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ANALYSIS OF THE TIME-DOMAIN PML PROBLEM FOR THE ELECTROMAGNETIC SCATTERING BY PERIODIC STRUCTURES*

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Abstract. This paper is concerned with the time-domain scattering of an electromagnetic plane wave by a periodic structure. An initial boundary value problem is formulated in a bounded domain by applying the perfectly matched layer (PML) technique to the scattering problem imposed in an unbounded domain. Based on the abstract inversion theorem of the Laplace transform and the analysis in the frequency domain, the well-posedness and stability are established for the truncated time-domain PML problem. Moreover, the exponential convergence of the solution for the truncated PML problem is proved by a careful study on the error for the Dirichlet-to-Neumann operators between the original scattering problem and the truncated PML problem.

Keywords. Time-domain Maxwell's equations; diffraction gratings; transparent boundary condition; perfectly matched layer; well-posedness and stability; convergence.

AMS subject classifications. 35Q61; 78A25; 78A45; 78M30.

1. Introduction

Scattering theory in periodic diffractive structures, known as diffraction gratings, has many important applications in micro-optics, which include the design and fabrication of diffractive optical elements such as corrective lenses and microsensors [28]. A good introduction to the topic can be found in [30, 31]. The basic electromagnetic theory of gratings has been well studied [35]. The mathematical problems that arise in diffractive optics modeling in industry can be found in [4]. Recent advances have been made due to the development of new mathematical and computational methods, such as the integral equation method [15, 29] and the variational method [1–3, 5, 23]. We refer to the monograph [6] for a comprehensive account of the main aspects on diffraction of electromagnetic waves by periodic gratings including mathematical modeling and analysis, numerical approximations and inverse problems. In this paper, we consider the time-domain electromagnetic scattering problem in one-dimensional periodic structures, where the well-posedness and stability of the solution were established in [25]. This work is concerned with the analysis of the time-domain perfectly matched layer (PML) problem for the electromagnetic scattering by periodic structures.

The PML was first introduced by Bérenger as a technique of domain truncation to numerically solve the time-domain Maxwell equations imposed in unbounded domains [10, 11]. Due to its effectiveness, simplicity and flexibility, the PML technique has been widely adopted in the field of computational wave propagation [21, 22, 32, 34]. Under the assumption that the exterior solution is composed of outgoing waves only, the basic idea of the PML method is to surround the domain of interest by a specially designed model

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medium that either slows down or attenuates all the waves that propagate from inside the domain. In practical simulation, the PML medium needs to be truncated into a layer of finite thickness and the artificial boundary may generate reflected waves which may pollute the solution in the domain of interest. Therefore, it is crucial to examine the error estimate between the solutions of the original scattering problem and the truncated PML problem. For time-harmonic scattering problems, convergence analysis of the PML method has been extensively investigated by many researchers. For example, the exponential convergence was established in terms of the thickness and medium parameter of the PML for the time-harmonic acoustic scattering problems in [19,26,27], for the three-dimensional time-harmonic electromagnetic obstacle scattering problems in [8, 12–14], and for the time-harmonic scattering problems in periodic structures in [7, 17].

Compared with the time-harmonic PML problems, the convergence analysis of the PML method for time-domain scattering problems is challenging due to the temporal dependence of the artificially designed absorbing medium. For the two-dimensional time-domain acoustic scattering problem, the exponential convergence in terms of the thickness and medium parameter of the PML was obtained in [16] for a circular PML method by taking advantage of the exponential decay of the modified Bessel functions, and in [18] for a uniaxial PML method by using the Laplace transform and complex coordinate stretching in the frequency domain. We refer to [9] for the stability and convergence analysis of the time-domain PML problem for the acoustic wave equation in waveguides.

In this work, we investigate the PML method for the time-domain electromagnetic scattering by periodic structures. The goal is twofold:

(1) Establish the well-posedness and stability of the time-domain PML problem;

(2) Prove the exponential convergence of the solution for the time-domain PML problem.

Specifically, we consider the scattering of an electromagnetic plane wave by a one-dimensional periodic structure in $\mathbb{R}^3$, which is assumed to be invariant in the y-axis and periodic in the x-axis. The electromagnetic fields can be decomposed into two fundamental polarizations: transverse electric (TE) polarization and transverse magnetic (TM) polarization, where the Maxwell equations can be reduced to the two-dimensional wave equation. We study the two-dimensional wave equation for both polarizations. Motivated by the uniqueness of the solution, we assume that the fields satisfy a certain translation property, which allows us to seek periodic solutions in the x direction for the electromagnetic fields under the change of variables. In the z direction, two rectangular PML regions, at the top and the bottom, are utilized to enclose the domain of interest. As an initial boundary value problem, the truncated PML problem is obtained in a bounded domain by imposing the Dirichlet boundary condition at the outer boundaries of the PML regions. Based on the abstract inversion theorem for the Laplace transform and the analysis in the frequency domain, the well-posedness and stability are established for the time-domain PML problem. Moreover, the exponential convergence of the solution is proved for the truncated PML problem. A key ingredient of the analysis is to examine the error of the Dirichlet-to-Neumann (DtN) operators between the truncated PML problem and the original scattering problem. The estimate shows that error decays exponentially by either enlarging the PML medium parameter or increasing the PML layer thickness.

The paper is organized as follows. In Section 2, we introduce the problem formulation and present an exact time-domain transparent boundary condition (TBC) to reduce the scattering problem into an initial boundary value problem in a bounded domain. Section 3 is devoted to the well-posedness and stability of the truncated time-domain PML problem, where a time-domain TBC is proposed for the PML problem. The exponential convergence of the PML method is established in Section 4. The paper is concluded with some remarks and directions for future work in Section 5.

2. Problem formulation

In this section, we introduce the model problem and define some necessary notation such as the Laplace transform and Sobolev spaces for the time-domain electromagnetic scattering by periodic structures. The well-posedness and stability are presented for the reduced scattering problem by using the exact time-domain transparent boundary condition.

2.1. Model equations. We consider the same model equations as those studied in [25]. First we specify the problem geometry, which is shown in Figure 2.1. Let $\boldsymbol{x}=(x,y,z)\in\mathbb{R}^3$. The structure is assumed to be invariant in the y-axis and periodic in the x-axis with period Λ. Due to the periodicity of the structure, the problem can be restricted into a single periodic cell where $x\in(0,\Lambda)$. Let S be the grating surface, which is embedded in the region $\Omega=\{(x,z)\in\mathbb{R}^2: 0<x<\Lambda, h_2<z<h_1\}$, where h_1 and h_2 are constants. Denote by $\Omega_1=\{(x,z)\in\mathbb{R}^2: 0<x<\Lambda, z>h_1\}$ and $\Omega_2=\{(x,z)\in\mathbb{R}^2: 0<x<\Lambda, z<h_2\}$ the domains above and below Ω, respectively. Denote by $\Gamma_1=\{(x,z)\in\mathbb{R}^2: 0<x<\Lambda, z=h_1\}$ and $\Gamma_2=\{(x,z)\in\mathbb{R}^2: 0<x<\Lambda, z=h_2\}$.

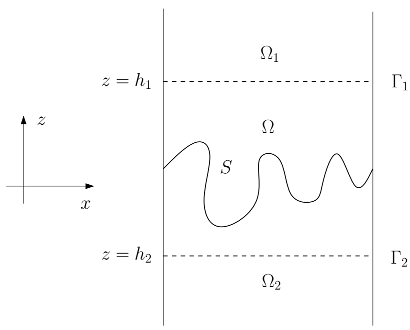

FIG. 2.1. *The problem geometry of the time-domain scattering by a periodic structure.*

Suppose that the whole space is filled with some material, which can be characterized by the dielectric permittivity ε and the magnetic permeability μ satisfying

$$0<\varepsilon_{\min}\le\varepsilon\le\varepsilon_{\max}<\infty,\quad 0<\mu_{\min}\le\mu\le\mu_{\max}<\infty,$$

where $\varepsilon_{\min},\varepsilon_{\max},\mu_{\min},\mu_{\max}$ are constants. Since the medium is assumed to be periodic in the x-axis with period Λ, we have

$$\varepsilon(x+n\Lambda,z)=\varepsilon(x,z),\quad \mu(x+n\Lambda,z)=\mu(x,z),\quad (x,z)\in\mathbb{R}^2, n\in\mathbb{Z}.$$

Furthermore, the medium is assumed to be homogeneous away from Ω where the medium can be inhomogeneous. Hence we may assume that there exist positive constants ε_j and μ_j such that

$$\varepsilon(x,z)=\varepsilon_j,\quad \mu(x,z)=\mu_j,\quad (x,z)\in\Omega_j,\quad j=1,2.$$

Throughout, we also assume that $\varepsilon\mu\ge\varepsilon_1\mu_1$, which is usually satisfied since ε_1 and μ_1 are the dielectric permittivity and the magnetic permeability in the free space Ω_1.

Consider the system of the time-domain Maxwell equations in $\mathbb{R}^3$ for $t>0$:

$$\nabla\times\boldsymbol{E}(\boldsymbol{x},t)+\mu\partial_t\boldsymbol{H}(\boldsymbol{x},t)=0,\quad \nabla\times\boldsymbol{H}(\boldsymbol{x},t)-\varepsilon\partial_t\boldsymbol{E}(\boldsymbol{x},t)=0, \tag{2.1}$$

where $\boldsymbol{E}$ and $\boldsymbol{H}$ are the electric field and the magnetic field, respectively. Since the structure is invariant in the y-axis, we consider two fundamental polarizations for the electromagnetic fields: TE polarization and TM polarization. In TE polarization, the electric and magnetic fields are

$$\boldsymbol{E}(x,y,z,t)=(0,E(x,z,t),0),\quad \boldsymbol{H}(x,y,z,t)=(H_1(x,z,t),0,H_3(x,z,t)).$$

Eliminating the magnetic field from (2.1), we get the wave equation for the electric field:

$$\varepsilon\partial_t^2E(x,z,t)=\nabla\cdot(\mu^{-1}\nabla E(x,z,t)). \tag{2.2}$$

In TM polarization, the electric and magnetic fields take the forms

$$\boldsymbol{E}(x,y,z,t)=(E_1(x,z,t),0,E_3(x,z,t)),\quad \boldsymbol{H}(x,z,t)=(0,H(x,z,t),0).$$

Eliminating the electric field from (2.1), we obtain the wave equation for the magnetic field:

$$\mu\partial_t^2H(x,z,t)=\nabla\cdot(\varepsilon^{-1}\nabla H(x,z,t)). \tag{2.3}$$

It can be seen from (2.2) and (2.3) that the electric field E and the magnetic field H satisfy the same wave equation. Hence it suffices to consider either (2.2) or (2.3). We shall only use (2.2) as the model equation to present the results in the rest of the paper.

Let $E^{\rm inc}$ be an incoming plane wave that is incident upon the structure from above. Explicitly, we have

$$E^{\rm inc}(x,z,t)=g(t-c_1x-c_2z),$$

where g is a smooth function and its regularity will be specified later, and $c_1=\cos\theta/c$, $c_2=\sin\theta/c$. Here θ is the incident angle satisfying $0<\theta<\pi$, and $c=(\varepsilon_1\mu_1)^{-1/2}$ is the wave speed in the free space. It can be verified that the incident field $E^{\rm inc}(x,z,t)$ satisfies the wave Equation (2.2) with $\varepsilon=\varepsilon_1,\mu=\mu_1$. Moreover, we assume that the incident plane wave $E^{\rm inc}$ vanishes for $t<0$.

Although the incident field $E^{\rm inc}$ may not be a periodic function in the x-axis, it is easy to note that

$$E^{\rm inc}(x+\Lambda,z,t)=E^{\rm inc}(x,z,t-c_1\Lambda)\quad \forall(x,z)\in\mathbb{R}^2.$$

Motivated by the uniqueness of the solution, we assume that the total field satisfies the same translation property, i.e.,

$$E(x+\Lambda,z,t)=E(x,z,t-c_1\Lambda)\quad \forall(x,z)\in\mathbb{R}^2. \tag{2.4}$$

Define the translated total field U and incident field $U^{\rm inc}$:

$$U(x,z,t)=E(x,z,t+c_1(x-\Lambda)),\quad U^{\rm inc}(x,z,t)=E^{\rm inc}(x,z,t+c_1(x-\Lambda)). \tag{2.5}$$

It follows from (2.5) and (2.4) that

$$U(x+\Lambda,z,t)=E(x+\Lambda,z,t+c_1x)=E(x,z,t+c_1x-c_1\Lambda)=U(x,z,t),$$

which shows that U is a periodic function in the x-axis with period Λ. A simple calculation yields

$$U^{\rm inc}(x,z,t) = E^{\rm inc}(x,z,t+c_1(x-\Lambda)) = g(t-c_2z-c_1\Lambda),$$

which implies that $U^{\rm inc}$ is also a periodic function of x since it does not depend on x.

Applying the change of variables, we get

$$\partial_t E = \partial_t U, \quad \partial_x E = \partial_x U - c_1\partial_t U.$$

Then the wave equation (2.2) can be written as

$$(\varepsilon - c_1^2\mu^{-1})\partial_t^2 U = \nabla\cdot(\mu^{-1}\nabla U) - c_1\left(\mu^{-1}\partial_{tx}U + \partial_x(\mu^{-1}\partial_t U)\right). \tag{2.6}$$

A simple calculation shows that

$$\begin{aligned}\varepsilon - c_1^2\mu^{-1} &= (\varepsilon\mu - \varepsilon_1\mu_1\cos^2\theta)\mu^{-1} \geq \varepsilon_1\mu_1(1-\cos^2\theta)\mu^{-1}\\ &= \varepsilon_1\mu_1\mu^{-1}\sin^2\theta > 0 \quad \forall\theta\in(0,\pi),\end{aligned}$$

which indicates that the wave equation (2.6) is well-defined. The Equation (2.6) is constrained by the initial conditions

$$U|_{t=0} = \partial_t U|_{t=0} = 0. \tag{2.7}$$

It is easy to verify that the incident field $U^{\rm inc}$ satisfies (2.6) with $\varepsilon=\varepsilon_1$, $\mu=\mu_1$, i.e.,

$$(\varepsilon_1 - c_1^2\mu_1^{-1})\partial_t^2 U^{\rm inc} = \nabla\cdot(\mu_1^{-1}\nabla U^{\rm inc}) - c_1\left(\mu_1^{-1}\partial_{tx}U^{\rm inc} + \partial_x(\mu_1^{-1}\partial_t U^{\rm inc})\right). \tag{2.8}$$

The same homogeneous initial conditions are imposed on the incident field $U^{\rm inc}$:

$$U^{\rm inc}|_{t=0} = \partial_t U^{\rm inc}|_{t=0} = 0. \tag{2.9}$$

2.2. Laplace transform and Sobolev spaces. For any $s = s_1 + {\rm i}s_2$ with $s_1, s_2\in\mathbb{R}, s_1>0$, define by $\breve{u}(x,z,s)$ the Laplace transform of $u(x,z,t)$ with respect to t, i.e.,

$$\breve{u}(x,z,s) = \mathscr{L}(u)(x,z,s) = \int_0^\infty e^{-st}u(x,z,t){\rm d}t.$$

It is easy to note from the Laplace transform that

$$\mathscr{L}(u_t)(x,z,s) = s\mathscr{L}(u)(x,z,s) - u(x,z,0), \tag{2.10}$$

and

$$\int_0^t u(x,z,\tau){\rm d}\tau = \mathscr{L}^{-1}(s^{-1}\breve{u})(x,z,t), \tag{2.11}$$

where $\mathscr{L}^{-1}$ denotes the inverse Laplace transform. It can also be verified from the inverse Laplace transform that

$$u(t) = \mathscr{F}^{-1}(e^{s_1t}\mathscr{L}(u)(s_1+{\rm i}s_2)), \tag{2.12}$$

where $\mathscr{F}^{-1}$ denotes the inverse Fourier transform with respect to s_2.

The following result concerns the Plancherel or Parseval identity for the Laplace transform (cf. [20, (2.46)]).

