}^{j_2+\frac{1}{2}}}\right]\frac{\mathbb{E}(f_i(y)f_i(z))}{|x-y|^{j_1+\frac{3}{2}}|x-z|^{j_2+\frac{1}{2}}}\mathrm{d}y\mathrm{d}z, \\
&\mathbb{E}(\tilde{u}_{i2}(x,\omega_1)\overline{\tilde{u}_{i2}(x,\omega_2)}) = \frac{1}{16}\sum_{j_1,j_2=0}^{3}\frac{a_{j_1}^{(1)}\overline{a_{j_2}^{(1)}}}{\omega_1^{j_1+\frac{3}{2}}\omega_2^{j_2+\frac{3}{2}}} \\
&\quad\times\int_{\mathbb{R}^4}\left[\frac{e^{\mathrm{i}(c_{\mathrm{s}}\omega_1|x-y|-c_{\mathrm{s}}\omega_2|x-z|)}}{c_{\mathrm{s}}^{j_1+j_2-1}}+\frac{e^{\mathrm{i}(c_{\mathrm{p}}\omega_1|x-y|-c_{\mathrm{p}}\omega_2|x-z|)}}{c_{\mathrm{p}}^{j_1+j_2-1}}\right. \\
&\qquad\left.-\frac{e^{\mathrm{i}(c_{\mathrm{s}}\omega_1|x-y|-c_{\mathrm{p}}\omega_2|x-z|)}}{c_{\mathrm{s}}^{j_1-\frac{1}{2}}c_{\mathrm{p}}^{j_2-\frac{1}{2}}}-\frac{e^{\mathrm{i}(c_{\mathrm{p}}\omega_1|x-y|-c_{\mathrm{s}}\omega_2|x-z|)}}{c_{\mathrm{p}}^{j_1-\frac{1}{2}}c_{\mathrm{s}}^{j_2-\frac{1}{2}}}\right]\frac{\mathbb{E}(f_i(y)f_i(z))}{|x-y|^{j_1+\frac{3}{2}}|x-z|^{j_2+\frac{3}{2}}}\mathrm{d}y\mathrm{d}z,
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}(\tilde{u}_{i2}(x,\omega_1)\overline{\tilde{u}_{i3}(x,\omega_2)}) &= \frac{e^{\frac{\pi}{2}\mathrm{i}}}{16}\sum_{j_1=0}^{3}\sum_{j_2=0}^{4}\frac{a_{j_1}^{(1)}\overline{a_{j_2}^{(2)}}}{\omega_1^{j_1+\frac{3}{2}}\omega_2^{j_2+\frac{1}{2}}}\\
&\times\int_{\mathbb{R}^4}\Bigg[-\frac{e^{\mathrm{i}(c_s\omega_1|x-y|-c_s\omega_2|x-z|)}}{c_s^{j_1+j_2-2}}-\frac{e^{\mathrm{i}(c_p\omega_1|x-y|-c_p\omega_2|x-z|)}}{c_p^{j_1+j_2-2}}\\
&\quad+\frac{e^{\mathrm{i}(c_s\omega_1|x-y|-c_p\omega_2|x-z|)}}{c_s^{j_1-\frac{1}{2}}c_p^{j_2-\frac{3}{2}}}+\frac{e^{\mathrm{i}(c_p\omega_1|x-y|-c_s\omega_2|x-z|)}}{c_p^{j_1-\frac{1}{2}}c_s^{j_2-\frac{3}{2}}}\Bigg]\frac{(x_i-z_i)^2\mathbb{E}(f_i(y)f_i(z))}{|x-y|^{j_1+\frac{3}{2}}|x-z|^{j_2+\frac{5}{2}}}\mathrm{d}y\mathrm{d}z,\\
\mathbb{E}(\tilde{u}_{i3}(x,\omega_1)\overline{\tilde{u}_{i1}(x,\omega_2)}) &= \frac{e^{-\pi\mathrm{i}}}{16\mu}\sum_{j_1=0}^{4}\sum_{j_2=0}^{2}\frac{a_{j_1}^{(2)}\overline{a_{j_2}^{(0)}}}{\omega_1^{j_1+\frac{1}{2}}\omega_2^{j_2+\frac{1}{2}}}\\
&\times\int_{\mathbb{R}^4}\Bigg[\frac{e^{\mathrm{i}(c_s\omega_1|x-y|-c_s\omega_2|x-z|)}}{c_s^{j_1+j_2-1}}-\frac{e^{\mathrm{i}(c_p\omega_1|x-y|-c_s\omega_2|x-z|)}}{c_p^{j_1-\frac{3}{2}}c_s^{j_2+\frac{1}{2}}}\Bigg]\frac{(x_i-y_i)^2\mathbb{E}(f_i(y)f_i(z))}{|x-y|^{j_1+\frac{5}{2}}|x-z|^{j_2+\frac{1}{2}}}\mathrm{d}y\mathrm{d}z,\\
\mathbb{E}(\tilde{u}_{i3}(x,\omega_1)\overline{\tilde{u}_{i2}(x,\omega_2)}) &= \frac{e^{-\frac{\pi}{2}\mathrm{i}}}{16}\sum_{j_1=0}^{4}\sum_{j_2=0}^{3}\frac{a_{j_1}^{(2)}\overline{a_{j_2}^{(1)}}}{\omega_1^{j_1+\frac{1}{2}}\omega_2^{j_2+\frac{3}{2}}}\\
&\times\int_{\mathbb{R}^4}\Bigg[-\frac{e^{\mathrm{i}(c_s\omega_1|x-y|-c_s\omega_2|x-z|)}}{c_s^{j_1+j_2-2}}-\frac{e^{\mathrm{i}(c_p\omega_1|x-y|-c_p\omega_2|x-z|)}}{c_p^{j_1+j_2-2}}\\
&\quad+\frac{e^{\mathrm{i}(c_s\omega_1|x-y|-c_p\omega_2|x-z|)}}{c_s^{j_1-\frac{3}{2}}c_p^{j_2-\frac{1}{2}}}+\frac{e^{\mathrm{i}(c_p\omega_1|x-y|-c_s\omega_2|x-z|)}}{c_p^{j_1-\frac{3}{2}}c_s^{j_2-\frac{1}{2}}}\Bigg]\frac{(x_i-y_i)^2\mathbb{E}(f_i(y)f_i(z))}{|x-y|^{j_1+\frac{5}{2}}|x-z|^{j_2+\frac{3}{2}}}\mathrm{d}y\mathrm{d}z,\\
\mathbb{E}(\tilde{u}_{i3}(x,\omega_1)\overline{\tilde{u}_{i3}(x,\omega_2)}) &= \frac{1}{16}\sum_{j_1,j_2=0}^{4}\frac{a_{j_1}^{(2)}\overline{a_{j_2}^{(2)}}}{\omega_1^{j_1+\frac{1}{2}}\omega_2^{j_2+\frac{1}{2}}}\\
&\times\int_{\mathbb{R}^4}\Bigg[\frac{e^{\mathrm{i}(c_s\omega_1|x-y|-c_s\omega_2|x-z|)}}{c_s^{j_1+j_2-3}}+\frac{e^{\mathrm{i}(c_p\omega_1|x-y|-c_p\omega_2|x-z|)}}{c_p^{j_1+j_2-3}}\\
&\quad-\frac{e^{\mathrm{i}(c_s\omega_1|x-y|-c_p\omega_2|x-z|)}}{c_s^{j_1-\frac{3}{2}}c_p^{j_2-\frac{3}{2}}}-\frac{e^{\mathrm{i}(c_p\omega_1|x-y|-c_s\omega_2|x-z|)}}{c_p^{j_1-\frac{3}{2}}c_s^{j_2-\frac{3}{2}}}\Bigg]\frac{(x_i-y_i)^2(x_i-z_i)^2\mathbb{E}(f_i(y)f_i(z))}{|x-y|^{j_1+\frac{5}{2}}|x-z|^{j_2+\frac{5}{2}}}\mathrm{d}y\mathrm{d}z,\\
\mathbb{E}(\tilde{u}_{i4}(x,\omega_1)\overline{\tilde{u}_{i4}(x,\omega_2)}) &= \frac{1}{16}\sum_{j_1,j_2=0}^{4}\frac{a_{j_1}^{(2)}\overline{a_{j_2}^{(2)}}}{\omega_1^{j_1+\frac{1}{2}}\omega_2^{j_2+\frac{1}{2}}}\\
&\times\int_{\mathbb{R}^4}\Bigg[\frac{e^{\mathrm{i}(c_s\omega_1|x-y|-c_s\omega_2|x-z|)}}{c_s^{j_1+j_2-3}}+\frac{e^{\mathrm{i}(c_p\omega_1|x-y|-c_p\omega_2|x-z|)}}{c_p^{j_1+j_2-3}}-\frac{e^{\mathrm{i}(c_s\omega_1|x-y|-c_p\omega_2|x-z|)}}{c_s^{j_1-\frac{3}{2}}c_p^{j_2-\frac{3}{2}}}\\
&\quad-\frac{e^{\mathrm{i}(c_p\omega_1|x-y|-c_s\omega_2|x-z|)}}{c_p^{j_1-\frac{3}{2}}c_s^{j_2-\frac{3}{2}}}\Bigg]\frac{(x_1-y_1)(x_2-y_2)(x_1-z_1)(x_2-z_2)}{|x-y|^{j_1+\frac{5}{2}}|x-z|^{j_2+\frac{5}{2}}}\mathbb{E}(f_k(y)f_k(z))\mathrm{d}y\mathrm{d}z,
\end{aligned}$$

