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STABILITY FOR THE ACOUSTIC INVERSE SOURCE PROBLEM IN INHOMOGENEOUS MEDIA*

PEIJUN LI†, JIAN ZHAI‡, AND YUE ZHAO§

Abstract. In this paper, we show for the first time the stability of the inverse source problem for the three-dimensional Helmholtz equation in an inhomogeneous background medium. The stability estimate consists of the Lipschitz type data discrepancy and the high frequency tail of the source function, where the latter decreases as the upper bound of the frequency increases. The analysis employs scattering theory to obtain the holomorphic domain and an upper bound for the resolvent of the elliptic operator.

Key words. inverse source problem, the Helmholtz equation, stability

AMS subject classifications. 35R30, 78A46

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1. Introduction.

The inverse source scattering problems arise in diverse scientific and industrial areas such as antenna design and synthesis, medical imaging [26]. Due to the significant applications, these problems have continuously attracted much attention by many researchers. Consequently, a great number of numerical and mathematical results are available [1, 3, 5, 8–12, 16, 20, 25, 29, 30, 37]. In general, it is known that there is no uniqueness for the inverse source problems at a fixed frequency due to the existence of nonradiating sources [7, 21, 28]. Computationally, a more serious issue is the lack of stability, i.e., a small variation of the data might lead to a huge error in the reconstruction. Hence it is crucial to study the stability of the inverse source problems. The first stability result was obtained in [14] for the inverse source problem of the Helmholtz equation by using multifrequency data. Later on, the increasing stability was studied for the inverse source problems of the acoustic, elastic, and electromagnetic wave equations [15, 18, 23, 24, 31]. A topic review can be found in [13] on the general inverse scattering problems with multifrequencies.

In many practical situations, the source, which needs to be identified, is usually embedded in an inhomogeneous background medium. For instance, in the photoacoustic imaging of the brain, it is important to incorporate the sudden change of sound speed across the skull [33, 35]. Moreover, it is possible to achieve some specified radiation pattern that would otherwise not be realistically possible for a source embedded in free space. This possibility has attracted research from time to time in the antenna community in designing antenna embedding materials or substrates, including plasmas, nonmagnetic dielectrics, magneto-dielectrics, and double negative meta-materials, to achieve specified electromagnetic radiation patterns. However, there are few works on the inverse source problems in inhomogeneous media and the

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†Department of Mathematics, Purdue University, West Lafayette, IN 47907 USA (lipeijun@math.purdue.edu).
‡Institute for Advanced Study, The Hong Kong University of Science and Technology, Kowloon, Hong Kong, China (jian.zhai@outlook.com).
§Corresponding author. School of Mathematics and Statistics, Central China Normal University, Wuhan 430079, China (zhaoy@mail.ccnu.edu.cn).

available results are mainly focused on uniqueness and numerics [1, 20, 29]. The stability issue is wide open to be investigated for the inverse source problems in inhomogeneous media.

In this paper, we consider the mathematical study on the stability of the acoustic inverse source problem in an inhomogeneous medium. Consider the three-dimensional Helmholtz equation

$$\Delta u(x,\kappa) + \frac{\kappa^2}{c^2(x)} u(x,\kappa) = f(x), \quad x \in \mathbb{R}^3, \tag{1.1}$$

where $\kappa > 0$ is the wavenumber, $c(x) > 0$ is known as the wave speed, and the source f stands for the electric current density and is assumed to have a compact support contained in $B_R = \{x \in \mathbb{R}^3 \ : \ |x| \le R\}$, where $R > 0$ is a constant. Let ∂B_R be the boundary of B_R. The Sommerfeld radiation condition is imposed to ensure the well-posedness of the problem

$$\lim_{r\to\infty} r(\partial_r u - \mathrm{i}\kappa u) = 0, \quad r = |x| \tag{1.2}$$

uniformly in all directions $\hat{x} = x/|x|$. The inverse source problem is to determine f from the boundary measurements $u(x,\kappa)|_{\partial B_R}$ corresponding to the wavenumber κ given in a finite interval.

The above inverse source problem is closely related to the problem of identifying the initial value of the hyperbolic wave equation, which arises from the photoacoustic tomography (PAT) and thermoacoustic tomography (TAT). The uniqueness and stability for the hyperbolic problem have been well studied by using the boundary control methods (cf. [2] and reference therein). The inverse source problems for the hyperbolic equations are also motivated partially by studying the recovery of the velocity $c(x)$ from boundary measurements. Such problems have been examined by using the Carleman estimates from the ideas of Bukhgeim and Klibanov (cf. [6, Chapter 5]).

Following [29], we consider an eigenvalue problem for the Helmholtz equation in an inhomogeneous medium and deduce integral equations, which connect the scattering data $u|_{\partial B_R}$ and the unknown source function f. To overcome the absence of the explicit Green function for the inhomogeneous Helmholtz equation, we adopt methods and techniques from the scattering theory (e.g., [22]) and study the corresponding resolvent of the elliptic operator to obtain a holomorphic domain of the data with respect to the complex wavenumber κ and the bound of the analytic continuation of the data from the given data to the higher frequency data. The stability estimate consists of the Lipschitz type of data discrepancy and the high frequency tail of the source function. The latter decreases as the frequency of the data increases, which implies that the inverse problem is more stable when the higher frequency data is used. We also mention that only the Dirichlet data is required for the analysis. This paper focuses on the three-dimensional problem due to the practical significance. We believe that the arguments should work in all odd dimensions.

The paper is organized as follows. In section 2, we briefly discuss the direct problem. Resolvent is introduced for the elliptic operator with a variable wave speed, and its holomorphic domain and upper bound are obtained. Section 3 is devoted to the stability analysis of the inverse source problem by using discrete multifrequency data.

