

An Adaptive Finite Element PML Method for the Open Cavity Scattering Problems

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Abstract. Consider the electromagnetic scattering of a time-harmonic plane wave by an open cavity which is embedded in a perfectly electrically conducting infinite ground plane. This paper is concerned with the numerical solutions of the transverse electric and magnetic polarizations of the open cavity scattering problems. In each polarization, the scattering problem is reduced equivalently into a boundary value problem of the two-dimensional Helmholtz equation in a bounded domain by using the transparent boundary condition (TBC). An a posteriori estimate based adaptive finite element method with the perfectly matched layer (PML) technique is developed to solve the reduced problem. The estimate takes account both of the finite element approximation error and the PML truncation error, where the latter is shown to decay exponentially with respect to the PML medium parameter and the thickness of the PML layer. Numerical experiments are presented and compared with the adaptive finite element TBC method for both polarizations to illustrate the competitive behavior of the proposed method.

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1 Introduction

The phenomena of electromagnetic scattering by open cavities have attracted much attention due to the significant industrial and military applications in such areas as antenna synthesis and stealth design. The underlying scattering problems have been extensively

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studied by many researchers in the engineering and applied mathematics communities. We refer to the survey [20] and the references cited therein for a comprehensive account on analysis, computation, and optimal design of the cavity scattering problems.

In applications, one of particular interests is the radar cross section (RCS) analysis, which aims at how to mitigate or amplify a signal. The RCS is a quantity which measures the detectability of a target by radar system. Deliberate control in the form of enhancement or reduction of the RCS of a target is of high importance in the electromagnetic interference, especially in the aircraft detection and the stealth design. Since the problems are imposed in open domains and the solutions may have singularities, it presents challenging and significant mathematical and computational questions on precise modeling and accurate computing for the cavity scattering problems in order to successfully implement any desired control of the RCS. This paper concerns the numerical solutions of the open cavity scattering problems. We intend to develop an adaptive finite element method with the perfect matched layer (PML) technique to overcome the difficulties.

The PML technique was first proposed by Bérenger for solving the time-dependent Maxwell equations [7]. Due to its effectiveness, simplicity and flexibility, the PML technique is widely used in computational wave propagation [14, 15, 24, 25, 29]. It has been recognized as one of the most important and popular approaches for the domain truncation. Under the assumption that the exterior solution is composed of outgoing waves only, the basic idea of the PML technique is to surround the domain of interest with a layer of finite thickness of a special medium, which is designed to either slow down or attenuate all the waves propagating into the PML layer from inside of the computational domain. As either the PML parameter or the thickness of the PML layer tends to infinity, the exponential convergence error estimate was obtained in [17, 19] between the solution of the PML problem and the solution of the Helmholtz-type scattering problem. The convergence analysis of the PML problems for the three-dimensional electromagnetic scattering was studied in [6, 8, 9, 21].

In practice, if we use a very thick PML layer and a uniform finite element mesh, it requires very excessive grids points and hence involves more computational cost. In contrast, if we choose a thin PML layer, it is inevitable to have a rapid variation of the PML medium property, which renders a very fine mesh in order to reach the desired accuracy. On the other hand, the solutions of the open cavity scattering problem may have singularities due to the existence of corners of cavities or the discontinuity of the dielectric coefficient for the filling medium. These singularities slow down the speed of convergence if uniform mesh refinements are applied. The a posteriori error estimate based adaptive finite element method is an ideal tool to handle these issues.

A posteriori error estimators are computable quantities in terms of numerical solutions and data. They measure the error between the numerical solution and the exact solution without requiring any a priori information of the exact solution. A reliable a posteriori error estimator plays a crucial role in an adaptive procedure for mesh modification such as refinement or coarsening. Since the work of Babuška and Rheinboldt [4], the study of adaptive method based on a posteriori error estimator has become an active

research topic in scientific computing. Some relevant work can be found in [1, 2, 5, 10, 11] on the adaptive finite element method. We refer to [12, 13, 22, 23] for studies on the scattering problems by using the a posteriori error estimate based adaptive finite element method.

Motivated by the work of Chen and Liu [12], we develop an adaptive procedure, which combines the finite element method and the PML technique, to solve the open cavity scattering problems. Specifically, we consider the electromagnetic scattering of a time-harmonic plane wave by an open cavity embedded in an infinite ground plane. The ground plane and the cavity wall are assumed to be perfect electric conductors. The cavity is assumed to be filled with some inhomogeneous medium, which may protrude out of the cavity to the upper half-space in a finite extent. The upper half-space above the ground plane and the protruding part of the cavity is assumed to be filled with some homogeneous medium. By assuming invariance of the cavity in the x_3 direction, we consider two fundamental polarizations: transverse magnetic (TM) and transverse electric (TE) polarizations, where the three-dimensional Maxwell equations can be reduced to the two-dimensional Helmholtz equation. We restrict our attention to the numerical solutions of the TM and TE polarizations. In each polarization, the scattering problem is reduced equivalently into a boundary value problem of the two-dimensional Helmholtz equation in a bounded domain by using the transparent boundary condition. Computationally, the PML technique is utilized to truncate the infinite half-space above the ground plane and the homogeneous Dirichlet boundary condition is imposed on the outer boundary of the PML layer. The a posteriori error estimate is deduced between the solution of the original scattering problem and the finite element solution of the truncated PML problem. The a posteriori error estimate takes account both of the finite element discretization error and the truncation error of the PML method. The PML truncation error has a nice feature of exponential decay in terms of the PML medium parameter and the thickness of the layer. Based on this property, the proper PML medium parameter and the thickness of the layer can be chosen to make the PML error negligible compared with the finite element discretization error. Once the PML region and the medium property are fixed, the finite element discretization error is used to design the adaptive strategy.

We point out a closely related work [28], where an adaptive finite element method with transparent boundary condition (TBC) was developed for solving the open cavity scattering problems. Since the nonlocal TBC is directly used to truncate the open domain, it does not require a layer of artificially designed absorbing medium to enclose the domain of interest, which makes the TBC method different from the PML approach. But the TBC is given as an infinite series and needs to be truncated into a sum of finitely many terms in computation. Due to the simplicity in the implementation of the PML method, this work provides a viable alternative to the adaptive finite element TBC method for solving the open cavity scattering problems. Numerical experiments are presented and compared with the adaptive finite element TBC method for both polarizations to illustrate the competitive behavior of the adaptive finite element PML method.

The outline of this paper is as follows. In Section 2, we introduce the problem for-

mulation, where the governing equations are given for the TM and TE polarizations. Sections 3 and 4 are devoted to the analysis of the TM and TE polarizations, respectively. Topics are organized to address the variational problem, the PML problem and its convergence, the finite element approximation, the a posteriori error analysis for the discrete truncated PML problem, and the adaptive finite element algorithm. In Section 5, some numerical examples are presented to illustrate the performance of the proposed method. The paper is concluded with some general remarks in Section 6.

2 Problem formulation

Let us first specify the problem geometry which is shown in Fig. 1. Denote by $D \subset \mathbb{R}^2$ the cross section of an x_3 -invariant cavity with a Lipschitz continuous boundary $\partial D = S \cup \Gamma$, where S refers to as the cavity wall and Γ is the opening of the cavity. We assume that the cavity wall S is a perfect electric conductor and the opening Γ is aligned with the perfectly electrically conducting infinite ground plane Γ_g . The cavity may be filled with some inhomogeneous medium, which can be characterized by the dielectric permittivity ϵ and the magnetic permeability μ . Moreover, the medium may protrude from the cavity into the upper half-space. In this case, the cavity is called an overfilled cavity. Let B_R^+ and B_ρ^+ be the upper half-discs with radii R and ρ , where $\rho > R > 0$. Denote by Γ_R^+ and Γ_ρ^+ the upper semi-circles. The radius R can be chosen large enough such that the upper half-disc B_R^+ can enclose the possibly protruding inhomogeneous medium from the cavity. The infinite exterior domain $\mathbb{R}^2 \setminus \overline{B_R^+}$ is assumed to be filled with some homogeneous medium with a constant dielectric permittivity ϵ_0 and a constant magnetic permeability μ_0 .

Since the structure is invariant the x_3 -axis, we consider two fundamental polarizations: transverse magnetic (TM) polarization and transverse electric (TE) polarization. The three-dimensional Maxwell equations can be reduced to the two-dimensional

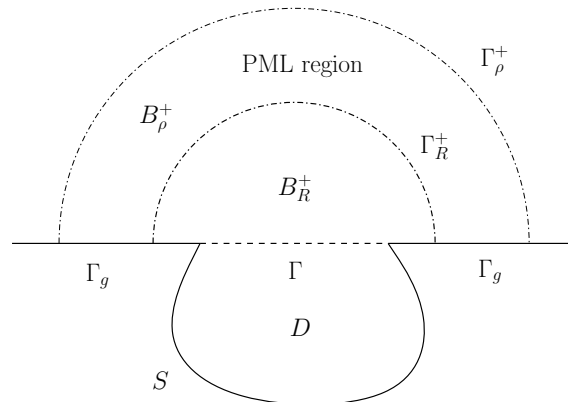


Figure 1: Schematic of the open cavity scattering problem.

Helmholtz equation under these two modes. In the TM polarization, the magnetic field is transverse to the x_3 -axis and the electric field has the form $E(x_1, x_2) = (0, 0, u(x_1, x_2))^T$, where the scalar function u satisfies

$$\begin{cases} \Delta u + \kappa^2 u = 0 & \text{in } \mathbb{R}_+^2 \cup D, \\ u = 0 & \text{on } \Gamma_g \cup S, \end{cases} \quad (2.1)$$

where $\kappa = \omega(\epsilon\mu)^{1/2}$ is the wave number and $\omega > 0$ is the angular frequency. In the TE polarization, the electric field is transverse to the x_3 -axis and the magnetic field takes the form $H(x_1, x_2) = (0, 0, u(x_1, x_2))^T$, where u satisfies

$$\begin{cases} \nabla \cdot (\kappa^{-2} \nabla u) + u = 0 & \text{in } \mathbb{R}_+^2 \cup D, \\ \partial_\nu u = 0 & \text{on } \Gamma_g \cup S, \end{cases} \quad (2.2)$$

where ν is the unit outward normal vector to $\Gamma_g \cup S$.

Consider the incidence of a plane wave

$$u^i(x_1, x_2) = e^{i(k_1 x_1 - k_2 x_2)},$$

which is sent from the above to impinge the cavity. Here $k_1 = \kappa_0 \sin\theta$, $k_2 = \kappa_0 \cos\theta$, $\theta \in (-\pi/2, \pi/2)$ is the angle of the incidence, and $\kappa_0 = \omega(\epsilon_0\mu_0)^{1/2}$ is the wavenumber in the free space. Due to the perfectly electrically conducting ground plane, the reflected field in the TM polarization is

$$u^r(x_1, x_2) = -e^{i(k_1 x_1 + k_2 x_2)},$$

while the reflected field in the TE polarization is

$$u^r(x_1, x_2) = e^{i(k_1 x_1 + k_2 x_2)}.$$

Let the reference field u^{ref} be the superposition of the incident field and the reflected field, i.e., $u^{\text{ref}} = u^i + u^r$. The total field u consists of the reference field u^{ref} and the scattered field u^s , i.e.,

$$u = u^{\text{ref}} + u^s.$$

In addition, the scattered field u^s is required to satisfy the Sommerfeld radiation condition

$$\lim_{r=|x| \rightarrow \infty} r^{1/2}(\partial_r u^s - i\kappa_0 u^s) = 0. \quad (2.3)$$

3 TM polarization

In this section, we consider the TM polarization. First the transparent boundary condition is introduced to reduce the open cavity problem into a boundary value problem in a bounded domain. Next the variational problem is described, and the PML problem and its convergence are discussed. Then the finite element approximation and the a posteriori error estimate are studied. Finally the adaptive finite element method with PML is presented for solving the discrete PML problem.

3.1 The variational problem

It can be verified from (2.1) that the scattered field u^s satisfies the Helmholtz equation

$$\Delta u^s + \kappa_0^2 u^s = 0 \quad \text{in } \mathbb{R}_+^2 \setminus \overline{B_R^+}. \tag{3.1}$$

Based on the radiation condition (2.3), we know that the solution of (3.1) has the Fourier series expansion

$$u^s(r, \phi) = \sum_{n=0}^{\infty} \frac{H_n^{(1)}(\kappa_0 r)}{H_n^{(1)}(\kappa_0 R)} (a_n \sin(n\phi) + b_n \cos(n\phi)), \quad r \geq R, \tag{3.2}$$

where $H_n^{(1)}$ is the Hankel function of the first kind with order n . Noting the fact $u=0$ and $u^{\text{ref}}=0$ on Γ_g , we have $u^s(r, 0) = u^s(r, \pi) = 0$, which implies $b_n = 0$ and (3.2) reduces to

$$u^s(r, \phi) = \sum_{n=1}^{\infty} \frac{H_n^{(1)}(\kappa_0 r)}{H_n^{(1)}(\kappa_0 R)} a_n \sin(n\phi), \quad r \geq R. \tag{3.3}$$

Taking the partial derivative of (3.3) with respect to r and evaluating it at $r = R$ yields

$$\partial_r u^s(R, \phi) = \kappa_0 \sum_{n=1}^{\infty} \frac{H_n^{(1)' }(\kappa_0 R)}{H_n^{(1)}(\kappa_0 R)} a_n \sin(n\phi). \tag{3.4}$$

Let $L_{\text{TM}}^2(\Gamma_R^+) := \{u \in L^2(\Gamma_R^+) : u(R, 0) = u(R, \pi) = 0\}$. For any $u \in L_{\text{TM}}^2(\Gamma_R^+)$, it has the Fourier series expansion

$$u(R, \phi) = \sum_{n=1}^{\infty} a_n \sin(n\phi), \quad a_n = \frac{2}{\pi} \int_0^\pi u(R, \phi) \sin(n\phi) d\phi.$$

Define the trace function space $H_{\text{TM}}^s(\Gamma_R^+) := \{u \in L_{\text{TM}}^2(\Gamma_R^+) : \|u\|_{H_{\text{TM}}^s(\Gamma_R^+)} \leq \infty\}$, where the $H_{\text{TM}}^s(\Gamma_R^+)$ norm is given by

$$\|u\|_{H_{\text{TM}}^s(\Gamma_R^+)} = \left(\sum_{n=1}^{\infty} (1+n^2)^s |a_n|^2 \right)^{1/2}.$$

It is clear that the dual space of $H_{\text{TM}}^s(\Gamma_R^+)$ is $H_{\text{TM}}^{-s}(\Gamma_R^+)$ with respect to the scalar product in $L^2(\Gamma_R^+)$ given by

$$\langle u, v \rangle_{\Gamma_R^+} = \int_{\Gamma_R^+} u \bar{v} ds.$$

Introduce the DtN operator

$$(\mathcal{B}_{\text{TM}} u)(R, \phi) = \kappa_0 \sum_{n=1}^{\infty} \frac{H_n^{(1)' }(\kappa_0 R)}{H_n^{(1)}(\kappa_0 R)} a_n \sin(n\phi) \quad \text{on } \Gamma_R^+. \tag{3.5}$$

It is shown in [27] that the boundary operator $\mathcal{B}_{\text{TM}}: H_{\text{TM}}^{1/2}(\Gamma_R^+) \rightarrow H_{\text{TM}}^{-1/2}(\Gamma_R^+)$ is continuous. Using (3.4)-(3.5), we obtain the transparent boundary condition for the scattered field u^s :

$$\partial_r u^s = \mathcal{B}_{\text{TM}} u^s \quad \text{on } \Gamma_R^+,$$

which can be equivalently imposed for the total field u :

$$\partial_r u = \mathcal{B}_{\text{TM}} u + f \quad \text{on } \Gamma_R^+,$$

where $f = \partial_r u^{\text{ref}} - \mathcal{B}_{\text{TM}} u^{\text{ref}}$.