for $i=1,\,2$ and $\{i,k\}=\{1,2\}$.

A direct application of Lemma 3.4 to each item on the right hand side of (4.8) gives the following lemma.

Lemma 4.5. *For $\omega_1\geq 1,\omega_2\geq 1$, the following estimates*

$$|\mathbb{E}(\tilde{\boldsymbol{u}}(x,\omega_1)\cdot\overline{\tilde{\boldsymbol{u}}(x,\omega_2)})|\leq c_n(\omega_1+\omega_2)^{-m-1}(1+|\omega_1-\omega_2|)^{-n}$$
$$|\mathbb{E}(\tilde{\boldsymbol{u}}(x,\omega_1)\cdot\tilde{\boldsymbol{u}}(x,\omega_2))|\leq c_n(\omega_1+\omega_2)^{-n}(1+|\omega_1-\omega_2|)^{-m}$$

holds uniformly for $x \in U$, where $n \in \mathbb{N}$ is arbitrary and $c_n > 0$ is a constant depending only on n.

To obtain the relation between the scattering data and the function in the principle symbol, it is required to estimate the order of $\mathbb{E}(\tilde{u}(x,\omega)\cdot\overline{\tilde{u}(x,\omega)})$. According to (4.8) where we set $\omega_1 = \omega_2 = \omega$, it reduces to estimate the order of $\mathbb{E}(\tilde{u}_{ij_1}(x,\omega)\overline{\tilde{u}_{ij_2}(x,\omega)})$ for $i = 1, 2, j_1, j_2 = 1, 2, 3, 4$. Applying Lemma 3.5 gives that

$$\begin{aligned}
&\mathbb{E}(\tilde{u}_{i1}(x,\omega)\overline{\tilde{u}_{i2}(x,\omega)}) = O(\omega^{-(m+2)}), \quad \mathbb{E}(\tilde{u}_{i2}(x,\omega)\overline{\tilde{u}_{i1}(x,\omega)}) = O(\omega^{-(m+2)}),\\
&\mathbb{E}(\tilde{u}_{i2}(x,\omega)\overline{\tilde{u}_{i2}(x,\omega)}) = O(\omega^{-(m+2)}), \quad \mathbb{E}(\tilde{u}_{i2}(x,\omega)\overline{\tilde{u}_{i3}(x,\omega)}) = O(\omega^{-(m+2)}),\\
&\mathbb{E}(\tilde{u}_{i3}(x,\omega)\overline{\tilde{u}_{i2}(x,\omega)}) = O(\omega^{-(m+2)}),
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}(\tilde{u}_{i1}(x,\omega)\overline{\tilde{u}_{i1}(x,\omega)}) &= N_{1i}^{(2)}(x)\omega^{-(m+1)} + O(\omega^{-(m+2)})\\
\mathbb{E}(\tilde{u}_{i1}(x,\omega)\overline{\tilde{u}_{i3}(x,\omega)}) &= N_{2i}^{(2)}(x,\omega)\omega^{-(m+1)} + O(\omega^{-(m+2)})\\
\mathbb{E}(\tilde{u}_{i3}(x,\omega)\overline{\tilde{u}_{i1}(x,\omega)}) &= N_{3i}^{(2)}(x,\omega)\omega^{-(m+1)} + O(\omega^{-(m+2)})\\
\mathbb{E}(\tilde{u}_{i3}(x,\omega)\overline{\tilde{u}_{i3}(x,\omega)}) &= N_{4i}^{(2)}(x,\omega)\omega^{-(m+1)} + O(\omega^{-(m+2)})\\
\mathbb{E}(\tilde{u}_{i4}(x,\omega)\overline{\tilde{u}_{i4}(x,\omega)}) &= N_{5i}^{(2)}(x,\omega)\omega^{-(m+1)} + O(\omega^{-(m+2)}),
\end{aligned}$$