2. Direct scattering problem.

Given $f \in L^2(B_R)$ and $c^{-2}(x) \in L^\infty(B_R)$, the scattering problem (1.1)–(1.2) is equivalent to the Lippmann–Schwinger integral equation

$$u(x,\kappa) = \kappa^2 \int_{B_R} G(x,y,\kappa)(c^{-2}(y)-1)u(y,\kappa)\mathrm{d}x + \int_{B_R} G(x,y,\kappa)f(y)\mathrm{d}y,$$

where G is the Green function of the three-dimensional Helmholtz equation and is given by

$$G(x,y,\kappa) = \frac{1}{4\pi}\frac{e^{\mathrm{i}\kappa|x-y|}}{|x-y|}.$$

It is known that the scattering problem (1.1)–(1.2) has a unique solution for all the wavenumbers, which can be stated in the following result. The proof may be found in many references (e.g., [13, Theorem 2.2]).

THEOREM 2.1. *For any $\kappa > 0$, the scattering problem* (1.1)–(1.2) *admits a unique weak solution $u \in H^1(B_R)$.*

Hereafter, the notation $a \lesssim b$ stands for $a \le Cb$, where $C > 0$ is a generic constant which may change step by step in the proofs.

We further assume that $c^{-2}(x) - 1 \in C_c^\infty(\mathbb{R}^3)$ has a compact support satisfying $\mathrm{supp}(c^{-2}(x)-1) \subset B_R$. Denote $P = -c^2\Delta$; it is easy to see that P is self-adjoint in the Hilbert space $\mathcal{H}$, where the inner product is given by

$$\langle u, v\rangle_{\mathcal{H}} = \int_{\mathbb{R}^3} c^{-2}(x)u(x)\bar{v}(x)\mathrm{d}x.$$

Introduce the Hilbert space $\mathcal{D}$ with norm characterized by

$$\|u\|_{\mathcal{D}}^2 := \|(P+I)u\|_{\mathcal{H}}^2,$$

where I is the identity operator. More generally, for any $\alpha \in \mathbb{R}$, we introduce the space $\mathcal{D}^\alpha$ with norm defined as

$$\|u\|_{\mathcal{D}^\alpha} := \|(P+I)^\alpha u\|_{\mathcal{H}}.$$

Let H_p be the Hamiltonian vector field of $p(x,\xi) = c^2(x)|\xi|^2$. Explicitly, we have

$$H_p = \sum_{j=1}^{3} \frac{\partial p}{\partial \xi_j}\frac{\partial}{\partial x_j} - \frac{\partial p}{\partial x_j}\frac{\partial}{\partial \xi_j}.$$

The Hamiltonian flow associated with H_p is defined by

$$\exp(tH_p) : T^*\mathbb{R}^3 \to T^*\mathbb{R}^3.$$

We make the following assumption on the wave speed.

Assumption 1. The metric $c^{-2}\mathrm{d}s^2$ is nontrapping; i.e., for any $(x,\xi) \in T^*\mathbb{R}^3 \setminus \{0\}$

$$\pi(\exp tH_p(x,\xi)) \to \infty, \quad t \to \pm\infty,$$

where $\pi : T^*\mathbb{R}^3 \to \mathbb{R}^3$ is the natural projection.

The assumption implies that for any $a > 0$ and $|x| < a$, $p(x,\xi) = 1$, there exists T_a such that

$$|\pi(\exp tH_p(x,\xi))| > a \quad \forall\, |t| > T_a.$$

We note that $P = -c^2\Delta$ can be viewed as a black box operator in the sense of [32]. We refer to [22, Chapter 4] for a detailed study of the resolvent of black box operators. For convenience, some important properties of P are summarized below. For $\operatorname{Im}\kappa > 0$, denote by $R(\kappa)$ the resolvent of P given by

$$R(\kappa) := (P - \kappa^2)^{-1}.$$

By [22, Theorem 4.4], $R(\kappa) : \mathcal{H} \to \mathcal{D}$ is meromorphic for $\operatorname{Im}\kappa > 0$ and can be extended to a meromorphic family

$$R(\kappa) : \mathcal{H}_{\mathrm{comp}} \to \mathcal{D}_{\mathrm{loc}}, \quad \kappa \in \mathbb{C},$$

where

$$\mathcal{H}_{\mathrm{comp}} := \{u \in \mathcal{H} :\ u|_{\mathbb{R}^3\setminus B_R} \in L^2_{\mathrm{comp}}(\mathbb{R}^3 \setminus B_R)\},$$
$$\mathcal{D}_{\mathrm{loc}} := \{u \in L^2_{\mathrm{loc}}(\mathbb{R}^3) :\ \chi \in C_c^\infty(\mathbb{R}^3), \chi|_{B_R} = 1 \Rightarrow \chi u \in \mathcal{D}\}.$$

The following lemma is a direct consequence of [22, Theorem 4.43].

LEMMA 2.2. *For any $M > 0$, there exits C_0 such that $R(\kappa)$ is holomorphic in the domain*

$$\Omega_M = \{\kappa \in \mathbb{C} :\ \operatorname{Im}\kappa \geq -M \log|\kappa|, \quad |\kappa| \geq C_0\}.$$

Moreover, the following estimate holds:

$$\|R(\kappa)h\|_{H^1(B_R)} \leq Ce^{T(\operatorname{Im}\kappa)_-}\|h\|_{L^2(B_R)} \tag{2.1}$$

for $\kappa \in \Omega_M$, and C and T are positive constants.

