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An Adaptive Finite Element DtN Method for Maxwell's Equations

Gang Bao[1], Mingming Zhang[1], Xue Jiang[2], Peijun Li[3,*]
and Xiaokai Yuan[4]

[1]*School of Mathematical Sciences, Zhejiang University, Hangzhou, Zhejiang 310027, China.*
[2]*Faculty of Science, Beijing University of Technology, Beijing 100124, China.*
[3]*Department of Mathematics, Purdue University, West Lafayette, Indiana 47907, USA.*
[4]*School of Mathematics, Jilin University, Changchun, Jilin 130012, China.*

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Dedicated to Professor Tao Tang on the occasion of his 60th birthday.

Abstract. We consider a numerical solution to the electromagnetic obstacle scattering problem in three dimensions. Based on the Dirichlet-to-Neumann (DtN) operator, the exterior problem is reduced into a boundary value problem in a bounded domain. An a posteriori error estimate is deduced to include both the finite element approximation error and the DtN operator truncation error, where the latter decays exponentially with respect to the number of truncation terms. The discrete problem is solved by the adaptive finite element method with the transparent boundary condition. The effectiveness of the method is illustrated by numerical experiments.

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Key words: Maxwell's equations, electromagnetic scattering problem, DtN operator, transparent boundary condition, adaptive finite element method, a posteriori error estimate.

1. Introduction

Scattering problems are concerned with the interaction between an inhomogeneous medium and an incident field. They have significant applications in many scientific areas including geophysical exploration, non-destructive testing, and medical imaging [13]. Motivated by significant applications, scattering problems have received great attention in both of the engineering and mathematical communities. A considerable amount of mathematical and numerical results are available for the scattering problems of acoustic, elastic,

*Corresponding author. *Email addresses:* `baog@zju.edu.cn` (G. Bao), `mmzaip@zju.edu.cn` (M. Zhang), `jxue@lsec.cc.ac.cn` (X. Jiang), `lipeijun@math.purdue.edu` (P. Li), `yuanxk@jlu.edu.cn` (X. Yuan)

and electromagnetic waves. We refer to the monographs [21, 28, 30] on comprehensive accounts of the electromagnetic scattering theory for Maxwell's equations.

In this paper, we consider a numerical solution to the electromagnetic obstacle scattering problem in three dimensions. In addition to the large scale computation of the three-dimensional problem, there are two other main challenges: the scattering problem is imposed in an unbounded domain and the solution may have local singularity due to the nonsmooth surface of the obstacle. The first issue is concerned with the domain truncation where a transparent boundary condition is preferred to avoid artificial wave reflection; the second difficulty can be resolved by using the adaptive finite element method to balance the accuracy and computational cost.

One of the most popular methods for domain truncation is the perfectly matched layer (PML) technique, which was proposed by Bérenger to simulate the electromagnetic wave propagation in unbounded domains [6]. The idea of PML is to put a layer of artificially absorbing media around the computational domain so that outgoing waves can be attenuated. Mathematically, it was proved in [12] that when the thickness of the layer is infinity, the PML solution in the domain of interest is the same as the solution of the original scattering problem. However, in practice, the layer needs to be truncated to finite thickness which inevitably introduces the truncation error. The overall error contains three parts when applying the finite element method to the PML problem: the truncation error of the PML layer, the discretization error in the PML layer, and the discretization error in the domain of interest. It was shown in [3] that the PML truncation error decays exponentially with respect to the thickness of the layer and the PML parameters. As is known, the artificial PML layer is constructed through the complex coordinate stretching [11], which makes the PML layer to be an inhomogeneous medium. It is difficult to balance the efficiency and accuracy if a uniform mesh refinement is used. If a thin PML layer is used to reduce the computational cost, then the discretization error is large since the medium is inhomogeneous in the layer; on the contrary, if the discretization error is controlled to be small, then a thick PML layer is preferred, which increases the cost. To handle this issue, the adaptive finite element method is effective, especially when combined with a posteriori error estimates. Based on numerical solutions, the a posteriori error estimates can be used for mesh modification such as refinement or coarsening [32]. The method can control the error and asymptotically optimize the approximation. Moreover, it can effectively deal with the issue that the solution has local singularities in the domain of interest. It is worth mentioning that even though the solution is smooth, the adaptive finite element method is still desirable due to the inhomogeneous medium in the PML layer. We refer to [8–10, 16, 18] for the discussion of adaptive finite element PML methods for scattering problems in different structures.