where

$$\begin{aligned}
N_{1i}^{(2)}(x) &= a_1\int_{\mathbb{R}^2}\frac{1}{|x-y|}\phi(y)\mathrm{d}y,\\
N_{2i}^{(2)}(x,\omega) &= \int_{\mathbb{R}^2}(a_2 e^{\mathrm{i}(c_{\mathrm{s}}-c_{\mathrm{p}})|x-y|\omega} - a_1)\frac{(x_i-y_i)^2}{|x-y|^3}\phi(y)\mathrm{d}y,\\
N_{3i}^{(2)}(x,\omega) &= \int_{\mathbb{R}^2}(a_2 e^{\mathrm{i}(c_{\mathrm{p}}-c_{\mathrm{s}})|x-y|\omega} - a_1)\frac{(x_i-y_i)^2}{|x-y|^3}\phi(y)\mathrm{d}y,\\
N_{4i}^{(2)}(x,\omega) &= \int_{\mathbb{R}^2}(a_3 - 2a_2\cos((c_{\mathrm{s}}-c_{\mathrm{p}})|x-y|\omega))\frac{(x_i-y_i)^4}{|x-y|^5}\phi(y)\mathrm{d}y,\\
N_{5i}^{(2)}(x,\omega) &= \int_{\mathbb{R}^2}(a_3 - 2a_2\cos((c_{\mathrm{s}}-c_{\mathrm{p}})|x-y|\omega))\frac{(x_1-y_1)^2(x_2-y_2)^2}{|x-y|^5}\phi(y)\mathrm{d}y.
\end{aligned}$$

Here, a_1, a_2, and a_3 are positive constants given by

$$a_1 = \frac{1}{32\pi}c_{\mathrm{s}}^{3-m}, \quad a_2 = \frac{(c_{\mathrm{s}}c_{\mathrm{p}})^{\frac{3}{2}}}{32\pi}\left(\frac{2}{c_{\mathrm{s}}+c_{\mathrm{p}}}\right)^m, \quad a_3 = \frac{1}{32\pi}\left(c_{\mathrm{s}}^{3-m} + c_{\mathrm{p}}^{3-m}\right).$$

By (4.8) and a simple calculation, we obtain

$$\mathbb{E}(\tilde{\boldsymbol{u}}(x,\omega)\cdot\overline{\tilde{\boldsymbol{u}}(x,\omega)}) = T_{\mathrm{E}}^{(2)}(x)\omega^{-(m+1)} + O(\omega^{-(m+2)}), \tag{4.9}$$

where

$$T_{\mathrm{E}}^{(2)}(x)=\sum_{j=1}^{5}\sum_{i=1}^{2}N_{ji}^{(2)}(x,\omega)=a_3\int_{\mathbb{R}^2}\frac{1}{|x-y|}\phi(y)dy. \tag{4.10}$$

Now we are ready to present the main result for elastic waves in the two dimensions.

Theorem 4.6. *Let the external source $\boldsymbol{f}$ be a microlocally isotropic Gaussian random vector field which satisfies Assumption C. Then for all $x\in U$, it holds almost surely that*

$$\lim_{Q\to\infty}\frac{1}{Q-1}\int_1^Q\omega^{m+1}|\boldsymbol{u}(x,\omega)|^2\mathrm{d}\omega=T_{\mathrm{E}}^{(2)}(x), \tag{4.11}$$

where $T_{\mathrm{E}}^{(2)}(x)$ is given in (4.10). Moreover, the scattering data $T_{\mathrm{E}}^{(2)}(x)$, for $x\in U$ uniquely determine the micro-correlation strength ϕ through the linear integral equation (4.10).

Proof. Since

$$\begin{aligned}
&\frac{1}{Q-1}\int_1^Q\omega^{m+1}|\boldsymbol{u}(x,\omega)|^2\mathrm{d}\omega\\
&=\frac{1}{Q-1}\int_1^Q\omega^{m+1}|\tilde{\boldsymbol{u}}(x,\omega)+\boldsymbol{u}(x,\omega)-\tilde{\boldsymbol{u}}(x,\omega)|^2\mathrm{d}\omega\\
&=\frac{1}{Q-1}\int_1^Q\omega^{m+1}|\tilde{\boldsymbol{u}}(x,\omega)|^2d\omega+\frac{1}{Q-1}\int_1^Q\omega^{m+1}|\boldsymbol{u}(x,\omega)-\tilde{\boldsymbol{u}}(x,\omega)|^2\mathrm{d}\omega\\
&\quad+\frac{2}{Q-1}\int_1^Q\omega^{m+1}\Re\left[\overline{\tilde{\boldsymbol{u}}(x,\omega)}(\boldsymbol{u}(x,\omega)-\tilde{\boldsymbol{u}}(x,\omega))\right]\mathrm{d}\omega
\end{aligned}$$

thus, (4.11) holds as long as we show that

$$\lim_{Q\to\infty}\frac{1}{Q-1}\int_1^Q\omega^{m+1}|\tilde{\boldsymbol{u}}(x,\omega)|^2\mathrm{d}\omega=T_{\mathrm{E}}^{(2)}(x), \tag{4.12}$$

$$\lim_{Q\to\infty}\frac{1}{Q-1}\int_1^Q\omega^{m+1}|\boldsymbol{u}(x,\omega)-\tilde{\boldsymbol{u}}(x,\omega)|^2\mathrm{d}\omega=0, \tag{4.13}$$

$$\lim_{Q\to\infty}\frac{2}{Q-1}\int_1^Q\omega^{m+1}\Re\left[\overline{\tilde{\boldsymbol{u}}(x,\omega)}(\boldsymbol{u}(x,\omega)-\tilde{\boldsymbol{u}}(x,\omega))\right]\mathrm{d}\omega=0. \tag{4.14}$$