LEMMA 2.1. *If* $\breve{u} = \mathscr{L}(u)$ *and* $\breve{v} = \mathscr{L}(v)$, *then*

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \breve{u}(s)\breve{v}(s)\mathrm{d}s_2 = \int_0^{\infty} e^{-2s_1 t} u(t)v(t)\mathrm{d}t$$

for all $s_1 > s_0$, *where* s_0 *is the abscissa of convergence for the Laplace transform of* u *and* v.

Hereafter, the expression $a \lesssim b$ stands for $a \le Cb$, where C is a generic positive constant whose precise value is not required but should be clear from the context.

We also recall the following result (cf. [33, Theorem 43.1]), which is an analogue of the Paley–Wiener–Schwarz theorem for the Fourier transform of the distributions with compact support in the case of the Laplace transform.

LEMMA 2.2. *Let* $\breve{u}$ *denote a holomorphic function in the half plane* $s_1 > s_0$, *valued in the Banach space* $\mathbb{E}$. *The following statements are equivalent:*

(1) *there is a distribution* $u \in \mathcal{D}'_+(\mathbb{E})$ *whose Laplace transform is equal to* $\breve{u}(s)$;

(2) *there is a* σ_1 *with* $s_0 \le \sigma_1 < \infty$ *and integer* $m \ge 0$ *such that for all complex numbers* s *with* $s_1 > \sigma_1$, *it holds that* $\|\breve{u}(s)\|_{\mathbb{E}} \lesssim (1+|s|)^m$,

where $\mathcal{D}'_+(\mathbb{E})$ *is the space of distributions on the real line which vanish identically in the open negative half line.*

Next, we introduce some Sobolev spaces which are used in this work. Define a weighted periodic function space

$$H^1_{s,\mathrm{p}}(\Omega) = \{u \in H^1(\Omega) : u(0,z) = u(\Lambda, z)\},$$

which is a Sobolev space with the norm characterized by

$$\|u\|^2_{H^1_{s,\mathrm{p}}(\Omega)} = \int_\Omega (|\nabla u|^2 + |s|^2|u|^2)\mathrm{d}x\mathrm{d}z.$$

For any $u \in H^1_{s,\mathrm{p}}(\Omega)$, it has the Fourier expansion with respect to x:

$$u(x,z) = \sum_{n\in\mathbb{Z}} u_n(z)e^{\mathrm{i}\alpha_n x}, \quad u_n(z) = \frac{1}{\Lambda}\int_0^\Lambda u(x,z)e^{-\mathrm{i}\alpha_n x}\mathrm{d}x,$$

where $\alpha_n = 2n\pi/\Lambda$. A simple calculation yields an equivalent norm in $H^1_{s,p}(\Omega)$ via the Fourier coefficients:

$$\|u\|^2_{H^1_{s,\mathrm{p}}(\Omega)} = \sum_{n\in\mathbb{Z}} (|s|^2 + \alpha_n^2)\int_{h_2}^{h_1} |u_n(z)|^2\mathrm{d}z + \sum_{n\in\mathbb{Z}}\int_{h_2}^{h_1} |u_n'(z)|^2\mathrm{d}z.$$

Denote by $H^\lambda_{s,\mathrm{p}}(\Gamma_j)$ the standard Sobolev trace space with the norm being characterized by

$$\|u\|^2_{H^\lambda_{s,\mathrm{p}}(\Gamma_j)} = \sum_{n\in\mathbb{Z}} (|s|^2 + \alpha_n^2)^\lambda |u_n|^2,$$

where $\lambda \in \mathbb{R}$. Clearly, the dual space of $H_{s,\mathrm{p}}^{1/2}(\Gamma_j)$ is $H_{s,\mathrm{p}}^{-1/2}(\Gamma_j)$ with respect to the scalar product in $L^2(\Gamma_j)$ defined by

$$\langle u,v\rangle_{\Gamma_j} = \int_{\Gamma_j} u\bar{v}\mathrm{d}x.$$

The weighted Sobolev spaces $H_{s,\mathrm{p}}^1(\Omega)$ and $H_{s,\mathrm{p}}^\lambda(\Gamma_j)$ are equivalent to the standard Sobolev spaces $H_\mathrm{p}^1(\Omega)$ and $H_\mathrm{p}^\lambda(\Gamma_j)$ since $s\neq 0$.

2.3. The reduced problem. We briefly present an exact time-domain TBC to reformulate the scattering problem into an equivalent initial boundary value problem in a bounded domain. We refer to [25] for the details on the derivation and properties of the TBC.

Subtracting (2.8) from (2.6), we obtain the equation for the scattered field U^{sc} in Ω_1 for $t>0$:

$$(\varepsilon_1 - c_1^2\mu_1^{-1})\partial_t^2 U^{\mathrm{sc}} = \nabla\cdot(\mu_1^{-1}\nabla U^{\mathrm{sc}}) - c_1\left(\mu_1^{-1}\partial_{tx}U^{\mathrm{sc}} + \partial_x(\mu_1^{-1}\partial_t U^{\mathrm{sc}})\right). \tag{2.13}$$

From (2.7) and (2.9), we get the initial conditions:

$$U^{\mathrm{sc}}|_{t=0} = \partial_t U^{\mathrm{sc}}|_{t=0} = 0 \quad \text{in } \Omega_1. \tag{2.14}$$

Taking the Laplace transform of (2.13) and using the initial conditions (2.14), we obtain

$$(\varepsilon_1 - c_1^2\mu_1^{-1})s^2\breve{U}^{\mathrm{sc}} = \nabla\cdot(\mu_1^{-1}\nabla\breve{U}^{\mathrm{sc}}) - c_1\left(\mu_1^{-1}s\partial_x\breve{U}^{\mathrm{sc}} + \partial_x(\mu_1^{-1}s\breve{U}^{\mathrm{sc}})\right) \quad \text{in } \Omega_1,$$

which is equivalent to

$$(\varepsilon_1\mu_1 - c_1^2)s^2\breve{U}^{\mathrm{sc}} = \Delta\breve{U}^{\mathrm{sc}} - 2c_1 s\partial_x\breve{U}^{\mathrm{sc}} \quad \text{in } \Omega_1. \tag{2.15}$$

Since $\breve{U}^{\mathrm{sc}}$ is a periodic function with respect to x, it has the Fourier series expansion

$$\breve{U}^{\mathrm{sc}}(x,z,s) = \sum_{n\in\mathbb{Z}} \breve{U}_n^{\mathrm{sc}}(z,s)e^{\mathrm{i}\alpha_n x}, \quad z>h_1.$$

Substituting the Fourier expansion of $\breve{U}^{\mathrm{sc}}$ into (2.15), we obtain an ordinary differential equation for the Fourier coefficients:

$$\begin{cases} \partial_z^2\breve{U}_n^{\mathrm{sc}} - (\beta_1^n)^2\breve{U}_n^{\mathrm{sc}} = 0, & z>h_1, \\ \breve{U}_n^{\mathrm{sc}} = \breve{U}_n^{\mathrm{sc}}(h_1,s), & z=h_1, \end{cases}$$

where

$$\beta_1^n(s) = (\varepsilon_1\mu_1 s^2 + (\alpha_n + \mathrm{i}c_1 s)^2)^{1/2}, \quad \Re\beta_1^n(s)>0.$$

It follows from the bounded and outgoing condition that

$$\breve{U}_n^{\mathrm{sc}}(z,s) = \breve{U}_n^{\mathrm{sc}}(h_1,s)e^{-\beta_1^n(s)(z-h_1)}.$$

Then we get the Rayleigh expansion for the scattered field in Ω_1:

$$\breve{U}^{\mathrm{sc}}(x,z,s) = \sum_{n\in\mathbb{Z}} \breve{U}_n^{\mathrm{sc}}(h_1,s)e^{-\beta_1^n(s)(z-h_1)}e^{\mathrm{i}\alpha_n x}, \quad z>h_1.$$

Taking the normal derivative of $\breve{U}^{\rm sc}$ on Γ_1, we have

$$\partial_{\nu_1}\breve{U}^{\rm sc}(x,h_1,s)=\sum_{n\in\mathbb{Z}}(-1)\beta_1^n(s)\breve{U}_n^{\rm sc}(h_1,s)e^{{\rm i}\alpha_n x},$$

where $\nu_1=(0,1)$ is the unit normal vector on Γ_1.

Similarly, in Ω_2, we may derive the Rayleigh expansion for the total field $\breve{U}$:

$$\breve{U}(x,z,s)=\sum_{n\in\mathbb{Z}}\breve{U}_n(h_2,s)e^{\beta_2^n(s)(z-h_2)}e^{{\rm i}\alpha_n x},$$

where

$$\beta_2^n(s)=(\varepsilon_2\mu_2 s^2+(\alpha_n+{\rm i}c_1 s)^2)^{1/2},\quad \Re\beta_2^n(s)>0.$$

Taking the normal derivative of $\breve{U}(x,z,s)$ on Γ_2 yields

$$\partial_{\nu_2}\breve{U}(x,h_2,s)=\sum_{n\in\mathbb{Z}}(-1)\beta_2^n(s)\breve{U}_n(h_2,s)e^{{\rm i}\alpha_n x},$$

where $\nu_2=(0,-1)$ is the unit normal vector on Γ_2. For any function $u(x,h_j)$ defined on Γ_j, we define the Dirichlet-to-Neumann (DtN) operators

$$(\mathscr{B}_j u)(x,h_j)=\sum_{n\in\mathbb{Z}}(-1)\beta_j^n u_n(h_j)e^{{\rm i}\alpha_n x},\quad u(x,h_j)=\sum_{n\in\mathbb{Z}}u_n(h_j)e^{{\rm i}\alpha_n x}. \tag{2.16}$$

We have the following results on the trace regularity and the boundary operators $\mathscr{B}_j$ (cf. [25, Lemmas 2.1, 2.2 and 2.3]).

LEMMA 2.3. *There exists a positive constant* C_1 *such that*

$$\|u\|_{H^{1/2}_{s,{\rm p}}(\Gamma_j)}\le C_1\|u\|_{H^1_{s,{\rm p}}(\Omega)}\quad \forall u\in H^1_{s,{\rm p}}(\Omega),$$

where $C_1^2=\max\{1+(h_1-h_2)^{-1}s_1^{-1},1+(2\pi)^{-1}(h_1-h_2)^{-1}\Lambda\}$.

LEMMA 2.4. *The DtN operator* $\mathscr{B}_j:H^{1/2}_{s,{\rm p}}(\Gamma_j)\to H^{-1/2}_{s,{\rm p}}(\Gamma_j)$ *is continuous, i.e.,*

$$\|\mathscr{B}_j u\|_{H^{-1/2}_{s,{\rm p}}(\Gamma_j)}\le C_2\|u\|_{H^{1/2}_{s,{\rm p}}(\Gamma_j)},$$

where C_2 *is a positive constant and is given by* $C_2^2=\max\{2,2c_1^2+\varepsilon_{\max}\mu_{\max}\}$.

LEMMA 2.5. *The following estimate holds:*

$$\Re\langle(s\mu_j)^{-1}\mathscr{B}_j u,u\rangle_{\Gamma_j}\le 0\quad \forall u\in H^{1/2}_{s,{\rm p}}(\Gamma_j).$$

Using the DtN operators (2.16), we obtain the following TBCs in the frequency domain:

$$\begin{cases}\partial_{\nu_1}\breve{U}=\mathscr{B}_1\breve{U}+\breve{f} & \text{on } \Gamma_1,\\ \partial_{\nu_2}\breve{U}=\mathscr{B}_2\breve{U} & \text{on } \Gamma_2,\end{cases} \tag{2.17}$$

where $\breve{f}=\partial_{\nu_1}\breve{U}^{\rm inc}-\mathscr{B}_1\breve{U}^{\rm inc}$. Taking the inverse Laplace transform of (2.17) yields the TBCs in the time domain:

$$\begin{cases}\partial_{\nu_1}U=\mathscr{T}_1 U+f & \text{on } \Gamma_1,\\ \partial_{\nu_2}U=\mathscr{T}_2 U & \text{on } \Gamma_2,\end{cases} \tag{2.18}$$

where f is inverse the Laplace transform of $\breve{f}$, i.e., $f=\mathscr{L}^{-1}(\breve{f})$, and $\mathscr{T}_j=\mathscr{L}^{-1}\circ\mathscr{B}_j\circ\mathscr{L}$.

Based on the time-domain TBCs given in (2.18), the scattering problem can be reformulated equivalently into the following initial boundary value problem in the bounded domain Ω:

$$\begin{cases}(\varepsilon-c_1^2\mu^{-1})\partial_t^2U=\nabla\cdot(\mu^{-1}\nabla U)-c_1\left(\mu^{-1}\partial_{tx}U+\partial_x(\mu^{-1}\partial_tU)\right) & \text{in } \Omega, t>0,\\ U|_{t=0}=\partial_tU|_{t=0}=0 & \text{in } \Omega,\\ \partial_{\nu_1}U=\mathscr{T}_1U+f & \text{on } \Gamma_1, t>0,\\ \partial_{\nu_2}U=\mathscr{T}_2U & \text{on } \Gamma_2, t>0.\end{cases} \tag{2.19}$$

It is shown in [25, Theorem 3.3] that the reduced problem is well-posed and the solution is stable, which are stated in the following theorem.

THEOREM 2.1. *The initial boundary value problem* (2.19) *has a unique solution, which satisfies*

$$U(x,z,t)\in L^2(0,T;H_{\rm p}^1(\Omega))\cap H^1(0,T;L^2(\Omega))$$

and the stability estimate

$$\max_{t\in[0,T]}\left(\|\partial_tU\|_{L^2(\Omega)}+\|\partial_t(\nabla U)\|_{L^2(\Omega)^2}\right)$$
$$\lesssim\|f\|_{L^1(0,T;H^{-1/2}(\Gamma_1))}+\max_{t\in[0,T]}\|\partial_tf\|_{H^{-1/2}(\Gamma_1)}+\|\partial_t^2f\|_{L^1(0,T;H^{-1/2}(\Gamma_1))}.$$

3. The time-domain PML problem

In this section, we introduce the PML formulation for the scattering problem and establish the well-posedness and stability of the time-domain PML problem. Moreover, we introduce the TBC for the truncated time-domain PML problem.

3.1. The PML formulation. Now we turn to the introduction of absorbing PML layers. The domain Ω is surrounded with two rectangular PML layers of thicknesses δ_1 and δ_2 in Ω_1 and Ω_2, respectively. Figure 3.1 shows the geometry of the PML problem. Define the PML regions

$$\Omega_1^{\rm PML}=\{(x,z)\in\mathbb{R}^2:0<x<\Lambda,h_1<z<h_1+\delta_1\},$$
$$\Omega_2^{\rm PML}=\{(x,z)\in\mathbb{R}^2:0<x<\Lambda,h_2-\delta_2<z<h_2\}.$$

Let

$$\Gamma_1^{\rm PML}=\{(x,z)\in\mathbb{R}^2:0<x<\Lambda,z=h_1+\delta_1\},$$
$$\Gamma_2^{\rm PML}=\{(x,z)\in\mathbb{R}^2:0<x<\Lambda,z=h_2-\delta_2\}$$

be the upper and bottom boundaries of the domain

$$\Omega_\delta=\{(x,z)\in\mathbb{R}^2:0<x<\Lambda,h_2-\delta_2<z<h_1+\delta_1\},$$

in which the truncated PML problem is formulated.