Another effective approach is to impose transparent boundary conditions to solve the scattering problems formulated in open domains. A key step of the method is to construct the Dirichlet-to-Neumann operator, which can be done via different manners such as the boundary integral equation [14], the Fourier transform or Fourier series expansions [17, 22]. In this paper, observing that the solution is analytical when it is away from the obstacle, we consider the Fourier series expansion of the solution on any sphere enclosing the obstacle. The DtN operator can be obtained by studying the resulting systems of ordinary differential equations for the Fourier coefficients. Compared to the PML technique, the DtN method does not introduce an auxiliary layer of inhomogeneous medium, which can reduce the cost. Defined as an infinite series, the DtN operator is nonlocal and needs to be truncated into a finite series in actual computation. It was shown in [1,15] that if the solution is smooth enough, the DtN operator truncation error decays exponentially with respect to the truncation number. When the solution has singularities, the convergence analysis is sophisticated. The a posteriori error estimate should take into account of the DtN operator truncation error and is able to determine the truncation number. Combined with the truncated DtN operator, the a posteriori error estimate based adaptive finite element method has been adopted for solving various scattering problems imposed in open domains, such as the acoustic wave scattering problems [4,17,20,27], the electromagnetic wave scattering problems [19,33], and the elastic wave scattering problems [2,25,26].

This work is a non-trivial extension of the adaptive finite element DtN method for the acoustic and elastic wave scattering problems by bounded obstacles. Compared to the acoustic and elastic scattering problems, the electromagnetic scattering problem is more involved. Computationally, it is challenging to solve the electromagnetic scattering problem in three-dimensions. In this paper, an a posteriori error estimate is deduced to include both the finite element approximation error and the DtN operator truncation error. Moreover, we show that the latter decay exponentially with respect to the number of truncation terms. One of the key steps in the analysis is to consider a new dual problem and to deduce its analytical solution. Based on the a posteriori error estimate, we develop an adaptive finite element DtN method. Numerical experiments are presented to demonstrate the competitive behavior of the proposed method. As an alternative to the PML method, this work provides an effective approach to solve the three-dimensional electromagnetic scattering problem. It is expected that the proposed method can also be used to solve scattering problems of many other types, especially for those imposed in open domains.

The outline of the paper is as follows. In Section 2, the problem formulation and the variational problems are introduced. Section 3 presents the finite element discretization with the truncated DtN operator and states the a posteriori error estimate. Section 4 is devoted to the proof of the error estimate and is the main part of the work. In Section 5, numerical experiments are presented to illustrate the effectiveness of the proposed method. Finally, some concluding remarks are given in Section 6.

2. Problem Formulation

Denote by D the obstacle with Lipschitz boundary ∂D. Let $R > 0$ be a sufficiently large constant so that the obstacle is contained in the ball $B_R = \{\boldsymbol{x} \in \mathbb{R}^3 : |\boldsymbol{x}| < R\}$ with boundary $\Gamma_R = \{\boldsymbol{x} \in \mathbb{R}^3 : |\boldsymbol{x}| = R\}$. Let $B_{R'}$ be the smallest ball centered at the origin with radius R' that also contains $\overline{D}$, i.e., $D \subset\subset B_{R'} \subset\subset B_R$ with $0 < R' < R$. Denote by $\Omega := B_R \setminus \overline{D}$ the bounded domain enclosed by Γ_R and ∂D. The exterior domain $\mathbb{R}^3 \setminus \overline{D}$ is assumed to be filled with a homogeneous medium characterized by the dielectric permittivity ϵ and the magnetic permeability μ. Without loss of generality, we may assume that $\epsilon = 1$ and $\mu = 1$.

Furthermore, we assume that the obstacle is a perfect electric conductor.

Let the incident electromagnetic field $(\boldsymbol{E}^{\rm inc},\boldsymbol{H}^{\rm inc})$ be either a plane wave or a point source. The total electromagnetic field $(\boldsymbol{E},\boldsymbol{H})$ is governed by the time-harmonic Maxwell equations

$$\begin{aligned} &\nabla\times\boldsymbol{E}-{\rm i}\kappa\boldsymbol{H}=0,\quad \nabla\times\boldsymbol{H}+{\rm i}\kappa\boldsymbol{E}=0 && \text{in } \mathbb{R}^3\setminus\overline{D},\\ &\boldsymbol{\nu}\times\boldsymbol{E}=0 && \text{on } \partial D,\\ &|\boldsymbol{x}|\,(\boldsymbol{E}^s-\boldsymbol{H}^s\times\hat{\boldsymbol{x}})\to 0 && \text{as } |\boldsymbol{x}|\to+\infty, \end{aligned}\tag{2.1}$$