Let $\Omega = B_R^+ \cup D$. The open cavity scattering problem can be reduced to the following boundary value problem:

$$\begin{cases} \Delta u + \kappa^2 u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_g \cup S, \\ \partial_r u = \mathcal{B}_{\text{TM}} u + f & \text{on } \Gamma_R^+, \end{cases}$$

which has the variational formulation: find $u \in H_S^1(\Omega) = \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_g \cup S\}$ such that

$$a_{\text{TM}}(u, v) = \langle f, v \rangle_{\Gamma_R^+} \quad \forall v \in H_S^1(\Omega), \tag{3.6}$$

where the sesquilinear form $a_{\text{TM}}(\cdot, \cdot) : H^1(\Omega) \times H^1(\Omega) \rightarrow \mathbb{C}$ is defined by

$$a_{\text{TM}}(u, v) = \int_{\Omega} (\nabla u \cdot \nabla \bar{v} - \kappa^2 u \bar{v}) \, dx - \langle \mathcal{B}_{\text{TM}} u, v \rangle_{\Gamma_R^+}. \tag{3.7}$$

The following result states the well-posedness of the variational problem (3.6). The proof can be found in [27].

Theorem 3.1. *The variational problem (3.6) has a unique weak solution in $H_S^1(\Omega)$, which satisfies the estimate*

$$\|u\|_{H^1(\Omega)} \leq C \|f\|_{H_{\text{TM}}^{-1/2}(\Gamma_R^+)},$$

where $C > 0$ is a constant.

It follows from the general theory in Babuška and Aziz [3, Chapter 5] that there exists a constant $C > 0$ such that the following inf-sup condition holds:

$$\sup_{0 \neq v \in H_S^1(\Omega)} \frac{|a_{\text{TM}}(u, v)|}{\|v\|_{H^1(\Omega)}} \geq C \|u\|_{H^1(\Omega)} \quad \forall u \in H_S^1(\Omega). \tag{3.8}$$

3.2 The PML problem

Let $\Omega^{\text{PML}} = \{x \in \mathbb{R}_+^2 : R < |x| < \rho\}$ be the PML region which encloses the bounded domain Ω in the upper half-space. Denote by $\Omega_\rho = B_\rho^+ \cup D$ the computational domain in which the truncated PML problem is formulated.

Define the PML parameters by using the complex coordinate stretching

$$\tilde{r} = \int_0^r \alpha(t) dt = r\beta(r), \tag{3.9}$$

where $\alpha(r) = 1 + i\sigma(r)$. In practice, σ is usually taken as a power function

$$\sigma(r) = \begin{cases} 0, & 0 \leq r < R, \\ \sigma_0 \left(\frac{r-R}{\rho-R}\right)^m, & r \geq R, \end{cases}$$

where σ_0 is a positive constant and $m \geq 1$ is an integer. It can be seen from (3.9) that

$$\beta(r) = 1 + i\hat{\sigma}(r), \quad \hat{\sigma}(r) = \frac{1}{r} \int_R^r \sigma(t) dt.$$

In the polar coordinates, the gradient and divergence operators can be written as

$$\nabla u = \partial_r u e_r + \frac{1}{r} \partial_\phi u e_\phi, \quad \nabla \cdot u = \frac{1}{r} \partial_r (ru_r) + \frac{1}{r} \partial_\phi u_\phi, \tag{3.10}$$

where $u = u_r e_r + u_\phi e_\phi$ and $e_r = (\cos\phi, \sin\phi)^\top$, $e_\phi = (-\sin\phi, \cos\phi)^\top$. By the chain rule and (3.9), a simple calculation yields

$$\partial_{\tilde{r}} u = \partial_r u \left(\frac{dr}{d\tilde{r}} \right) = \frac{1}{\alpha(r)} \partial_r u. \tag{3.11}$$

Combining (3.10) and (3.11), we introduce the modified gradient operator

$$\tilde{\nabla} u = \frac{1}{\alpha(r)} \partial_r u e_r + \frac{1}{r\beta(r)} \partial_\phi u e_\phi.$$

It is easy to verify

$$\tilde{\Delta} u = \frac{1}{r\alpha(r)\beta(r)} \partial_r \left(\frac{r\beta(r)}{\alpha(r)} \partial_r u \right) + \frac{1}{r\beta(r)} \partial_\phi \left(\frac{1}{r\beta(r)} \partial_\phi u \right) = \frac{1}{\alpha\beta} \nabla \cdot (A \nabla u),$$

where

$$A = \begin{bmatrix} \frac{\beta(r)}{\alpha(r)} \cos^2 \phi + \frac{\alpha(r)}{\beta(r)} \sin^2 \phi & \left(\frac{\beta(r)}{\alpha(r)} - \frac{\alpha(r)}{\beta(r)} \right) \sin \phi \cos \phi \\ \left(\frac{\beta(r)}{\alpha(r)} - \frac{\alpha(r)}{\beta(r)} \right) \sin \phi \cos \phi & \frac{\beta(r)}{\alpha(r)} \sin^2 \phi + \frac{\alpha(r)}{\beta(r)} \cos^2 \phi \end{bmatrix}.$$

Hence we obtain the PML equation for the scattered field $u^{\text{s,PML}}$:

$$\nabla \cdot (A \nabla u^{\text{s,PML}}) + \kappa_0^2 \alpha \beta u^{\text{s,PML}} = 0 \quad \text{in } \mathbb{R}_+^2 \setminus \overline{B_R^+},$$

where $u^{s,\text{PML}}$ is required to be uniformly bounded as $r = |x| \rightarrow \infty$. In practice, the open domain $\mathbb{R}_+^2 \setminus \overline{B_R^+}$ needs to be truncated into a bounded domain. Replacing r with \tilde{r} in (3.3) and noting the exponential decay of the Hankel functions with a complex argument, we can observe that the scattered field $u^{s,\text{PML}}$ decays exponentially in $\mathbb{R}_+^2 \setminus \overline{B_R^+}$. Hence it is reasonable to impose the Dirichlet boundary condition

$$u^{s,\text{PML}} = 0 \quad \text{on } \Gamma_\rho^+.$$

We obtain the truncated PML problem

$$\begin{cases} \nabla \cdot (A \nabla u^{\text{PML}}) + \kappa^2 \alpha \beta u^{\text{PML}} = F & \text{in } \Omega_\rho, \\ u^{\text{PML}} = 0 & \text{on } \Gamma_g \cup S, \\ u^{\text{PML}} = u^{\text{ref}} & \text{on } \Gamma_\rho^+, \end{cases} \quad (3.12)$$

where

$$F = \begin{cases} \nabla \cdot (A \nabla u^{\text{ref}}) + \kappa_0^2 \alpha \beta u^{\text{ref}} & \text{in } \Omega^{\text{PML}}, \\ 0 & \text{otherwise.} \end{cases}$$

Introduce another DtN operator $\hat{\mathcal{B}}_{\text{TM}}: H_{\text{TM}}^{1/2}(\Gamma_R^+) \rightarrow H_{\text{TM}}^{-1/2}(\Gamma_R^+)$ which defined as follows: given $\zeta \in H_{\text{TM}}^{1/2}(\Gamma_R^+)$,

$$\hat{\mathcal{B}}_{\text{TM}} \zeta = \partial_r \zeta|_{\Gamma_R^+},$$

where $\zeta \in H^1(\Omega^{\text{PML}})$ satisfies

$$\begin{cases} \nabla \cdot (A \nabla \zeta) + \kappa^2 \alpha \beta \zeta = 0 & \text{in } \Omega^{\text{PML}}, \\ \zeta = \zeta & \text{on } \Gamma_R^+, \\ \zeta = 0 & \text{on } \Gamma_g \cup \Gamma_\rho^+. \end{cases}$$

Using the boundary condition

$$\partial_r (u^{\text{PML}} - u^{\text{ref}})|_{\Gamma_R^+} = \hat{\mathcal{B}}_{\text{TM}} (u^{\text{PML}} - u^{\text{ref}}),$$

and noting $A = I$, $\alpha = \beta = 1$ in Ω , we reformulate (3.12) equivalently into the following boundary value problem:

$$\begin{cases} \Delta u^{\text{PML}} + \kappa^2 u^{\text{PML}} = 0 & \text{in } \Omega, \\ u^{\text{PML}} = 0 & \text{on } \Gamma_g \cup S, \\ \partial_r u^{\text{PML}} = \hat{\mathcal{B}}_{\text{TM}} u^{\text{PML}} + \hat{f} & \text{on } \Gamma_R^+, \end{cases} \quad (3.13)$$

where $\hat{f} = \partial_r u^{\text{ref}} - \hat{\mathcal{B}}_{\text{TM}} u^{\text{ref}}$. The weak formulation of the problem (3.13) is to find $u^{\text{PML}} \in H_S^1(\Omega)$ such that

$$\hat{a}_{\text{TM}}(u^{\text{PML}}, v) = \langle \hat{f}, v \rangle_{\Gamma_R^+} \quad \forall v \in H_S^1(\Omega), \quad (3.14)$$

where the sesquilinear form $\hat{a}_{\text{TM}}(\cdot, \cdot): H^1(\Omega) \times H^1(\Omega) \rightarrow \mathbb{C}$ is defined as

$$\hat{a}_{\text{TM}}(u, v) = \int_{\Omega} (\nabla u \cdot \nabla \bar{v} - \kappa^2 u \bar{v}) \, dx - \langle \hat{\mathcal{B}}_{\text{TM}} u, v \rangle_{\Gamma_R^+}.$$

3.3 Convergence of the PML problem

Consider a boundary value problem of the PML equation in Ω^{PML} :

$$\begin{cases} \nabla \cdot (A \nabla w) + \kappa_0^2 \alpha \beta w = 0 & \text{in } \Omega^{\text{PML}}, \\ w = 0 & \text{on } \Gamma_g \cup \Gamma_R^+, \\ w = q & \text{on } \Gamma_\rho^+. \end{cases} \tag{3.15}$$

Define $H_0^1(\Omega^{\text{PML}}) = \{u \in H^1(\Omega^{\text{PML}}) : u = 0 \text{ on } \Gamma_g \cup \Gamma_R^+ \cup \Gamma_\rho^+\}$. Given $q \in H_{\text{TM}}^{1/2}(\Gamma_\rho^+)$, the weak formulation of (3.15) is to find $w \in H^1(\Omega^{\text{PML}})$ such that $w = 0$ on $\Gamma_g \cup \Gamma_R^+$, $w = q$ on Γ_ρ^+ and

$$\hat{b}(w, v) = 0 \quad \forall v \in H_0^1(\Omega^{\text{PML}}), \tag{3.16}$$

where the sesquilinear form $\hat{b}(\cdot, \cdot) : H^1(\Omega^{\text{PML}}) \times H^1(\Omega^{\text{PML}}) \rightarrow \mathbf{C}$ is

$$\hat{b}(u, v) = \int_R^\rho \int_0^\pi \left(\frac{\beta r}{\alpha} \partial_r u \partial_r \bar{v} + \frac{\alpha}{\beta r} \partial_\phi u \partial_\phi \bar{v} - \kappa_0^2 \alpha \beta r u \bar{v} \right) dr d\phi.$$

As is discussed in [16], in general, the uniqueness of (3.16) can not be guaranteed due to the possible existence of eigenvalues which form a discrete set. Since our focus is on the convergence analysis, we simply assume that the PML problem (3.16) has a unique solution in the PML region.

For any $u \in H^1(\Omega^{\text{PML}})$, define

$$\|u\|_{*, \Omega^{\text{PML}}} = \left[\int_R^\rho \int_0^\pi \left(\left(\frac{1 + \sigma \hat{\sigma}}{1 + \sigma^2} \right) r |\partial_r u|^2 + \left(\frac{1 + \sigma \hat{\sigma}}{1 + \hat{\sigma}^2} \right) \frac{1}{r} |\partial_\phi u|^2 + (1 + \sigma \hat{\sigma}) \kappa_0^2 r |u|^2 \right) dr d\phi \right]^{1/2}.$$

It is easy to show that the norm $\|\cdot\|_{*, \Omega^{\text{PML}}}$ is equivalent to the usual $H^1(\Omega^{\text{PML}})$ -norm. An application of the general theory in [3, Chapter 5] implies that there exists a positive constant \hat{C} depending on Ω^{PML} and κ_0 such that

$$\sup_{0 \neq v \in H_0^1(\Omega^{\text{PML}})} \frac{|\hat{b}(u, v)|}{\|v\|_{*, \Omega^{\text{PML}}}} \geq \hat{C} \|u\|_{*, \Omega^{\text{PML}}} \quad \forall u \in H_0^1(\Omega^{\text{PML}}). \tag{3.17}$$

The following results play an important role in the convergence analysis. The proof is similar to that of [12, Theorem 2.4] for solving the obstacle scattering problem and is omitted here for brevity.

Theorem 3.2. *There exists a constant $C > 0$ independent of κ_0, R, ρ , and σ_0 such that the following estimates are satisfied:*

$$\|\alpha|^{-1} \nabla w\|_{L^2(\Omega^{\text{PML}})} \leq C \hat{C}^{-1} (1 + \kappa_0 R) |\alpha_0| \|q\|_{H_{\text{TM}}^{1/2}(\Gamma_\rho^+)}, \tag{3.18}$$

$$\|\partial_r w\|_{H_{\text{TM}}^{-1/2}(\Gamma_R^+)} \leq C \hat{C}^{-1} (1 + \kappa_0 R)^2 |\alpha_0|^2 \|q\|_{H_{\text{TM}}^{1/2}(\Gamma_\rho^+)}, \tag{3.19}$$

where \hat{C} is given in (3.17) and $\alpha_0 = 1 + i\sigma_0$.