Define the PML medium property as $\zeta(z)=1+s^{-1}\sigma(z)$, which satisfies $\sigma\in L^\infty(\mathbb{R}),0\le\sigma\le\sigma_0$ and $\sigma(z)=0$ for $h_2<z<h_1$, where σ_0 is a positive constant. Following the complex coordinate stretching for the time-domain PML problems (cf. [9, 16]), we introduce the change of variable

$$\tilde{z}=\int_0^z\zeta(\tau){\rm d}\tau.$$

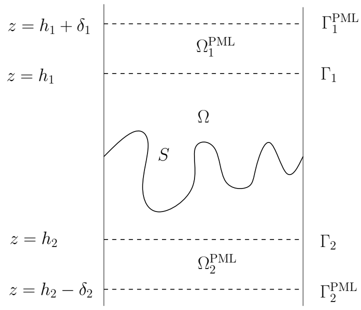

FIG. 3.1. *Geometry of the PML problem.*

Using the change of variables and imposing the homogeneous Dirichlet boundary condition for the scattered field, we obtain the PML equation for the total field in the frequency domain:

$$\begin{aligned}(\varepsilon - c_1^2\mu^{-1})s(1+\frac{\sigma}{s})\breve{U}^{\rm P} =& \partial_z\left((s\mu)^{-1}(1+\frac{\sigma}{s})^{-1}\partial_z\breve{U}^{\rm P}\right) + (1+\frac{\sigma}{s})\partial_x\left((s\mu)^{-1}\partial_x\breve{U}^{\rm P}\right)\\ &- c_1(1+\frac{\sigma}{s})\left(\mu^{-1}\partial_x\breve{U}^{\rm P} + \partial_x(\mu^{-1}\breve{U}^{\rm P})\right) + s^{-1}\breve{F} \qquad \text{in } \Omega_\delta \end{aligned} \tag{3.1}$$

and the boundary conditions

$$\begin{cases}\breve{U}^{\rm P} = \breve{U}^{\rm inc} & \text{on } \Gamma_1^{\rm PML},\\ \breve{U}^{\rm P} = 0 & \text{on } \Gamma_2^{\rm PML},\end{cases} \tag{3.2}$$

where

$$\begin{aligned}\breve{F} =& (\varepsilon_1 - c_1^2\mu_1^{-1})s^2(1+\frac{\sigma}{s})\breve{U}^{\rm inc} - \partial_z\left(\mu_1^{-1}(1+\frac{\sigma}{s})^{-1}\partial_z\breve{U}^{\rm inc}\right) - (1+\frac{\sigma}{s})\partial_x\left(\mu_1^{-1}\partial_x\breve{U}^{\rm inc}\right)\\ &+ c_1(1+\frac{\sigma}{s})\left(\mu_1^{-1}s\partial_x\breve{U}^{\rm inc} + \partial_x(\mu_1^{-1}s\breve{U}^{\rm inc})\right) \qquad \text{in } \Omega_1^{\rm PML},\\ \breve{F} =& 0 \qquad \text{in } \Omega_\delta \setminus \overline{\Omega_1^{\rm PML}}.\end{aligned}$$

We define the following initial boundary value problem for $(U^{\rm P}, \boldsymbol{\phi} = (\phi_x, \phi_z)^\top)$, which is referred to as the PML problem in the rest of this paper,

$$\begin{cases}(\varepsilon - c_1^2\mu^{-1})\partial_t^2 U^{\rm P} + (\varepsilon - c_1^2\mu^{-1})\sigma\partial_t U^{\rm P}\\ \quad = \nabla\cdot(\mu^{-1}\nabla U) + \nabla\cdot\boldsymbol{\phi}\\ \qquad + c_1\left(\mu^{-1}\partial_{tx}U^{\rm P} + \partial_x(\mu^{-1}\partial_t U^{\rm P})\right)\\ \qquad + c_1\sigma\left(\mu^{-1}\partial_x U^{\rm P} + \partial_x(\mu^{-1}U^{\rm P})\right) + F & \text{in } \Omega_\delta\times(0,T),\\ \partial_t\phi_x = \mu^{-1}\sigma\partial_x U^{\rm P} & \text{in } \Omega_\delta\times(0,T),\\ \partial_t\phi_z + \sigma\phi_z + \mu^{-1}\sigma\partial_z U^{\rm P} = 0 & \text{in } \Omega_\delta\times(0,T),\\ \partial_\nu U^{\rm P} + \boldsymbol{\phi}\cdot\nu = 0 & \text{on } \Gamma_j^{\rm PML}\times(0,T), \quad j=1,2,\\ U^{\rm P} = U^{\rm inc} & \text{on } \Gamma_1^{\rm PML}\times(0,T),\\ U^{\rm P} = 0 & \text{on } \Gamma_2^{\rm PML}\times(0,T),\\ U^{\rm P}|_{t=0} = 0, \quad \partial_t U^{\rm P}|_{t=0} = 0, \quad \boldsymbol{\phi}|_{t=0} = 0 & \text{in } \Omega_\delta,\end{cases} \tag{3.3}$$

where $F = \mathscr{L}^{-1}(\breve{F})$.

Now we make some additional assumptions on the incident field by requiring that

$$U^{\rm inc} \in H^3(0,T; H_{\rm p}^2(\Omega_1^{\rm PML})), \quad \partial_t^j U^{\rm inc}(x,z,0) = 0, \quad j = 0,1,2 \tag{3.4}$$

and that $U^{\rm inc}$ can be extended to infinity in time so that

$$U^{\rm inc} \in H^3(0,\infty; H_{\rm p}^2(\Omega_1^{\rm PML}))$$

and

$$\|U^{\rm inc}\|_{H^3(0,\infty;H_{\rm p}^2(\Omega_1^{\rm PML}))} \le C\|U^{\rm inc}\|_{H^3(0,T;H_{\rm p}^2(\Omega_1^{\rm PML}))}. \tag{3.5}$$

Under the above assumptions, it is clear that

$$F \in H^1(0,\infty; L^2(\Omega_1^{\rm PML})), \quad F(x,z,0) = 0.$$

3.2. Well-posedness and stability. Define $\mathring{H}_{s,\rm p}^1(\Omega_\delta) := \{u \in H_{s,\rm p}^1(\Omega_\delta) : u = 0 \text{ on } \Gamma_1^{\rm PML} \cup \Gamma_2^{\rm PML}\}$. Multiplying (3.1) by the complex conjugate of a test function $V \in \mathring{H}_{s,\rm p}^1(\Omega_\delta)$ and using the integration by parts, we arrive at the variational problem: find $\breve{U}^{\rm P} \in H_{s,\rm p}^1(\Omega_\delta)$ such that $\breve{U}^{\rm P} = \breve{U}^{\rm inc}$ on $\Gamma_1^{\rm PML}, \breve{U}^{\rm P} = 0$ on $\Gamma_2^{\rm PML}$ and

$$a_{\rm PML}(\breve{U}^{\rm P}, V) = \int_{\Omega_\delta} s^{-1}\breve{F}\overline{V}{\rm d}x{\rm d}z \quad \forall V \in \mathring{H}_{s,\rm p}^1(\Omega_\delta), \tag{3.6}$$

where the sesquilinear form

$$\begin{aligned} a_{\rm PML}(W,V) = \int_{\Omega_\delta} \Big(&(s\mu)^{-1}(1+\frac{\sigma}{s})^{-1}\partial_z W \partial_z \overline{V} + (s\mu)^{-1}(1+\frac{\sigma}{s})\partial_x W \partial_x \overline{V} \\ &+ (\varepsilon - c_1^2\mu^{-1})(1+\frac{\sigma}{s})sW\overline{V} \\ &+ c_1(1+\frac{\sigma}{s})\left(\mu^{-1}\partial_x W + \partial_x(\mu^{-1}W)\right)\overline{V}\Big){\rm d}x{\rm d}z. \end{aligned} \tag{3.7}$$

LEMMA 3.1. *The variational problem* (3.6) *has a unique solution* $\breve{U}^{\rm P} \in H_{s,\rm p}^1(\Omega_\delta)$, *which satisfies*

$$\begin{aligned} &\|\nabla \breve{U}^{\rm P}\|_{L^2(\Omega_\delta)^2} + \|s\breve{U}^{\rm P}\|_{L^2(\Omega_\delta)} \\ &\lesssim s_1^{-1}(1+s_1^{-1}\sigma_0)^4\Big(\|s\breve{U}^{\rm inc}\|_{H_{\rm p}^{1/2}(\Gamma_1^{\rm PML})} + \|s\breve{U}^{\rm inc}\|_{H_{\rm p}^{-1/2}(\Gamma_1^{\rm PML})} \\ &\quad + \|s^2\breve{U}^{\rm inc}\|_{H_{\rm p}^{-1/2}(\Gamma_1^{\rm PML})} + \|\breve{F}\|_{L^2(\Omega_1^{\rm PML})}\Big). \end{aligned}$$

Proof. Let $\zeta \in H^1(0,T; H_{\rm p}^1(\Omega_\delta)) \cap H^2(0,T; L^2(\Omega_\delta))$ be the function such that $\zeta = 0$ on $\Gamma_2^{\rm PML} \times (0,T)$, $\zeta = U^{\rm inc}$ on $\Gamma_1^{\rm PML} \times (0,T)$. By testing (3.1) with the complex conjugate of $\breve{U}^{\rm P} - \breve{\zeta} \in \mathring{H}_{\rm p}^1(\Omega_\delta)$ and integrating by parts we obtain

$$a_{\rm PML}(\breve{U}^{\rm P}, \breve{U}^{\rm P}) = a_{\rm PML}(\breve{U}^{\rm P}, \breve{\zeta}) + \int_{\Omega_1^{\rm PML}} s^{-1}\breve{F}(\breve{U}^{\rm P} - \breve{\zeta}){\rm d}x{\rm d}z. \tag{3.8}$$

It suffices to show the coercivity of $a_{\rm PML}$, since the continuity follows directly from the Cauchy–Schwarz inequality. Using (3.7), we have

$$a_{\rm PML}(\breve{U}^{\rm P}, (1+s^{-1}\sigma)\breve{U}^{\rm P}) = \int_{\Omega_\delta} \Big((s\mu)^{-1}(1+\frac{\sigma}{s})^{-1}(1+\frac{\sigma}{\bar{s}})|\partial_z \breve{U}^{\rm P}|^2$$

$$+(s\mu)^{-1}(1+\frac{\sigma}{s})(1+\frac{\sigma}{\bar{s}})|\partial_x\breve{U}^{\rm P}|^2$$
$$+(\varepsilon-c_1^2\mu^{-1})(1+\frac{\sigma}{s})s(1+\frac{\sigma}{\bar{s}})|\breve{U}^{\rm P}|^2$$
$$+c_1(1+\frac{\sigma}{s})(1+\frac{\sigma}{\bar{s}})\left(\mu^{-1}\partial_x\breve{U}^{\rm P}+\partial_x(\mu^{-1}\breve{U}^{\rm P})\right)\overline{\breve{U}^{\rm P}}\Big){\rm d}x{\rm d}z.$$

Taking the real part of the above equation yields

$$\begin{aligned}\Re a_{\rm PML}(\breve{U}^{\rm P},(1+s^{-1}\sigma)\breve{U}^{\rm P})=&\int_{\Omega_\delta}\Big(\frac{(s_1+2\sigma)|s|^2+\sigma^2 s_1}{\mu|s|^2|s+\sigma|^2}|\partial_z\breve{U}^{\rm P}|^2+\frac{|s+\sigma|^2 s_1}{\mu|s|^4}|\partial_x\breve{U}^{\rm P}|^2\\&+(\varepsilon-c_1^2\mu^{-1})\frac{|s+\sigma|^2 s_1}{|s|^2}|\breve{U}^{\rm P}|^2\Big){\rm d}x{\rm d}z\\&+\Re\int_{\Omega_\delta}c_1\frac{|s+\sigma|^2}{|s|^2}\left(\mu^{-1}\partial_x\breve{U}^{\rm P}+\partial_x(\mu^{-1}\breve{U}^{\rm P})\right)\overline{\breve{U}^{\rm P}}{\rm d}x{\rm d}z.\end{aligned}$$

Since μ and $\breve{U}^{\rm P}$ are periodic functions of period Λ with respect to x, we have from the integration by parts that

$$\int_{\Omega_\delta}\left(\mu^{-1}\partial_x\breve{U}^{\rm P}+\partial_x(\mu^{-1}\breve{U}^{\rm P})\right)\overline{\breve{U}^{\rm P}}{\rm d}x{\rm d}z+\int_{\Omega_\delta}\left(\mu^{-1}\partial_x\overline{\breve{U}^{\rm P}}+\partial_x(\mu^{-1}\overline{\breve{U}^{\rm P}})\right)\breve{U}^{\rm P}{\rm d}x{\rm d}z=0,$$

which implies

$$\Re\int_{\Omega_\delta}\left(\mu^{-1}\partial_x\breve{U}^{\rm P}+\partial_x(\mu^{-1}\breve{U}^{\rm P})\right)\overline{\breve{U}^{\rm P}}{\rm d}x{\rm d}z=0.$$

Combining the above estimates, we obtain

$$\Re a_{\rm PML}(\breve{U}^{\rm P},(1+s^{-1}\sigma)\breve{U}^{\rm P})\gtrsim\frac{s_1}{|s+\sigma_0|^2}\left(\|\nabla\breve{U}^{\rm P}\|^2_{L^2(\Omega_\delta)^2}+\|s\breve{U}^{\rm P}\|^2_{L^2(\Omega_\delta)}\right),$$

which implies that

$$|a_{\rm PML}(\breve{U}^{\rm P},\breve{U}^{\rm P})|\gtrsim\frac{|s|s_1}{|s+\sigma_0|^3}\left(\|\nabla\breve{U}^{\rm P}\|^2_{L^2(\Omega_\delta)^2}+\|s\breve{U}^{\rm P}\|^2_{L^2(\Omega_\delta)}\right).\tag{3.9}$$

It follows from the Lax–Milgram theorem and the coercivity of the sesquilinear form $a_{\rm PML}(\cdot,\cdot)$ that the variational problem (3.6) has a unique solution for each $s=s_1+{\rm i}s_2$ with $s_1,s_2\in\mathbb{R},\Re s=s_1>0$.

Moreover, we have from (3.8) that

$$\begin{aligned}|a_{\rm PML}(\breve{U}^{\rm P},\breve{U}^{\rm P})|\le|s|^{-2}(1+s_1^{-1}\sigma_0)&\left(\|\nabla\breve{U}^{\rm P}\|^2_{L^2(\Omega_\delta)^2}+\|s\breve{U}^{\rm P}\|^2_{L^2(\Omega_\delta)}\right)^{1/2}\\&\times\left(\|s\nabla\breve{\zeta}\|^2_{L^2(\Omega_\delta)^2}+\|s^2\breve{\zeta}\|^2_{L^2(\Omega_\delta)}\right)^{1/2}\\&+|s|^{-2}\|\breve{F}\|_{L^2(\Omega_1^{\rm PML})}\|s(\breve{U}^{\rm P}-\breve{\zeta})\|_{L^2(\Omega_1^{\rm PML})}.\end{aligned}\tag{3.10}$$

Combining (3.9) and (3.10) yields

$$\begin{aligned}\|\nabla\breve{U}^{\rm P}\|^2_{L^2(\Omega_\delta)^2}+\|s\breve{U}^{\rm P}\|^2_{L^2(\Omega_\delta)}&\lesssim s_1^{-1}(1+s_1^{-1}\sigma_0)^4\left(\|\nabla\breve{U}^{\rm P}\|^2_{L^2(\Omega_\delta)^2}+\|s\breve{U}^{\rm P}\|^2_{L^2(\Omega_\delta)}\right)^{1/2}\\\times\left(\|s\nabla\breve{\zeta}\|^2_{L^2(\Omega_\delta)^2}+\|s^2\breve{\zeta}\|^2_{L^2(\Omega_\delta)}\right)^{1/2}&+s_1^{-1}(1+s_1^{-1}\sigma_0)^3\|\breve{F}\|_{L^2(\Omega_1^{\rm PML})}\|s(\breve{U}^{\rm P}-\breve{\zeta})\|_{L^2(\Omega_1^{\rm PML})},\end{aligned}$$

which completes the proof after applying the Cauchy–Schwarz inequality and the trace theorem. ∎

The well-posedness and stability of the PML problem (3.3) can be easily established by using Lemma 3.1.