where $\boldsymbol{\nu}$ is the unit normal vector to ∂D pointing to the exterior of D, and $\boldsymbol{E}^s=\boldsymbol{E}-\boldsymbol{E}^{\rm inc}$ and $\boldsymbol{H}^s=\boldsymbol{H}-\boldsymbol{H}^{\rm inc}$ are the scattered electric and magnetic fields, respectively. We may eliminate the magnetic field $\boldsymbol{H}$ from (2.1) and obtain the scattering problem for the electric field $\boldsymbol{E}$

$$\begin{aligned} &\nabla\times(\nabla\times\boldsymbol{E})-\kappa^2\boldsymbol{E}=0 && \text{in } \mathbb{R}^3\setminus\overline{D},\\ &\boldsymbol{\nu}\times\boldsymbol{E}=0 && \text{on } \partial D,\\ &|\boldsymbol{x}|\,[(\nabla\times\boldsymbol{E}^s)\times\hat{\boldsymbol{x}}-{\rm i}\kappa\boldsymbol{E}^s]\to 0 && \text{as } |\boldsymbol{x}|\to+\infty. \end{aligned}\tag{2.2}$$

We introduce a transparent boundary condition (TBC) to reduce the boundary value problem (2.2) into the bounded domain Ω and discuss its variational formulation.

Given a tangential vector $\boldsymbol{\phi}$ on Γ_R, it has the Fourier series expansion

$$\boldsymbol{\phi}=\sum_{n\in\mathbb{N}}\sum_{|m|\le n}\phi_{1n}^m\boldsymbol{U}_n^m+\phi_{2n}^m\boldsymbol{V}_n^m,\tag{2.3}$$

where $\{(\boldsymbol{U}_n^m,\boldsymbol{V}_n^m):|m|\le n, n=0,1,2,\ldots\}$ is an orthonormal basis for $TL(\Gamma_R)$ given in (B.1)-(B.2). The Calderón operator $\mathscr{T}:\boldsymbol{H}^{-1/2}({\rm curl},\Gamma_R)\to\boldsymbol{H}^{-1/2}({\rm div},\Gamma_{\rm R})$ is defined by

$$\mathscr{T}\boldsymbol{\phi}=\sum_{n\in\mathbb{N}}\sum_{|m|\le n}\frac{{\rm i}\kappa R}{1+z_n^{(1)}(\kappa R)}\phi_{1n}^m\boldsymbol{U}_n^m+\frac{1+z_n^{(1)}(\kappa R)}{{\rm i}\kappa R}\phi_{2n}^m\boldsymbol{V}_n^m,\tag{2.4}$$

where $z_n^{(1)}(z)=zh_n^{(1)'}(z)/h_n^{(1)}(z)$ and $h_n^{(1)}(z)$ is the spherical Hankel function of the first kind with order n. In [3], it is shown that the solution $\boldsymbol{E}$ of (2.2) satisfies the following TBC on Γ_R:

$$(\nabla\times\boldsymbol{E})\times\boldsymbol{\nu}-{\rm i}\kappa\mathscr{T}\boldsymbol{E}_{\Gamma_R}=\boldsymbol{f},\tag{2.5}$$

where $\boldsymbol{E}_{\Gamma_R}=\boldsymbol{e}_\rho\times(\boldsymbol{E}\times\boldsymbol{e}_\rho)$ is the tangential component of $\boldsymbol{E}$ and

$$\boldsymbol{f}=(\nabla\times\boldsymbol{E}^{\rm inc})\times\boldsymbol{\nu}-{\rm i}\kappa\mathscr{T}\boldsymbol{E}_{\Gamma_R}^{\rm inc}.$$

Based on (2.5), the boundary value problem (2.2) can be equivalently reduced into the bounded domain Ω. The corresponding variational problem is to find $\boldsymbol{E}\in\boldsymbol{H}_{\partial D}({\rm curl},\Omega)$ such that

$$a(\boldsymbol{E},\boldsymbol{\psi})=\int_{\Gamma_R}\boldsymbol{f}\cdot\overline{\boldsymbol{\psi}}_{\Gamma_R}{\rm d}s,\quad\forall\boldsymbol{\psi}\in\boldsymbol{H}_{\partial D}({\rm curl},\Omega),\tag{2.6}$$

where the sesquilinear form $a : \boldsymbol{H}(\mathrm{curl},\Omega)\times\boldsymbol{H}(\mathrm{curl},\Omega)\to\mathbb{C}$ is defined by