Following the idea in [19], for any function $f \in H_{\text{TM}}^{1/2}(\Gamma_R^+)$, we introduce the propagation operator $\mathcal{P}_{\text{TM}}: H_{\text{TM}}^{1/2}(\Gamma_R^+) \rightarrow H_{\text{TM}}^{1/2}(\Gamma_\rho^+)$ defined by

$$\mathcal{P}_{\text{TM}}(f) = \sum_{n=1}^{\infty} \frac{H_n^{(1)}(\kappa_0 \tilde{\rho})}{H_n^{(1)}(\kappa_0 R)} f_n \sin(n\phi), \quad f_n = \frac{2}{\pi} \int_0^\pi f(R, \phi) \sin(n\phi) d\phi.$$

As shown in [12], the operator $\mathcal{P}_{\text{TM}}: H_{\text{TM}}^{1/2}(\Gamma_R^+) \rightarrow H_{\text{TM}}^{1/2}(\Gamma_\rho^+)$ is well defined and satisfies the estimate

$$\|\mathcal{P}_{\text{TM}}(f)\|_{H_{\text{TM}}^{1/2}(\Gamma_\rho^+)} \leq e^{-\kappa_0 \Im(\tilde{\rho}) \left(1 - \frac{R^2}{|\tilde{\rho}|^2}\right)^{1/2}} \|f\|_{H_{\text{TM}}^{1/2}(\Gamma_R^+)} \quad \forall \rho \geq R. \tag{3.20}$$

Lemma 3.1. For any $f \in H_{\text{TM}}^{1/2}(\Gamma_R^+)$, we have

$$\|(\mathcal{B}_{\text{TM}} - \hat{\mathcal{B}}_{\text{TM}})f\|_{H_{\text{TM}}^{-1/2}(\Gamma_R^+)} \leq C \hat{C}^{-1} (1 + \kappa_0 R)^2 |\alpha_0|^2 e^{-\kappa_0 \Im(\tilde{\rho}) \left(1 - \frac{R^2}{|\tilde{\rho}|^2}\right)^{1/2}} \|f\|_{H_{\text{TM}}^{1/2}(\Gamma_R^+)}.$$

Proof. For any $f \in H_{\text{TM}}^{1/2}(\Gamma_R^+)$, it follows the definitions of \mathcal{B}_{TM} and $\hat{\mathcal{B}}_{\text{TM}}$ that

$$(\mathcal{B}_{\text{TM}} - \hat{\mathcal{B}}_{\text{TM}})f = \partial_r w|_{\Gamma_R^+},$$

where $w \in H^1(\Omega^{\text{PML}})$ satisfies

$$\begin{cases} \nabla \cdot (A \nabla w) + \kappa_0^2 \alpha \beta w = 0 & \text{in } \Omega^{\text{PML}}, \\ w = 0 & \text{on } \Gamma_g \cup \Gamma_R^+, \\ w = \mathcal{P}_{\text{TM}}(f) & \text{on } \Gamma_\rho^+. \end{cases}$$

Using (3.19)-(3.20) yields

$$\begin{aligned} \|\partial_r w\|_{H_{\text{TM}}^{-1/2}(\Gamma_R^+)} &\leq C \hat{C}^{-1} (1 + \kappa_0 R)^2 |\alpha_0|^2 \|\mathcal{P}_{\text{TM}}(f)\|_{H_{\text{TM}}^{1/2}(\Gamma_\rho^+)} \\ &\leq C \hat{C}^{-1} (1 + \kappa_0 R)^2 |\alpha_0|^2 e^{-\kappa_0 \Im(\tilde{\rho}) \left(1 - \frac{R^2}{|\tilde{\rho}|^2}\right)^{1/2}} \|f\|_{H_{\text{TM}}^{1/2}(\Gamma_R^+)}, \end{aligned}$$

which completes the proof. □

Theorem 3.3. For sufficiently large $\sigma_0 > 0$, the PML problem (3.14) has a unique solution $u^{\text{PML}} \in H_\zeta^1(\Omega)$. Moreover, we have the following estimate:

$$\|u - u^{\text{PML}}\|_{H^1(\Omega)} \leq C \hat{C}^{-1} (1 + \kappa_0 R)^2 |\alpha_0|^2 e^{-\kappa_0 \Im(\tilde{\rho}) \left(1 - \frac{R^2}{|\tilde{\rho}|^2}\right)^{1/2}} \|u^{\text{PML}} - u^{\text{ref}}\|_{H_{\text{TM}}^{1/2}(\Gamma_R^+)}.$$

Proof. The existence of a unique solution can be shown by following the same arguments in [13, Theorem 2.4]. Furthermore, by (3.6) and (3.14), we have for any $\varphi \in H^1_S(\Omega)$ that

$$\begin{aligned} a_{\text{TM}}(u - u^{\text{PML}}, \varphi) &= a_{\text{TM}}(u, \varphi) - a_{\text{TM}}(u^{\text{PML}}, \varphi) \\ &= \langle f, \varphi \rangle_{\Gamma_R^+} - a_{\text{TM}}(u^{\text{PML}}, \varphi) \\ &= \langle f - \hat{f}, \varphi \rangle_{\Gamma_R^+} + \langle \hat{f}, \varphi \rangle_{\Gamma_R^+} - a(u^{\text{PML}}, \varphi) \\ &= \langle (\hat{\mathcal{B}}_{\text{TM}} - \mathcal{B}_{\text{TM}})u^{\text{ref}}, \varphi \rangle_{\Gamma_R^+} + \hat{a}_{\text{TM}}(u^{\text{PML}}, \varphi) - a_{\text{TM}}(u^{\text{PML}}, \varphi) \\ &= \langle (\mathcal{B}_{\text{TM}} - \hat{\mathcal{B}}_{\text{TM}})(u^{\text{PML}} - u^{\text{ref}}), \varphi \rangle_{\Gamma_R^+}, \end{aligned}$$

which completes the proof after using Lemma 3.1 and (3.8). □

3.4 Finite element approximation

Define $H^1_S(\Omega_\rho) = \{u \in H^1(\Omega_\rho) : u = 0 \text{ on } \Gamma_g \cup S\}$. The weak formulation of (3.12) is to find $u^{\text{PML}} \in H^1_S(\Omega_\rho)$ and $u^{\text{PML}} = u^{\text{ref}}$ on Γ_ρ^+ such that

$$b(u^{\text{PML}}, v) = - \int_{\Omega_\rho} F \bar{v} dx \quad \forall v \in H^1_0(\Omega_\rho), \tag{3.21}$$

where $H^1_0(\Omega_\rho) = \{u \in H^1(\Omega_\rho) : u = 0 \text{ on } \Gamma_g \cup S \cup \Gamma_\rho^+\}$ and the sesquilinear form $b(\cdot, \cdot) : H^1(\Omega_\rho) \times H^1(\Omega_\rho) \rightarrow \mathbb{C}$ is given by

$$b(u, v) = \int_{\Omega_\rho} (A \nabla u \cdot \nabla \bar{v} - \kappa^2 \alpha \beta u \bar{v}) dx. \tag{3.22}$$

Let \mathcal{M}_h be a regular triangulation of Ω_ρ , where h denotes the maximum diameter of all the elements in \mathcal{M}_h . To avoid being distracted from the main focus of the a posteriori error analysis, we assume for simplicity that Γ_ρ^+ is polygonal to keep from using the isoparametric finite element space and deriving the approximation error of the boundary Γ_ρ^+ .

Let V_h be the a conforming finite element space, i.e.,

$$V_h = \{v_h \in C(\bar{\Omega}_\rho) : v_h|_K \in P_m(K), \forall K \in \mathcal{M}_h\},$$

where m is a positive integer and $P_m(K)$ denotes the set of all polynomials of degree no more than m . The finite element approximation to the variational problem (3.21) is to find $u_h \in V_h$ with $u_h = u^{\text{ref}}$ on Γ_ρ^+ such that

$$b(u_h, \psi_h) = - \int_{\Omega_\rho} F \bar{\psi}_h dx \quad \forall \psi_h \in V_{S,h}, \tag{3.23}$$

where $V_{S,h} = \{v_h \in V_h : v_h = 0 \text{ on } \Gamma_g \cup S\}$.

For sufficiently small h , the discrete inf-sup condition of the sesquilinear form b can be established by an argument of Schatz [26]. It follows from the general theory in [3] that the truncated variational problem (3.23) admits a unique solution. Since our focus is the a posteriori error analysis and the associated adaptive algorithm, we assume that the discrete problem (3.23) has a unique solution $u_h \in V_h$.

3.5 A posteriori error analysis

For any triangular element $K \in \mathcal{M}_h$, denote by h_K its diameter. Let \mathcal{B}_h denote the set of all the edges that do not lie on $\partial\Omega_\rho$. For any $e \in \mathcal{B}_h$, h_e denotes its length. For any $K \in \mathcal{M}_h$, we introduce the residual

$$R_K(u) = \nabla \cdot (A \nabla u|_K) + \kappa^2 \alpha \beta u|_K.$$

For any interior edge e , which is the common side of triangular elements $K_1, K_2 \in \mathcal{M}_h$, we define the jump residual across e as

$$J_e = -(A \nabla u_h|_{K_1} \cdot \nu_1 + A \nabla u_h|_{K_2} \cdot \nu_2),$$

where ν_j is the unit outward normal vector on the boundary of K_j , $j = 1, 2$. Let

$$\tilde{R}_K = \begin{cases} R_K(u_h) & \text{if } K \in \mathcal{M}_h \cap \Omega, \\ R_K(u_h - u^{\text{ref}}) & \text{if } K \in \mathcal{M}_h \cap \Omega^{\text{PML}}. \end{cases}$$

For any triangle $K \in \mathcal{M}_h$, denote by η_K the local error estimator as follows:

$$\eta_K = \max_{x \in K} w(x) \left(\|h_K \tilde{R}_K\|_{L^2(K)}^2 + \frac{1}{2} \sum_{e \in \partial K \cap \mathcal{B}_h} \|h_e^{1/2} J_e\|_{L^2(e)}^2 \right)^{1/2},$$

where the rescaling function

$$w(x) = \begin{cases} 1 & \text{if } x \in \bar{\Omega}, \\ |\frac{\alpha}{\alpha_0}| e^{-\kappa \Im \bar{r}} \left(1 - \frac{r^2}{|r|^2}\right)^{1/2} & \text{if } x \in \Omega^{\text{PML}}. \end{cases}$$

For any $\varphi \in H^1(\Omega)$, let $\tilde{\varphi}$ be its extension in Ω^{PML} such that

$$\begin{cases} \nabla \cdot (\bar{A} \nabla \tilde{\varphi}) + \kappa_0^2 \bar{\alpha} \bar{\beta} \tilde{\varphi} = 0 & \text{in } \Omega^{\text{PML}}, \\ \tilde{\varphi} = \varphi & \text{on } \Gamma_R^+, \\ \tilde{\varphi} = 0 & \text{on } \Gamma_g \cup \Gamma_\rho^+. \end{cases} \tag{3.24}$$

Repeating essentially the proofs of those in [12, Lemmas 4.1 and 4.4], we may obtain the following two results on the extension.

Lemma 3.2. For any $\varphi, \psi \in H^1(\Omega^{\text{PML}})$, the following identity holds:

$$\langle \hat{\mathcal{B}}_{\text{TM}}\varphi, \psi \rangle_{\Gamma_R^+} = \langle \hat{\mathcal{B}}_{\text{TM}}\bar{\psi}, \bar{\varphi} \rangle_{\Gamma_R^+}.$$

Lemma 3.3. For any $\varphi \in H^1(\Omega)$, let $\tilde{\varphi}$ be its extension in $H^1(\Omega^{\text{PML}})$ according to (3.24). Then there exists a constant $C > 0$ independent of κ_0, R, ρ and σ_0 such that

$$\| |\alpha|^{-1} \gamma \nabla \tilde{\varphi} \|_{L^2(\Omega^{\text{PML}})} \leq C \hat{C}^{-1} (1 + \kappa_0 R) |\alpha_0| \| \varphi \|_{H^{1/2}(\Gamma_R^+)},$$

where $\gamma(r) = e^{\kappa_0 \Im \tilde{r} (1 - \frac{r^2}{|\tilde{r}|^2})^{1/2}}$.

The following lemma is needed in order to present the error representation formula.

Lemma 3.4. For any $\varphi \in H^1(\Omega)$, let $\tilde{\varphi}$ be its extension in $H^1(\Omega_\rho)$ according to (3.24). Then we have for any $\xi \in H_0^1(\Omega_\rho)$ that

$$\int_{\Omega^{\text{PML}}} (A \nabla \xi \cdot \nabla \tilde{\varphi} - \kappa_0^2 \alpha \beta \xi \tilde{\varphi}) \, dx = - \langle \hat{\mathcal{B}}_{\text{TM}} \xi, \varphi \rangle_{\Gamma_R^+}.$$

Proof. Multiplying the first equation of (3.24) by $\xi \in H_0^1(\Omega_\rho)$, using the integration by parts, and noting $A = I$ on Γ_R^+ , we deduce

$$\int_{\Omega^{\text{PML}}} (\bar{A} \nabla \tilde{\varphi} \cdot \nabla \bar{\xi} - \kappa_0^2 \overline{\alpha \beta} \tilde{\varphi} \bar{\xi}) \, dx = \int_{\partial \Omega^{\text{PML}}} (\bar{A} \nabla \tilde{\varphi}) \cdot \nu \bar{\xi} \, ds = - \int_{\Gamma_R^+} \partial_\nu \tilde{\varphi} \bar{\xi} \, ds,$$

where ν is the outward normal vector to Γ_R^+ pointing to the outside of Ω . Taking the complex conjugate on both sides of the above equation yields

$$\int_{\Omega^{\text{PML}}} (A \nabla \xi \cdot \nabla \tilde{\varphi} - \kappa_0^2 \alpha \beta \xi \tilde{\varphi}) \, dx = - \int_{\Gamma_R^+} \partial_\nu \tilde{\varphi} \bar{\xi} \, ds.$$

It follows from the definition of $\hat{\mathcal{B}}_{\text{TM}}: H^{1/2}(\Gamma_R^+) \rightarrow H^{-1/2}(\Gamma_R^+)$ that

$$\partial_\nu \tilde{\varphi}|_{\Gamma_R^+} = \hat{\mathcal{B}}_{\text{TM}} \tilde{\varphi}.$$

Combining the above two equations leads to

$$\int_{\Omega^{\text{PML}}} (A \nabla \xi \cdot \nabla \tilde{\varphi} - \kappa_0^2 \alpha \beta \xi \tilde{\varphi}) \, dx = - \langle \hat{\mathcal{B}}_{\text{TM}} \tilde{\varphi}, \bar{\xi} \rangle_{\Gamma_R^+}.$$

By Lemma 3.2, we have

$$\int_{\Omega^{\text{PML}}} (A \nabla \xi \cdot \nabla \tilde{\varphi} - \alpha \beta \kappa^2 \xi \tilde{\varphi}) \, dx = - \langle \hat{\mathcal{B}}_{\text{TM}} \xi, \varphi \rangle_{\Gamma_R^+},$$

which completes the proof. □

The following lemma gives the error representation formula.