THEOREM 3.1. *The time-domain PML problem* (3.3) *has a unique weak solution* $U^{\rm P}(x,z,t)$ *which satisfies*

$$U^{\rm P}(x,z,t)\in L^2(0,T;H^1_{\rm p}(\Omega_\delta))\cap H^1(0,T;L^2(\Omega_\delta))$$

and the stability estimate

$$\begin{aligned}&\|\partial_t U^{\rm P}\|_{L^2(0,T;L^2(\Omega_\delta))}+\|\nabla U^{\rm P}\|_{L^2(0,T;L^2(\Omega_\delta)^2)}\\ \le& CT(1+\sigma T)^4\Big(\|\partial_t U^{\rm inc}\|_{L^2(0,T;H^{1/2}_{\rm p}(\Gamma_1^{\rm PML}))}+\|\partial_t U^{\rm inc}\|_{L^2(0,T;H^{-1/2}_{\rm p}(\Gamma_1^{\rm PML}))}\\ &+\|\partial_t^2 U^{\rm inc}\|_{L^2(0,T;H^{-1/2}_{\rm p}(\Gamma_1^{\rm PML}))}+\|F\|_{L^2(0,T;L^2(\Omega_1^{\rm PML}))}\Big).\end{aligned}\tag{3.11}$$

Proof. Since

$$\begin{aligned}&\int_0^T\left(\|\nabla U^{\rm P}\|^2_{L^2(\Omega_\delta)^2}+\|\partial_t U^{\rm P}\|^2_{L^2(\Omega_\delta)}\right){\rm d}t\\ \le&\int_0^T e^{-2s_1(t-T)}\left(\|\nabla U^{\rm P}\|^2_{L^2(\Omega_\delta)^2}+\|\partial_t U^{\rm P}\|^2_{L^2(\Omega_\delta)}\right){\rm d}t\\ =&e^{2s_1T}\int_0^T e^{-2s_1t}\left(\|\nabla U^{\rm P}\|^2_{L^2(\Omega_\delta)^2}+\|\partial_t U^{\rm P}\|^2_{L^2(\Omega_\delta)}\right){\rm d}t\\ \lesssim&\int_0^\infty e^{-2s_1t}\left(\|\nabla U^{\rm P}\|^2_{L^2(\Omega_\delta)^2}+\|\partial_t U^{\rm P}\|^2_{L^2(\Omega_\delta)}\right){\rm d}t,\end{aligned}$$

it suffices to estimate the integral

$$\int_0^\infty e^{-2s_1t}\left(\|\nabla U^{\rm P}\|^2_{L^2(\Omega_\delta)^2}+\|\partial_t U^{\rm P}\|^2_{L^2(\Omega_\delta)}\right){\rm d}t.$$

Taking the Laplace transform of (3.3) and eliminating $\breve{\phi}$, we obtain

$$\begin{aligned}(\varepsilon-c_1^2\mu^{-1})s(1+\frac{\sigma}{s})\breve{U}^{\rm P}=&\partial_z\left((s\mu)^{-1}(1+\frac{\sigma}{s})^{-1}\partial_z\breve{U}^{\rm P}\right)+(1+\frac{\sigma}{s})\partial_x\left((s\mu)^{-1}\partial_x\breve{U}^{\rm P}\right)\\ &-c_1(1+\frac{\sigma}{s})\left(\mu^{-1}\partial_x\breve{U}^{\rm P}+\partial_x(\mu^{-1}\breve{U}^{\rm P})\right)+s^{-1}\breve{F}\qquad\text{in }\Omega_\delta\end{aligned}$$

and the boundary conditions

$$\begin{cases}\breve{U}^{\rm P}=\breve{U}^{\rm inc}&\text{on }\Gamma_1^{\rm PML},\\ \breve{U}^{\rm P}=0&\text{on }\Gamma_2^{\rm PML}.\end{cases}$$

By Lemma 3.1, we have

$$\begin{aligned}\|\nabla\breve{U}^{\rm P}\|_{L^2(\Omega_\delta)^2}+\|s\breve{U}^{\rm P}\|_{L^2(\Omega_\delta)}\lesssim& s_1^{-1}(1+s_1^{-1}\sigma_0)^4\Big(\|s\breve{U}^{\rm inc}\|_{H^{1/2}_{\rm p}(\Gamma_1^{\rm PML})}+\|s\breve{U}^{\rm inc}\|_{H^{-1/2}_{\rm p}(\Gamma_1^{\rm PML})}\\ &+\|s^2\breve{U}^{\rm inc}\|_{H^{-1/2}_{\rm p}(\Gamma_1^{\rm PML})}+\|\breve{F}\|_{L^2(\Omega_1^{\rm PML})}\Big).\end{aligned}\tag{3.12}$$

It follows from [33, Lemma 44.1] that $\breve{U}^{\rm P}$ is a holomorphic function of s on the half plane $s_1>s_0>0$, where s_0 is any positive constant. Hence we have from Lemma 2.2 that the inverse Laplace transform of $\breve{U}^{\rm P}$ exists and is supported in $[0,\infty)$.

Denote $U^{\rm P}=\mathscr{L}^{-1}(\breve{U}^{\rm P})$. It follows from (2.12) that

$$\breve{U}^{\rm P}=\mathscr{L}(U^{\rm P})=\mathscr{F}(e^{-s_1t}U^{\rm P}),$$

where $\mathscr{F}$ denotes the Fourier transform with respect to s_2. Hence we have from the Parseval identity and (3.12) that

$$\begin{aligned}&\int_0^\infty e^{-2s_1t}\left(\|\nabla U^{\rm P}\|^2_{L^2(\Omega_\delta)^2}+\|\partial_t U^{\rm P}\|^2_{L^2(\Omega_\delta)}\right){\rm d}t\\ =&2\pi\int_{-\infty}^{\infty}\left(\|\nabla\breve{U}^{\rm P}\|^2_{L^2(\Omega_\delta)^2}+\|s\breve{U}^{\rm P}\|^2_{L^2(\Omega_\delta)}\right){\rm d}s_2\\ \lesssim& s_1^{-2}(1+s_1^{-1}\sigma_0)^8\int_{-\infty}^{\infty}\left(\|s\breve{U}^{\rm inc}\|^2_{H_{\rm p}^{1/2}(\Gamma_1^{\rm PML})}+\|s\breve{U}^{\rm inc}\|^2_{H_{\rm p}^{-1/2}(\Gamma_1^{\rm PML})}\right.\\ &\qquad\left.+\|s^2\breve{U}^{\rm inc}\|^2_{H_{\rm p}^{-1/2}(\Gamma_1^{\rm PML})}+\|\breve{F}\|^2_{L^2(\Omega_1^{\rm PML})}\right){\rm d}s_2.\end{aligned}$$

Using the Parseval identity again, we obtain

$$\begin{aligned}&\int_0^\infty e^{-2s_1t}\left(\|\nabla U^{\rm P}\|^2_{L^2(\Omega_\delta)^2}+\|\partial_t U^{\rm P}\|^2_{L^2(\Omega_\delta)}\right){\rm d}t\\ \lesssim& s_1^{-2}(1+s_1^{-1}\sigma_0)^8\int_0^\infty e^{-2s_1t}\left(\|\partial_t U^{\rm inc}\|^2_{H_{\rm p}^{1/2}(\Gamma_1^{\rm PML})}+\|\partial_t U^{\rm inc}\|^2_{H_{\rm p}^{-1/2}(\Gamma_1^{\rm PML})}\right.\\ &\left.+\|\partial_t^2 U^{\rm inc}\|^2_{H_{\rm p}^{-1/2}(\Gamma_1^{\rm PML})}+\|F\|^2_{L^2(\Omega_1^{\rm PML})}\right){\rm d}t,\end{aligned}\tag{3.13}$$

which shows that

$$U^{\rm P}(x,z,t)\in L^2(0,T;H_{\rm p}^1(\Omega_\delta))\cap H^1(0,T;L^2(\Omega_\delta)).$$

On the other hand, (3.13) implies that

$$\begin{aligned}&\int_0^T\left(\|\nabla U^{\rm P}\|^2_{L^2(\Omega_\delta)^2}+\|\partial_t U^{\rm P}\|^2_{L^2(\Omega_\delta)}\right){\rm d}t\\ \lesssim& s_1^{-2}(1+s_1^{-1}\sigma_0)^8e^{2s_1T}\int_0^T\left(\|\partial_t U^{\rm inc}\|^2_{H_{\rm p}^{1/2}(\Gamma_1^{\rm PML})}+\|\partial_t U^{\rm inc}\|^2_{H_{\rm p}^{-1/2}(\Gamma_1^{\rm PML})}\right.\\ &\left.+\|\partial_t^2 U^{\rm inc}\|^2_{H_{\rm p}^{-1/2}(\Gamma_1^{\rm PML})}+\|F\|^2_{L^2(\Omega_1^{\rm PML})}\right){\rm d}t.\end{aligned}$$

The stability estimate (3.11) in the theorem is obtained by taking $s_1=1/T$. □

3.3. Transparent boundary condition. We introduce the transparent boundary condition for the PML problem so that it can be reformulated into an equivalent initial boundary problem in the domain Ω.

It follows from (3.1) that the scattered electric field $\breve{U}^{\rm sc,P}=\breve{U}^{\rm P}-\breve{U}^{\rm inc}$ satisfies

$$(\varepsilon_1\mu_1-c_1^2)s^2\breve{U}^{\rm sc,P}=\partial_{\tilde z}^2\breve{U}^{\rm sc,P}+\partial_x^2\breve{U}^{\rm sc,P}-2c_1s\partial_x\breve{U}^{\rm sc,P}\quad\text{in }\Omega_1^{\rm PML}.\tag{3.14}$$

Since $\breve{U}^{\rm sc,P}$ is a periodic function of x, it has the Fourier series expansion

$$\breve{U}^{\rm sc,P}(x,\tilde z,s)=\sum_{n\in\mathbb{Z}}\breve{U}_n^{\rm sc,P}(\tilde z,s)e^{{\rm i}\alpha_n x},\quad z>h_1.\tag{3.15}$$

Substituting the above expansion into (3.14), we obtain an ordinary differential problem for the Fourier coefficients:

$$\begin{cases}\partial_{\tilde z}^2 \breve U_n^{\rm sc,P} - (\beta_1^n)^2 \breve U_n^{\rm sc,P} = 0, \quad h_1 < z < h_1+\delta_1,\\ \breve U_n^{\rm sc,P}(\tilde z(h_1), s) = \breve U_n^{\rm sc,P}(h_1,s),\\ \breve U_n^{\rm sc,P}(\tilde z(h_1+\delta_1), s) = 0.\end{cases} \tag{3.16}$$

The general solution of (3.16) is a linear combination of the fundamental solution:

$$\breve U_n^{\rm sc,P}(z,s) = A_n e^{\beta_1^n(s)\tilde z(z)} + B_n e^{-\beta_1^n(s)\tilde z(z)},$$

where $A_n, B_n \in \mathbb{C}$ are two constants. Using the boundary conditions in (3.16), we have

$$\begin{pmatrix} e^{\beta_1^n(s)\tilde z(h_1)} & e^{-\beta_1^n(s)\tilde z(h_1)} \\ e^{\beta_1^n(s)\tilde z(h_1+\delta_1)} & e^{-\beta_1^n(s)\tilde z(h_1+\delta_1)}\end{pmatrix}\begin{pmatrix} A_n\\ B_n\end{pmatrix} = \begin{pmatrix}\breve U_n^{\rm sc,P}(h_1,s)\\ 0\end{pmatrix}.$$

A straightforward computation gives

$$A_n = \frac{e^{-\beta_1^n(s)(\gamma_1+h_1)}e^{-\beta_1^n(s)\gamma_1}}{e^{-2\beta_1^n(s)\gamma_1}-1}\breve U_n^{\rm sc,P}(h_1,s),$$
$$B_n = -\frac{e^{\beta_1^n(s)(\gamma_1+h_1)}e^{-\beta_1^n(s)\gamma_1}}{e^{-2\beta_1^n(s)\gamma_1}-1}\breve U_n^{\rm sc,P}(h_1,s),$$

where

$$\gamma_1 = \tilde z(h_1+\delta_1) - \tilde z(h_1) = \delta_1\left(1+s^{-1}\bar\sigma_1\right), \quad \bar\sigma_1 = \delta_1^{-1}\int_{h_1}^{h_1+\delta_1}\sigma(z')\mathrm{d}z'. \tag{3.17}$$

Hence we have the following representation for $\breve U_n^{\rm sc,P}(\tilde z, s)$:

$$\breve U_n^{\rm sc,P}(\tilde z,s) = \frac{e^{-\beta_1^n(s)(\gamma_1+h_1)}e^{-\beta_1^n(s)\gamma_1}}{e^{-2\beta_1^n(s)\gamma_1}-1}\breve U_n^{\rm sc,P}(h_1,s)e^{\beta_1^n(s)\tilde z(z)}$$
$$- \frac{e^{\beta_1^n(s)(\gamma_1+h_1)}e^{-\beta_1^n(s)\gamma_1}}{e^{-2\beta_1^n(s)\gamma_1}-1}\breve U_n^{\rm sc,P}(h_1,s)e^{-\beta_1^n(s)\tilde z(z)},$$

which together with (3.15) implies

$$\breve U^{\rm sc,P}(x,\tilde z,s) = \sum_{n\in\mathbb{Z}}\Big(\frac{e^{-\beta_1^n(s)(\gamma_1+h_1)}e^{-\beta_1^n(s)\gamma_1}}{e^{-2\beta_1^n(s)\gamma_1}-1}\breve U_n^{\rm sc,P}(h_1,s)e^{\beta_1^n(s)\tilde z(z)}$$
$$- \frac{e^{\beta_1^n(s)(\gamma_1+h_1)}e^{-\beta_1^n(s)\gamma_1}}{e^{-2\beta_1^n(s)\gamma_1}-1}\breve U_n^{\rm sc,P}(h_1,s)e^{-\beta_1^n(s)\tilde z(z)}\Big)e^{\mathrm{i}\alpha_n x}.$$

Taking the normal derivative of the above equation on Γ_1 yields

$$\partial_{\nu_1}\breve U^{\rm sc,P}(h_1,s) = \sum_{n\in\mathbb{Z}}\beta_1^n(s)\frac{e^{-2\beta_1^n(s)\gamma_1}+1}{e^{-2\beta_1^n(s)\gamma_1}-1}\breve U_n^{\rm sc,P}(h_1,s)e^{\mathrm{i}\alpha_n x},$$

where $\nu_1 = (0,1)$ is the unit normal vector on Γ_1.

Similarly, we have the following representation for the total field $\breve U^{\rm P}(x,\tilde z,s)$ in $\Omega_2^{\rm PML}$:

$$\breve U^{\rm P}(x,\tilde z,s) = \sum_{n\in\mathbb{Z}}\Big(-\frac{e^{\beta_2^n(s)(\gamma_2-h_2)}e^{-\beta_2^n(s)\gamma_2}}{e^{-2\beta_2^n(s)\gamma_2}-1}\breve U_n^{\rm P}(h_2,s)e^{\beta_2^n(s)\tilde z(z)}$$

$$+\frac{e^{-\beta_2^n(s)(\gamma_2-h_2)}e^{-\beta_2^n(s)\gamma_2}}{e^{-2\beta_2^n(s)\gamma_2}-1}\breve{U}_n^{\rm P}(h_2,s)e^{-\beta_2^n(s)\tilde{z}(z)}\Big)e^{{\rm i}\alpha_n x},$$

where

$$\gamma_2=\tilde{z}(h_2)-\tilde{z}(h_2-\delta_2)=\delta_2\left(1+s^{-1}\bar{\sigma}_2\right),\quad \bar{\sigma}_2=\delta_2^{-1}\int_{h_2-\delta_2}^{h_2}\sigma(z'){\rm d}z'. \tag{3.18}$$

Taking the normal derivative of the above equation on Γ_2 yields

$$\partial_{\nu_2}\breve{U}^{\rm P}(h_2,s)=\sum_{n\in\mathbb{Z}}\beta_2^n(s)\frac{e^{-2\beta_2^n(s)\gamma_2}+1}{e^{-2\beta_2^n(s)\gamma_2}-1}\breve{U}_n^{\rm P}(h_2,s)e^{{\rm i}\alpha_n x},$$

where $\nu_2=(0,-1)$ is the unit normal vector on Γ_2.