To prove (4.12), we denote $Y(x,\omega)=\omega^{m+1}(|\tilde{\boldsymbol{u}}(x,\omega)|^2-\mathbb{E}(|\tilde{\boldsymbol{u}}(x,\omega)|^2))$, which yields

$$\int_1^Q\omega^{m+1}|\tilde{\boldsymbol{u}}(x,\omega)|^2\mathrm{d}\omega=\int_1^Q\omega^{m+1}\mathbb{E}(|\tilde{\boldsymbol{u}}(x,\omega)|^2)\mathrm{d}\omega+\int_1^Q Y(x,\omega)\mathrm{d}\omega.$$

Hence, (4.12) holds as long as we prove

$$\lim_{Q\to\infty}\frac{1}{Q-1}\int_1^Q\omega^{m+1}\mathbb{E}(|\tilde{\boldsymbol{u}}(x,\omega)|^2)\mathrm{d}\omega=T_{\mathrm{E}}^{(2)}(x),\qquad\lim_{Q\to\infty}\frac{1}{Q-1}\int_1^Q Y(x,\omega)\mathrm{d}\omega=0. \tag{4.15}$$

Multiplying (4.9) by ω^{m+1} and integrating with respect to the frequency ω in the internal $(1,Q)$, we arrive at

$$\frac{1}{Q-1}\int_1^Q\omega^{m+1}\mathbb{E}(|\tilde{\boldsymbol{u}}(x,\omega)|^2)\mathrm{d}\omega=\frac{1}{Q-1}\int_1^Q\left(T_{\mathrm{E}}^{(2)}(x)+O(\omega^{m-3})\right)\mathrm{d}\omega.$$

It is clear to note that

$$\left|\frac{1}{Q-1}\int_1^Q O(\omega^{m-3})\mathrm{d}\omega\right| \lesssim \frac{1}{Q-1}\int_1^Q \omega^{m-3}\mathrm{d}\omega \to 0 \quad \text{as } Q\to\infty,$$

where we use the fact that $m\in[d,d+\frac{1}{2})$. Thus, the first equation in (4.15) holds. Now we focus on the second equation in (4.15) and want to show that

$$\lim_{Q\to\infty}\frac{1}{Q-1}\int_1^Q Y(x,\omega)\mathrm{d}\omega=0.$$

By the definition of $Y(x,\omega)$,

$$\begin{aligned}Y(x,\omega)&=\omega^{m+1}(|\tilde{\boldsymbol{u}}(x,\omega)|^2-\mathbb{E}(|\tilde{\boldsymbol{u}}(x,\omega)|^2))\\&=\omega^{m+1}((\Re\tilde{\boldsymbol{u}}(x,\omega))^2-\mathbb{E}(\Re\tilde{\boldsymbol{u}}(x,\omega))^2+(\Im\tilde{\boldsymbol{u}}(x,\omega))^2-\mathbb{E}(\Im\tilde{\boldsymbol{u}}(x,\omega))^2).\end{aligned}$$

Therefore,

$$\mathbb{E}(Y(x,\omega_1)Y(x,\omega_2))=I_{E,1}+I_{E,2}+I_{E,3}+I_{E,4},$$

where

$$\begin{aligned}I_{E,1}&=\omega_1^{m+1}\omega_2^{m+1}\mathbb{E}\big[((\Re\tilde{\boldsymbol{u}}(x,\omega_1))^2-\mathbb{E}(\Re\tilde{\boldsymbol{u}}(x,\omega_1))^2)((\Re\tilde{\boldsymbol{u}}(x,\omega_2))^2-\mathbb{E}(\Re\tilde{\boldsymbol{u}}(x,\omega_2))^2)\big],\\I_{E,2}&=\omega_1^{m+1}\omega_2^{m+1}\mathbb{E}\big[((\Re\tilde{\boldsymbol{u}}(x,\omega_1))^2-\mathbb{E}(\Re\tilde{\boldsymbol{u}}(x,\omega_1))^2)((\Im\tilde{\boldsymbol{u}}(x,\omega_2))^2-\mathbb{E}(\Im\tilde{\boldsymbol{u}}(x,\omega_2))^2)\big],\\I_{E,3}&=\omega_1^{m+1}\omega_2^{m+1}\mathbb{E}\big[((\Im\tilde{\boldsymbol{u}}(x,\omega_1))^2-\mathbb{E}(\Im\tilde{\boldsymbol{u}}(x,\omega_1))^2)((\Re\tilde{\boldsymbol{u}}(x,\omega_2))^2-\mathbb{E}(\Re\tilde{\boldsymbol{u}}(x,\omega_2))^2)\big],\\I_{E,4}&=\omega_1^{m+1}\omega_2^{m+1}\mathbb{E}\big[((\Im\tilde{\boldsymbol{u}}(x,\omega_1))^2-\mathbb{E}(\Im\tilde{\boldsymbol{u}}(x,\omega_1))^2)((\Im\tilde{\boldsymbol{u}}(x,\omega_2))^2-\mathbb{E}(\Im\tilde{\boldsymbol{u}}(x,\omega_2))^2)\big].\end{aligned}$$

Combing the expression of $\tilde{\boldsymbol{u}}(x,\omega)$ and the assumption $\mathbb{E}(f_1)=0,\mathbb{E}(f_2)=0$ gives that $\Re\tilde{\boldsymbol{u}}(x,\omega)$ and $\Im\tilde{\boldsymbol{u}}(x,\omega)$ are zero-mean Gaussian random variables. Applying Lemmas 2.3 and 4.5 leads to

$$\begin{aligned}I_{E,1}&=2\omega_1^{m+1}\omega_2^{m+1}[\mathbb{E}(\Re\tilde{\boldsymbol{u}}(x,\omega_1)\Re\tilde{\boldsymbol{u}}(x,\omega_2))]^2\\&=\frac{1}{2}\omega_1^{m+1}\omega_2^{m+1}\Big[\mathbb{E}(\Re(\tilde{\boldsymbol{u}}(x,\omega_1)\tilde{\boldsymbol{u}}(x,\omega_2))+\Re(\tilde{\boldsymbol{u}}(x,\omega_1)\overline{\tilde{\boldsymbol{u}}(x,\omega_2)}))\Big]^2\\&\lesssim\left[\frac{\omega_1^{\frac{m+1}{2}}\omega_2^{\frac{m+1}{2}}}{(\omega_1+\omega_2)^n(1+|\omega_1-\omega_2|)^m}+\frac{\omega_1^{\frac{m+1}{2}}\omega_2^{\frac{m+1}{2}}}{(\omega_1+\omega_2)^{m+1}(1+|\omega_1-\omega_2|)^n}\right]^2\\&\lesssim\left[\frac{1}{(1+|\omega_1-\omega_2|)^m}+\frac{1}{(1+|\omega_1-\omega_2|)^n}\right]^2.\end{aligned}$$