$$a(\boldsymbol{\varphi},\boldsymbol{\psi})=\int_\Omega(\nabla\times\boldsymbol{\varphi})\cdot(\nabla\times\overline{\boldsymbol{\psi}})\mathrm{d}\boldsymbol{x}-\kappa^2\int_\Omega\boldsymbol{\varphi}\cdot\overline{\boldsymbol{\psi}}\mathrm{d}\boldsymbol{x}-\mathrm{i}\kappa\int_{\Gamma_R}\mathscr{T}\boldsymbol{\varphi}_{\Gamma_R}\cdot\overline{\boldsymbol{\psi}}_{\Gamma_R}\mathrm{d}s \tag{2.7}$$

and $\boldsymbol{H}_{\partial D}(\mathrm{curl},\Omega)=\{\boldsymbol{\phi}\in\boldsymbol{H}(\mathrm{curl},\Omega):\boldsymbol{\nu}\times\boldsymbol{\phi}=0\,\mathrm{on}\,\partial D\}$.

The well-posedness of the variational problem (2.6) is discussed in [28, 30]. For simplicity, we assume that the variational problem (2.6) admits a unique weak solution $\boldsymbol{E}\in\boldsymbol{H}_{\partial D}(\mathrm{curl},\Omega)$. It follows from [28, Lemma 10.9] that there exists a constant $\gamma>0$ depending on κ and R but independent of $\boldsymbol{\varphi}$ such that the inf-sup condition holds

$$\sup_{0\neq\boldsymbol{\psi}\in\boldsymbol{H}_{\partial D}(\mathrm{curl},\Omega)}\frac{|a(\boldsymbol{\varphi},\boldsymbol{\psi})|}{\|\boldsymbol{\psi}\|_{\boldsymbol{H}(\mathrm{curl},\Omega)}}\geq\gamma\|\boldsymbol{\varphi}\|_{\boldsymbol{H}(\mathrm{curl},\Omega)},\quad\forall\boldsymbol{\varphi}\in\boldsymbol{H}_{\partial D}(\mathrm{curl},\Omega). \tag{2.8}$$

In computation, the Calderón operator needs to be truncated into a finite series. Define the truncated Calderón operator

$$\mathscr{T}^N\boldsymbol{\phi}=\sum_{n\leq N}\sum_{|m|\leq n}\frac{\mathrm{i}\kappa R}{1+z_n^{(1)}(\kappa R)}\phi_{1n}^m\boldsymbol{U}_n^m+\frac{1+z_n^{(1)}(\kappa R)}{\mathrm{i}\kappa R}\phi_{2n}^m\boldsymbol{V}_n^m, \tag{2.9}$$

where $N>0$ is a sufficiently large integer and $\boldsymbol{\phi}$ is a tangent vector on Γ_R with the Fourier series expansion (2.3).

Replacing $\mathscr{T}$ in (2.6) by $\mathscr{T}^N$, we obtain the truncated variational problem which is to find $\boldsymbol{E}^N\in\boldsymbol{H}_{\partial D}(\mathrm{curl},\Omega)$ such that

$$a^N(\boldsymbol{E}^N,\boldsymbol{\psi})=\int_{\Gamma_R}\boldsymbol{f}^N\cdot\overline{\boldsymbol{\psi}}_{\Gamma_R}\mathrm{d}s,\quad\forall\boldsymbol{\psi}\in\boldsymbol{H}_{\partial D}(\mathrm{curl},\Omega), \tag{2.10}$$

where the sesquilinear form $a^N : \boldsymbol{H}(\mathrm{curl},\Omega)\times\boldsymbol{H}(\mathrm{curl},\Omega)\to\mathbb{C}$ is defined by

$$a^N(\boldsymbol{\varphi},\boldsymbol{\psi})=\int_\Omega(\nabla\times\boldsymbol{\varphi})\cdot(\nabla\times\overline{\boldsymbol{\psi}})\mathrm{d}\boldsymbol{x}-\kappa^2\int_\Omega\boldsymbol{\varphi}\cdot\overline{\boldsymbol{\psi}}\mathrm{d}\boldsymbol{x}-\mathrm{i}\kappa\int_{\Gamma_R}\mathscr{T}^N\boldsymbol{\varphi}_{\Gamma_R}\cdot\overline{\boldsymbol{\psi}}_{\Gamma_R}\mathrm{d}s \tag{2.11}$$

and

$$\boldsymbol{f}^N:=(\nabla\times\boldsymbol{E}^{\mathrm{inc}})\times\boldsymbol{\nu}-\mathrm{i}\kappa\mathscr{T}^N\boldsymbol{E}^{\mathrm{inc}}_{\Gamma_R}.$$

Following the argument in [15], we may show that the truncated problem (2.10) has a unique weak solution $\boldsymbol{E}^N\in\boldsymbol{H}_{\partial D}(\mathrm{curl},\Omega)$ when the truncation number N is sufficiently large. The detail is omitted since our focus is to deduce the a posteriori error estimate and develop a corresponding adaptive finite element method for the solution of the truncated problem.