Proof. Take $\chi \in C_c^\infty(\mathbb{R}^3)$ such that $\chi = 1$ near B_R. By [22, Theorem 4.43] and the related remarks, the following estimates hold:

$$\|\chi R(\kappa)\chi\|_{\mathcal{H}\to\mathcal{D}^\alpha} \leq C|\kappa|^{2\alpha-1}e^{T(\operatorname{Im}\kappa)_-}, \quad \alpha = 0, \frac{1}{2}, 1, \tag{2.2}$$

for $\kappa \in \Omega_M$, where C, T are positive constants and $(\operatorname{Im}\kappa)_- := \max(0, -\operatorname{Im}\kappa)$. Consequently, by a direct application of (2.2) and letting $\alpha = \frac{1}{2}$ we obtain that

$$\|R(\kappa)h\|_{H^1(B_R)} \leq Ce^{T(\operatorname{Im}\kappa)_-}\|h\|_{L^2(B_R)}, \qquad \square$$

which completes the proof.

LEMMA 2.3. *The meromorphically continued resolvent $R(\kappa)$ has no poles on $\mathbb{R} \setminus \{0\}$*

Proof. We follow the lines in the proof of [19, Lemma 4.1]. Suppose by contradiction that $\kappa_0 \in \mathbb{R}\setminus\{0\}$ is a pole of $R(\kappa)$; then by [34, Theorem 5.3], κ_0^2 is an eigenvalue of P, and there exists a compactly supported eigenfunction u_0 associated to the eigenvalue κ_0^2. Then $c^2\Delta u_0 + \kappa_0^2 u_0 = 0$, where u_0 is not identically zero. However, since u_0 is compactly supported, it must vanish by unique continuation principle. This leads to a contradiction and proves the lemma. $\square$

Therefore, for $\kappa \in \mathbb{R} \setminus \{0\}$, it follows from [22, Theorem 3.37] which may be modified for the operator P (see the remark above [22, Definition 4.16]), the solution to the problem (1.1)–(1.2) can be expressed as

$$u(\cdot, \kappa) = R(\kappa)(-c^2 f). \tag{2.3}$$

Then we can let $u(\cdot, \kappa)$ be defined by (2.3) for κ in all of $\mathbb{C}$ except at the poles of $R(\kappa)$. By Lemma 2.2, we have

$$\|u(\cdot, \kappa)\|_{H^1(B_R)} \leq Ce^{T(\operatorname{Im}\kappa)_-}\|f\|_{L^2(B_R)} \tag{2.4}$$

for $\kappa \in \Omega_M$.

3. Inverse scattering problem. In this section, we discuss the uniqueness and stability of the inverse problem. Firstly, we study the the spectrum of the operator P with the Dirichlet boundary condition. Let $\{\mu_j, \phi_j\}_{j=1}^{\infty}$ be the increasing Dirichlet eigenvalues and eigenfunctions of P in B_R, where ϕ_j and μ_j satisfy

$$\begin{cases} -c^2(x)\Delta\phi_j(x) = \mu_j\phi_j(x) & \text{in } B_R, \\ \phi_j(x) = 0 & \text{on } \partial B_R. \end{cases}$$

Let $\mu_j = \kappa_j^2$ such that $\kappa_j > 0$, and assume that ϕ_j is normalized such that

$$\int_{B_R} c^{-2}(x)|\phi_j(x)|^2 \mathrm{d}x = 1.$$

We obtain the spectral decomposition of $c^2 f$:

$$c^2(x)f(x) = \sum_{j=1}^{\infty} f_j\phi_j(x),$$

where

$$f_j = \langle c^2 f, \phi_j\rangle_{\mathcal{H}} = \int_{B_R} f(x)\bar{\phi}_j(x)\mathrm{d}x.$$

It is clear that

$$c_1\sum_j |f_j|^2 \le \|f\|^2_{L^2(B_R)} \le c_2\sum_j |f_j|^2,$$

where c_1, c_2 are two positive constants. Denote $\kappa_j^2 = \mu_j$. Let $u(x, \kappa_j)$ be the solution to (1.1)–(1.2) with $\kappa = \kappa_j$.

Lemma 3.1. *The following estimate holds:*

$$|f_j|^2 \lesssim \kappa_j^2\|u(x,\kappa_j)\|^2_{L^2(\partial B_R)}$$

for $j = 1, 2, 3, \ldots$.

Proof. Multiplying both sides of (1.1) by $\bar{\phi}_j$ and using the integration by parts yield

$$\int_{B_R} f(x)\bar{\phi}_j(x)\mathrm{d}x = -\int_{\partial B_R} u(x,\kappa_j)\partial_\nu\bar{\phi}_j(x)\mathrm{d}s. \tag{3.1}$$

The proof is completed by using Lemma A.2 and the Schwartz inequality. ∎

Lemma 3.2. *Let f be a real-valued function and $\|f\|_{L^2(B_R)} \le Q$. Then there exist positive constants d and A, A_1 satisfying $C_0 < A < A_1$, which do not depend on f and Q, such that*

$$\kappa^2\|u(x,\kappa)\|^2_{L^2(\partial B_R)} \lesssim Q^2 e^{c\kappa}\epsilon_1^{2\mu(\kappa)} \quad \forall\, \kappa \in (A_1, +\infty),$$

where C_0 is specified in Lemma 2.2, c is any positive constant, and

$$\epsilon_1^2 := \sup_{\kappa\in(A,A_1)}\kappa^2\|u\|^2_{L^2(\partial B_R)}, \quad \mu(\kappa) \ge \frac{64ad}{3\pi^2(a^2+4d^2)}e^{\frac{\pi}{2d}(\frac{a}{2}-\kappa)}.$$

Here $a = A_1 - A$.