Lemma 3.5 (error representation formula). *For any $\varphi \in H^1(\Omega)$, let $\tilde{\varphi}$ be its extension in $H^1(\Omega_\rho)$ according to (3.24). For any $\varphi_h \in V_{S,h}$, the following identity holds:*

$$a_{\text{TM}}(u - u_h, \varphi) = \langle \mathcal{B}_{\text{TM}}(u_h - u^{\text{ref}}) - \hat{\mathcal{B}}_{\text{TM}}(u_h - u^{\text{ref}}), \varphi \rangle_{\Gamma_R^+} - b(u_h, \varphi - \varphi_h) - \int_{\Omega^{\text{PML}}} (\nabla \cdot (A \nabla u^{\text{ref}}) + \kappa_0^2 \alpha \beta u^{\text{ref}}) (\tilde{\varphi} - \bar{\varphi}_h) dx.$$

Proof. It follows from (3.6) that

$$a_{\text{TM}}(u - u_h, \varphi) = a_{\text{TM}}(u, \varphi) - a_{\text{TM}}(u_h, \varphi) = \langle f, \varphi \rangle_{\Gamma_R^+} - b(u_h, \varphi - \varphi_h) + b(u_h, \varphi) - b(u_h, \varphi_h) - a_{\text{TM}}(u_h, \varphi). \tag{3.25}$$

Using (4.18) and the integration by parts, we obtain

$$\begin{aligned} b(u_h, \varphi_h) &= - \int_{\Omega^{\text{PML}}} F \bar{\varphi}_h dx \\ &= - \int_{\Omega^{\text{PML}}} (\nabla \cdot (A \nabla u^{\text{ref}}) + \kappa_0^2 \alpha \beta u^{\text{ref}}) \bar{\varphi}_h dx \\ &= \int_{\Omega^{\text{PML}}} (\nabla \cdot (A \nabla u^{\text{ref}}) + \kappa_0^2 \alpha \beta u^{\text{ref}}) (\tilde{\varphi} - \bar{\varphi}_h) dx \\ &\quad + \int_{\Omega^{\text{PML}}} (A \nabla u^{\text{ref}} \cdot \nabla \tilde{\varphi} + \kappa_0^2 \alpha \beta u^{\text{ref}} \tilde{\varphi}) dx \\ &\quad + \int_{\Gamma_R^+} \partial_\nu u^{\text{ref}} \tilde{\varphi} ds. \end{aligned} \tag{3.26}$$

By the definition of the sesquilinear form (3.22), we have

$$b(u_h, \varphi) = \int_{\Omega} (A \nabla u_h \cdot \nabla \tilde{\varphi} - \kappa^2 \alpha \beta u_h \tilde{\varphi}) dx + \int_{\Omega^{\text{PML}}} (A \nabla u_h \cdot \nabla \tilde{\varphi} - \kappa_0^2 \alpha \beta u_h \tilde{\varphi}) dx. \tag{3.27}$$

It is easy to get from (3.7) that

$$a_{\text{TM}}(u_h, \varphi) = \int_{\Omega} (A \nabla u_h \cdot \nabla \tilde{\varphi} - \kappa^2 \alpha \beta u_h \tilde{\varphi}) dx - \langle \mathcal{B}_{\text{TM}} u_h, \varphi \rangle_{\Gamma_R^+}. \tag{3.28}$$

Using (3.26)-(3.28) yields

$$\begin{aligned} b(u_h, \varphi) - b(u_h, \varphi_h) - a_{\text{TM}}(u_h, \varphi) &= - \int_{\Omega^{\text{PML}}} (\nabla \cdot (A \nabla u^{\text{ref}}) + \kappa_0^2 \alpha \beta u^{\text{ref}}) (\tilde{\varphi} - \bar{\varphi}_h) dx \\ &\quad + \int_{\Omega^{\text{PML}}} (A \nabla (u_h - u^{\text{ref}}) \cdot \nabla \tilde{\varphi} - \alpha \beta \kappa^2 (u_h - u^{\text{ref}}) \tilde{\varphi}) dx \\ &\quad - \int_{\Gamma_R^+} \partial_\nu u^{\text{ref}} \tilde{\varphi} ds + \langle \mathcal{B}_{\text{TM}} u_h, \varphi \rangle_{\Gamma_R^+}, \end{aligned}$$

which together with Lemma 3.4 implies

$$\begin{aligned}
 & b(u_h, \varphi) - b(u_h, \varphi_h) - a_{\text{TM}}(u_h, \varphi) \\
 &= - \int_{\Omega^{\text{PML}}} (\nabla \cdot (A \nabla u^{\text{ref}}) + \kappa_0^2 \alpha \beta u^{\text{ref}}) (\bar{\varphi} - \bar{\varphi}_h) dx \\
 & \quad + \langle \mathcal{B}_{\text{TM}} u_h - \hat{\mathcal{B}}_{\text{TM}} u_h, \varphi \rangle_{\Gamma_R^+} + \langle -\partial_\nu u^{\text{ref}} + \hat{\mathcal{B}}_{\text{TM}} u^{\text{ref}}, \varphi \rangle_{\Gamma_R^+}.
 \end{aligned} \tag{3.29}$$

Substituting (3.29) into (3.25), we have

$$\begin{aligned}
 a_{\text{TM}}(u - u_h, \varphi) &= \langle \partial_\nu u^{\text{ref}} - \mathcal{B}_{\text{TM}} u^{\text{ref}}, \varphi \rangle_{\Gamma_R^+} - b(u_h, \varphi - \varphi_h) \\
 & \quad - \int_{\Omega^{\text{PML}}} (\nabla \cdot (A \nabla u^{\text{ref}}) + \kappa_0^2 \alpha \beta u^{\text{ref}}) (\bar{\varphi} - \bar{\varphi}_h) dx \\
 & \quad + \langle \mathcal{B}_{\text{TM}} u_h - \hat{\mathcal{B}}_{\text{TM}} u_h, \varphi \rangle_{\Gamma_R^+} + \langle -\partial_\nu u^{\text{ref}} + \hat{\mathcal{B}}_{\text{TM}} u^{\text{ref}}, \varphi \rangle_{\Gamma_R^+} \\
 &= \langle \mathcal{B}_{\text{TM}}(u_h - u^{\text{ref}}) - \hat{\mathcal{B}}_{\text{TM}}(u_h - u^{\text{ref}}), \varphi \rangle_{\Gamma_R^+} - b(u_h, \varphi - \varphi_h) \\
 & \quad - \int_{\Omega^{\text{PML}}} (\nabla \cdot (A \nabla u^{\text{ref}}) + \kappa_0^2 \alpha \beta u^{\text{ref}}) (\bar{\varphi} - \bar{\varphi}_h) dx,
 \end{aligned}$$

which completes the proof. □

Let $\Pi_h : H_S^1(\Omega_\rho) \rightarrow V_{S,h}$ be the Clement-type interpolation operator. It can be verified that the operator enjoys the following estimates: for any $v \in H_S^1(\Omega_\rho)$,

$$\|v - \Pi_h v\|_{L^2(K)} \leq Ch_K \|\nabla v\|_{L^2(\tilde{K})}, \quad \|v - \Pi_h v\|_{L^2(e)} \leq Ch_e^{1/2} \|\nabla v\|_{L^2(\tilde{e})},$$

where \tilde{K} and \tilde{e} are the union of all elements in \mathcal{M}_h having nonempty intersection with $K \in \mathcal{M}_h$ and the side e , respectively.

The following theorem presents the a posteriori error estimate and is the main result for the TM polarization.

Theorem 3.4. *Let u and u_h be the solutions of (3.6) and (3.23), respectively. There exists a constant C depending only on the minimum angle of the mesh \mathcal{M}_h such that the following a posterior error estimate holds:*

$$\begin{aligned}
 \|u - u_h\|_{H^1(\Omega)} &\leq C \hat{C}^{-1} (1 + \kappa R) \left(\sum_{K \in \mathcal{M}_h} \eta_K^2 \right)^{1/2} \\
 & \quad + C \hat{C}^{-1} (1 + \kappa_0 R)^2 |\alpha_0|^2 e^{-\kappa_0 \Im(\hat{\rho}) (1 - \frac{R^2}{|\hat{\rho}|^2})^{1/2}} \|u_h - u^{\text{ref}}\|_{H_{\text{TM}}^{1/2}(\Gamma_R^+)}.
 \end{aligned}$$

Proof. Taking $\varphi_h = \Pi_h \varphi$ and using Lemma 3.5, we have

$$\begin{aligned}
 a_{\text{TM}}(u - u_h, \varphi) &= \langle \mathcal{B}_{\text{TM}}(u_h - u^{\text{ref}}) - \hat{\mathcal{B}}_{\text{TM}}(u_h - u^{\text{ref}}), \varphi \rangle_{\Gamma_R^+} - b(u_h, \varphi - \Pi_h \varphi) \\
 & \quad - \int_{\Omega^{\text{PML}}} (\nabla \cdot (A \nabla u^{\text{ref}}) + \kappa_0^2 \alpha \beta u^{\text{ref}}) (\bar{\varphi} - \Pi_h \bar{\varphi}) dx \\
 & := I_1 + I_2 + I_3.
 \end{aligned}$$

It follows from Lemma 3.1 that

$$\begin{aligned} I_1 &= \langle \mathcal{B}_{\text{TM}}(u_h - u^{\text{ref}}) - \hat{\mathcal{B}}_{\text{TM}}(u_h - u^{\text{ref}}), \varphi \rangle_{\Gamma_R^+} \\ &\leq C \hat{C}^{-1} (1 + \kappa_0 R)^2 |\alpha_0|^2 e^{-\kappa_0 \Im(\hat{\rho}) \left(1 - \frac{R^2}{|\hat{\rho}|^2}\right)^{1/2}} \|u_h - u^{\text{ref}}\|_{H_{\text{TM}}^{1/2}(\Gamma_R^+)} \|\varphi\|_{H_{\text{TM}}^{1/2}(\Gamma_R^+)}. \end{aligned}$$

Using the integration by parts yields

$$\begin{aligned} I_2 + I_3 &= \sum_{K \in \mathcal{M}_h \cap \Omega} \left(\int_K R_K(u_h - u^{\text{ref}}) (\bar{\varphi} - \Pi_h \bar{\varphi}) dx + \sum_{e \in \partial K \cap \mathcal{B}_h} \frac{1}{2} \int_e J_e (\bar{\varphi} - \Pi_h \bar{\varphi}) ds \right) \\ &\quad + \sum_{K \in \mathcal{M}_h \cap \Omega^{\text{PML}}} \left(\int_K R_K(u_h) (\bar{\varphi} - \Pi_h \bar{\varphi}) dx + \sum_{e \in \partial K \cap \mathcal{B}_h} \frac{1}{2} \int_e J_e (\bar{\varphi} - \Pi_h \bar{\varphi}) ds \right). \end{aligned}$$

It follows from the Cauchy–Schwarz inequality, the interpolation estimates and Lemma 3.3 that

$$\begin{aligned} |I_2 + I_3| &\leq C \sum_{K \in \mathcal{M}_h} \left(\|h_K \tilde{R}_K\|_{L^2(K)}^2 + \frac{1}{2} \sum_{e \in \partial K \cap \mathcal{B}_h} \|h_e^{1/2} J_e\|_{L^2(e)}^2 \right)^{1/2} \|\nabla \varphi\|_{L^2(\bar{K})} \\ &\leq C \sum_{K \in \mathcal{M}_h} \eta_K \|w^{-1} \nabla \varphi\|_{L^2(\bar{K})} \\ &\leq C \hat{C}^{-1} (1 + \kappa R) \left(\sum_{K \in \mathcal{M}_h} \eta_K^2 \right)^{1/2} \|\varphi\|_{H^{1/2}(\Gamma_R^+)}. \end{aligned}$$

Using the inf-sup condition (3.8) and combining the above estimates, we get

$$\begin{aligned} \|u - u_h\|_{H^1(\Omega)} &\leq C \sup_{0 \neq \varphi \in H_S^1(\Omega)} \frac{|a_{\text{TM}}(u - u_h, \varphi)|}{\|\varphi\|_{H^1(\Omega)}} \\ &\leq C \hat{C}^{-1} (1 + \kappa R) \left(\sum_{K \in \mathcal{M}_h} \eta_K^2 \right)^{1/2} \\ &\quad + C \hat{C}^{-1} (1 + \kappa_0 R)^2 |\alpha_0|^2 e^{-\kappa_0 \Im(\hat{\rho}) \left(1 - \frac{R^2}{|\hat{\rho}|^2}\right)^{1/2}} \|u_h - u^{\text{ref}}\|_{H_{\text{TM}}^{1/2}(\Gamma_R^+)}, \end{aligned}$$

which completes the proof. □

3.6 Adaptive FEM algorithm

It can be seen from the Theorem 3.4 that the a posteriori error estimate consists of two parts: the finite element approximation error ε_h and the truncation error of the PML method ε_{PML} , where

$$\varepsilon_h = \left(\sum_{K \in \mathcal{M}_h} \eta_K^2 \right)^{1/2}, \quad \varepsilon_{\text{PML}} = e^{-\kappa_0 \Im(\hat{\rho}) \left(1 - \frac{R^2}{|\hat{\rho}|^2}\right)^{1/2}} \|u_h - u^{\text{ref}}\|_{H_{\text{TM}}^{1/2}(\Gamma_R^+)}.$$

Table 1: The adaptive finite element PML method for TM polarization.

- (1) Given the tolerance $\varepsilon > 0$ and the parameter $\tau \in (0, 1)$.
- (2) Choose σ_0 and ρ such that $\varepsilon_{\text{PML}} \leq 10^{-8}$.
- (3) Construct an initial triangulation \mathcal{M}_h over Ω_ρ and compute error estimators.
- (4) While $\varepsilon_h > \varepsilon$ do
- (5) refine \mathcal{M}_h according to the strategy
- (6) if $\eta_{\hat{K}} > \tau \max_{K \in \mathcal{M}_h} \eta_K$, refine the element $\hat{K} \in \mathcal{M}_h$;
- (7) obtain a new mesh denoted still by \mathcal{M}_h ;
- (8) solve (3.23) on the new mesh \mathcal{M}_h and compute the error estimators.
- (9) End while.

In the implementation, we may first choose σ_0 and ρ to make sure that the PML error ε_{PML} is small enough, for instance $\varepsilon_{\text{PML}} \leq 10^{-8}$, such that the PML error is negligible compared with the finite element approximation error. Next we design the adaptive strategy to modify the mesh according to the estimate ε_h . Table 1 shows the algorithm of the adaptive finite element PML method for solving the open cavity scattering problem in the TM polarization.

4 TE polarization

In this section, we consider the TE polarization. Since the discussions are similar to the TM polarization, we briefly present the parallel results without providing the details.