We can obtain the same estimates for $I_{E,2}, I_{E,3}$, and $I_{E,4}$ by the similar arguments. Thus, an application of Lemma 2.4 gives

$$\lim_{Q\to\infty}\frac{1}{Q-1}\int_1^Q Y(x,\omega)\mathrm{d}\omega=0.$$

To prove (4.13), from lemma 4.4, we obtain

$$\begin{aligned}&\left|\frac{1}{Q-1}\int_1^Q \omega^{m+1}\right|\boldsymbol{u}(x,\omega)-\tilde{\boldsymbol{u}}(x,\omega)|^2\mathrm{d}\omega|\lesssim\frac{1}{Q-1}\int_1^Q\omega^{m+1}\omega^{-7}\mathrm{d}\omega\\&\lesssim\frac{1}{Q-1}\int_1^Q\omega^{m-6}\mathrm{d}\omega\lesssim\frac{1}{m-5}\frac{Q^{m-5}-1}{Q-1}\to 0\quad \text{as}Q\to\infty.\end{aligned}$$

To prove (4.14), by the Hölder inequality, we have

$$\begin{aligned}
&\left|\frac{2}{Q-1}\int_1^Q \omega^{m+1}\Re\left[\overline{\tilde{\boldsymbol{u}}(x,\omega)}(\boldsymbol{u}(x,\omega)-\tilde{\boldsymbol{u}}(x,\omega))\right]\mathrm{d}\omega\right| \\
&\lesssim \frac{2}{Q-1}\int_1^Q \omega^{m+1}|\tilde{\boldsymbol{u}}(x,\omega)||\boldsymbol{u}(x,\omega)-\tilde{\boldsymbol{u}}(x,\omega)|\mathrm{d}\omega \\
&\lesssim 2\left[\frac{1}{Q-1}\int_1^Q \omega^{m+1}|\tilde{\boldsymbol{u}}(x,\omega)|^2\mathrm{d}\omega\right]^{\frac{1}{2}}\left[\frac{1}{Q-1}\int_1^Q \omega^{m+1}|\boldsymbol{u}(x,\omega)-\tilde{\boldsymbol{u}}(x,\omega)|^2\mathrm{d}\omega\right]^{\frac{1}{2}} \\
&\to 2T_{\mathrm{E}}^{(2)}(x)^{\frac{1}{2}}\cdot 0 = 0 \quad \text{as} Q\to\infty.
\end{aligned}$$

The unique determination of by $T_{\mathrm{E}}^{(2)}(x)$ for $x\in U$ is a direct consequence of Lemma 3.6. □

4.3. The three-dimensional case

To derive the linear relation between the scattering data and the function in the principle symbol, it is required to express the wave field more explicitly than (4.7). Substituting (4.6) into (4.3) gives the wave field $\boldsymbol{u}(x,\omega)=(u_1(x,\omega),u_2(x,\omega),u_3(x,\omega))^\top$ where each component $u_i(x,\omega)$ is given by

$$u_i(x,\omega)=u_{i1}(x,\omega)+u_{i2}(x,\omega)+u_{i3}(x,\omega).$$

Here

$$\begin{aligned}
u_{i1}(x,\omega) &= \frac{1}{4\pi\mu}\int_{\mathbb{R}^3}\frac{e^{\mathrm{i}\kappa_s|x-y|}}{|x-y|}f_i(y)\mathrm{d}y, \\
u_{i2}(x,\omega) &= \frac{1}{4\pi\omega^2}\int_{\mathbb{R}^3}\frac{1}{|x-y|^3}\left[e^{\mathrm{i}\kappa_s|x-y|}(\mathrm{i}\kappa_s|x-y|-1)-e^{\mathrm{i}\kappa_p|x-y|}(\mathrm{i}\kappa_p|x-y|-1)\right]f_i(y)\mathrm{d}y, \\
u_{i3}(x,\omega) &= -\frac{1}{4\pi\omega^2}\int_{\mathbb{R}^3}\left[\left(\frac{3(\mathrm{i}\kappa_s|x-y|-1)}{|x-y|^2}+\kappa_s^2\right)e^{\mathrm{i}\kappa_s|x-y|}\right. \\
&\qquad \left. -\left(\frac{3(\mathrm{i}\kappa_p|x-y|-1)}{|x-y|^2}+\kappa_p^2\right)e^{\mathrm{i}\kappa_p|x-y|}\right]\frac{(x_i-y_i)}{|x-y|^3}(x-y)\cdot\boldsymbol{f}(y)\mathrm{d}y.
\end{aligned}$$

As mentioned above, we need to derive the relationship between the scattering data and the function ϕ in the principle symbol. For this end, it is required to calculate the exception $\mathbb{E}(\boldsymbol{u}(x,\omega_1)\cdot\overline{\boldsymbol{u}(x,\omega_2)})$. Noting that $\boldsymbol{u}(x,\omega)\in\mathbb{C}^3$ and each component has been decomposed into three parts, we obtain

$$\begin{aligned}
&\mathbb{E}(\boldsymbol{u}(x,\omega)\cdot\overline{\boldsymbol{u}(x,\omega)}) \\
&=\mathbb{E}(u_1(x,\omega)\overline{u_1(x,\omega)}+u_2(x,\omega)\overline{u_2(x,\omega)}+u_3(x,\omega)\overline{u_3(x,\omega)}) \\
&=\sum_{i,j=1}^3\mathbb{E}\Big(u_{1i}(x,\omega)\overline{u_{1j}(x,\omega)}+u_{2i}(x,\omega)\overline{u_{2j}(x,\omega)}+u_{3i}(x,\omega)\overline{u_{3j}(x,\omega)}\Big).
\end{aligned} \tag{4.16}$$