Proof. Let

$$I(\kappa) := \kappa^2 \int_{\partial B_R} u(x,\kappa)u(x,-\kappa)\mathrm{d}s, \quad \kappa \in \mathbb{C}.$$

Since $f(x)$ is a real-valued function, we have $\overline{u(x,\kappa)} = u(x,-\kappa)$ for $\kappa \in \mathbb{R}$, which gives

$$I(\kappa) = \kappa^2 \|u(x,\kappa)\|^2_{L^2(\partial B_R)}, \quad \kappa \in \mathbb{R}.$$

It follows from Lemma 2.2 that $I(\kappa)$ is analytic in the domain

$$\widetilde{\Omega}_M = \{\kappa \in \mathbb{C} : -M\log|\kappa| \leq \operatorname{Im}\kappa \leq M\log|\kappa|, \, |\kappa| \geq C_0\},$$

which is symmetric with respect to the origin. Hence, there exists $d > 0$ such that $\mathcal{R} = (A,+\infty) \times (-d,d) \subset \widetilde{\Omega}_M$. The geometry of domain $\mathcal{R}$ is shown in Figure 1. By (2.4) we have for $\kappa \in \mathcal{R}$ that

$$\begin{aligned}|\kappa|\|u(x,\pm\kappa)\|_{L^2(\partial B_R)} &\lesssim |\kappa|\|u(x,\pm\kappa)\|_{H^{1/2}(B_R)} \lesssim |\kappa|\|u(x,\pm\kappa)\|_{H^1(B_R)} \\ &\lesssim |\kappa|e^{T(\operatorname{Im}(\pm\kappa))_-}\|f\|_{L^2(B_R)} \lesssim |\kappa|e^{Td}\|f\|_{L^2(B_R)},\end{aligned}$$

which shows that

$$|\kappa|\|u(x,\pm\kappa)\|_{L^2(\partial B_R)} \lesssim |\kappa|\|f\|_{L^2(B_R)}, \quad \kappa \in \mathcal{R}.$$

Since

$$|I(\kappa)| \leq |\kappa|\|u(x,\kappa)\|_{L^2(\partial B_R)}|\kappa|\|u(x,-\kappa)\|_{L^2(\partial B_R)} \lesssim |\kappa|^2\|f\|^2_{L^2(B_R)}, \quad \kappa \in \mathcal{R},$$

we have

$$|e^{-c\kappa}I(\kappa)| \lesssim Q^2, \quad \kappa \in \mathcal{R},$$

for any positive constant c. An application of Lemma A.1 leads to

$$\left|e^{-c\kappa}I(\kappa)\right| \lesssim Q^2\epsilon_1^{2\mu(\kappa)} \quad \forall\, \kappa \in (A_1,+\infty),$$

where

$$\mu(\kappa) \geq \frac{64ad}{3\pi^2(a^2+4d^2)}e^{\frac{\pi}{2d}(\frac{a}{2}-\kappa)},$$

which completes the proof. ∎

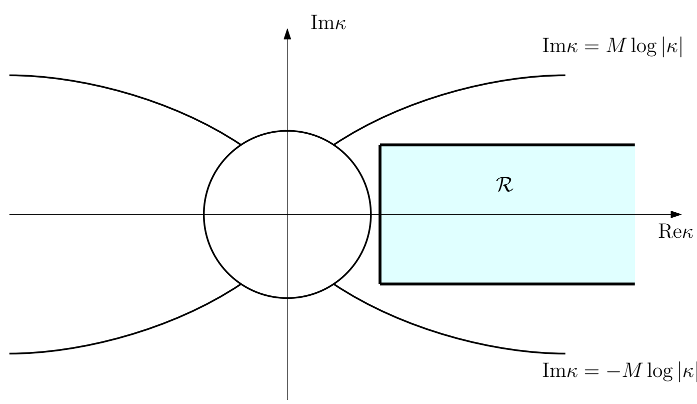

FIG. 1. *The region $\mathcal{R}$.*

Here we state a simple uniqueness result for the inverse problem.

THEOREM 3.3. *Let $f \in L^2(B_R)$ and $I := (C_0, C_0 + \delta) \subset \mathbb{R}^+$ be an open interval, where C_0 is the constant given in the definition of $\widetilde{\Omega}_M$ in Lemma* 3.2 *and δ is any positive constant. Then the source term f can be uniquely determined by the multiple-frequency data $\{u(x,\kappa) : x \in \partial B_R, \kappa \in I\} \cup \{u(x,\kappa_j) : x \in \partial B_R, \kappa_j \in (0, C_0]\}$.*

Proof. Let $u(x,\kappa) = 0$ for $x \in \partial B_R$ and $\kappa \in I \cup \{\kappa_j : \kappa_j \in (0, C_0])\}$. It suffices to show that $f(x) = 0$. Since $u(x,\kappa)$ is analytic in $\widetilde{\Omega}_M$ for $x \in \partial B_R$, it holds that $u(x,\kappa) = 0$ for all eigenvalues $\kappa > C_0$. Then we have that $u(x,\kappa_j) = 0$ for all $\kappa_j, j = 1, 2, 3, \ldots$. Hence, it follows from (3.1) that

$$\int_{B_R} f(x)\bar{\phi}_j(x)\mathrm{d}x = 0, \quad j = 1, 2, 3, \ldots,$$

which implies $f = 0$. ∎

The following lemma is important in the stability analysis.

LEMMA 3.4. *Let $f \in H^{n+1}(B_R)$ and $\|f\|_{H^{n+1}(B_R)} \le Q$. It holds that*

$$\sum_{j\ge s} |f_j|^2 \lesssim \frac{Q^2}{s^{\frac{2}{3}(n+1)}}.$$

Proof. A simple calculation yields

$$\sum_{j\ge s} |f_j|^2 \le \sum_{j\ge s} \frac{\kappa_j^{2n+2}}{\kappa_s^{2n+2}} |f_j|^2 \le \frac{1}{\kappa_s^{2n+2}} \sum_{j\ge s} \kappa_j^{2n+2} |f_j|^2 \lesssim \frac{M^2}{\kappa_s^{2n+2}}.$$

Noting

$$\|f\|^2_{H^s(B_R)} \cong \sum_{j=1}^{\infty} (\kappa_j^2 + 1)^s |f_j|^2,$$

and using the Weyl-type inequality in Lemma A.2, we have $\kappa_s^2 \ge E_2 s^{\frac{2}{3}}$ and complete the proof. ∎

Define a real-valued functional space

$$\mathcal{C}_Q = \{f \in H^{n+1}(B_R) : \|f\|_{H^{n+1}(B_R)} \le Q,\ \mathrm{supp} f \subset B_R,\ f : B_R \to \mathbb{R}\}.$$

Now we are in the position to discuss the inverse source problem. Let $f \in \mathcal{C}_Q$. The inverse source problem is to determine f from the boundary data $u(x,\kappa)$, $x \in \partial B_R$, $\kappa \in (A, A_1) \cup \cup_{j=1}^N \kappa_j$, where $1 \le N \in \mathbb{N}$ and $\kappa_N > A_1$. Here A and A_1 are the constants specified in Lemma 3.2.