4.1 Variational problem

It can be verified from (2.2) that the scattered field u^s satisfies the Helmholtz equation

$$\Delta u^s + \kappa_0^2 u^s = 0 \quad \text{in } \mathbb{R}_+^2 \setminus \overline{B_R^+}. \tag{4.1}$$

By the radiation condition (2.3), the solution of (4.1) has the Fourier series expansion

$$u^s(r, \phi) = \sum_{n=0}^{\infty} \frac{H_n^{(1)}(\kappa_0 r)}{H_n^{(1)}(\kappa_0 R)} (a_n \sin(n\phi) + b_n \cos(n\phi)), \quad r \geq R. \tag{4.2}$$

Using the fact $\partial_\nu u = 0$ and $\partial_\nu u^{\text{ref}} = 0$ on Γ_g , we have $\partial_\phi u^s(r, 0) = \partial_\phi u^s(r, \pi) = 0$. Hence $a_n = 0$ and (4.2) reduces to

$$u^s(r, \phi) = \sum_{n=0}^{\infty} \frac{H_n^{(1)}(\kappa_0 r)}{H_n^{(1)}(\kappa_0 R)} b_n \cos(n\phi), \quad r \geq R, \tag{4.3}$$

which gives

$$\partial_r u^s(R, \phi) = \kappa_0 \sum_{n=0}^{\infty} \frac{H_n^{(1)' }(\kappa_0 R)}{H_n^{(1)}(\kappa_0 R)} b_n \cos(n\phi).$$

Let $L^2_{TE}(\Gamma_R^+) := \{u \in L^2(\Gamma_R^+) : \partial_\phi u(R, 0) = \partial_\phi u(R, \pi) = 0\}$. For any $u \in L^2_{TE}(\Gamma_R^+)$, it has the Fourier series expansion

$$u(R, \phi) = \sum_{n=0}^{\infty} b_n \cos(n\phi),$$

where

$$b_0 = \frac{1}{\pi} \int_0^\pi u(R, \phi) d\phi, \quad b_n = \frac{2}{\pi} \int_0^\pi u(R, \phi) \cos(n\phi) d\phi.$$

Define the trace function space $H^s_{TE}(\Gamma_R^+) := \{u \in L^2_{TE}(\Gamma_R^+) : \|u\|_{H^s_{TE}(\Gamma_R^+)} \leq \infty\}$, where the $H^s_{TE}(\Gamma_R^+)$ norm is given by

$$\|u\|_{H^s_{TE}(\Gamma_R^+)} = \left(\sum_{n=0}^{\infty} (1+n^2)^s |b_n|^2 \right)^{1/2}.$$

It is clear that the dual space of $H^s_{TE}(\Gamma_R^+)$ is $H^{-s}_{TE}(\Gamma_R^+)$ with respect to the scalar product in $L^2(\Gamma_R^+)$ given by

$$\langle u, v \rangle_{\Gamma_R^+} = \int_{\Gamma_R^+} u \bar{v} ds.$$

We introduce a DtN operator on Γ_R^+ :

$$(\mathcal{B}_{TE}u)(R, \phi) = \kappa_0 \sum_{n=0}^{\infty} \frac{H_n^{(1)' }(\kappa_0 R)}{H_n^{(1)}(\kappa_0 R)} b_n \cos(n\phi). \tag{4.4}$$

It is shown [27, Lemma 3.1] that the DtN operator $\mathcal{B}_{TE} : H^{1/2}_{TE}(\Gamma_R^+) \rightarrow H^{-1/2}_{TE}(\Gamma_R^+)$ is continuous. Using the boundary operator (4.4), we obtain the transparent boundary condition for the TE polarization:

$$\partial_r u^s = \mathcal{B}_{TE}u^s \quad \text{on } \Gamma_R^+,$$

which can be equivalently written for the total field u :

$$\partial_r u = \mathcal{B}_{TE}u + g \quad \text{on } \Gamma_R^+,$$

where $g = \partial_r u^{\text{ref}} - \mathcal{B}_{TE}u^{\text{ref}}$.

In the TE polarization, the open cavity scattering problem can be reduced to the following boundary value problem:

$$\begin{cases} \nabla \cdot (\kappa^{-2} \nabla u) + u = 0 & \text{in } \Omega, \\ \partial_\nu u = 0 & \text{on } \Gamma_g \cup S, \\ \partial_r u = \mathcal{B}_{TE}u + g & \text{on } \Gamma_R^+, \end{cases}$$

which has the variational formulation: find $u \in H^1(\Omega)$ such that

$$a_{TE}(u, v) = \langle \kappa_0^{-2} g, v \rangle_{\Gamma_R^+} \quad \forall v \in H^1(\Omega). \tag{4.5}$$

Here the sesquilinear form $a_{TE}(\cdot, \cdot) : H^1(\Omega) \times H^1(\Omega) \rightarrow \mathbb{C}$ is given by

$$a_{TE}(u, v) = \int_{\Omega} (\kappa^{-2} \nabla u \cdot \nabla \bar{v} - u \bar{v}) \, dx - \langle \kappa_0^{-2} \mathcal{B}_{TE} u, v \rangle_{\Gamma_R^+}.$$

The following theorem concerns the well-posedness for the variational problem (4.5) and the proof can be found in [20].

Theorem 4.1. *The variational problem (4.5) has a unique weak solution in $H^1(\Omega)$, which satisfies the estimate*

$$\|u\|_{H^1(\Omega)} \lesssim \|g\|_{H_{TE}^{-1/2}(\Gamma_R^+)}.$$

The general theory in Babuška and Aziz [3, Chapter 5] implies that there exists a constant $C > 0$ such that the following inf-sup condition holds:

$$\sup_{0 \neq v \in H^1(\Omega)} \frac{|a_{TE}(u, v)|}{\|v\|_{H^1(\Omega)}} \geq C \|u\|_{H^1(\Omega)} \quad \forall u \in H^1(\Omega). \tag{4.6}$$

4.2 The PML problem

Using the complex coordinate stretching (3.9), we may similarly obtain the truncated PML problem in the TE polarization:

$$\begin{cases} \nabla \cdot (\kappa^{-2} A \nabla u^{PML}) + \alpha \beta u^{PML} = G & \text{in } \Omega_{\rho}, \\ (A \nabla u^{PML}) \cdot \nu = 0 & \text{on } \Gamma_g \cup S, \\ u^{PML} = u^{ref} & \text{on } \Gamma_{\rho}^+, \end{cases} \tag{4.7}$$

where

$$G = \begin{cases} \nabla \cdot (\kappa_0^{-2} A \nabla u^{ref}) + \alpha \beta u^{ref} & \text{in } \Omega^{PML}, \\ 0 & \text{otherwise.} \end{cases}$$

A DtN operator $\hat{\mathcal{B}}_{TE} : H_{TE}^{1/2}(\Gamma_R^+) \rightarrow H_{TE}^{-1/2}(\Gamma_R^+)$ is defined as follows: given $f \in H_{TE}^{1/2}(\Gamma_R^+)$,

$$\hat{\mathcal{B}}_{TE} f = \partial_r \xi|_{\Gamma_R^+},$$

where $\xi \in H^1(\Omega^{PML})$ satisfies

$$\begin{cases} \nabla \cdot (\kappa^{-2} A \nabla \xi) + \alpha \beta \xi = 0 & \text{in } \Omega^{PML}, \\ \xi = f & \text{on } \Gamma_R^+, \\ \xi = 0 & \text{on } \Gamma_{\rho}^+, \\ (A \nabla \xi) \cdot \nu = 0 & \text{on } \Gamma_g. \end{cases}$$

By imposing the boundary condition

$$\partial_r(u^{\text{PML}} - u^{\text{ref}}) = \hat{\mathcal{B}}_{\text{TE}}(u^{\text{PML}} - u^{\text{ref}}) \quad \text{on } \Gamma_R^+,$$

the problem (4.7) can be reformulated as

$$\begin{cases} \nabla \cdot (\kappa^{-2} A \nabla u^{\text{PML}}) + \alpha \beta u^{\text{PML}} = 0 & \text{in } \Omega, \\ (A \nabla u^{\text{PML}}) \cdot \nu = 0 & \text{on } \Gamma_g \cup S, \\ \partial_r u^{\text{PML}} = \hat{\mathcal{B}}_{\text{TE}} u^{\text{PML}} + \hat{g} & \text{on } \Gamma_R^+, \end{cases} \quad (4.8)$$

where $\hat{g} = \partial_r u^{\text{ref}} - \hat{\mathcal{B}}_{\text{TE}} u^{\text{ref}}$. The weak formulation of the problem (4.8) is to find $u^{\text{PML}} \in H^1(\Omega)$ such that

$$\hat{a}_{\text{TE}}(u^{\text{PML}}, v) = \langle \kappa_0^{-2} \hat{g}, v \rangle_{\Gamma_R^+} \quad \forall v \in H^1(\Omega), \quad (4.9)$$

where the sesquilinear form $\hat{a}_{\text{TE}}(\cdot, \cdot) : H^1(\Omega) \times H^1(\Omega) \rightarrow \mathbb{C}$ is defined as

$$\hat{a}_{\text{TE}}(u, v) = \int_{\Omega} (\kappa^{-2} A \nabla u \cdot \nabla \bar{v} - \alpha \beta u \bar{v}) \, dx - \langle \kappa_0^{-2} \hat{\mathcal{B}}_{\text{TE}} u, v \rangle_{\Gamma_R^+}.$$

4.3 Convergence of the PML problem

Consider a Dirichlet boundary value problem of the PML equation in the PML layer Ω^{PML} :

$$\begin{cases} \nabla \cdot (\kappa_0^{-2} A \nabla w) + \alpha \beta w = 0 & \text{in } \Omega^{\text{PML}}, \\ w = 0 & \text{on } \Gamma_R^+, \\ w = q & \text{on } \Gamma_{\rho}^+, \\ (A \nabla w) \cdot \nu = 0 & \text{on } \Gamma_g, \end{cases} \quad (4.10)$$

where $q \in H_{\text{TE}}^{1/2}(\Gamma_{\rho}^+)$.

Define $H_{R\rho}^1(\Omega^{\text{PML}}) = \{v \in H^1(\Omega^{\text{PML}}) : v = 0 \text{ on } \Gamma_R^+ \text{ and } \Gamma_{\rho}^+\}$. The weak formulation of (4.10) reads as follows: given $q \in H_{\text{TE}}^{1/2}(\Gamma_{\rho}^+)$, find $w \in H^1(\Omega^{\text{PML}})$ such that $w = 0$ on Γ_R^+ , $w = q$ on Γ_{ρ}^+ and

$$\hat{b}(w, v) = 0 \quad \forall v \in H_{R\rho}^1(\Omega^{\text{PML}}), \quad (4.11)$$

where

$$\hat{b}(u, v) = \int_R^{\rho} \int_0^{\pi} \left(\kappa_0^{-2} \left(\frac{\beta r}{\alpha} \partial_r u \partial_r \bar{v} + \frac{\alpha}{\beta r} \partial_{\phi} u \partial_{\phi} \bar{v} \right) - \alpha \beta r u \bar{v} \right) \, dr d\phi.$$

Here we also assume that the PML problem (4.11) admits a unique weak solution.

For any $u \in H^1(\Omega^{\text{PML}})$, define

$$\|u\|_{*, \Omega^{\text{PML}}} = \left[\int_R^{\rho} \int_0^{\pi} \left(\left(\frac{1 + \sigma \hat{\sigma}}{1 + \sigma^2} \right) r |\partial_r u|^2 + \left(\frac{1 + \sigma \hat{\sigma}}{1 + \hat{\sigma}^2} \right) \frac{1}{r} |\partial_{\phi} u|^2 + (1 + \sigma \hat{\sigma}) \kappa_0^2 r |u|^2 \right) \, dr d\phi \right]^{1/2}.$$

It is easy to see that $\|\cdot\|_{*,\Omega^{\text{PML}}}$ is an equivalent norm on $H^1(\Omega^{\text{PML}})$. By using the general theory in [3, Chapter 5], there exists a positive constant \hat{C} such that

$$\sup_{0 \neq v \in H_0^1(\Omega^{\text{PML}})} \frac{|\hat{b}(u,v)|}{\|v\|_{*,\Omega^{\text{PML}}}} \geq \hat{C} \|u\|_{*,\Omega^{\text{PML}}} \quad \forall u \in H_{R\rho}^1(\Omega^{\text{PML}}).$$

The constant \hat{C} depends on the domain Ω^{PML} and the wave number κ_0 .

The following results concern the estimates of the solution for the boundary value problem (4.10) and are crucial for the convergence analysis. The proof is essentially the same as that in [12, Theorem 2.4] and is omitted for brevity.

Theorem 4.2. *There exists a constant $C > 0$ independent of κ_0, R, ρ , and σ_0 such that the following estimates are satisfied:*

$$\|\alpha_0^{-1} \nabla w\|_{L^2(\Omega^{\text{PML}})} \leq C \hat{C}^{-1} \kappa_0^{-2} (1 + \kappa_0 R) |\alpha_0| \|q\|_{H_{\text{TE}}^{-1/2}(\Gamma_\rho^+)}, \tag{4.12}$$

$$\|\partial_r w\|_{H_{\text{TM}}^{-1/2}(\Gamma_R^+)} \leq C \hat{C}^{-1} \kappa_0^{-2} (1 + \kappa_0 R)^2 |\alpha_0|^2 \|q\|_{H_{\text{TE}}^{-1/2}(\Gamma_\rho^+)}, \tag{4.13}$$

where $\alpha_0 = 1 + i\sigma_0$.

Similarly, for any function $f \in H_{\text{TE}}^{1/2}(\Gamma_R^+)$, we introduce the propagation operator $\mathcal{P}_{\text{TE}}: H_{\text{TE}}^{1/2}(\Gamma_R^+) \rightarrow H_{\text{TE}}^{1/2}(\Gamma_\rho^+)$ as follows:

$$\mathcal{P}_{\text{TE}}(f) = \sum_{n=0}^{\infty} \frac{H_n^{(1)}(\kappa_0 \tilde{\rho})}{H_n^{(1)}(\kappa_0 R)} f_n \cos(n\phi),$$

where

$$f_0 = \frac{1}{\pi} \int_0^\pi f(R, \phi) d\phi, \quad f_n = \frac{2}{\pi} \int_0^\pi f(R, \phi) \cos(n\phi) d\phi.$$

As discussed in [12], the operator $\mathcal{P}_{\text{TE}}: H_{\text{TE}}^{1/2}(\Gamma_R^+) \rightarrow H_{\text{TE}}^{1/2}(\Gamma_\rho^+)$ is well defined and satisfies the estimate

$$\|\mathcal{P}_{\text{TE}}(f)\|_{H_{\text{TE}}^{1/2}(\Gamma_\rho^+)} \leq e^{-\kappa_0 \Im(\tilde{\rho})} \left(1 - \frac{R^2}{|\tilde{\rho}|^2}\right)^{1/2} \|f\|_{H_{\text{TE}}^{1/2}(\Gamma_R^+)} \quad \forall \rho \geq R. \tag{4.14}$$

Lemma 4.1. *For any $f \in H_{\text{TE}}^{1/2}(\Gamma_R^+)$, the following estimate holds:*

$$\|(\mathcal{B}_{\text{TE}} - \hat{\mathcal{B}}_{\text{TE}})f\|_{H_{\text{TE}}^{1/2}(\Gamma_R^+)} \leq C \hat{C}^{-1} \kappa_0^{-2} (1 + \kappa_0 R)^2 |\alpha_0|^2 e^{-\kappa_0 \Im(\tilde{\rho})} \left(1 - \frac{R^2}{|\tilde{\rho}|^2}\right)^{1/2} \|f\|_{H_{\text{TE}}^{1/2}(\Gamma_R^+)}. \tag{4.15}$$

Proof. For any $f \in H_{\text{TE}}^{1/2}(\Gamma_R^+)$, we have

$$(\mathcal{B}_{\text{TE}} - \hat{\mathcal{B}}_{\text{TE}})f = \partial_r w|_{\Gamma_R^+}, \tag{4.16}$$

where $w \in H^1(\Omega^{\text{PML}})$ satisfies

$$\begin{cases} \nabla \cdot (\kappa_0^{-2} A \nabla w) + \alpha \beta w = 0 & \text{in } \Omega^{\text{PML}}, \\ w = 0 & \text{on } \Gamma_R^+, \\ w = \mathcal{P}_{\text{TE}}(f) & \text{on } \Gamma_\rho^+, \\ (A \nabla w) \cdot \nu = 0 & \text{on } \Gamma_g. \end{cases}$$

It follows from (4.13)-(4.14) that

$$\begin{aligned} \|\partial_r w\|_{H_{\text{TM}}^{-1/2}(\Gamma_R^+)} &\leq C \hat{C}^{-1} \kappa_0^{-2} (1 + \kappa_0 R)^2 |\alpha_0|^2 \|\mathcal{P}_{\text{TE}}(f)\|_{H_{\text{TM}}^{1/2}(\Gamma_\rho^+)} \\ &\leq C \hat{C}^{-1} \kappa_0^{-2} (1 + \kappa_0 R)^2 |\alpha_0|^2 e^{-\kappa_0 \Im(\hat{\rho}) \left(1 - \frac{R^2}{|\hat{\rho}|^2}\right)^{1/2}} \|f\|_{H_{\text{TM}}^{1/2}(\Gamma_R^+)}, \end{aligned}$$

which completes the proof. □

Below is the main result of this subsection.