For any function $u(x,h_j)$ defined on Γ_j with the Fourier expansion $u(x,h_j)=\sum_{n\in\mathbb{Z}}u_n(h_j)e^{{\rm i}\alpha_n x}$, we define the DtN operators

$$(\mathscr{B}_j^{\rm PML}u)(x,h_j)=\sum_{n\in\mathbb{Z}}\beta_j^n(s)\frac{e^{-2\beta_j^n(s)\gamma_j}+1}{e^{-2\beta_j^n(s)\gamma_j}-1}u_n(h_j)e^{{\rm i}\alpha_n x}. \tag{3.19}$$

Using the DtN operators (3.19), we obtain the following TBCs for the PML problem in the frequency domain:

$$\begin{cases}\partial_{\nu_1}\breve{U}^{\rm P}=\mathscr{B}_1^{\rm PML}\breve{U}^{\rm P}+\breve{f}^{\rm P} & \text{on } \Gamma_1,\\ \partial_{\nu_2}\breve{U}^{\rm P}=\mathscr{B}_2^{\rm PML}\breve{U}^{\rm P} & \text{on } \Gamma_2,\end{cases} \tag{3.20}$$

where $\breve{f}^{\rm P}=\partial_{\nu_1}\breve{U}^{\rm inc}-\mathscr{B}_1^{\rm PML}\breve{U}^{\rm inc}$. Taking the inverse Laplace transform of (3.20) yields the TBCs for the time-domain PML problem:

$$\begin{cases}\partial_{\nu_1}U^{\rm P}=\mathscr{T}_1^{\rm PML}U^{\rm P}+f^{\rm P} & \text{on } \Gamma_1,\\ \partial_{\nu_2}U^{\rm P}=\mathscr{T}_2^{\rm PML}U^{\rm P} & \text{on } \Gamma_2,\end{cases} \tag{3.21}$$

where $f^{\rm P}$ is the inverse Laplace transform of $\breve{f}^{\rm P}$, i.e., $f^{\rm P}=\mathscr{L}^{-1}(\breve{f}^{\rm P})$, and $\mathscr{T}_j^{\rm PML}=\mathscr{L}^{-1}\circ\mathscr{B}_j^{\rm PML}\circ\mathscr{L}, j=1,2$.

4. Convergence analysis

This section is devoted to the convergence analysis of the time-domain PML problem. We derive an error estimate for the solutions between the original scattering problem and the PML problem.

Using the time-domain TBCs (3.21), eliminating ϕ and noting $F=0$ in Ω, we reformulate the time-domain PML problem (3.3) into an equivalent initial boundary value problem in Ω:

$$\begin{cases}(\varepsilon-\frac{c_1^2}{\mu})\partial_t^2U^{\rm P}=\nabla\cdot(\mu^{-1}\nabla U^{\rm P})-c_1\left(\mu^{-1}\partial_{tx}U^{\rm P}+\partial_x(\mu^{-1}\partial_t U^{\rm P})\right) & \text{in } \Omega, t>0,\\ U^{\rm P}|_{t=0}=\partial_t U^{\rm P}|_{t=0}=0 & \text{in } \Omega,\\ \partial_{\nu_1}U^{\rm P}=\mathscr{T}_1^{\rm PML}U^{\rm P}+f^{\rm P} & \text{on } \Gamma_1, t>0,\\ \partial_{\nu_2}U^{\rm P}=\mathscr{T}_2^{\rm PML}U^{\rm P} & \text{on } \Gamma_2, t>0.\end{cases} \tag{4.1}$$

Apparently, the key to the convergence analysis is to estimate the error of the DtN operators $\mathscr{B}_j$ and $\mathscr{B}_j^{\rm PML}$, which can be written as follows:

$$
\begin{aligned}
(\mathcal{B}_j u)(x,h_j) &= (\mathscr{B}_j u)(x,h_j) - (\mathscr{B}_j^{\rm PML} u)(x,h_j) \\
&= \sum_{n\in\mathbb{Z}} (-1)\beta_j^n(s) u_n(h_j) e^{{\rm i}\alpha_n x} - \sum_{n\in\mathbb{Z}} \beta_j^n(s) \frac{e^{-2\beta_j^n(s)\gamma_j}+1}{e^{-2\beta_j^n(s)\gamma_j}-1} u_n(h_j) e^{{\rm i}\alpha_n x} \\
&= \sum_{n\in\mathbb{Z}} \beta_j^n(s) \frac{2e^{-2\beta_j^n(s)\gamma_j}}{1-e^{-2\beta_j^n(s)\gamma_j}} u_n(h_j) e^{{\rm i}\alpha_n x}, \quad j=1,2. \qquad (4.2)
\end{aligned}
$$

Let

$$
\beta_j^n(s) = \left(\varepsilon_j\mu_j s^2 + (\alpha_n + {\rm i}c_1 s)^2\right)^{1/2} := a_j^n(s) + {\rm i}b_j^n(s), \quad a_j^n(s) > 0.
$$

It is easy to note that

$$
(a_j^n(s))^2 - (b_j^n(s))^2 = (\varepsilon_j\mu_j - c_1^2)(s_1^2 - s_2^2) + \alpha_n^2 - 2\alpha_n c_1 s_2 \qquad (4.3)
$$

and

$$
a_j^n(s) b_j^n(s) = (\varepsilon_j\mu_j - c_1^2) s_1 s_2 + \alpha_n c_1 s_1. \qquad (4.4)
$$

It follows from a straightforward calculation that

$$
(a_j^n(s))^2 = \frac{1}{2}\left((\varepsilon_j\mu_j - c_1^2)(s_1^2 - s_2^2) + \alpha_n^2 - 2\alpha_n c_1 s_2\right) + \frac{1}{2}\sqrt{\Delta_1}, \qquad (4.5)
$$

$$
(b_j^n(s))^2 = -\frac{1}{2}\left((\varepsilon_j\mu_j - c_1^2)(s_1^2 - s_2^2) + \alpha_n^2 - 2\alpha_n c_1 s_2\right) + \frac{1}{2}\sqrt{\Delta_1}, \qquad (4.6)
$$

where

$$
\Delta_1 = \left((\varepsilon_j\mu_j - c_1^2)(s_1^2 - s_2^2) + \alpha_n^2 - 2\alpha_n c_1 s_2\right)^2 + 4((\varepsilon_j\mu_j - c_1^2) s_1 s_2 + \alpha_n c_1 s_1)^2.
$$

Let

$$
s^{-1}\beta_j^n(s) = c_j^n(s) + {\rm i}d_j^n(s).
$$

By a simple calculation, we have

$$
(c_j^n(s))^2 - (d_j^n(s))^2 = \frac{(\varepsilon_j\mu_j - c_1^2)|s|^4 + \alpha_n^2(s_1^2 - s_2^2) + 2\alpha_n c_1 |s|^2 s_2}{|s|^4} \qquad (4.7)
$$

and

$$
c_j^n(s) d_j^n(s) = \frac{-\alpha_n^2 s_1 s_2 + \alpha_n c_1 |s|^2 s_1}{|s|^4}. \qquad (4.8)
$$

It follows from (4.7) and (4.8) that

$$
(c_j^n(s))^2 = \frac{(\varepsilon_j\mu_j - c_1^2)|s|^4 + \alpha_n^2(s_1^2 - s_2^2) + 2\alpha_n c_1 |s|^2 s_2 + \sqrt{\Delta_2}}{2|s|^4}, \qquad (4.9)
$$

where

$$
\Delta_2 = \left((\varepsilon_j\mu_j - c_1^2)|s|^4 + \alpha_n^2(s_1^2 - s_2^2) + 2\alpha_n c_1 |s|^2 s_2\right)^2 + 4\left(-\alpha_n^2 s_1 s_2 + \alpha_n c_1 |s|^2 s_1\right)^2.
$$

Lemma 4.1. *For any $s=s_1+\mathrm{i}s_2$ with $s_1,s_2\in\mathbb{R}, s_1>0$, $\Re\beta_j^n(s)\geq(\varepsilon_j\mu_j-c_1^2)^{1/2}\Re s$.*

Proof. Using (4.5), we have

$$(a_j^n(s))^2-(\varepsilon_j\mu_j-c_1^2)s_1^2=\frac{1}{2}\left(-(\varepsilon_j\mu_j-c_1^2)(s_1^2+s_2^2)+\alpha_n^2-2\alpha_nc_1s_2\right)+\frac{1}{2}\sqrt{\Delta_1}. \tag{4.10}$$

By a simple calculation, we can rewrite Δ in the following form:

$$\begin{aligned}\Delta_1&=\left((\varepsilon_j\mu_j-c_1^2)(s_1^2-s_2^2)\right)^2+\left(\alpha_n^2-2\alpha_nc_1s_2\right)^2+2(\varepsilon_j\mu_j-c_1^2)(s_1^2-s_2^2)(\alpha_n^2-2\alpha_nc_1s_2)\\&\quad+4((\varepsilon_j\mu_j-c_1^2)s_1s_2+\alpha_nc_1s_1)^2\\&=\left((\varepsilon_j\mu_j-c_1^2)(s_1^2+s_2^2)\right)^2+\left(\alpha_n^2-2\alpha_nc_1s_2\right)^2-2(\varepsilon_j\mu_j-c_1^2)(s_1^2+s_2^2)(\alpha_n^2-2\alpha_nc_1s_2)\\&\quad-4((\varepsilon_j\mu_j-c_1^2)s_1s_2)^2+4(\varepsilon_j\mu_j-c_1^2)s_1^2(\alpha_n^2-2\alpha_nc_1s_2)\\&\quad+4((\varepsilon_j\mu_j-c_1^2)s_1s_2+\alpha_nc_1s_1)^2\\&=\left((\varepsilon_j\mu_j-c_1^2)(s_1^2+s_2^2)-(\alpha_n^2-2\alpha_nc_1s_2)\right)^2+4\varepsilon_j\mu_js_1^2\alpha_n^2,\end{aligned}$$

which together with (4.10) implies

$$(a_j^n(s))^2-(\varepsilon_j\mu_j-c_1^2)s_1^2\geq 0.$$

Since $a_j^n(s)>0$, we get that $a_j^n(s)\geq(\varepsilon_j\mu_j-c_1^2)^{1/2}s_1$, which completes the proof. ∎

Lemma 4.2. *For any $s=s_1+\mathrm{i}s_2$ with $s_1,s_2\in\mathbb{R}, s_1>0$, $\Re s^{-1}\beta_j^n(s)>0$.*

Proof. It is easy to note that

$$c_j^n(s)=\frac{s_1a_j^n(s)+s_2b_j^n(s)}{|s|^2}. \tag{4.11}$$

It follows from (4.4) that

$$s_1a_j^n(s)+s_2b_j^n(s)=\frac{s_1}{a_j^n(s)}\left((a_j^n(s))^2+(\varepsilon_j\mu_j-c_1^2)s_2^2+\alpha_nc_1s_2\right). \tag{4.12}$$

Plugging (4.3) into (4.12) yields

$$s_1a_j^n(s)+s_2b_j^n(s)=\frac{s_1}{a_j^n(s)}\left((b_j^n(s))^2+(\varepsilon_j\mu_j-c_1^2)s_1^2+\alpha_n^2-\alpha_nc_1s_2\right). \tag{4.13}$$

Adding (4.12) and (4.13), we obtain

$$s_1a_j^n(s)+s_2b_j^n(s)=\frac{s_1}{2a_j^n(s)}\left((a_j^n(s))^2+(b_j^n(s))^2+(\varepsilon_j\mu_j-c_1^2)(s_1^2+s_2^2)+\alpha_n^2\right)>0, \tag{4.14}$$

which completes the proof by combining (4.11). ∎

Let $\delta=\min\{\delta_1,\delta_2\}$ and $\bar{\sigma}=\min\{\bar{\sigma}_1,\bar{\sigma}_2\}$. The following lemma plays a key role in the subsequent analysis.

Lemma 4.3. *For any $u,v\in H_{s,\mathrm{p}}^{1/2}(\Gamma_j)$, the following estimate holds:*

$$\left|\langle\mathcal{B}_ju,v\rangle_{\Gamma_j}\right|\lesssim C_3C_4e^{-\eta\delta\bar{\sigma}}\|u\|_{H_{s,\mathrm{p}}^{1/2}(\Gamma_j)}\|v\|_{H_{s,\mathrm{p}}^{1/2}(\Gamma_j)},$$

where η is positive constant independent of s_2 and α_n, $C_3=\max\{1,(2\delta(\varepsilon_{\min}\mu_{\min}-c_1^2)^{1/2}s_1)^{-1}\}$ and $C_4=\max\{\sqrt{2},(2c_1^2+\varepsilon_{\max}\mu_{\max})^{1/2}\}$.

Proof. It is easy to note that

$$|e^{-\beta_j^n(s)\gamma_j}| = e^{-a_j^n(s)\delta_j - c_j^n(s)\delta_j\bar{\sigma}_j}. \tag{4.15}$$

A direct calculation yields

$$\begin{aligned}
\Delta_2 &= |s|^4\Big((\varepsilon_j\mu_j - c_1^2)^2|s|^4 + \alpha_n^4 + 4\alpha_n^2c_1^2s_1^2 + 4\alpha_n^2c_1^2s_2^2 - 2(\varepsilon_j\mu_j - c_1^2)\alpha_n^2(s_1^2+s_2^2)\\
&\quad +4(\varepsilon_j\mu_j - c_1^2)\alpha_n^2s_1^2 + 4(\varepsilon_j\mu_j - c_1^2)\alpha_nc_1|s|^2s_2 - 4\alpha_n^3c_1s_2\Big)\\
&= |s|^4\Big((\varepsilon_j\mu_j - c_1^2)^2(s_1^2+s_2^2)^2 + (\alpha_n^2 - 2\alpha_nc_1s_2)^2\\
&\quad -2\big((\varepsilon_j\mu_j - c_1^2)(s_1^2+s_2^2)(\alpha_n^2 - 2\alpha_nc_1s_2)\big) + 4\varepsilon_j\mu_j\alpha_n^2s_1^2\Big)\\
&= |s|^4\Delta_1.
\end{aligned} \tag{4.16}$$

It follows from (4.9) and (4.16) that

$$(c_j^n(s))^2 = \frac{(\varepsilon_j\mu_j - c_1^2)|s|^4 + \alpha_n^2(s_1^2 - s_2^2) + 2\alpha_nc_1|s|^2s_2 + |s|^2\sqrt{\Delta_1}}{2|s|^4}$$

For any fixed α_n, we have

$$\lim_{|s_2|\to\infty} c_j^n(s) = (\varepsilon_j\mu_j - c_1^2)^{1/2}. \tag{4.17}$$

On the other hand, using (4.11) and (4.14), we obtain

$$c_j^n(s) = \frac{s_1}{2a_j^n(s)|s|^2}\big((a_j^n(s))^2 + (b_j^n(s))^2 + (\varepsilon_j\mu_j - c_1^2)(s_1^2+s_2^2) + \alpha_n^2\big) \to +\infty$$

as $|\alpha_n|\to\infty$. So there exists a positive constant η independent of s_2,α_n such that

$$c_j^n(s) > \eta \quad \forall s_2,\alpha_n. \tag{4.18}$$

Using (4.15), (4.18) and Lemma 4.1, we have

$$|e^{-\beta_j^n(s)\gamma_j}| \ge e^{-\eta\delta_j\bar{\sigma}_j}. \tag{4.19}$$

Applying Lemmas 4.1 and 4.2 leads to

$$|e^{-\beta_j^n(s)\gamma_j}| = e^{-\delta_j(\varepsilon_j\mu_j - c_1^2)^{1/2}s_1}, \tag{4.20}$$

which deduces that

$$|1 - e^{-2\beta_j^n(s)\gamma_j}| \ge 1 - e^{-2\delta_j(\varepsilon_j\mu_j - c_1^2)^{1/2}s_1}.$$

For $x>0$, it is easy to verify that

$$1 - e^{-x} > \frac{1}{2}\min\{1,x\}.$$

Thus

$$|1 - e^{-2\beta_j^n(s)\gamma_j}|^{-1} \le 2\max\{1,(2\delta_j(\varepsilon_j\mu_j - c_1^2)^{1/2}s_1)^{-1}\}. \tag{4.21}$$

Combining(4.19) and (4.21) yields

$$\left|\frac{2e^{-2\beta_j^n(s)\gamma_j}}{1-e^{-2\beta_j^n(s)\gamma_j}}\right| \le 4C_3 e^{-\eta\delta_j\bar{\sigma}_j}, \tag{4.22}$$

where $C_3 = \max\{1, (2\delta(\varepsilon_{\min}\mu_{\min} - c_1^2)^{1/2} s_1)^{-1}\}$.