The following stability estimate is the main result of this paper.

THEOREM 3.5. *Let $u(x,\kappa)$ be the solution of the scattering problem* (1.1)–(1.2) *corresponding to the source $f \in \mathcal{C}_Q$. Then for ϵ_1 sufficiently small,*

$$\|f\|^2_{L^2(B_R)} \lesssim \epsilon^2 + \frac{Q^2}{N^{\frac{1}{3}(n+1)} (\ln|\ln \epsilon_1|)^{\frac{1}{3}(n+1)}}, \tag{3.2}$$

where

$$\epsilon^2 = \sum_{j=1}^{N} \kappa_j^2 \|u(x,\kappa_j)\|^2_{L^2(\partial B_R)}, \quad \epsilon_1^2 = \sup\nolimits_{\kappa\in(A,A_1)} \kappa^2 \|u(x,\kappa)\|^2_{L^2(\partial B_R)}.$$

Proof. We can assume that $\epsilon_1 \le e^{-1}$, otherwise the estimate is obvious.

First, we link the data $\kappa^2\|u(x,\kappa)\|^2_{L^2(\partial B_R)}$ for large wavenumber κ satisfying $\kappa \le L$ with the given data ϵ_1 of small wavenumber by using the analytic continuation in Lemma 3.2, where L is some large positive integer to be determined later. By Lemma 3.2, we obtain

$$
\begin{aligned}
&\kappa^2\|u(x,\kappa)\|^2_{L^2(\partial B_R)}\\
&\quad\lesssim Q^2 e^{c|\kappa|}\epsilon_1^{\mu(\kappa)}\\
&\quad\lesssim Q^2\exp\left\{c\kappa - \frac{c_2 a}{a^2+c_3}e^{c_1\left(\frac{a}{2}-\kappa\right)}|\ln\epsilon_1|\right\}\\
&\quad\lesssim Q^2\exp\left\{-\frac{c_2 a}{a^2+c_3}e^{c_1\left(\frac{a}{2}-\kappa\right)}|\ln\epsilon_1|\left(1-\frac{c_4\kappa(a^2+c_3)}{a}e^{c_1\left(\kappa-\frac{a}{2}\right)}|\ln\epsilon_1|^{-1}\right)\right\}\\
&\quad\lesssim Q^2\exp\left\{-\frac{c_2 a}{a^2+c_3}e^{c_1\left(\frac{a}{2}-L\right)}|\ln\epsilon_1|\left(1-\frac{c_4 L(a^2+c_3)}{a}e^{c_1\left(L-\frac{a}{2}\right)}|\ln\epsilon_1|^{-1}\right)\right\}\\
&\quad\lesssim Q^2\exp\left\{-b_0 e^{-c_1 L}|\ln\epsilon_1|\left(1-b_1 L e^{c_1 L}|\ln\epsilon_1|^{-1}\right)\right\},
\end{aligned}
$$

where $c, c_i, i=1,2$ and b_0, b_1 are constants. Let

$$
L=\begin{cases}\left[\frac{1}{2c_1}\ln|\ln\epsilon_1|\right], & N\le \frac{1}{2c_1}\ln|\ln\epsilon_1|,\\ N, & N>\frac{1}{2c_1}\ln|\ln\epsilon_1|.\end{cases}
$$

If $N\le \frac{1}{2c_1}\ln|\ln\epsilon_1|$, we obtain for ϵ_1 sufficiently small that

$$
\begin{aligned}
\kappa^2\|u(x,\kappa)\|^2_{L^2(\partial B_R)} &\lesssim Q^2\exp\left\{-b_0 e^{-c_1 L}|\ln\epsilon_1|\left(1-b_1 L e^{c_1 L}|\ln\epsilon_1|^{-1}\right)\right\}\\
&\lesssim Q^2\exp\left\{-\frac{1}{2}b_0 e^{-c_1 L}|\ln\epsilon_1|\right\}.
\end{aligned}
$$

Noting $e^{-x}\le \frac{(2n+3)!}{x^{2n+3}}$ for $x>0$, we obtain

$$
\sum_{j=N+1}^{L}\kappa_j^2\|u(x,\kappa_j)\|^2_{L^2(\partial B_R)} \lesssim Q^2 L e^{(2n+3)c_1 L}|\ln\epsilon_1|^{-(2n+3)}.
$$