Theorem 4.3. *For sufficiently large $\sigma_0 > 0$, the PML problem (4.8) has a unique solution $u^{\text{PML}} \in H^1(\Omega_\rho)$. Moreover, the following estimate holds:*

$$\|u - u^{\text{PML}}\|_{H^1(\Omega)} \leq C \hat{C}^{-1} (\kappa_0^{-2} + \kappa_0^{-1} R)^2 |\alpha_0|^2 e^{-\kappa_0 \Im(\hat{\rho}) \left(1 - \frac{R^2}{|\hat{\rho}|^2}\right)^{1/2}} \|u^{\text{PML}} - u^{\text{ref}}\|_{H_{\text{TM}}^{1/2}(\Gamma_R^+)}.$$

Proof. By (4.5) and (4.9), for any $\varphi \in H^1(\Omega)$, we have

$$\begin{aligned} a_{\text{TE}}(u - u^{\text{PML}}, \varphi) &= a_{\text{TE}}(u, \varphi) - a_{\text{TE}}(u^{\text{PML}}, \varphi) \\ &= \langle \kappa_0^{-2} g, \varphi \rangle_{\Gamma_R^+} - a_{\text{TE}}(u^{\text{PML}}, \varphi) \\ &= \langle \kappa_0^{-2} (g - \hat{g}), \varphi \rangle_{\Gamma_R^+} + \langle \kappa_0^{-2} \hat{g}, \varphi \rangle_{\Gamma_R^+} - a(u^{\text{PML}}, \varphi) \\ &= \langle \kappa_0^{-2} (\hat{\mathcal{B}}_{\text{TE}} - \mathcal{B}_{\text{TE}}) u^{\text{ref}}, \varphi \rangle_{\Gamma_R^+} + \hat{a}_{\text{TE}}(u^{\text{PML}}, \varphi) - a_{\text{TE}}(u^{\text{PML}}, \varphi) \\ &= \langle \kappa_0^{-2} (\mathcal{B}_{\text{TE}} - \hat{\mathcal{B}}_{\text{TE}}) (u^{\text{PML}} - u^{\text{ref}}), \varphi \rangle_{\Gamma_R^+}, \end{aligned}$$

which implies the desired estimate by using Lemma 4.1 and (4.6). □

4.4 Finite element approximation

Let $b(\cdot, \cdot) : H^1(\Omega_\rho) \times H^1(\Omega_\rho) \rightarrow \mathbb{C}$ be the sesquilinear form given by

$$b(u, v) = \int_{\Omega_\rho} (\kappa^{-2} A \nabla u \cdot \nabla \bar{v} - \alpha \beta u \bar{v}) dx.$$

Define $H_\rho^1(\Omega_\rho) = \{u \in H^1(\Omega_\rho) : u = 0 \text{ on } \Gamma_\rho^+\}$. The the weak formulation of (4.7) is to find $u^{\text{PML}} \in H^1(\Omega_\rho)$ and $u^{\text{PML}} = u^{\text{ref}}$ on Γ_ρ^+ such that

$$b(u^{\text{PML}}, v) = - \int_{\Omega_\rho} G \bar{v} dx \quad \forall v \in H_\rho^1(\Omega_\rho). \tag{4.17}$$

Let V_h be a conforming finite element space of $H^1(\Omega_\rho)$, i.e.,

$$V_h = \{v_h \in C(\bar{\Omega}_\rho) : v_h|_K \in P_m(K), \forall K \in \mathcal{M}_h\}.$$

Denote

$$V_{\rho,h} = \{v_h \in V_h : v_h = 0 \text{ on } \Gamma_\rho^+\}.$$

The finite element approximation to the variational problem (4.17) is to find $u_h \in V_h$ with $u_h = u^{\text{ref}}$ on Γ_ρ^+ such that

$$b(u_h, v_h) = - \int_{\Omega_\rho} G \bar{v}_h dx \quad \forall v_h \in V_{\rho,h}. \tag{4.18}$$

4.5 A posteriori error analysis

For any $K \in \mathcal{M}_h$, we introduce the residual

$$R_K(u) := \nabla \cdot (\kappa^{-2} A \nabla u|_K) + \alpha \beta u|_K.$$

Let \mathcal{B}_h denote the set of all the edges that do not lie on $\partial\Gamma_\rho^+$. For any interior edge e , which is the common side of triangular elements $K_1, K_2 \in \mathcal{M}_h$, we define the jump residual across e as

$$J_e := -(\kappa^{-2} A \nabla u_h|_{K_1} \cdot \nu_1 + \kappa^{-2} A \nabla u_h|_{K_2} \cdot \nu_2),$$

where ν_j is the unit outward normal vector on the boundary of $K_j, j=1,2$. If $e = \partial K \cap (\Gamma_g \cup S)$ for some $K \in \mathcal{M}_h$, then define the jump residual

$$J_e := 2(\kappa^{-2} A \nabla u_h|_K \cdot \nu).$$

Let

$$\tilde{R}_K := \begin{cases} R_K(u_h) & \text{if } K \in \mathcal{M}_h \cap \Omega, \\ R_K(u_h - u^{\text{ref}}) & \text{if } K \in \mathcal{M}_h \cap \Omega^{\text{PML}}. \end{cases}$$

For any triangle $K \in \mathcal{M}_h$, denote by η_K the local error estimator as follows:

$$\eta_K = \max_{x \in \bar{K}} w(x) \left(\|h_K \tilde{R}_K\|_{L^2(K)}^2 + \frac{1}{2} \sum_{e \in K} \|h_e^{1/2} J_e\|_{L^2(e)}^2 \right)^{1/2},$$

where

$$w(x) = \begin{cases} 1 & \text{if } x \in \bar{\Omega}, \\ |\frac{\alpha}{\alpha_0}| e^{-\kappa_0 \Im \tilde{r} (1 - \frac{r^2}{|\tilde{r}|^2})^{1/2}} & \text{if } x \in \Omega^{\text{PML}}. \end{cases}$$

For any $\varphi \in H^1(\Omega_R)$, let $\tilde{\varphi}$ be its extension in Ω^{PML} such that

$$\begin{cases} \nabla \cdot (\kappa_0^{-2} \bar{A} \nabla \tilde{\varphi}) + \bar{\alpha} \bar{\beta} \tilde{\varphi} = 0 & \text{in } \Omega^{\text{PML}}, \\ \tilde{\varphi} = \varphi & \text{on } \Gamma_R^+, \\ \tilde{\varphi} = 0 & \text{on } \Gamma_\rho^+, \\ (\bar{A} \nabla \tilde{\varphi}) \cdot \nu = 0 & \text{on } \Gamma_g. \end{cases} \tag{4.19}$$

The proofs of Lemmas 4.2 and 4.3 are essentially the same as those in [12, Lemmas 4.1 and 4.4], while the proof of Lemma 4.4 is also similar to Lemma 3.4.

Lemma 4.2. For any $\varphi \in H^1(\Omega)$, let $\tilde{\varphi}$ be its extension in $H^1(\Omega^{\text{PML}})$ according to (4.19). Then there exists a constant $C > 0$ independent of κ_0, R, ρ and σ_0 such that

$$\| |\alpha|^{-1} \gamma \nabla \tilde{\varphi} \|_{L^2(\Omega^{\text{PML}})} \leq C \hat{C}^{-1} \kappa_0^{-2} (1 + \kappa_0 R) |\alpha_0| \| \varphi \|_{H^{1/2}(\Gamma_R^+)},$$

where $\gamma(r) = e^{\kappa_0 \Im \tilde{r} (1 - \frac{r^2}{|\tilde{r}|^2})^{1/2}}$.

Lemma 4.3. For any $\varphi, \psi \in H^1(\Omega^{\text{PML}})$, we have

$$\langle \hat{\mathcal{B}}_{\text{TE}} \varphi, \psi \rangle = \langle \hat{\mathcal{B}}_{\text{TE}} \tilde{\varphi}, \tilde{\psi} \rangle.$$

Lemma 4.4. For any $\varphi \in H^1(\Omega)$, let $\tilde{\varphi}$ be its extension in $H^1(\Omega_\rho)$ according to (4.19). For $\xi \in H_0^1(\Omega_\rho)$, the following identity holds:

$$\int_{\Omega^{\text{PML}}} (\kappa_0^{-2} A \nabla \xi \cdot \nabla \tilde{\varphi} - \alpha \beta \xi \tilde{\varphi}) \, dx = - \langle \kappa_0^{-2} \hat{\mathcal{B}}_{\text{TE}} \xi, \varphi \rangle_{\Gamma_R^+}.$$

The following result present the error representation formula for the TE polarization.

Lemma 4.5 (error representation formula). For any $\varphi \in H^1(\Omega)$, let $\tilde{\varphi}$ be its extension in $H^1(\Omega_\rho)$ according to (4.19). The for any $\varphi_h \in V_h$, the following identity holds:

$$\begin{aligned} a_{\text{TE}}(u - u_h, \varphi) &= \langle \kappa^{-2} \mathcal{B}_{\text{TE}}(u_h - u^{\text{ref}}) - \kappa^{-2} \hat{\mathcal{B}}_{\text{TE}}(u_h - u^{\text{ref}}), \varphi \rangle_{\Gamma_R^+} - b(u_h, \varphi - \varphi_h) \\ &\quad - \int_{\Omega^{\text{PML}}} (\nabla \cdot (\kappa_0^{-2} A \nabla u^{\text{ref}}) + \alpha \beta u^{\text{ref}}) (\tilde{\varphi} - \tilde{\varphi}_h) \, dx. \end{aligned}$$

Proof. It follows from (4.5) that

$$\begin{aligned} a_{\text{TE}}(u - u_h, \varphi) &= a_{\text{TE}}(u, \varphi) - a_{\text{TE}}(u_h, \varphi) \\ &= \langle \kappa^{-2} g, \varphi \rangle_{\Gamma_R^+} - b(u_h, \varphi - \varphi_h) \\ &\quad + b(u_h, \varphi) - b(u_h, \varphi_h) - a_{\text{TE}}(u_h, \varphi). \end{aligned} \tag{4.20}$$

By the definition of the sesquilinear form b , we have

$$\begin{aligned} b(u_h, \varphi) &= \int_{\Omega_\rho} (\kappa^{-2} A \nabla u_h \cdot \nabla \tilde{\varphi} - \alpha \beta u_h \tilde{\varphi}) \, dx \\ &= \int_{\Omega} (\kappa^{-2} A \nabla u_h \cdot \nabla \tilde{\varphi} - \alpha \beta u_h \tilde{\varphi}) \, dx + \int_{\Omega^{\text{PML}}} (\kappa_0^{-2} A \nabla u_h \cdot \nabla \tilde{\varphi} - \alpha \beta u_h \tilde{\varphi}) \, dx. \end{aligned}$$

We also get from the definition of the sesquilinear form a_{TE} that

$$a_{\text{TE}}(u_h, \varphi) = \int_{\Omega} (\kappa^{-2} A \nabla u_h \cdot \nabla \tilde{\varphi} - \alpha \beta u_h \tilde{\varphi}) \, dx - \langle \kappa_0^{-2} \mathcal{B}_{\text{TE}} u_h, \varphi \rangle_{\Gamma_R^+}.$$

Using (4.17) and the integration by parts yields

$$\begin{aligned}
 b(u_h, \varphi_h) &= - \int_{\Omega^{\text{PML}}} G \bar{\varphi}_h dx \\
 &= - \int_{\Omega^{\text{PML}}} (\nabla \cdot (\kappa_0^{-2} A \nabla u^{\text{ref}}) + \alpha \beta u^{\text{ref}}) \bar{\varphi}_h dx \\
 &= \int_{\Omega^{\text{PML}}} (\nabla \cdot (\kappa_0^{-2} A \nabla u^{\text{ref}}) + \alpha \beta u^{\text{ref}}) (\bar{\varphi} - \bar{\varphi}_h) dx \\
 &\quad + \int_{\Omega^{\text{PML}}} (\kappa_0^{-2} A \nabla u^{\text{ref}} \cdot \nabla \bar{\varphi} - \alpha \beta u^{\text{ref}} \bar{\varphi}) dx \\
 &\quad + \int_{\Gamma_R^+} \kappa_0^{-2} \partial_\nu u^{\text{ref}} \bar{\varphi} ds.
 \end{aligned}$$

Combining the above equations leads to

$$\begin{aligned}
 &b(u_h, \varphi) - b(u_h, \varphi_h) - a_{\text{TE}}(u_h, \varphi) \\
 &= - \int_{\Omega^{\text{PML}}} (\nabla \cdot (\kappa_0^{-2} A \nabla u^{\text{ref}}) + \alpha \beta u^{\text{ref}}) (\bar{\varphi} - \bar{\varphi}_h) dx \\
 &\quad + \int_{\Omega^{\text{PML}}} (\kappa_0^{-2} A \nabla (u_h - u^{\text{ref}}) \cdot \nabla \bar{\varphi} - \alpha \beta (u_h - u^{\text{ref}}) \bar{\varphi}) dx \\
 &\quad - \int_{\Gamma_R^+} \kappa_0^{-2} \partial_\nu u^{\text{ref}} \bar{\varphi} ds + \langle \kappa_0^{-2} \mathcal{B}_{\text{TE}} u_h, \varphi \rangle_{\Gamma_R^+}.
 \end{aligned}$$

By Lemma 4.4, we have

$$\begin{aligned}
 &b(u_h, \varphi) - b(u_h, \varphi_h) - a_{\text{TE}}(u_h, \varphi) \\
 &= - \int_{\Omega^{\text{PML}}} (\nabla \cdot (\kappa_0^{-2} A \nabla u^{\text{ref}}) + \alpha \beta u^{\text{ref}}) (\bar{\varphi} - \bar{\varphi}_h) dx \\
 &\quad + \langle \kappa_0^{-2} (\mathcal{B}_{\text{TE}} - \hat{\mathcal{B}}_{\text{TE}}) u_h, \varphi \rangle_{\Gamma_R^+} - \langle \kappa_0^{-2} (\partial_\nu u^{\text{ref}} - \hat{\mathcal{B}}_{\text{TE}} u^{\text{ref}}), \varphi \rangle_{\Gamma_R^+}. \tag{4.21}
 \end{aligned}$$

Substituting (4.21) into (4.20), we obtain

$$\begin{aligned}
 a_{\text{TE}}(u - u_h, \varphi) &= \langle \kappa_0^{-2} \mathcal{B}_{\text{TE}}(u_h - u^{\text{ref}}) - \kappa_0^{-2} \hat{\mathcal{B}}_{\text{TE}}(u_h - u^{\text{ref}}), \varphi \rangle_{\Gamma_R^+} - b(u_h, \varphi - \varphi_h) \\
 &\quad - \int_{\Omega^{\text{PML}}} (\nabla \cdot (\kappa_0^{-2} A \nabla u^{\text{ref}}) + \alpha \beta u^{\text{ref}}) (\bar{\varphi} - \bar{\varphi}_h) dx,
 \end{aligned}$$

which completes the proof. □

The following theorem presents the a posteriori error estimate and is the main result for the TE polarization.