It can be verified that

$$|\beta_j^n(s)|^2 = |\varepsilon_j\mu_j s^2 + (\alpha_n + \mathrm{i}c_1 s)^2| \le \varepsilon_j\mu_j|s|^2 + 2(\alpha_n^2 + c_1^2|s|^2) \le C_4^2(|s|^2 + \alpha_n^2), \tag{4.23}$$

where

$$C_4 = \max\{\sqrt{2}, (2c_1^2 + \varepsilon_{\max}\mu_{\max})^{1/2}\}.$$

It follows from (4.2), (4.22) and (4.23) that

$$\begin{aligned}\left|\langle \mathcal{B}_j u, v\rangle_{\Gamma_j}\right| &\le \sum_{n\in\mathbb{Z}} |\beta_j^n(s)| \left|\frac{2e^{-2\beta_j^n(s)\gamma_j}}{1-e^{-2\beta_j^n(s)\gamma_j}}\right| |u_n(h_j)||v_n(h_j)| \\ &\lesssim \sum_{n\in\mathbb{Z}} C_4(|s|^2+\alpha_n^2)^{1/2} C_3 e^{-\eta\delta_j\bar{\sigma}_j}|u_n(h_j)||v_n(h_j)| \\ &\lesssim C_3C_4 e^{-\eta\delta\bar{\sigma}} \|u\|_{H^{1/2}_{s,\mathrm{p}}(\Gamma_j)} \|v\|_{H^{1/2}_{s,\mathrm{p}}(\Gamma_j)},\end{aligned}$$

which completes the proof. ∎

Let $\mathcal{T}_j = \mathscr{L}^{-1}\circ\mathcal{B}_j\circ\mathscr{L}$ and $V = U - U^{\mathrm{P}}$. It follows from (2.19) and (4.1) that

$$\begin{cases} (\varepsilon - c_1^2\mu^{-1})\partial_t^2 V = \nabla\cdot(\mu^{-1}\nabla V) - c_1\left(\mu^{-1}\partial_{tx}V + \partial_x(\mu^{-1}\partial_t V)\right) & \text{in } \Omega, t>0, \\ V|_{t=0} = \partial_t V|_{t=0} = 0 & \text{in } \Omega, \\ \partial_{\nu_1} V = \mathscr{T}_1 V + \mathcal{T}_1 U^{\mathrm{P}} - \mathcal{T}_1 U^{\mathrm{inc}} & \text{on } \Gamma_1, t>0, \\ \partial_{\nu_2} V = \mathscr{T}_2 V + \mathcal{T}_2 U^{\mathrm{P}} & \text{on } \Gamma_2, t>0. \end{cases} \tag{4.24}$$

The variational problem of (4.24) is to find $V \in H^1_{s,\mathrm{p}}(\Omega)$ for all $t>0$ such that

$$\begin{aligned}&\int_\Omega (\varepsilon - c_1^2\mu^{-1})\partial_t^2 V \bar{w}\,\mathrm{d}x\mathrm{d}z \\ &= -\int_\Omega \mu^{-1}\nabla V\cdot\nabla\bar{w}\,\mathrm{d}x\mathrm{d}z + \sum_{j=1}^2 \int_{\Gamma_j} \mu_j^{-1}\mathscr{T}_j V\bar{w}\,\mathrm{d}x \\ &\quad + \int_{\Gamma_1} \mu_1^{-1}\mathcal{T}_1 U^{\mathrm{P}}\bar{w}\,\mathrm{d}x - \int_{\Gamma_1}\mu_1^{-1}\mathcal{T}_1 U^{\mathrm{inc}}\bar{w}\,\mathrm{d}x + \int_{\Gamma_2}\mu_2^{-1}\mathcal{T}_2 U^{\mathrm{P}}\bar{w}\,\mathrm{d}x \\ &\quad - c_1\int_\Omega \left(\mu^{-1}\partial_{tx}V + \partial_x(\mu^{-1}\partial_t V)\right)\bar{w}\,\mathrm{d}x\mathrm{d}z \quad \forall w\in H^1_{s,\mathrm{p}}(\Omega).\end{aligned} \tag{4.25}$$

We are now ready to prove the exponential convergence of the time-domain PML method, as stated in the following theorem. A similar proof is used in [25, Theorem 3.4] to show the stability of the solution to the time-domain grating problem.

THEOREM 4.1. *Let U and U^{P} be the solutions of the problems (2.19) and (4.1) with $s_1 = 1/T$, respectively. If the Assumptions (3.4) and (3.5) are satisfied, then*

$$\|U - U^{\mathrm{P}}\|_{L^\infty(0,T;L^2(\Omega))} + \|\nabla(U-U^{\mathrm{P}})\|_{L^\infty(0,T;L^2(\Omega)^2)} + \|\partial_t(U-U^{\mathrm{P}})\|_{L^\infty(0,T;L^2(\Omega))}$$

$$\lesssim C(T)e^{-\eta\delta\bar{\sigma}}\Big(\|F\|_{L^1(0,T;L^2(\Omega_1^{\rm PML}))}+\|\partial_t F\|_{L^1(0,T;L^2(\Omega_1^{\rm PML}))}+\|U^{\rm inc}\|_{L^1(0,T;H_{s,{\rm p}}^{1/2}(\Gamma_1))}$$
$$+\|\partial_t U^{\rm inc}\|_{L^1(0,T;H_{s,{\rm p}}^{1/2}(\Gamma_1))}+\|\partial_t^2 U^{\rm inc}\|_{L^1(0,T;H_{s,{\rm p}}^{1/2}(\Gamma_1))}+\|\partial_t U^{\rm inc}\|_{L^1(0,T;H_{s,{\rm p}}^{1/2}(\Gamma_1^{\rm PML}))}$$
$$+\|\partial_t^2 U^{\rm inc}\|_{L^1(0,T;H_{s,{\rm p}}^{1/2}(\Gamma_1^{\rm PML}))}+\|\partial_t U^{\rm inc}\|_{L^1(0,T;H_{s,{\rm p}}^{-1/2}(\Gamma_1^{\rm PML}))}$$
$$+\|\partial_t^2 U^{\rm inc}\|_{L^1(0,T;H_{s,{\rm p}}^{-1/2}(\Gamma_1^{\rm PML}))}+\|\partial_t^3 U^{\rm inc}\|_{L^1(0,T;H_{s,{\rm p}}^{-1/2}(\Gamma_1^{\rm PML}))}\Big),$$

where

$$C(T)=\max\{1,T\}\max\{1,T(\delta\eta)^{-1}\}\max\{1,T(1+T\sigma_0)^4\}$$
$$\times\max\{1+(h_1-h_2)^{-1}T,1+(2\pi)^{-1}(h_1-h_2)^{-1}\Lambda\}.$$

Proof. Let $0<\xi<T$ and define an auxiliary function

$$\psi_1(x,z,t)=\int_t^\xi V(x,z,\tau){\rm d}\tau,\quad (x,z)\in\Omega,\ 0<t<\xi.$$

It is easy to check that

$$\psi_1(x,z,\xi)=0,\quad \partial_t\psi_1(x,z,t)=-V(x,z,t). \tag{4.26}$$

For any $\phi(x,z,t)\in L^2(0,\xi;L^2(\Omega))$, it follows from [25, Eq. (3.14)] that

$$\int_0^\xi \phi(x,z,t)\overline{\psi_1}(x,z,t){\rm d}t=\int_0^\xi\left(\int_0^t\phi(x,z,\tau){\rm d}\tau\right)\overline{V}(x,z,t){\rm d}t. \tag{4.27}$$

Taking the test function $w=\psi_1$ in (4.25), then integrating from $t=0$ to $t=\xi$ and taking the real part, we obtain

$$\Re\int_0^\xi\int_\Omega(\varepsilon-c_1^2\mu^{-1})\partial_t^2V\overline{\psi_1}{\rm d}x{\rm d}z{\rm d}t+\Re\int_0^\xi\int_\Omega\mu^{-1}\nabla V\cdot\nabla\overline{\psi_1}{\rm d}x{\rm d}z{\rm d}t$$
$$=\Re\int_0^\xi\sum_{j=1}^2\int_{\Gamma_j}\mu_j^{-1}\mathscr{T}_jV\overline{\psi_1}{\rm d}x{\rm d}t+\Re\int_0^\xi\int_{\Gamma_1}\mu_1^{-1}\mathcal{T}_1U^{\rm P}\overline{\psi_1}{\rm d}x{\rm d}t$$
$$-\Re\int_0^\xi\int_{\Gamma_1}\mu_1^{-1}\mathcal{T}_1U^{\rm inc}\overline{\psi_1}{\rm d}x{\rm d}t+\Re\int_0^\xi\int_{\Gamma_2}\mu_2^{-1}\mathcal{T}_2U^{\rm P}\overline{\psi_1}{\rm d}x{\rm d}t$$
$$-c_1\Re\int_0^\xi\int_\Omega\left(\mu^{-1}\partial_{tx}V+\partial_x(\mu^{-1}\partial_tV)\right)\overline{\psi_1}{\rm d}x{\rm d}z{\rm d}t.$$

It follows from (4.26) and the initial conditions in (4.24) that

$$\Re\int_0^\xi\int_\Omega(\varepsilon-c_1^2\mu^{-1})\partial_t^2V\overline{\psi_1}{\rm d}x{\rm d}z{\rm d}t$$
$$=\Re\int_\Omega\int_0^\xi\left(\partial_t((\varepsilon-c_1^2\mu^{-1})\partial_tV\overline{\psi_1})+(\varepsilon-c_1^2\mu^{-1})\partial_tV\overline{V}\right){\rm d}x{\rm d}z{\rm d}t$$
$$=\Re\int_\Omega\left(((\varepsilon-c_1^2\mu^{-1})\partial_tV\overline{\psi_1})\,\Big|_0^\xi+\frac12(\varepsilon-c_1^2\mu^{-1})|V|^2\Big|_0^\xi\right){\rm d}x{\rm d}z$$
$$=\frac12\|(\varepsilon-c_1^2\mu^{-1})^{1/2}V(\cdot,\xi)\|_{L^2(\Omega)}^2.$$

Using (4.26), we have

$$\begin{aligned}\Re\int_0^\xi\int_\Omega \mu^{-1}\nabla V\cdot\nabla\overline{\psi_1}\mathrm{d}x\mathrm{d}z\mathrm{d}t&=-\Re\int_\Omega\int_0^\xi \mu^{-1}\partial_t(\nabla\psi_1)\cdot\nabla\overline{\psi_1}\mathrm{d}t\mathrm{d}x\mathrm{d}z\\&=-\frac{1}{2}\int_\Omega\int_0^\xi \mu^{-1}\partial_t|\nabla\psi_1|^2\mathrm{d}t\mathrm{d}x\mathrm{d}z\\&=-\frac{1}{2}\int_\Omega \mu^{-1}|\nabla\psi_1|^2\Big|_0^\xi\mathrm{d}x\mathrm{d}z\\&=\frac{1}{2}\int_\Omega\mu^{-1}\Big|\int_0^\xi\nabla V(\cdot,t)\mathrm{d}t\Big|^2\mathrm{d}x\mathrm{d}z.\end{aligned}$$

Combining the above estimates yields

$$\begin{aligned}&\frac{1}{2}\|(\varepsilon-c_1^2\mu^{-1})^{1/2}V(\cdot,\xi)\|^2_{L^2(\Omega)}+\frac{1}{2}\int_\Omega\mu^{-1}\Big|\int_0^\xi\nabla V(\cdot,t)\mathrm{d}t\Big|^2\mathrm{d}x\mathrm{d}z\\&=\Re\int_0^\xi\sum_{j=1}^2\int_{\Gamma_j}\mu_j^{-1}\mathscr{T}_jV\overline{\psi_1}\mathrm{d}x\mathrm{d}t+\Re\int_0^\xi\int_{\Gamma_1}\mu_1^{-1}\mathcal{T}_1U^{\mathrm{p}}\overline{\psi_1}\mathrm{d}x\mathrm{d}t\\&\quad-\Re\int_0^\xi\int_{\Gamma_1}\mu_1^{-1}\mathcal{T}_1U^{\mathrm{inc}}\overline{\psi_1}\mathrm{d}x\mathrm{d}t+\Re\int_0^\xi\int_{\Gamma_2}\mu_2^{-1}\mathcal{T}_2U^{\mathrm{P}}\overline{\psi_1}\mathrm{d}x\mathrm{d}t\\&\quad-c_1\Re\int_0^\xi\int_\Omega\big(\mu^{-1}\partial_{tx}V+\partial_x(\mu^{-1}\partial_tV)\big)\overline{\psi_1}\mathrm{d}x\mathrm{d}z\mathrm{d}t.\end{aligned}\tag{4.28}$$

In what following, we will estimate each term on the right-hand side of (4.28) separately.

Using (4.27), we have

$$\begin{aligned}\Re\int_0^\xi\int_{\Gamma_j}\mu_j^{-1}\mathscr{T}_jV\overline{\psi_1}\mathrm{d}x\mathrm{d}t&=\Re\int_{\Gamma_j}\int_0^\xi\mu_j^{-1}\mathscr{T}_jV\overline{\psi_1}\mathrm{d}t\mathrm{d}x\\&=\Re\int_{\Gamma_j}\int_0^\xi\left(\int_0^t\mu_j^{-1}\mathscr{T}_jV(x,z,\tau)\mathrm{d}\tau\right)\overline{V}(x,z,t)\mathrm{d}t\mathrm{d}x\\&=\Re\int_0^\xi\int_{\Gamma_j}\left(\int_0^t\mu_j^{-1}\mathscr{T}_jV(x,z,\tau)\mathrm{d}\tau\right)\overline{V}(x,z,t)\mathrm{d}x\mathrm{d}t.\end{aligned}$$

Let $\tilde V$ be the extension of V with respect to t in $\mathbb{R}$ such that $\tilde V=0$ outside the interval $[0,\xi]$. We obtain from the Parseval identity, (2.11) and Lemma 2.5 that

$$\begin{aligned}&\Re\int_0^\xi e^{-2s_1t}\int_{\Gamma_j}\left(\int_0^t\mu_j^{-1}\mathscr{T}_jV(x,z,\tau)\mathrm{d}\tau\right)\overline{V}(x,z,t)\mathrm{d}x\mathrm{d}t\\&=\Re\int_{\Gamma_j}\int_0^\xi e^{-2s_1t}\left(\int_0^t\mu_j^{-1}\mathscr{T}_jV(x,z,\tau)\mathrm{d}\tau\right)\overline{V}(x,z,t)\mathrm{d}t\mathrm{d}x\\&=\Re\int_{\Gamma_j}\int_0^\infty e^{-2s_1t}\left(\int_0^t\mu_j^{-1}\mathscr{T}_j\tilde V(x,z,\tau)\mathrm{d}\tau\right)\overline{\tilde V}(x,z,t)\mathrm{d}t\mathrm{d}x\\&=\Re\int_{\Gamma_j}\int_0^\infty e^{-2s_1t}\left(\mathscr{L}^{-1}\circ(s\mu_j)^{-1}\mathscr{B}_j\circ\mathscr{L}\tilde V(x,z,t)\right)\overline{\tilde V}(x,z,t)\mathrm{d}t\mathrm{d}x\\&=\frac{1}{2\pi}\int_{-\infty}^\infty\Re\langle(s\mu_j)^{-1}\mathscr{B}_j\breve{\tilde V},\breve{\tilde V}\rangle_{\Gamma_j}\mathrm{d}s_2\le 0.\end{aligned}$$

Letting $s_1 \to 0$, we obtain

$$\Re \int_0^\xi \sum_{j=1}^2 \int_{\Gamma_j} \mu_j^{-1} \mathscr{T}_j V \overline{\psi_1} \mathrm{d}x\mathrm{d}t \le 0. \tag{4.29}$$