Taking $L=\frac{1}{2c_1}\ln|\ln\epsilon_1|$, combining the above estimates and Lemma 3.4, we get

$$
\begin{aligned}
&\|f\|^2_{L^2(B_R)}\\
&\quad\lesssim \sum_{j=1}^{N}|f_j|^2+\sum_{j=N+1}^{L}|f_j|^2+\sum_{j=L+1}^{+\infty}|f_j|^2\\
&\quad\lesssim \sum_{j=1}^{N}\kappa_j^2\|u(x,\kappa_j)\|^2_{L^2(\partial B_R)}+\sum_{j=N+1}^{L}\kappa_j^2\|u(x,\kappa_j)\|^2_{L^2(\partial B_R)}+\frac{1}{L^{\frac{2}{3}(n+1)}}\|f\|^2_{H^{n+1}(B_R)}\\
&\quad\lesssim \epsilon^2 + LQ^2 e^{(2n+3)c_1 L}|\ln\epsilon_1|^{-(2n+3)}+\frac{Q^2}{L^{\frac{2}{3}(n+1)}}\\
&\quad\lesssim \epsilon^2 + Q^2\left((\ln|\ln\epsilon_1|)|\ln\epsilon_1|^{\frac{2n+3}{2}}|\ln\epsilon_1|^{-(2n+3)}+(\ln|\ln\epsilon_1|)^{-\frac{2}{3}(n+1)}\right)\\
&\quad\lesssim \epsilon^2 + Q^2\left((\ln|\ln\epsilon_1|)|\ln\epsilon_1|^{-\frac{2n+3}{2}}+(\ln|\ln\epsilon_1|)^{-\frac{2}{3}(n+1)}\right)\\
&\quad\lesssim \epsilon^2 + Q^2(\ln|\ln\epsilon_1|)^{-\frac{2}{3}(n+1)}\\
&\quad\lesssim \epsilon^2 + \frac{Q^2}{N^{\frac{1}{3}(n+1)}(\ln|\ln\epsilon_1|)^{\frac{1}{3}(n+1)}},
\end{aligned}
$$

where we have used $|\ln\epsilon_1|^{1/2}\ge \ln|\ln\epsilon_1|$ for sufficiently small ϵ_1.

If $N > \frac{1}{2c_1}\ln|\ln\epsilon_1|$, we have from Lemma 3.4 that

$$\begin{aligned}\|f\|^2_{L^2(B_R)} &\lesssim \sum_{j=1}^{N}|f_j|^2 + \sum_{j=N+1}^{+\infty}|f_j|^2\\ &\lesssim \epsilon^2 + \frac{Q^2}{N^{\frac{2}{3}(n+1)}}\\ &\lesssim \epsilon^2 + \frac{Q^2}{N^{\frac{1}{3}(n+1)}(\ln|\ln\epsilon_1|)^{\frac{1}{3}(n+1)}},\end{aligned}$$

which completes the proof. ∎

The stability (3.2) consists of two parts: the data discrepancy and the high frequency tail. The former is of the Lipschitz type. The latter decreases as N increases which makes the problem have an almost Lipschitz stability. The result reveals that the problem becomes more stable when higher frequency data is used.

4. Conclusion. We have presented a stability result for the inverse source problem of time-harmonic acoustic waves in inhomogeneous background media. The analysis requires the Dirichlet data only at multiple discrete frequencies without resorting to the Dirichlet-to-Neumann map which was considered in [15, 31]. The increasing stability is achieved to reconstruct the source term, and it consists of the data discrepancy and the high frequency tail of the source function. The result shows that the ill-posedness of the inverse source problem decreases as the frequency increases for the data. A possible continuation of this work is to extend the stability to the two-dimensional case. Due to the absence of the Huygens principle, the scattering theory is not so obvious as that for the three-dimensional Helmholtz equation. Another interesting direction is to study the stability of the inverse source problems for elastic and electromagnetic waves in inhomogeneous media, where the properties of the corresponding resolvent need to be analyzed for the associated second order operators. For the Maxwell system, additional difficulties arise from their spectral analysis, and the present method may not be directly applicable. A related but more challenging problem is to study the stability of the inverse medium problem which is to determine the scatterer q. A recent progress can be found in [17] on a stability result of the inverse medium problem for the one-dimensional Helmholtz equation.

Appendix A. Two useful lemmas. The following lemma gives a link between the values of an analytical function for small and large arguments.

LEMMA A.1. *Let $p(z)$ be analytic in the infinite rectangular slab*

$$R = \{z \in \mathbb{C} : (A, +\infty) \times (-d, d)\},$$

where A is a positive constant, and continuous in $\overline{R}$ satisfying

$$\begin{cases}|p(z)| \le \epsilon, & z \in (A, A_1],\\ |p(z)| \le M, & z \in R,\end{cases}$$

where A, A_1, ϵ and M are positive constants. Then there exists a function $\mu(z)$ with $z \in (A_1, +\infty)$ satisfying

$$\mu(z) \ge \frac{64ad}{3\pi^2(a^2+4d^2)}e^{\frac{\pi}{2d}(\frac{a}{2}-z)}, \tag{A.1}$$

where $a = A_1 - A$, such that

$$|p(z)| \le M\epsilon^{\mu(z)} \quad \forall\, z \in (A_1, +\infty). \tag{A.2}$$

Proof. By applying [4, Lemma 3.1] to the domain

$$\mathcal{R} \setminus \gamma := \{(A, b+A) \times (-d, d)\} \setminus \{(A, A_1) \times \{0\}\}$$

for any $b > A_1 - A$, we obtain that there exists a harmonic measure $\mu(z)$ of γ with respect to $\mathcal{R}\setminus\gamma$ such that

$$\mu(x, 0) \ge C \sinh\left[\frac{\pi a}{2d}\left(\frac{x - \tilde{b}}{a}\right)\right], \quad A_1 < x < \tilde{b} := b + A, \tag{A.3}$$

where

$$C = \frac{64ad \coth\left[\frac{\pi d}{a}\right]}{3\pi^2(a^2 + 4d^2)\sinh\left[\frac{\pi a}{2d}\left(\frac{a-2\tilde{b}}{2a}\right)\right]}.$$

Noting the following asymptotics as $b \to +\infty$, which means $\tilde{b} \to +\infty$,

$$\sinh\left[\frac{\pi a}{2d}\left(\frac{x-\tilde{b}}{2a}\right)\right] \sim -\frac{e^{\frac{\pi}{2d}(\tilde{b}-x)}}{2}, \quad \sinh\left[\frac{\pi a}{2d}\left(\frac{a-2\tilde{b}}{2a}\right)\right] \sim -\frac{e^{\frac{\pi}{2d}(\tilde{b}-\frac{a}{2})}}{2},$$

and the inequality $\coth[\frac{\pi d}{a}] \ge 1$, we obtain (A.1) by letting $\tilde{b} \to +\infty$ in (A.3).