To calculate the expectation $\mathbb{E}(\boldsymbol{u}(x,\omega_1)\cdot\overline{\boldsymbol{u}(x,\omega_2)})$, it is required to calculate the items on the right hand side of (4.16). Using the expression of $u_{ij}(x,\omega)(i,j=1,2,3)$, we have from direct calculations that

$$\mathbb{E}(u_{i1}(x,\omega_1)\overline{u_{i1}(x,\omega_2)}) = \frac{1}{16\pi^2\mu^2}\int_{\mathbb{R}^6}\frac{e^{\mathrm{i}(c_s\omega_1|x-y|-c_s\omega_2|x-z|)}}{|x-y||x-z|}\mathbb{E}(f_i(y)f_i(z))\mathrm{d}y\mathrm{d}z,$$

$$\mathbb{E}(u_{i1}(x,\omega_1)\overline{u_{i2}(x,\omega_2)}) = -\frac{1}{16\pi^2\mu\omega_2^2}\int_{\mathbb{R}^6}[e^{\mathrm{i}(c_s\omega_1|x-y|-c_s\omega_2|x-z|)}(\mathrm{i}c_s\omega_2|x-z|+1)$$
$$-e^{\mathrm{i}(c_s\omega_1|x-y|-c_p\omega_2|x-z|)}(\mathrm{i}c_p\omega_2|x-z|+1)]\frac{\mathbb{E}(f_i(y)f_i(z))}{|x-y||x-z|^3}\mathrm{d}y\mathrm{d}z,$$

$$\mathbb{E}(u_{i1}(x,\omega_1)\overline{u_{i3}(x,\omega_2)}) = \frac{1}{16\pi^2\mu\omega_2^2}\int_{\mathbb{R}^6}\Big[(\frac{3}{|x-z|^2}(\mathrm{i}c_s\omega_2|x-z|+1)-c_s^2\omega_2^2)\times$$
$$e^{\mathrm{i}(c_s\omega_1|x-y|-c_s\omega_2|x-z|)} - \left(\frac{3}{|x-z|^2}(\mathrm{i}c_p\omega_2|x-z|+1)-c_p^2\omega_2^2\right)e^{\mathrm{i}(c_s\omega_1|x-y|-c_p\omega_2|x-z|)}\Big]$$
$$\times\frac{(x_i-z_i)^2}{|x-y||x-z|^3}\mathbb{E}(f_i(y)f_i(z))\mathrm{d}y\mathrm{d}z,$$

$$\mathbb{E}(u_{i2}(x,\omega_1)\overline{u_{i1}(x,\omega_2)}) = \frac{1}{16\pi^2\mu\omega_1^2}\int_{\mathbb{R}^6}\Big[(\mathrm{i}c_s\omega_1|x-y|-1)e^{\mathrm{i}(c_s\omega_1|x-y|-c_s\omega_2|x-z|)}$$
$$-(\mathrm{i}c_p\omega_1|x-y|-1)e^{\mathrm{i}(c_p\omega_1|x-y|-c_s\omega_2|x-z|)}\Big]\frac{\mathbb{E}(f_i(y)f_i(z))}{|x-y|^3|x-z|}\mathrm{d}y\mathrm{d}z,$$

$$\mathbb{E}(u_{i2}(x,\omega_1)\overline{u_{i2}(x,\omega_2)}) = \frac{1}{16\pi^2\omega_1^2\omega_2^2}\times$$
$$\int_{\mathbb{R}^6}\Big[(\mathrm{i}c_s\omega_1|x-y|-1)(-\mathrm{i}c_s\omega_2|x-z|-1)e^{\mathrm{i}(c_s\omega_1|x-y|-c_s\omega_2|x-z|)}$$
$$+(\mathrm{i}c_p\omega_1|x-y|-1)(-\mathrm{i}c_p\omega_2|x-z|-1)e^{\mathrm{i}(c_p\omega_1|x-y|-c_p\omega_2|x-z|)}$$
$$-(\mathrm{i}c_s\omega_1|x-y|-1)(-\mathrm{i}c_p\omega_2|x-z|-1)e^{\mathrm{i}(c_s\omega_1|x-y|-c_p\omega_2|x-z|)}$$
$$-(\mathrm{i}c_p\omega_1|x-y|-1)(-\mathrm{i}c_s\omega_2|x-z|-1)e^{\mathrm{i}(c_p\omega_1|x-y|-c_s\omega_2|x-z|)}\Big]\frac{\mathbb{E}(f_i(y)f_i(z))}{|x-y|^3|x-z|^3}\mathrm{d}y\mathrm{d}z,$$

$$\mathbb{E}(u_{i2}(x,\omega_1)\overline{u_{i3}(x,\omega_2)}) = \frac{1}{16\pi^2\omega_1^2\omega_2^2}\times$$
$$\int_{\mathbb{R}^6}[(\mathrm{i}c_s\omega_1|x-y|-1)\left(\frac{3}{|x-z|^2}(\mathrm{i}c_s\omega_2|x-z|+1)+c_s^2\omega_2^2\right)e^{\mathrm{i}(c_s\omega_1|x-y|-c_s\omega_2|x-z|)}$$
$$+(\mathrm{i}c_p\omega_1|x-y|-1)\left(\frac{3}{|x-z|^2}(\mathrm{i}c_p\omega_2|x-z|+1)+c_p^2\omega_2^2\right)e^{\mathrm{i}(c_p\omega_1|x-y|-c_p\omega_2|x-z|)}$$
$$-(\mathrm{i}c_s\omega_1|x-y|-1)\left(\frac{3}{|x-z|^2}(\mathrm{i}c_p\omega_2|x-z|+1)+c_p^2\omega_2^2\right)e^{\mathrm{i}(c_s\omega_1|x-y|-c_p\omega_2|x-z|)}$$
$$-(\mathrm{i}c_p\omega_1|x-y|-1)\left(\frac{3}{|x-z|^2}(\mathrm{i}c_s\omega_2|x-z|+1)+c_s^2\omega_2^2\right)e^{\mathrm{i}(c_p\omega_1|x-y|-c_s\omega_2|x-z|)}]$$
$$\times\frac{(x_i-z_i)^2}{|x-y|^3|x-z|^3}\mathbb{E}(f_i(y)f_i(z))\mathrm{d}y\mathrm{d}z,$$