Using the integration by parts, (4.26) and the initial conditions in (4.24), we have

$$\begin{aligned}
&\int_0^\xi \int_\Omega \left(\mu^{-1}\partial_{tx} V + \partial_x(\mu^{-1}\partial_t V)\right) \overline{\psi_1} \mathrm{d}x\mathrm{d}z\mathrm{d}t \\
&= \int_0^\xi \int_\Omega \mu^{-1}\partial_t(\partial_x V)\overline{\psi_1}\mathrm{d}x\mathrm{d}z\mathrm{d}t + \int_0^\xi \int_\Omega \partial_x(\mu^{-1}\partial_t V)\overline{\psi_1}\mathrm{d}x\mathrm{d}z\mathrm{d}t \\
&= \int_\Omega \left(\mu^{-1}\partial_x V \overline{\psi_1}\right)\Big|_0^\xi \mathrm{d}x\mathrm{d}z - \int_\Omega \int_0^\xi \mu^{-1}\partial_x V \partial_t \overline{\psi_1} \mathrm{d}t\mathrm{d}x\mathrm{d}z \\
&\quad + \int_\Omega \left(\partial_x(\mu^{-1}V)\overline{\psi_1}\right)\Big|_0^\xi \mathrm{d}x\mathrm{d}z - \int_\Omega \int_0^\xi \partial_x(\mu^{-1}V)\partial_t\overline{\psi_1}\mathrm{d}t\mathrm{d}x\mathrm{d}z \\
&= \int_0^\xi \int_\Omega \left(\mu^{-1}\partial_x V + \partial_x(\mu^{-1}V)\right)\overline{V}\mathrm{d}x\mathrm{d}z\mathrm{d}t.
\end{aligned}$$

Since μ and V are periodic functions of x, we have from the integration by parts that

$$\int_0^\xi \int_\Omega \left(\mu^{-1}\partial_x V + \partial_x(\mu^{-1}V)\right)\overline{V}\mathrm{d}x\mathrm{d}z\mathrm{d}t + \int_0^\xi \int_\Omega \left(\mu^{-1}\partial_x \overline{V} + \partial_x(\mu^{-1}\overline{V})\right) V \mathrm{d}x\mathrm{d}z\mathrm{d}t = 0,$$

which implies

$$\Re \int_0^\xi \int_\Omega \left(\mu^{-1}\partial_x V + \partial_x(\mu^{-1}V)\right)\overline{V}\mathrm{d}x\mathrm{d}z\mathrm{d}t = 0.$$

Hence we have

$$\Re \int_0^\xi \int_\Omega \left(\mu^{-1}\partial_{tx} V + \partial_x(\mu^{-1}\partial_t V)\right)\overline{\psi_1}\mathrm{d}x\mathrm{d}z\mathrm{d}t = 0. \tag{4.30}$$

It follows from (4.27) that

$$\begin{aligned}
\left|\int_0^\xi \int_{\Gamma_j} \mu_j^{-1} \mathcal{T}_j U^{\mathrm{P}} \overline{\psi_1}\mathrm{d}x\mathrm{d}t\right| &= \left|\int_{\Gamma_j}\int_0^\xi \mu_j^{-1}\mathcal{T}_j U^{\mathrm{P}}\overline{\psi_1}\mathrm{d}t\mathrm{d}x\right| \\
&= \left|\int_{\Gamma_j}\int_0^\xi \left(\int_0^t \mu_j^{-1}\mathcal{T}_j U^{\mathrm{P}}(x,z,\tau)\mathrm{d}\tau\right)\overline{V}(x,z,t)\mathrm{d}t\mathrm{d}x\right| \\
&= \left|\int_0^\xi \int_{\Gamma_j} \left(\int_0^t \mu_j^{-1}\mathcal{T}_j U^{\mathrm{P}}(x,z,\tau)\mathrm{d}\tau\right)\overline{V}(x,z,t)\mathrm{d}x\mathrm{d}t\right|.
\end{aligned}$$

Let $\tilde{U}^{\mathrm{P}}$ be the extension of U^{P} with respect to t in $\mathbb{R}$ such that $\tilde{U}^{\mathrm{P}} = 0$ outside the interval $[0,\xi]$. We obtain from the Parseval identity and (2.11) that

$$\left|\int_0^\xi \int_{\Gamma_j} \mu_j^{-1}\mathcal{T}_j U^{\mathrm{P}}\overline{\psi_1}\mathrm{d}x\mathrm{d}t\right| = \left|\int_0^\infty \int_{\Gamma_j}\left(\int_0^t \mu_j^{-1}\mathcal{T}_j \tilde{U}^{\mathrm{P}}(x,z,\tau)\mathrm{d}\tau\right)\overline{\tilde{V}}\mathrm{d}x\mathrm{d}t\right|$$

$$
\begin{aligned}
&\leq e^{2s_1T}\left|\int_0^\infty e^{-2s_1t}\int_{\Gamma_j}\left(\int_0^t \mu_j^{-1}\mathcal{T}_j\tilde{U}^{\mathrm{P}}(x,z,\tau)\mathrm{d}\tau\right)\overline{\tilde{V}}(x,z,t)\mathrm{d}x\mathrm{d}t\right|\\
&= e^{2s_1T}\left|\int_{\Gamma_j}\int_0^\infty e^{-2s_1t}\left(\mathscr{L}^{-1}\circ(s\mu_j)^{-1}\mathcal{B}_j\circ\mathscr{L}\tilde{U}^{\mathrm{P}}(x,z,t)\right)\overline{\tilde{V}}(x,z,t)\mathrm{d}t\mathrm{d}x\right|\\
&\leq \frac{e^{2s_1T}}{2\pi}\int_{-\infty}^\infty\left|\langle(s\mu_j)^{-1}\mathcal{B}_j\breve{\tilde{U}}^{\mathrm{P}},\breve{\tilde{V}}\rangle_{\Gamma_j}\right|\mathrm{d}s_2.
\end{aligned}
$$

Let $\tilde{U}^{\mathrm{inc}}$ and $\tilde{F}$ be the extension of U^{inc} and F with respect to t in $\mathbb{R}$ such that $\tilde{U}^{\mathrm{inc}}=0$ and $\tilde{F}=0$ outside the interval $[0,T]$, respectively. For brevity, let us introduce the notations

$$
\begin{aligned}
&\breve{D}=\|s\breve{\tilde{U}}^{\mathrm{inc}}\|_{H_{\mathrm{p}}^{1/2}(\Gamma_1^{\mathrm{PML}})}+\|s\breve{\tilde{U}}^{\mathrm{inc}}\|_{H_{\mathrm{p}}^{-1/2}(\Gamma_1^{\mathrm{PML}})}+\|s^2\breve{\tilde{U}}^{\mathrm{inc}}\|_{H_{\mathrm{p}}^{-1/2}(\Gamma_1^{\mathrm{PML}})}+\|\breve{\tilde{F}}\|_{L^2(\Omega_1^{\mathrm{PML}})},\\
&D=\|\partial_t\tilde{U}^{\mathrm{inc}}\|_{H_{\mathrm{p}}^{1/2}(\Gamma_1^{\mathrm{PML}})}+\|\partial_t\tilde{U}^{\mathrm{inc}}\|_{H_{\mathrm{p}}^{-1/2}(\Gamma_1^{\mathrm{PML}})}+\|\partial_t^2\tilde{U}^{\mathrm{inc}}\|_{H_{\mathrm{p}}^{-1/2}(\Gamma_1^{\mathrm{PML}})}+\|\tilde{F}\|_{L^2(\Omega_1^{\mathrm{PML}})},\\
&D_{L^1(0,T)}=\|\partial_tU^{\mathrm{inc}}\|_{L^1(0,T;H_{\mathrm{p}}^{1/2}(\Gamma_1^{\mathrm{PML}}))}+\|\partial_tU^{\mathrm{inc}}\|_{L^1(0,T;H_{\mathrm{p}}^{-1/2}(\Gamma_1^{\mathrm{PML}}))}\\
&\qquad\qquad+\|\partial_t^2U^{\mathrm{inc}}\|_{L^1(0,T;H_{\mathrm{p}}^{-1/2}(\Gamma_1^{\mathrm{PML}}))}+\|F\|_{L^1(0,T;L^2(\Omega_1^{\mathrm{PML}}))}.
\end{aligned}
$$

It follows from Lemmas 4.3, 2.3, 3.1 and the Parseval identity that

$$
\begin{aligned}
&\left|\int_0^\xi\int_{\Gamma_j}\mu_j^{-1}\mathcal{T}_jU^{\mathrm{P}}\overline{\psi_1}\mathrm{d}x\mathrm{d}t\right|\leq\frac{e^{2s_1T}}{2\pi}\int_{-\infty}^\infty\left|\langle(s\mu_j)^{-1}\mathcal{B}_j\breve{\tilde{U}}^{\mathrm{P}},\breve{\tilde{V}}\rangle_{\Gamma_j}\right|\mathrm{d}s_2\\
&\lesssim\frac{e^{2s_1T}}{2\pi}\frac{1}{s_1\mu_{\min}}C_3C_4e^{-\eta\delta\bar{\sigma}}\int_{-\infty}^\infty\|\breve{\tilde{U}}^{\mathrm{P}}\|_{H_{s,\mathrm{p}}^{1/2}(\Gamma_j)}\|\breve{\tilde{V}}\|_{H_{s,\mathrm{p}}^{1/2}(\Gamma_j)}\mathrm{d}s_2\\
&\lesssim\frac{e^{2s_1T}}{2\pi}\frac{1}{s_1\mu_{\min}}C_3C_4e^{-\eta\delta\bar{\sigma}}C_1^2\int_{-\infty}^\infty\|\breve{\tilde{U}}^{\mathrm{P}}\|_{H_{s,p}^1(\Omega)}\|\breve{\tilde{V}}\|_{H_{s,p}^1(\Omega)}\mathrm{d}s_2\\
&\lesssim C_5e^{-\eta\delta\bar{\sigma}}\int_{-\infty}^\infty\breve{D}\left(\|\nabla\breve{\tilde{V}}\|_{L^2(\Omega)^2}+\|s\breve{\tilde{V}}\|_{L^2(\Omega)}\right)\mathrm{d}s_2\\
&\lesssim C_5e^{-\eta\delta\bar{\sigma}}\int_0^\infty D\left(\|\nabla\tilde{V}\|_{L^2(\Omega)^2}+\|\partial_t\tilde{V}\|_{L^2(\Omega)}\right)\mathrm{d}t\\
&\lesssim C_5e^{-\eta\delta\bar{\sigma}}D_{L^1(0,T)}\left(\|\nabla V\|_{L^\infty(0,T;L^2(\Omega)^2)}+\|\partial_tV\|_{L^\infty(0,T;L^2(\Omega))}\right),
\end{aligned}
$$

where $C_5=e^{2s_1T}(s_1\mu_{\min})^{-1}C_3C_4C_1^2s_1^{-1}(1+s_1^{-1/2}\sigma_0)^4$.

Similarly, we have

$$
\begin{aligned}
&\left|\int_0^\xi\int_{\Gamma_1}\mu_1^{-1}\mathcal{T}_1U^{\mathrm{inc}}\overline{\psi_1}\mathrm{d}x\mathrm{d}t\right|\leq\frac{e^{2s_1T}}{2\pi}\int_{-\infty}^\infty\left|\langle(s\mu_1)^{-1}\mathcal{B}_1\breve{U}^{\mathrm{inc}},\breve{\tilde{V}}\rangle_{\Gamma_1}\right|\mathrm{d}s_2\\
&\lesssim\frac{e^{2s_1T}}{2\pi}\frac{1}{s_1\mu_{\min}}C_3C_4e^{-\eta\delta\bar{\sigma}}\int_{-\infty}^\infty\|\breve{U}^{\mathrm{inc}}\|_{H_{s,\mathrm{p}}^{1/2}(\Gamma_1)}\|\breve{\tilde{V}}\|_{H_{s,\mathrm{p}}^{1/2}(\Gamma_1)}\mathrm{d}s_2\\
&\lesssim\frac{e^{2s_1T}}{2\pi}\frac{1}{s_1\mu_{\min}}C_3C_4e^{-\eta\delta\bar{\sigma}}C_1\int_{-\infty}^\infty\|\breve{U}^{\mathrm{inc}}\|_{H_{s,p}^1(\Gamma_1)}\|\breve{\tilde{V}}\|_{H_{s,p}^1(\Omega)}\mathrm{d}s_2\\
&\lesssim C_6e^{-\eta\delta\bar{\sigma}}\int_0^\infty\left(\|U^{\mathrm{inc}}\|_{H_{\mathrm{p}}^{1/2}(\Gamma_1)}+\|\partial_tU^{\mathrm{inc}}\|_{H_{\mathrm{p}}^{1/2}(\Gamma_1)}\right)\left(\|\nabla\tilde{V}\|_{L^2(\Omega)^2}+\|\partial_t\tilde{V}\|_{L^2(\Omega)}\right)\mathrm{d}t\\
&\lesssim C_6e^{-\eta\delta\bar{\sigma}}\left(\|U^{\mathrm{inc}}\|_{L^1(0,T;H_{\mathrm{p}}^{1/2}(\Gamma_1))}+\|\partial_tU^{\mathrm{inc}}\|_{L^1(0,T;H_{\mathrm{p}}^{1/2}(\Gamma_1))}\right)\\
&\qquad\times\left(\|\nabla V\|_{L^\infty(0,T;L^2(\Omega)^2)}+\|\partial_tV\|_{L^\infty(0,T;L^2(\Omega))}\right),
\end{aligned}
$$

where $C_6 = e^{2s_1 T}(s_1\mu_{\min})^{-1}C_3C_4C_1$.

Combining the above estimates and setting $s_1 = 1/T$, we obtain

$$\begin{aligned}
&\frac{1}{2}\|(\varepsilon - c_1^2\mu^{-1})^{1/2}V(\cdot,\xi)\|^2_{L^2(\Omega)} + \frac{1}{2}\int_\Omega \mu^{-1}\Big|\int_0^\xi \nabla V(\cdot,t)\mathrm{d}t\Big|^2 \mathrm{d}x\mathrm{d}z \\
&\lesssim C_7 e^{-\eta\delta\bar{\sigma}}\left(D_{L^1(0,T)} + \|U^{\mathrm{inc}}\|_{L^1(0,T;H^{1/2}_{\mathrm{p}}(\Gamma_1))} + \|\partial_t U^{\mathrm{inc}}\|_{L^1(0,T;H^{1/2}_{\mathrm{p}}(\Gamma_1))}\right) \\
&\qquad \times \left(\|\nabla V\|_{L^\infty(0,T;L^2(\Omega)^2)} + \|\partial_t V\|_{L^\infty(0,T;L^2(\Omega))}\right), \qquad (4.31)
\end{aligned}$$

where

$$\begin{aligned}
C_7 = {} & T\max\{1, T(\delta\eta)^{-1}\}\max\{1, T(1+T\sigma_0)^4\} \\
& \times \max\{1 + (h_1-h_2)^{-1}T, 1 + (2\pi)^{-1}(h_1-h_2)^{-1}\Lambda\}.
\end{aligned}$$

Taking the derivative of (4.24) with respect to t, we know that $\partial_t V$ satisfies the same equation with U^{P} and U^{inc} being replaced by $\partial_t U^{\mathrm{P}}$ and $\partial_t U^{\mathrm{inc}}$, respectively. Define an auxiliary function

$$\psi_2(x,z,t) = \int_t^\xi \partial_t V(x,z,\tau)\mathrm{d}\tau, \quad (x,z)\in\Omega,\ 0<t<\xi.$$

It is clear that

$$\psi_2(x,z,\xi) = 0, \quad \partial_t\psi_2(x,z,t) = -\partial_t V(x,z,t) \qquad (4.32)$$

and for any $\phi(x,z,t)\in L^2(0,\xi;L^2(\Omega))$

$$\int_0^\xi \phi(x,z,t)\overline{\psi_2}(x,z,t)\mathrm{d}t = \int_0^\xi \left(\int_0^t \phi(x,z,\tau)\mathrm{d}\tau\right)\partial_t\overline{V}(x,z,t)\mathrm{d}t. \qquad (4.33)$$

We may follow the same steps as those for proving (4.28) to obtain

$$\begin{aligned}
&\frac{1}{2}\|(\varepsilon - c_1^2\mu^{-1})^{1/2}\partial_t V(\cdot,\xi)\|^2_{L^2(\Omega)} + \frac{1}{2}\|\mu^{-1/2}\nabla V(\cdot,\xi)\|^2_{L^2(\Omega)^2} \\
&= \Re\int_0^\xi \sum_{j=1}^2 \int_{\Gamma_j} \mu_j^{-1}\mathscr{T}_j\partial_t V\overline{\psi_2}\mathrm{d}x\mathrm{d}t + \Re\int_0^\xi\int_{\Gamma_1}\mu_1^{-1}\mathcal{T}_1\partial_t U^{\mathrm{P}}\overline{\psi_2}\mathrm{d}x\mathrm{d}t \\
&\quad - \Re\int_0^\xi\int_{\Gamma_1}\mu_1^{-1}\mathcal{T}_1\partial_t U^{\mathrm{inc}}\overline{\psi_2}\mathrm{d}x\mathrm{d}t + \Re\int_0^\xi\int_{\Gamma_2}\mu_2^{-1}\mathcal{T}_2\partial_t U^{\mathrm{P}}\overline{\psi_2}\mathrm{d}x\mathrm{d}t \\
&\quad - c_1\Re\int_0^\xi\int_\Omega \left(\mu^{-1}\partial_{ttx}V + \partial_x(\mu^{-1}\partial_t^2 V)\right)\overline{\psi_2}\mathrm{d}x\mathrm{d}z\mathrm{d}t. \qquad (4.34)
\end{aligned}$$

The first and the last terms on the right hand of (4.34) can be estimated similarly as (4.29) and (4.30), respectively. We only need to consider the other three terms on the right hand of (4.34).