Theorem 4.4. *Let u and u_h be the solution of (4.5) and (4.18), respectively. Then there exists a positive constant C depending only on the minimum angle of the mesh \mathcal{M}_h such that the following*

the a posteriori error estimate holds:

$$\begin{aligned} \|u - u_h\|_{H^1(\Omega)} &\leq C\hat{C}^{-1}\kappa_0^{-2}(1 + \kappa R) \left(\sum_{K \in \mathcal{M}_h} \eta_K^2 \right)^{1/2} \\ &\quad + C\hat{C}^{-1}\kappa_0^{-2}(1 + \kappa_0 R)^2 |\alpha_0|^2 e^{-\kappa_0 \Im(\bar{\rho}) \left(1 - \frac{R^2}{|\bar{\rho}|^2}\right)^{1/2}} \|u_h - u^{\text{ref}}\|_{H_{\text{TE}}^{1/2}(\Gamma_R^+)}. \end{aligned}$$

Proof. Taking $\varphi_h = \Pi_h \varphi$ and using Lemma 4.5, we have

$$\begin{aligned} a_{\text{TE}}(u - u_h, \varphi) &= \langle \kappa_0^{-2} \mathcal{B}_{\text{TE}}(u_h - u^{\text{ref}}) - \kappa_0^{-2} \hat{\mathcal{B}}_{\text{TE}}(u_h - u^{\text{ref}}), \varphi \rangle_{\Gamma_R^+} - b(u_h, \varphi - \varphi_h) \\ &\quad - \int_{\Omega^{\text{PML}}} (\nabla \cdot (\kappa_0^{-2} A \nabla u^{\text{ref}}) + \alpha \beta u^{\text{ref}}) (\bar{\varphi} - \bar{\varphi}_h) dx \\ &:= I_1 + I_2 + I_3. \end{aligned}$$

By Lemma 4.15, we get

$$\begin{aligned} |I_1| &= \left| \langle \kappa_0^{-2} \mathcal{B}_{\text{TE}}(u_h - u^{\text{ref}}) - \kappa_0^{-2} \hat{\mathcal{B}}_{\text{TE}}(u_h - u^{\text{ref}}), \varphi \rangle_{\Gamma_R^+} \right| \\ &\leq C\hat{C}^{-1}\kappa_0^{-2}(1 + \kappa_0 R)^2 |\alpha_0|^2 e^{-\kappa_0 \Im(\bar{\rho}) \left(1 - \frac{R^2}{|\bar{\rho}|^2}\right)^{1/2}} \|u_h - u^{\text{ref}}\|_{H_{\text{TE}}^{1/2}(\Gamma_R^+)} \|\varphi\|_{H_{\text{TE}}^{1/2}(\Gamma_R^+)}. \end{aligned}$$

It is easy to see that

$$\begin{aligned} I_2 + I_3 &= \sum_{K \in \mathcal{M}_h \cap \Omega} \left(\int_K R_K(u_h) (\bar{\varphi} - \Pi_h \bar{\varphi}) dx + \sum_{e \in \partial K} \frac{1}{2} \int_e J_e (\bar{\varphi} - \Pi_h \bar{\varphi}) ds \right) \\ &\quad + \sum_{K \in \mathcal{M}_h \cap \Omega^{\text{PML}}} \left(\int_K R_K(u_h - u^{\text{ref}}) (\bar{\varphi} - \Pi_h \bar{\varphi}) dx + \sum_{e \in \partial K \cap \mathcal{B}_h} \frac{1}{2} \int_e J_e (\bar{\varphi} - \Pi_h \bar{\varphi}) ds \right). \end{aligned}$$

It follows from the Cauchy-Schwarz inequality, the interpolation estimates, and Lemma 4.2 that

$$\begin{aligned} |I_2 + I_3| &\leq C \sum_{K \in \mathcal{M}_h} \left(\|h_K \tilde{R}_K\|_{L^2(K)}^2 + \frac{1}{2} \sum_{e \in \partial K \cap \mathcal{B}_h} \|h_e^{1/2} J_e\|_{L^2(e)}^2 \right)^{1/2} \|\nabla \varphi\|_{L^2(\bar{K})} \\ &\leq C \sum_{K \in \mathcal{M}_h} \eta_K \|w^{-1} \nabla \varphi\|_{L^2(\bar{K})} \\ &\leq C\hat{C}^{-1}\kappa_0^{-2}(1 + \kappa_0 R) \left(\sum_{K \in \mathcal{M}_h} \eta_K^2 \right)^{1/2} \|\varphi\|_{H^{1/2}(\Gamma_R^+)}. \end{aligned}$$

By the inf-sup condition (4.6), we obtain

$$\begin{aligned} \|u - u_h\|_{H^1(\Omega)} &\leq C \sup_{0 \neq \varphi \in H_0^1(\Omega)} \frac{|a_{\text{TE}}(u - u_h, \varphi)|}{\|\varphi\|_{H^1(\Omega)}} \\ &\leq C \hat{C}^{-1} \kappa_0^{-2} (1 + \kappa R) \left(\sum_{K \in \mathcal{M}_h} \eta_K^2 \right)^{1/2} \\ &\quad + C \hat{C}^{-1} \kappa_0^{-2} (1 + \kappa_0 R)^2 |\alpha_0|^2 e^{-\kappa_0 \Im(\hat{\rho}) (1 - \frac{R^2}{|\hat{\rho}|^2})^{1/2}} \|u_h - u^{\text{ref}}\|_{H_{\text{TE}}^{1/2}(\Gamma_R^+)}, \end{aligned}$$

which completes the proof. \square

As can be seen from the Theorem 4.4, the a posteriori error estimate also consists of two parts: the finite element approximation error and the PML error which decays exponentially with respect to the PML parameters. In practice, we may choose the PML parameters appropriately such that the PML error is negligible compared with the finite element approximation error. The algorithm of the adaptive finite element PML method for the TE case is similar to that of the method for the TM polarized open cavity scattering problem described in Table 1.

5 Numerical experiments

In this section, we present some numerical examples to demonstrate the performance of the adaptive finite element PML method. The method is validated and compared with the adaptive finite element method with the transparent boundary condition (TBC) which is proposed in [28]. In the following examples, the PML parameters are $\rho = 3R$, $\sigma_0 = 20$ and $m = 2$.

The physical quantity of interest associated with the cavity scattering is the radar cross section (RCS), which measures the detectability of a target by a radar system [18]. When the incident angle and the observation angle are the same, the RCS is called the backscatter RCS. The specific formulas can be found in [28] for the backscatter RCS on both polarized wave fields.

5.1 Example 1

We consider a benchmark example for the TM polarized wave fields [18]. The cavity has a rectangular shape with width λ and depth 0.25λ . Fig. 2 shows the geometry of the cavity and the PML setting. The wavenumber in the free space is $\kappa_0 = 32\pi$ and wavelength $\lambda = 2\pi/\kappa_0 = 1/16$. The PML layer is a semi-annulus region and is imposed above the cavity with $R = 1/2\lambda$. Two cases are considered in this example: an empty cavity with no fillings inside of the cavity and a cavity filled by a lossy medium with the electric permittivity $\epsilon = 4 + i$ and the magnetic permeability $\mu = 1$. First, we compute the backscatter RCS

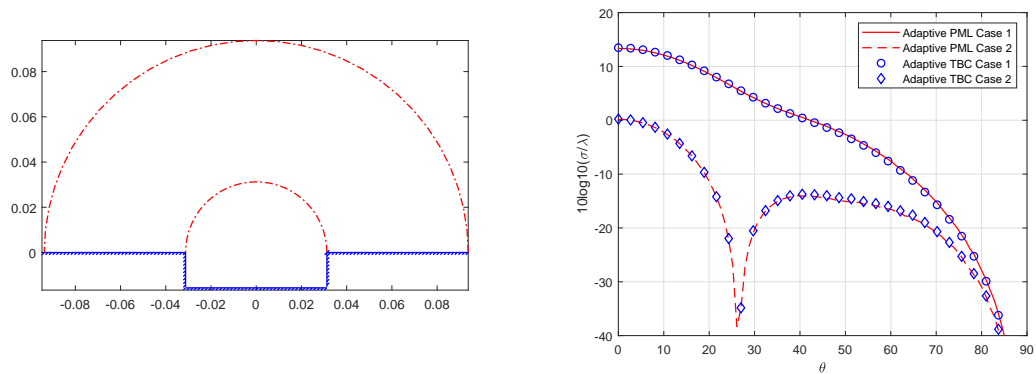


Figure 2: Example 1: (left) the cavity geometry; (right) the backscatter RCS for both cases by using the adaptive PML method and the adaptive TBC method.

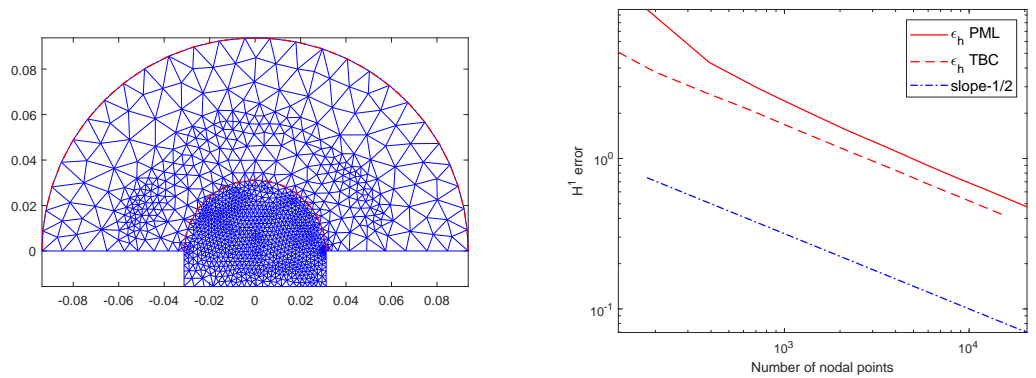


Figure 3: Example 1: (left) the adaptive mesh after 3 iterations with a total number of nodal points 1259; (right) the quasi-optimality of the a posteriori error estimates for both of the adaptive PML and TBC methods.

by using the adaptive finite element PML method and TBC method. For both methods, the adaptive mesh refinements are stopped once the total number of nodal points are over 15000. The backscatter RCS is shown as red solid lines and blue circles in Fig. 2 for the adaptive PML method and adaptive TBC method, respectively. It is clear to note that the results obtained by both methods are consistent with each other. Using the incident angle $\theta = \pi/4$ as a representative example in case 1, we present the adaptively refined mesh after 3 iterations with a total number of nodal points 1259 in Fig. 3. As expected, the mesh is refined near the two corners of the cavity and keep relatively coarse near the outer boundary of the PML layer, since the solution has singularity around the two L-shaped corners and is smooth and flat in the PML region, particularly in the part which is close to the outer boundary of the PML layer. The a posteriori error estimates for case 1 at incident angle $\theta = \pi/4$ are plotted in Fig. 3 to show the convergence rate of the method.

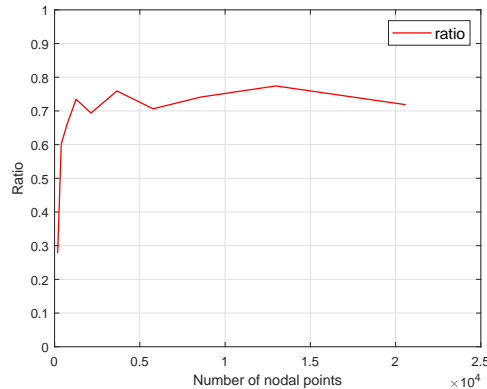


Figure 4: Example 1: the ratio of DoF_h between the physical domain and the whole computational domain.

It indicates that the meshes and the associated numerical complexity are quasi-optimal, i.e., $\epsilon_h = \mathcal{O}(\text{DoF}_h^{-1/2})$ holds asymptotically, where DoF_h is the degree of freedom or the total number of nodal points for the mesh \mathcal{M}_h . As a comparison, the a posteriori error estimates are also plotted for the adaptive TBC method in the red dashed line. Clearly, the method also preserves the quasi-optimality. It can be observed that the TBC method gives a smaller error than the PML method does for the same number of nodal points. There are two reasons: the TBC method does not require an artificial absorbing layer to enclose the physical domain which may reduce the size of the computational domain; the a posteriori error estimates may not be sharp for both methods as the lower bounds are not given. But the PML method is simpler than the TBC method from the implementation point of view. The PML method only involves the local Dirichlet boundary condition while the TBC method has to handle the nonlocal TBC. Fig. 4 plots the ratio of DoF_h between the physical domain and the whole computational domain. It shows that the physical domain asymptotically accounts for 70% of the total number of nodal points which illustrates that most of the nodal points are concentrated inside the physical domain and the a posteriori error estimate is effective for the PML method.

5.2 Example 2

In this example, we also consider the TM polarization. The backscatter RCS for a coated rectangular cavity with width 2.4λ and depth 1.6λ is computed. The each vertical side of the cavity wall is coated with a thin layer of some absorbing material. Fig. 5 illustrates the geometry of the cavity and PML setting. The coating on both sides has thickness 0.024λ and is made of a homogeneous absorbing material with a relative permittivity $\epsilon_r = 12 + 0.144i$ and a relative permeability $\mu_r = 1.74 + 3.306i$. This example has a multi-scale feature and is an interesting benchmark example to test the adaptive method. We take the same stopping rule as that in Example 1: the adaptive method is stopped once the number of nodal points is over 15000. Fig. 5 plots the backscatter RCS by using

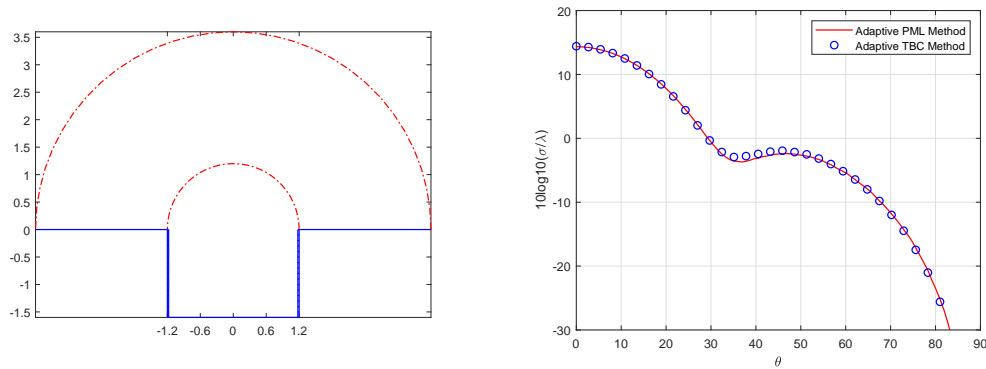


Figure 5: Example 2: (left) the cavity geometry; (right) the backscatter RCS by using the adaptive PML method and the adaptive TBC method.