$$
\begin{aligned}
&\mathbb{E}(u_{i3}(x,\omega_1)\overline{u_{i1}(x,\omega_2)}) = -\frac{1}{16\pi^2\mu\omega_1^2}\times \\
&\int_{\mathbb{R}^6}\Bigg[\left(\frac{3}{|x-y|^2}(\mathrm{i}c_\mathrm{s}\omega_1|x-y|-1)+c_\mathrm{s}^2\omega_1^2\right)e^{\mathrm{i}(c_\mathrm{s}\omega_1|x-y|-c_\mathrm{s}\omega_2|x-z|)} \\
&\qquad -\left(\frac{3}{|x-y|^2}(\mathrm{i}c_\mathrm{p}\omega_1|x-y|-1)+c_\mathrm{p}^2\omega_1^2\right)e^{\mathrm{i}(c_\mathrm{p}\omega_1|x-y|-c_\mathrm{s}\omega_2|x-z|)}\Bigg] \\
&\qquad \times\frac{(x_i-y_i)^2}{|x-y|^3|x-z|}\mathbb{E}(f_i(y)f_i(z))\mathrm{d}y\mathrm{d}z, \\
&\mathbb{E}(u_{i3}(x,\omega_1)\overline{u_{i2}(x,\omega_2)}) = \frac{1}{16\pi^2\omega_1^2\omega_2^2}\times \\
&\int_{\mathbb{R}^6}[(\mathrm{i}c_\mathrm{s}\omega_2|x-z|+1)\left(\frac{3}{|x-y|^2}(\mathrm{i}c_\mathrm{s}\omega_1|x-y|-1)+c_\mathrm{s}^2\omega_1^2\right)e^{\mathrm{i}(c_\mathrm{s}\omega_1|x-y|-c_\mathrm{s}\omega_2|x-z|)} \\
&+(\mathrm{i}c_\mathrm{p}\omega_2|x-z|+1)\left(\frac{3}{|x-y|^2}(\mathrm{i}c_\mathrm{p}\omega_1|x-y|-1)+c_\mathrm{p}^2\omega_1^2\right)e^{\mathrm{i}(c_\mathrm{p}\omega_1|x-y|-c_\mathrm{p}\omega_2|x-z|)} \\
&-(\mathrm{i}c_\mathrm{p}\omega_2|x-z|+1)\left(\frac{3}{|x-y|^2}(\mathrm{i}c_\mathrm{s}\omega_1|x-y|-1)+c_\mathrm{s}^2\omega_1^2\right)e^{\mathrm{i}(c_\mathrm{s}\omega_1|x-y|-c_\mathrm{p}\omega_2|x-z|)} \\
&-(\mathrm{i}c_\mathrm{s}\omega_2|x-z|+1)\left(\frac{3}{|x-y|^2}(\mathrm{i}c_\mathrm{p}\omega_1|x-y|-1)+c_\mathrm{p}^2\omega_1^2\right)e^{\mathrm{i}(c_\mathrm{p}\omega_1|x-y|-c_\mathrm{s}\omega_2|x-z|)}] \\
&\times\frac{(x_i-y_i)^2}{|x-y|^3|x-z|^3}\mathbb{E}(f_i(y)f_i(z))\mathrm{d}y\mathrm{d}z,
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{E}(u_{i3}(x,\omega_1)\overline{u_{i3}(x,\omega_2)}) = \frac{1}{16\pi^2\omega_1^2\omega_2^2}\int_{\mathbb{R}^6}\Bigg[\left(\frac{3}{|x-y|^2}(\mathrm{i}c_\mathrm{s}\omega_1|x-y|-1)+c_\mathrm{s}^2\omega_1^2\right)\times \\
&\quad\left(\frac{3}{|x-z|^2}(-\mathrm{i}c_\mathrm{s}\omega_2|x-z|-1)+c_\mathrm{s}^2\omega_2^2\right)e^{\mathrm{i}(c_\mathrm{s}\omega_1|x-y|-c_\mathrm{s}\omega_2|x-z|)} \\
&+\left(\frac{3}{|x-y|^2}(\mathrm{i}c_\mathrm{p}\omega_1|x-y|-1)+c_\mathrm{p}^2\omega_1^2\right)\times \\
&\quad\left(\frac{3}{|x-z|^2}(-\mathrm{i}c_\mathrm{p}\omega_2|x-z|-1)+c_\mathrm{p}^2\omega_2^2\right)e^{\mathrm{i}(c_\mathrm{p}\omega_1|x-y|-c_\mathrm{p}\omega_2|x-z|)} \\
&-\left(\frac{3}{|x-y|^2}(\mathrm{i}c_\mathrm{s}\omega_1|x-y|-1)+c_\mathrm{s}^2\omega_1^2\right)\times \\
&\quad\left(\frac{3}{|x-z|^2}(-\mathrm{i}c_\mathrm{p}\omega_2|x-z|-1)+c_\mathrm{p}^2\omega_2^2\right)e^{\mathrm{i}(c_\mathrm{s}\omega_1|x-y|-c_\mathrm{p}\omega_2|x-z|)} \\
&-(\frac{3}{|x-y|^2}(\mathrm{i}c_\mathrm{p}\omega_1|x-y|-1)+c_\mathrm{p}^2\omega_1^2)\times \\
&\quad\left(\frac{3}{|x-z|^2}(-\mathrm{i}c_\mathrm{s}\omega_2|x-z|-1)+c_\mathrm{s}^2\omega_2^2\right)e^{\mathrm{i}(c_\mathrm{p}\omega_1|x-y|-c_\mathrm{s}\omega_2|x-z|)}\Bigg] \\
&\times\frac{(x_i-y_i)(x_i-z_i)}{|x-y|^3|x-z|^3}\sum_{j=1}^{3}(x_j-y_j)(x_j-z_j)\mathbb{E}(f_j(y)f_j(z))\mathrm{d}y\mathrm{d}z.
\end{aligned}
$$

Observe the above expressions, it is easy to see that $\mathbb{E}(\boldsymbol{u}(x,\omega_1)\cdot\overline{\boldsymbol{u}(x,\omega_2)})$ is a linear combination of $I(x,\omega_1,\omega_2)$ which is defined by (3.15). A direct application of

Lemma 3.4 leads to the following lemma which plays an important role in the proof of the main results.

Lemma 4.7. *For $\omega_1 \geq 1, \omega_2 \geq 1,$ the estimates*

$$|\mathbb{E}(\boldsymbol{u}(x,\omega_1)\cdot\overline{\boldsymbol{u}(x,\omega_2)})| \leq c_n(\omega_1+\omega_2)^{-m}(1+|\omega_1-\omega_2|)^{-n}$$
$$|\mathbb{E}(\boldsymbol{u}(x,\omega_1)\cdot\boldsymbol{u}(x,\omega_2))| \leq c_n(\omega_1+\omega_2)^{-n}(1+|\omega_1-\omega_2|)^{-m}$$

holds uniformly for $x \in U$, where $n \in \mathbb{N}$ is arbitrary and $c_n \geq 0$ is a constant depending only on n.