Using (4.33), (2.11) and the Parseval identity, we have

$$\begin{aligned}
&\left|\int_0^\xi\int_{\Gamma_j}\mu_j^{-1}\mathcal{T}_j\partial_t U^{\mathrm{P}}\overline{\psi_2}\mathrm{d}x\mathrm{d}t\right| \\
&= \left|\int_{\Gamma_j}\int_0^\xi\left(\int_0^t \mu_j^{-1}\mathcal{T}_j\partial_t U^{\mathrm{P}}(\cdot,\tau)\mathrm{d}\tau\right)\partial_t\overline{V}(x,z,t)\mathrm{d}t\mathrm{d}x\right|
\end{aligned}$$

$$
\begin{aligned}
&\leq e^{2s_1T}\left|\int_0^\infty e^{-2s_1t}\int_{\Gamma_j}\left(\int_0^t \mu_j^{-1}\mathcal{T}_j\partial_t\tilde{U}^{\rm P}(x,z,\tau){\rm d}\tau\right)\partial_t\overline{\tilde{V}}(x,z,t){\rm d}x{\rm d}t\right|\\
&=e^{2s_1T}\left|\int_{\Gamma_j}\int_0^\infty e^{-2s_1t}\left(\mathscr{L}\circ(s\mu_j)^{-1}\mathcal{B}_j\circ\mathscr{L}\partial_t\tilde{U}^{\rm P}(x,z,t)\right)\partial_t\overline{\tilde{V}}(x,z,t){\rm d}t{\rm d}x\right|\\
&\leq\frac{e^{2s_1T}}{2\pi}\int_{-\infty}^\infty\left|\langle\mu_j^{-1}\mathcal{B}_j(s\breve{\tilde{U}}^{\rm P}),\breve{\tilde{V}}\rangle_{\Gamma_j}\right|{\rm d}s_2.
\end{aligned}
$$

Let us introduce the notations

$$
\begin{aligned}
\breve{E}=&\|s^2\breve{\tilde{U}}^{\rm inc}\|_{H_{\rm p}^{1/2}(\Gamma_1^{\rm PML})}+\|s^2\breve{\tilde{U}}^{\rm inc}\|_{H_{\rm p}^{-1/2}(\Gamma_1^{\rm PML})}\\
&+\|s^3\breve{\tilde{U}}^{\rm inc}\|_{H_{\rm p}^{-1/2}(\Gamma_1^{\rm PML})}+\|s\breve{\tilde{F}}\|_{L^2(\Omega_1^{\rm PML}))},\\
E=&\|\partial_t^2\tilde{U}^{\rm inc}\|_{H_{\rm p}^{1/2}(\Gamma_1^{\rm PML})}+\|\partial_t^2\tilde{U}^{\rm inc}\|_{H_{\rm p}^{-1/2}(\Gamma_1^{\rm PML})}\\
&+\|\partial_t^3\tilde{U}^{\rm inc}\|_{H_{\rm p}^{-1/2}(\Gamma_1^{\rm PML})}+\|\partial_t\tilde{F}\|_{L^2(\Omega_1^{\rm PML})},
\end{aligned}
$$

and

$$
\begin{aligned}
E_{L^1(0,T)}=&\|\partial_t^2U^{\rm inc}\|_{L^1(0,T;H_{\rm p}^{1/2}(\Gamma_1^{\rm PML}))}+\|\partial_t^2U^{\rm inc}\|_{L^1(0,T;H_{\rm p}^{-1/2}(\Gamma_1^{\rm PML}))}\\
&+\|\partial_t^3U^{\rm inc}\|_{L^1(0,T;H_{\rm p}^{-1/2}(\Gamma_1^{\rm PML}))}+\|\partial_tF\|_{L^1(0,T;L^2(\Omega_1^{\rm PML}))}.
\end{aligned}
$$

It follows from Lemmas 4.3, 2.3, 3.1 and the Parseval identity that

$$
\begin{aligned}
&\left|\int_0^\xi\int_{\Gamma_j}\mu_j^{-1}\mathcal{T}_j\partial_tU^{\rm P}\overline{\psi_2}{\rm d}x{\rm d}t\right|\leq\frac{e^{2s_1T}}{2\pi}\int_{-\infty}^\infty\left|\langle\mu_j^{-1}\mathcal{B}_j(s\breve{\tilde{U}}^{\rm P}),\breve{\tilde{V}}\rangle_{\Gamma_j}\right|{\rm d}s_2\\
&\lesssim\frac{e^{2s_1T}}{2\pi}\frac{1}{\mu_{\min}}C_3C_4e^{-\eta\delta\bar{\sigma}}\int_{-\infty}^\infty\|s\breve{\tilde{U}}^{\rm P}\|_{H_{s,{\rm p}}^{1/2}(\Gamma_j)}\|\breve{\tilde{V}}\|_{H_{s,{\rm p}}^{1/2}(\Gamma_j)}{\rm d}s_2\\
&\lesssim\frac{e^{2s_1T}}{2\pi}\frac{1}{\mu_{\min}}C_3C_4e^{-\eta\delta\bar{\sigma}}C_1^2\int_{-\infty}^\infty\|s\breve{\tilde{U}}^{\rm P}\|_{H_{s,{\rm p}}^1(\Omega)}\|\breve{\tilde{V}}\|_{H_{s,{\rm p}}^1(\Omega)}{\rm d}s_2\\
&\lesssim\frac{e^{2s_1T}}{2\pi}\frac{1}{\mu_{\min}}C_3C_4e^{-\eta\delta\bar{\sigma}}C_1^2s_1^{-1/2}(1+s_1^{-1}\sigma_0)^2\int_{-\infty}^\infty\breve{E}\left(\|\nabla\breve{\tilde{V}}\|_{L^2(\Omega)^2}+\|s\breve{\tilde{V}}\|_{L^2(\Omega)}\right){\rm d}s_2\\
&\lesssim e^{2s_1T}\frac{1}{\mu_{\min}}C_3C_4e^{-\eta\delta\bar{\sigma}}C_1^2s_1^{-1/2}(1+s_1^{-1}\sigma_0)^2\int_0^\infty E\left(\|\nabla\tilde{V}\|_{L^2(\Omega)^2}+\|\partial_t\tilde{V}\|_{L^2(\Omega)}\right){\rm d}t\\
&\lesssim s_1C_5e^{-\eta\delta\bar{\sigma}}E_{L^1(0,T)}\left(\|\nabla V\|_{L^\infty(0,T;L^2(\Omega)^2)}+\|\partial_tV\|_{L^\infty(0,T;L^2(\Omega))}\right).
\end{aligned}
$$

Similarly, we may show that

$$
\begin{aligned}
&\left|\int_0^\xi\int_{\Gamma_1}\mu_1^{-1}\mathcal{T}_1\partial_tU^{\rm inc}\overline{\psi_2}{\rm d}x{\rm d}t\right|\leq\frac{e^{2s_1T}}{2\pi}\int_{-\infty}^\infty\left|\langle\mu_1^{-1}\mathcal{B}_1(s\breve{U}^{\rm inc}),\breve{\tilde{V}}\rangle_{\Gamma_1}\right|{\rm d}s_2\\
&\lesssim\frac{e^{2s_1T}}{2\pi}\frac{1}{\mu_{\min}}C_3C_4e^{-\eta\delta\bar{\sigma}}\int_{-\infty}^\infty\|s\breve{U}^{\rm inc}\|_{H_{s,{\rm p}}^{1/2}(\Gamma_1)}\|\breve{\tilde{V}}\|_{H_{s,{\rm p}}^{1/2}(\Gamma_1)}{\rm d}s_2\\
&\lesssim\frac{e^{2s_1T}}{2\pi}\frac{1}{\mu_{\min}}C_3C_4e^{-\eta\delta\bar{\sigma}}C_1\int_{-\infty}^\infty\|s\breve{U}^{\rm inc}\|_{H_{s,{\rm p}}^{1/2}(\Gamma_1)}\|\breve{\tilde{V}}\|_{H_{s,{\rm p}}^1(\Omega)}{\rm d}s_2\\
&\lesssim s_1C_6e^{-\eta\delta\bar{\sigma}}\int_0^\infty\left(\|\partial_tU^{\rm inc}\|_{H_{\rm p}^{1/2}(\Gamma_1)}+\|\partial_t^2U^{\rm inc}\|_{H_{\rm p}^{1/2}(\Gamma_1)}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\times\left(\|\nabla\tilde V\|_{L^2(\Omega)^2}+\|\partial_t\tilde V\|_{L^2(\Omega)}\right)\mathrm{d}t\\
\lesssim\, & s_1C_6e^{-\eta\delta\bar\sigma}\left(\|\partial_tU^{\rm inc}\|_{L^1(0,T;H^{1/2}_{s,\rm p}(\Gamma_1))}+\|\partial_t^2U^{\rm inc}\|_{L^1(0,T;H^{1/2}_{s,\rm p}(\Gamma_1))}\right)\\
&\times\left(\|\nabla V\|_{L^\infty(0,T;L^2(\Omega)^2)}+\|\partial_tV\|_{L^\infty(0,T;L^2(\Omega))}\right).
\end{aligned}
$$

Combining the above estimates and setting $s_1=1/T$, we deduce that

$$
\begin{aligned}
&\frac12\|(\varepsilon-c_1^2\mu^{-1})^{1/2}\partial_tV(\cdot,\xi)\|^2_{L^2(\Omega)}+\frac12\|\mu^{-1/2}\nabla V(\cdot,\xi)\|^2_{L^2(\Omega)^2}\\
\lesssim\, & C_8e^{-\eta\delta\bar\sigma}\left(E_{L^1(0,T)}+\|\partial_tU^{\rm inc}\|_{L^1(0,T;H^{1/2}_{s,\rm p}(\Gamma_1))}+\|\partial_t^2U^{\rm inc}\|_{L^1(0,T;H^{1/2}_{s,\rm p}(\Gamma_1))}\right)\\
&\times\left(\|\nabla V\|_{L^\infty(0,T;L^2(\Omega)^2)}+\|\partial_tV\|_{L^\infty(0,T;L^2(\Omega))}\right),
\end{aligned}\tag{4.35}
$$

where

$$
\begin{aligned}
C_8=\,&\max\{1,T^{1/2}(1+T\sigma_0)^2\}\max\{1,T(\delta\eta)^{-1}\}\\
&\times\max\{1+(h_1-h_2)^{-1}T,1+(2\pi)^{-1}(h_1-h_2)^{-1}\Lambda\}.
\end{aligned}
$$

It follows from (4.31) and (4.35) that

$$
\begin{aligned}
&\|V(\cdot,\xi)\|^2_{L^2(\Omega)}+\|\nabla V(\cdot,\xi)\|^2_{L^2(\Omega)^2}+\|\partial_tV(\cdot,\xi)\|^2_{L^2(\Omega)}\\
\lesssim\, & C(T)e^{-\eta\delta\bar\sigma}\Big(D_{L^1(0,T)}+\|U^{\rm inc}\|_{L^1(0,T;H^{1/2}_{s,\rm p}(\Gamma_1))}+\|\partial_tU^{\rm inc}\|_{L^1(0,T;H^{1/2}_{s,\rm p}(\Gamma_1))}\\
&+E_{L^1(0,T)}+\|\partial_tU^{\rm inc}\|_{L^1(0,T;H^{1/2}_{s,\rm p}(\Gamma_1))}+\|\partial_t^2U^{\rm inc}\|_{L^1(0,T;H^{1/2}_{s,\rm p}(\Gamma_1))}\Big)\\
&\times\left(\|\nabla V\|_{L^\infty(0,T;L^2(\Omega)^2)}+\|\partial_tV\|_{L^\infty(0,T;L^2(\Omega))}\right),
\end{aligned}
$$

where

$$
\begin{aligned}
C(T)=\,&\max\{1,T\}\max\{1,T(\delta\eta)^{-1}\}\max\{1,T(1+T\sigma_0)^4\}\\
&\times\max\{1+(h_1-h_2)^{-1}T,1+(2\pi)^{-1}(h_1-h_2)^{-1}\Lambda\}.
\end{aligned}
$$

The proof is completed by taking the L^∞ norm with respect to ξ and using Cauchy–Schwarz inequality. ∎

It can be seen from Theorem 4.1 that the parameters $\delta=\min\{\delta_1,\delta_2\}$ and $\bar\sigma=\min\{\bar\sigma_1,\bar\sigma_2\}$ control the error of the PML solution. The error approaches zero exponentially as $\delta\bar\sigma$ tends to infinity. By (3.17) and (3.18), the quantity $\bar\sigma$ can be calculated by the medium function $\sigma(z)$, which may be taken as a power function:

$$
\sigma(z)=\begin{cases}\sigma_1\left(\frac{z-h_1}{\delta_1}\right)^m & \text{if } z\ge h_1,\\ \sigma_2\left(\frac{h_2-z}{\delta_2}\right)^m & \text{if } z\le h_2,\end{cases}\qquad m\ge1,
$$

where σ_1,σ_2 are positive constants and known as the medium parameters. Thus, we have from a simple calculation that

$$
\bar\sigma_1=\delta_1^{-1}\int_{h_1}^{h_1+\delta_1}\sigma(z')\mathrm{d}z'=\frac{\sigma_1}{m+1},\quad \bar\sigma_2=\delta_2^{-1}\int_{h_2-\delta_2}^{h_2}\sigma(z')\mathrm{d}z'=\frac{\sigma_2}{m+1}.
$$

Obviously, the error can be reduced by either enlarging the medium parameters σ_1,σ_2 or increasing the PML layer thickness δ_1,δ_2.

5. Conclusions

In this paper, we have the PML method for the time-domain electromagnetic scattering problem in one-dimensional periodic structures. Under some proper assumptions on the medium parameter of the PML, the truncated PML problem is shown to attain a unique solution. The well-posedness and stability of the truncated time-domain PML problem are established by using the abstract inversion theorem of the Laplace transform and the energy method. Based on the error estimate of the DtN operators between the truncated PML problem and the original scattering problem, we prove that the PML solution converges exponentially to the scattering solution by increasing either the PML medium parameter or the PML layer thickness. Computationally, the variational approach together with the PML technique reported here leads naturally to a class of finite element methods. As a time-dependent problem, a fast and accurate marching technique shall also be developed to deal with the temporal discretization [24]. We hope to report the progress on the numerical analysis and computation for the scattering problem elsewhere in the future.

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