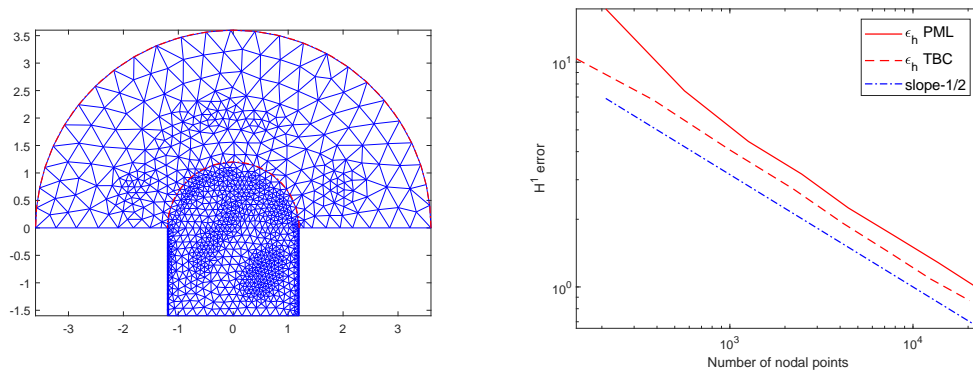


Figure 6: Example 2: (left) the adaptive mesh after 2 iterations with a total number of nodal points 1263; (right) the quasi-optimality of the a posteriori error estimates for both of the PML and TBC methods.

the adaptive PML and TBC methods, where the red solid line stands for the results of the PML method and the blue circles stand for the results of the TBC method. Clearly, these two methods are consistent with each other. Using a representative example of incident angle $\theta = \pi/4$, we present the refined mesh after 2 iterations with 1263 DoF_h and the a posteriori error estimates in Fig. 6. It is clear to note that the method can capture the behavior of the numerical solution in the two thin absorbing layers and displays the quasi-optimality between the meshes and the associated numerical complexity, i.e., $\epsilon_h = \mathcal{O}(\text{DoF}_h^{-1/2})$ holds asymptotically. As a comparison, we also show the a posteriori error estimates of the adaptive TBC method in the red dashed line. The quasi-optimality is also observed for the adaptive TBC method. Fig. 7 shows the ratio of DoF_h between the physical domain and the whole computational domain. Again, we see that most nodal points are concentrated in the physical domain, which illustrates the effectiveness of the adaptivity.

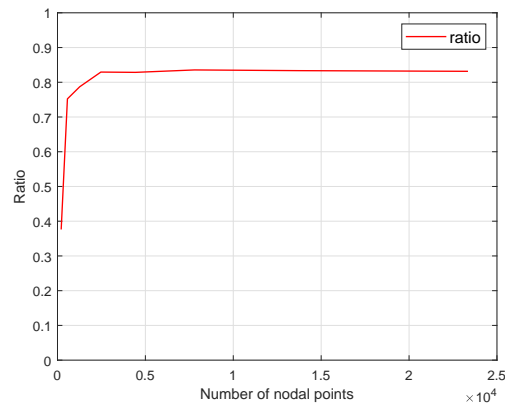


Figure 7: Example 2: the ratio of DoF_h between the physical domain and the whole computational domain.

5.3 Example 3

This example is still concerned with the TM polarization but some part of the structure for the cavity sticks out above the ground plane. The width and depth of the base cavity is 1.2λ and 0.8λ , respectively. There are two thin rectangular PEC humps in the middle of the cavity. Their width is $\frac{1}{20}\lambda$ and height is $\frac{16}{15}\lambda$ and $\frac{8}{15}\lambda$, respectively. The geometry of the cavity is shown in the left hand side of Fig. 8. The backscatter RCS is computed by using the adaptive PML and the adaptive TBC method. We also use the red solid line for the PML method and blue circles for the TBC method. Using the incident angle $\theta = \pi/4$ as an example, we show the refined mesh after three iterations with the number of nodal points 1261 and the a posteriori error estimates in Fig. 9. The adaptive PML method is able to refined meshes around the corners of the cavity where the solution has a singularity. The quasi-optimality is also obtained for the a posteriori error estimates. As a comparison,

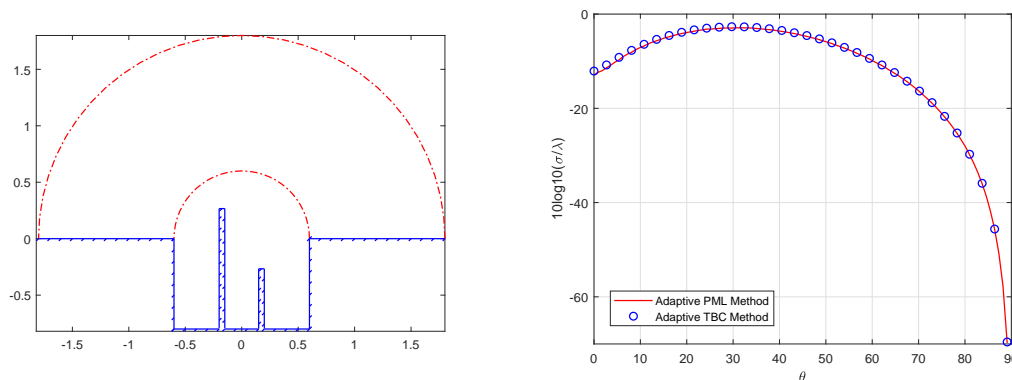


Figure 8: Example 3: (left) the cavity geometry; (right) the backscatter RCS by using the adaptive PML method and the adaptive TBC method.

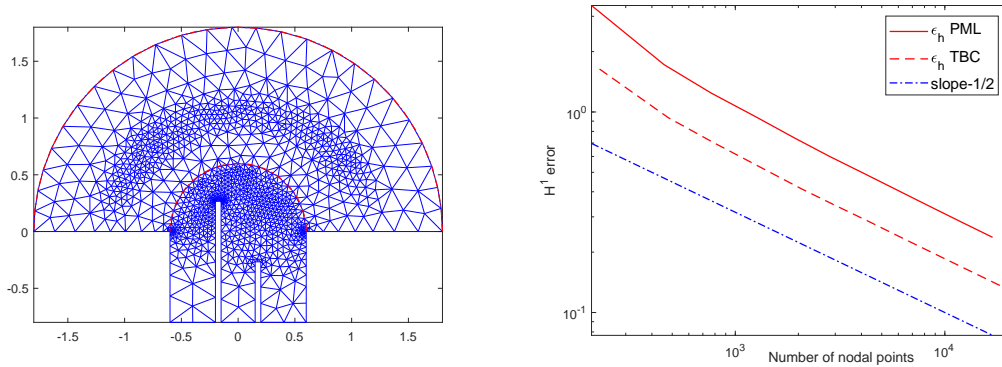


Figure 9: Example 3: (left) the adaptive mesh after 3 iterations with a total number of nodal points 1261; (right) the quasi-optimality of the a posteriori error estimates for both of the PML and TBC methods.

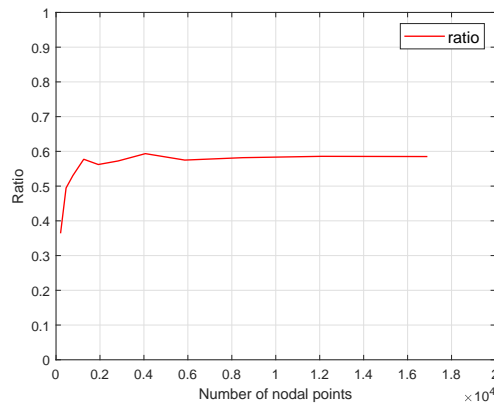


Figure 10: Example 3: the ratio of DoF_h between the physical domain and the whole computational domain.

we show the a posteriori error estimates for the adaptive TBC method in the red dashed line. The observation of the a posteriori error estimates for both methods is the same as that in Examples 1 and 2. Finally, Fig. 10 shows the ratio of DoF_h between the physical domain and the whole computational domain. Once again, the example confirms the effectiveness of the adaptive method.

5.4 Example 4

In this example, we consider the TE polarized cavity scattering problem. The cavity is a rectangle with a fixed width 0.025 m and a fixed depth 0.015 m. The cavity is empty with no filling materials. Instead of considering the illumination by a plane wave with a fixed frequency, we compute the backscatter RCS with the frequency ranging from 2 GHz to 18 GHz. Correspondingly, the range of the aperture of cavity is from $\frac{1}{6}\lambda$ to 1.5λ . The

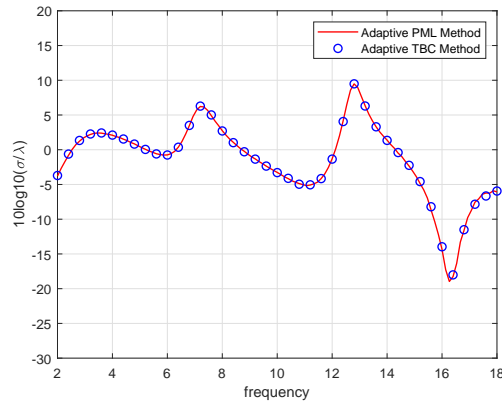


Figure 11: Example 4: the backscatter RCS by using the adaptive PML method and the adaptive TBC method.

incident angle is fixed to be $\frac{4}{9}\pi$. Fig. 11 shows the backward RCS by using the adaptive PML and the adaptive TBC method, where the red solid line and blue circles show their results, respectively. The stopping criterion is that the mesh refinement is stopped when the number of nodal points is over 25000. Once again, both methods are consistent with each other very well.

6 Conclusion

In this paper, we have presented an adaptive finite element PML method for solving the open cavity scattering problems. The a posteriori error analysis is carried out for both of the TM and TE polarizations. In each polarization, the estimate takes account of the finite element discretization error and the truncation error of PML method. The latter is shown to decay exponentially with respect to the PML medium parameter and the thickness of the layer. A possible future work is to extend our analysis to the adaptive finite element PML method for solving the three-dimensional cavity scattering problem, where the wave propagation is governed by Maxwell's equations.

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References

- [1] M. Ainsworth and A. W. Craig, A posteriori error estimators in the finite element method, *Numer. Math.*, 60 (1991), 429-463.

- [2] M. Ainsworth and J. T. Oden, A unified approach to a posteriori error estimation using element residual methods, *Numer. Math.*, 65 (1993), 23-50.
- [3] I. Babuška and A. Aziz, Survey lectures on Mathematical Foundation of the Finite Element Method, in the *Mathematical Foundations of the Finite Element Method with Application to the Partial Differential Equations*, ed. by A. Aziz, Academic Press, New York, 1973, 5-359.
- [4] I. Babuška and W. C. Rheinboldt, Error estimates for adaptive finite element, *SIAM J. Numer. Anal.*, 15 (1978), 736-754.
- [5] R. E. Bank and A. Weiser, Some a posteriori error estimators for elliptic partial differential equations, *Math. Comp.*, 44 (1985), 283-301.
- [6] G. Bao and H. Wu, Convergence analysis of the PML problems for time-harmonic Maxwell's equations, *SIAM J. Numer. Anal.*, 43 (2005), 2121-2143.
- [7] J.-P. Bérenger, A perfectly matched layer for the absorption of electromagnetic waves, *J. Comput. Phys.*, 114 (1994), 185-200.
- [8] J. Bramble and J. Pasciak, Analysis of a finite PML approximation for the three dimensional time-harmonic Maxwell's and acoustic scattering problems, *Math. Comp.*, 76 (2007), 597-614.
- [9] J. Chen and Z. Chen, An adaptive perfectly matched layer technique for 3-D time-harmonic electromagnetic scattering problems, *Math. Comp.*, 76 (2008), 673-698.
- [10] Z. Chen and S. Dai, Adaptive Galerkin methods with error control for a dynamical Landau model in superconductivity, *SIAM J. Numer. Anal.*, 38 (2001), 1961-1985.
- [11] Z. Chen and S. Dai, On the efficiency of adaptive finite element methods for with discontinuous coefficients, *SIAM J. Sci. Comput.*, 24 (2002), 443-462.
- [12] Z. Chen and X. Liu, An adaptive perfectly matched layer technique for time-harmonic scattering problems, *SIAM J. Numer. Anal.*, 43 (2005), 645-671.
- [13] Z. Chen and H. Wu, An adaptive finite element method with perfectly matched absorbing layer for the wave scattering by periodic structures, *SIAM J. Numer. Anal.*, 41 (2003), 799-826.
- [14] Z. Chen and W. Zheng, PML Method for electromagnetic scattering problem in a two-layered medium, *SIAM J. Numer. Anal.*, 55 (2017), 2050-2084.
- [15] F. Collino and P. Monk, Optimizing the perfectly matched layer, *Comput. Methods Appl. Mech. Engrg.*, 164 (1998), 157-171.
- [16] F. Collino and P. Monk, The perfectly matched layer in curvilinear coordinates, *SIAM J. Sci. Comput.*, 19(1998), 2061-2090.
- [17] T. Hohage, F. Schmidt and L. Zschiedrich, Solving time-harmonic scattering problems based on the pole condition, II. Convergence of the PML methods, *SIAM J. Appl. Math.*, 35 (2003), 547-560.
- [18] J. Jin, *The Finite Element Method in Electromagnetics*, Wiley & Son, New York, 2002.
- [19] M. Lassas and E. Somersalo, On the existence and convergence of the solution of the PML equations, *Computing*, 60 (1998), 229-241.
- [20] P. Li, A survey of open cavity scattering problems, *J. Comp. Math.*, 36 (2018), 1-16.
- [21] P. Li, H. Wu, and W. Zheng, An overfilled cavity problem for Maxwell's equations, *Math. Meth. Appl. Sci.*, 15 (2012), 1951-1979.
- [22] P. Monk, A posteriori error indicators for Maxwell's equations, *J. Comput. Appl. Math.*, 100 (1998), 173-190.
- [23] P. Monk and E. Soli, The adaptive computation of far-field patterns estimation of linear functionals, *SIAM J. Numer. Anal.*, 36 (1998), 251-274.
- [24] F. Teixeira and W. Chew, Systematic derivation of anisotropic PML absorbing media in cylin-

- drical and spherical coordinates, *IEEE Microw. Guided Wave Lett.*, 7 (1997), 371-373.
- [25] E. Turkel and A. Yefet, Absorbing PML boundary layers for wave-like equations, *Appl. Numer. Math.*, 27 (1998), 533-557.
 - [26] A. H. Schatz, An observation concerning Ritz-Galerkin methods with indefinite bilinear forms, *Math. Comp.*, 28 (1974), 959-962.
 - [27] A. Wood, Analysis of electromagnetic scattering from an overfilled cavity in the ground plane, *J. Comput. Phys.*, 215 (2006), 630-641.
 - [28] X. Yuan, G. Bao, and P. Li, An adaptive finite element DtN method for the open cavity scattering problems, *SIAM Trans. Appl. Math.*, 1 (2020), 316-345.
 - [29] D. Zhang, F. Ma, and M. Fang, Finite element method with perfectly matched absorbing layers for wave scattering from a cavity, *Chinese Journal of Computational Physics*, 25 (2008), 301-308.