Now we are ready to compute the order of $\mathbb{E}(|u(x,\omega)|^2)$. Let $\omega_1=\omega_2=\omega$ in $\mathbb{E}(u_{ij}(x,\omega_1)\overline{u_{ik}(x,\omega_1)})$ for $i,j,k=1,2,3,$ a direct application of Lemma 3.5 gives that

$$\mathbb{E}(u_{i1}(x,\omega)\overline{u_{i2}(x,\omega)}) = O(\omega^{-(m+1)}), \quad \mathbb{E}(u_{i2}(x,\omega)\overline{u_{i1}(x,\omega)}) = O(\omega^{-(m+1)}),$$
$$\mathbb{E}(u_{i2}(x,\omega)\overline{u_{i2}(x,\omega)}) = O(\omega^{-(m+1)}), \quad \mathbb{E}(u_{i2}(x,\omega)\overline{u_{i3}(x,\omega)}) = O(\omega^{-(m+1)}),$$
$$\mathbb{E}(u_{i3}(x,\omega)\overline{u_{i2}(x,\omega)}) = O(\omega^{-(m+1)}),$$

and

$$\mathbb{E}(u_{i1}(x,\omega)\overline{u_{i1}(x,\omega)}) = N_{1i}^{(3)}(x)\omega^{-m} + O(\omega^{-(m+1)}),$$
$$\mathbb{E}(u_{i1}(x,\omega)\overline{u_{i3}(x,\omega)}) = N_{2i}^{(3)}(x)(x,\omega)\omega^{-m} + O(\omega^{-(m+1)}),$$
$$\mathbb{E}(u_{i3}(x,\omega)\overline{u_{i1}(x,\omega)}) = N_{3i}^{(3)}(x,\omega)\omega^{-m} + O(\omega^{-(m+1)}),$$
$$\mathbb{E}(u_{i3}(x,\omega)\overline{u_{i3}(x,\omega)}) = \sum_{j=1}^{3} N_{4i,j}^{(3)}(x,\omega)\omega^{-m} + O(\omega^{-(m+1)}),$$

where

$$N_{1i}^{(3)}(x) = b_1\int_{\mathbb{R}^3}\frac{1}{|x-y|^2}\phi(y)\mathrm{d}y,$$
$$N_{2i}^{(3)}(x,\omega) = \int_{\mathbb{R}^3}(b_2e^{\mathrm{i}(c_{\mathrm{s}}-c_{\mathrm{p}})|x-y|\omega}-b_1)\frac{(x_i-y_i)^2}{|x-y|^4}\phi(y)\mathrm{d}y,$$
$$N_{3i}^{(3)}(x,\omega) = \int_{\mathbb{R}^3}(b_2e^{\mathrm{i}(c_{\mathrm{p}}-c_{\mathrm{s}})|x-y|\omega}-b_1)\frac{(x_i-y_i)^2}{|x-y|^4}\phi(y)\mathrm{d}y,$$
$$N_{4i,j}^{(3)}(x,\omega) = \int_{\mathbb{R}^3}(b_3-2b_2\cos((c_{\mathrm{s}}-c_{\mathrm{p}})|x-y|\omega))\frac{(x_i-y_i)^2(x_j-y_j)^2}{|x-y|^6}\phi(y)\mathrm{d}y.$$

Here b_1, b_2, b_3 are positive constants given by

$$b_1 = \frac{1}{128\pi^2}c_{\mathrm{s}}^{4-m}, \quad b_2 = \frac{(c_{\mathrm{s}}c_{\mathrm{p}})^2}{128\pi^2}\left(\frac{2}{c_{\mathrm{s}}+c_{\mathrm{p}}}\right)^m, \quad b_3 = \frac{1}{128\pi^2}(c_{\mathrm{s}}^{4-m}+c_{\mathrm{p}}^{4-m}).$$

Therefore

$$\mathbb{E}(|\boldsymbol{u}(x,\omega)|^2) = T_{\mathrm{E}}^{(3)}(x)\omega^{-m} + O(\omega^{-(m+1)}), \tag{4.17}$$

where

$$T_{\mathrm{E}}^{(3)}(x) = (b_3 + b_1)\int_{\mathbb{R}^3} \frac{1}{|x-y|^2}\phi(y)\mathrm{d}y. \tag{4.18}$$

Now we are ready to present the main result of elastic waves for the three-dimensional case.

Theorem 4.8. *Let the external source $\boldsymbol{f}$ be a microlocally isotropic Gaussian random vector field which satisfies Assumption C. Then for all $x \in U$, it holds almost surely that*

$$\lim_{Q\to\infty} \frac{1}{Q-1}\int_1^Q \omega^m |\boldsymbol{u}(x,\omega)|^2 \mathrm{d}\omega = T_{\mathrm{E}}^{(3)}(x).$$

Moreover, the scattering data $T_{\mathrm{E}}^{(3)}(x)$, for $x \in U$, uniquely determine the micro-correlation strength ϕ through the linear relation (4.18).

Proof. Using Lemma 3.6 and (4.17), we may follow the same proof as that for the two-dimensional case. The details are omitted here for brevity. □

5. Conclusion

We have studied an inverse source scattering problem for the two- and three-dimensional Helmholtz equation and Navier equation. The source is assumed to be a generalized Gaussian random function whose covariance operator is a classical pseudo-differential operator. By an exact expression of the random wave field and microlocal analysis, we derive a linear integral equation which connects the principle symbol of the covariance operator and the amplitude of the scattering data generated from a single realization of the random source. Based on this relationship, we obtain the uniqueness for the recovery of the principle symbol of the random source for the Helmholtz and Navier equations. A possible continuation of this work is to investigate the uniqueness for Maxwell's equations with a distributional source. Since the Green tensor has a higher singularity for the Maxwell equations, a new technique must be developed. Another interesting direction is to study the uniqueness for the inverse random source problems in inhomogeneous media, where the analytical Green function or tensor is not available any more and the present method may not be directly applicable. We hope to be able to report the progress on these problems in the future.

Funding

The work of TH was funded by the Academy of Finland via the Academy Research Fellow project, decision number 326961. The research of PL was supported in part by the NSF grant DMS-1912704.

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