Finally, by fundamental application of the harmonic measure for stability estimates of holomorphic continuation [4, Theorem 2.4] we obtain (A.2) ∎

LEMMA A.2. *The following estimate holds:*

$$\|\partial_\nu \phi_j\|_{L^2(\partial B_R)} \le C\kappa_j, \tag{A.4}$$

where the positive constant C is independent of j. Moreover, we have the following Weyl-type inequality for the Dirichlet eigenvalues $\{\mu_n\}_{n=1}^{\infty}$:

$$E_1 n^{2/3} \le \mu_n \le E_2 n^{2/3}, \tag{A.5}$$

where E_1 and E_2 are two positive constants independent of n.

Proof. We begin with the estimate (A.4) for the eigenfunctions on the boundary. Let u be a Dirichlet eigenfunction of $P = -c^2\Delta$ in B_R. For any differential operator A, by [27, Lemma 2.1], we have the Rellich-type identity

$$\int_{B_R} c^{-2} u [P, A] u \mathrm{d}x = \int_{\partial B_R} \partial_\nu u A u \mathrm{d}s, \tag{A.6}$$

where $[P, A] = PA - AP$. In fact, let λ be the eigenvalue corresponding to the eigenfunction u. We have $[P, A] = [P - \lambda, A]$ and $(P - \lambda)u = 0$. A simple calculation yields

$$
\begin{aligned}
&\int_{B_R} c^{-2} u[P,A]u\mathrm{d}x \\
&\quad= \int_{B_R} \left[u(-\Delta A u) - \lambda c^{-2} u A u\right] \mathrm{d}x \\
&\quad= \int_{B_R} \left[-\Delta u A u - \lambda c^{-2} u A u\right] \mathrm{d}x + \int_{\partial B_R} \partial_\nu u A u \mathrm{d}s \\
&\quad= \int_{B_R} c^{-2}(P-\lambda) u A u \mathrm{d}x + \int_{\partial B_R} \partial_\nu u A u \mathrm{d}s \\
&\quad= \int_{\partial B_R} \partial_\nu u A u \mathrm{d}s.
\end{aligned}
$$

Now let u be a normalized Dirichlet eigenfunction with eigenvalue λ. Choose local coordinates (r, y) near the boundary ∂B_R such that r is the distance to the boundary. Assume that there is a small number δ such that $c^2 = 1$ for $r \le \delta$. Take $\chi \in C_c^\infty(\mathbb{R})$ such that $\chi \equiv 1$ near 0 and vanishes for $r \ge \delta$. Choose $A = \chi(r)\partial_r$. It is clear that the right-hand side of (A.6) is exactly $\|\partial_\nu u\|_{L^2(\partial B_R)}$. It follows from the integration by parts that the left-hand side of (A.6) can be written as

$$\int_{B_R} c^{-2}(B_1 u)(B_2 u)\mathrm{d}x,$$

where B_1 and B_2 are two first order differential operators. Using the Poincaré inequality, we obtain

$$\int_{B_R} c^{-2}(B_1 u)(B_2 u)\mathrm{d}x \le C\|u\|^2_{H^1(B_R)} \le C\int_{B_R} \nabla u \cdot \nabla u \,\mathrm{d}x = C\lambda,$$

where the positive constant C does not depend on λ.

Next, we prove the Weyl-type inequality (A.5). Assume $\mu_1 < \mu_2 < \cdots$ are the Dirichlet eigenvalues of the operator $-c^2(x)\Delta$. Then we have following min-max principle:

$$\mu_n = \sup_{\varphi_1,\ldots,\varphi_{n-1}} \inf_{\substack{\psi\in[\varphi_1,\ldots,\varphi_{n-1}]^\perp \\ \psi\in H_0^1(B_R)}} \frac{\int_{B_R} |\nabla\psi|^2 \mathrm{d}x}{\int_{B_R} c^{-2}\psi^2 \mathrm{d}x}.$$

Assume $C_1 < c^2(x) < C_2$ on B_R, where C_1, C_2 are two constants. Assume $\mu_1^{(j)} < \mu_2^{(j)} < \cdots$ are the eigenvalues for the operator $-C_j\Delta$ for $j = 1, 2$. By the min-max principle, we have

$$\mu_n^{(2)} < \mu_n < \mu_n^{(1)}, \quad n = 1, 2, \ldots.$$

We have from Weyl's law [36] for $-C_j\Delta$ that

$$\lim_{n\to+\infty} \frac{\mu_n^{(j)}}{n^{2/3}} = D_j,$$

where D_j is a constant. Therefore there exist two constants E_1 and E_2 such that

$$E_1 n^{2/3} \le \mu_n \le E_2 n^{2/3},$$

which completes the proof. $\square$

REFERENCES

[1] S. Acosta, S. Chow , J. Taylor, and V. Villamizar, *On the multi-frequency inverse source problem in heterogeneous media*, Inverse Problems, 28 (2012), 075013.

[2] S. Acosta and C. Montalto, *Multiwave imaging in an enclosure with variable wave speed*, Inverse Problems, 31 (2015), 065009.

[3] R. Albanese and P. Monk, *The inverse source problem for Maxwell's equations*, Inverse Problems, 22 (2006), pp. 1023–1035.

[4] G. Alessandrini and F. Alberto, *Sharp stability estimates of harmonic continuation along lines*, Math. Meth. Appl. Sci., 23 (2000), pp. 1037–1056.

[5] H. Ammari, G. Bao, and J. Fleming, *An inverse source problem for Maxwell's equations in magnetoencephalography*, SIAM J. Appl. Math., 62 (2002), pp. 1369–1382.

[6] M. Bellassoued and M. Yamamoto, *Carleman Estimates and Applications to Inverse Problems for Hyperbolic Systems*, Springer, Tokyo, 2017.

[7] N. Bleistein and J. K. Cohen, *Nonuniqueness in the inverse source problem in acoustics and electromagnetics*, J. Math. Phys., 18 (1977), pp. 194–201.

[8] A. Badia and T. Nara, *An inverse source problem for Helmholtz's equation from the Cauchy data with a single wave number*, Inverse Problems, 27 (2011), 105001.

[9] A. Badia and T. Nara, *Inverse dipole source problem for time-harmonic Maxwell equations: Algebraic algorithm and Holder stability*, Inverse Problems, 29 (2013), 015007.

[10] G. Bao, C. Chen, and P. Li, *Inverse random source scattering problems in higher dimensions*, SIAM/ASA J. Uncertainty Quantification, 4 (2016), pp. 1263–1287.

[11] G. Bao, C. Chen, and P. Li, *Inverse random source scattering for elastic waves*, SIAM J. Numer. Anal., 55 (2017), pp. 2616–2643.

[12] G. Bao, S.-N. Chow, P. Li, and H.-M. Zhou, *An inverse random source problem for the Helmholtz equation*, Math. Comp., 83 (2014), pp. 215–233.

[13] G. Bao, P. Li, J. Lin, and F. Triki, *Inverse scattering problems with multi-frequencies*, Inverse Problems, 31 (2015), 093001.

[14] G. Bao, J. Lin, and F. Triki, *A multi-frequency inverse source problem*, J. Differential Equations, 249 (2010), pp. 3443–3465.

[15] G. Bao, P. Li, and Y. Zhao, *Stability for the inverse source problems in elastic and electromagnetic waves*, J. Math. Pures Appl., 134 (2020), pp. 122–178.

[16] G. Bao, S. Lu, W. Rundell, and B. Xu, *A recursive algorithm for multifrequency acoustic inverse source problems*, SIAM J. Numer. Anal., 53 (2015), pp. 1608–1628.

[17] G. Bao and F. Triki, *Stability for the multifrequency inverse medium problem*, J. Differential Equations, 269 (2020), pp. 7106–7128.

[18] J. Cheng, V. Isakov, and S. Lu, *Increasing stability in the inverse source problem with many frequencies*, J. Differential Equations, 260 (2016), pp. 4786–4804.

[19] K. Datchev and M. V. de Hoop, *Iterative reconstruction of the wave speed for the wave equation with bounded frequency boundary data*, Inverse Problems, 32 (2016), 025008.

[20] A. Devaney, E. Marengo, and M. Li, *Inverse source problem in nonhomogeneous background media*, SIAM J. Appl. Math., 67 (2007), pp. 1353–1378.

[21] A. Devaney and G. Sherman, *Nonuniqueness in inverse source and scattering problems*, IEEE Trans. Antennas Propag., 30 (1982), pp. 1034–1037.

[22] S. Dyatlov and M. Zworski, *Mathematical Theory of Scattering Resonances*, vol. 200, American Mathematical Soc., 2019.

[23] M. Entekhabi and V. Isakov, *On increasing stability in the two dimensional inverse source scattering problem with many frequencies*, Inverse Problems, 34 (2018), 055005.

[24] M. Entekhabi and V. Isakov, *Increasing Stability in Acoustic and Elastic Inverse Source Problems*, preprint.

[25] M. Eller and N. Valdivia, *Acoustic source identification using multiple frequency information*, Inverse Problems, 25 (2009), 115005.

[26] A. Fokas, Y. Kurylev, and V. Marinakis, *The unique determination of neuronal currents in the brain via magnetoencephalography*, Inverse Problems, 20 (2004), pp. 1067–1082.

[27] A. Hassell and T. Tao, *Upper and Lower Bounds for Normal Derivatives of Dirichlet Eigenfunctions*, arXiv preprint math/0202140, 2002.

[28] K. Hauer, L. Kuhn, and R. Potthast, *On uniqueness and non-uniqueness for current reconstruction from magnetic fields*, Inverse Problems, 21 (2005), pp. 955–967.

[29] M. Li, C. Chen, and P. Li, *Inverse random source scattering for the Helmholtz equation in inhomogeneous media*, Inverse Problems, 34 (2018), 015003.

[30] P. Li, *An inverse random source scattering problem in inhomogeneous media*, Inverse Problems, 27 (2011), 035004.

[31] P. Li and G. Yuan, *Increasing stability for the inverse source scattering problem with multi-frequencies*, Inverse Problems Imag., 11 (2017), pp. 745–759.

[32] J. Sjöstrand and M. Zworski, *Complex scaling and the distribution of scattering poles*, J. Amer. Math. Soc., 4 (1991), pp. 729–769.

[33] P. Stefanov and G. Uhlmann,*Thermoacoustic tomography with variable sound speed*, Inverse Problems, 25 (2009), 075011.

[34] P. Stefanov, *Scattering and Inverse Scattering*, *Lecture Notes*, https://www.math.purdue.edu/~stefanop/publications/SCATTERING.pdf.

[35] P. Stefanov and G. Uhlmann, *Thermoacoustic tomography arising in brain imaging*, Inverse Problems, 27 (2011), 045004.

[36] H. Weyl, *Das asymptotische verteilungsgesetz der eigenwerte linear partieller differentialgleichungen*, Math. Ann., 71 (1911), pp. 441–479.

[37] D. Zhang and Y. Guo, *Fourier method for solving the multi-frequency inverse acoustic source problem for the Helmholtz equation*, Inverse Problems, 31 (2015), 035007.