3. The a Posteriori Error Estimate

In this section, we introduce the finite element approximation of variational problem (2.10), present the a posteriori error estimate, and state the main result of this paper.

Denote by $\mathscr{M}_h$ a regular tetrahedral mesh of the domain Ω, where h stands for the maximum mesh size of the tetrahedra in $\mathscr{M}_h$. For any tetrahedral element which has more than one vertex on Γ_R, we may adopt the mapping T proposed in [28, Section 8.3.1] which maps the corresponding edge or face of the tetrahedral element exactly on Γ_R.

Denote by $R_{\hat{K}}$ the local subspace on the straight tetrahedron $\hat{K}$

$$R_{\hat{K}} = \left\{ \boldsymbol{v} = \boldsymbol{a}_{\hat{K}} + \boldsymbol{b}_{\hat{K}} \times \boldsymbol{x}, \text{ where } \boldsymbol{a}_{\hat{K}}, \boldsymbol{b}_{\hat{K}} \in \mathbb{C}^3 \right\}$$

and denoted by R_K the corresponding subspace on the curved tetrahedron K under the mapping $T_K = T|_K$

$$R_K = \left\{ \boldsymbol{v} : \boldsymbol{v} \circ T_K = (dT_K)^{-\top} \hat{v} \text{ for some } \hat{v} \in R_{\hat{K}} \right\},$$

where dT_K is the Jacobian matrix for T_K.

Introduce the lowest order Nédélec edge element space under the mapping T by

$$\boldsymbol{V}_h := \left\{ \boldsymbol{v} \in \boldsymbol{H}_{\partial D}(\mathrm{curl}, \Omega) : \boldsymbol{v}|_K \in R_K \text{ for some } K \in \mathscr{M}_h \right\}.$$

The discrete problem of (2.10) is to find $\boldsymbol{E}_h^N \in \boldsymbol{V}_h$ such that

$$a^N\left(\boldsymbol{E}_h^N, \boldsymbol{\psi}_h\right) = \int_{\Gamma_R} \boldsymbol{f}^N \cdot \overline{\boldsymbol{\psi}}_h, \quad \forall \boldsymbol{\psi}_h \in \boldsymbol{V}_h. \tag{3.1}$$

Following [28, Lemma 10.11], we may show that the discrete inf-sup condition of the sesquilinear form (2.11) holds for sufficiently small h. The detail is omitted here for brevity.

We introduce the a posteriori error estimate in the tetrahedral elements and across their faces. For any tetrahedral element $K \in \mathscr{M}_h$, we define the local residuals by

$$R_K^{(1)} := \kappa^2 \boldsymbol{E}_h^N\big|_K - \nabla \times \left(\nabla \times \boldsymbol{E}_h^N\right)\big|_K, \quad R_K^{(2)} := -\kappa^2 \nabla \cdot \boldsymbol{E}_h^N\big|_K.$$

Denote by $\mathscr{F}_h$ the set of all faces of tetrahedra in $\mathscr{M}_h$. If $F \in \mathscr{F}_h$ is an interior face and is the common face of elements K_1 and K_2, then we define the jump residuals across F as

$$J_F^{(1)} := \left(\nabla \times \boldsymbol{E}_h^N\big|_{K_1} - \nabla \times \boldsymbol{E}_h^N\big|_{K_2}\right) \times \boldsymbol{\nu}, \quad J_F^{(2)} := \kappa^2 \left(\boldsymbol{E}_h^N\big|_{K_1} - \boldsymbol{E}_h^N\big|_{K_2}\right) \cdot \boldsymbol{\nu},$$

where the unit normal vector $\boldsymbol{\nu}$ on F points from K_2 to K_1. Given a face $F \in \mathscr{F}_h \cap \Gamma_R$, define the residuals by

$$\begin{aligned} J_F^{(1)} &:= 2\left[-\left(\nabla \times \boldsymbol{E}_h^N\right) \times \boldsymbol{\nu} + \mathrm{i}\kappa \mathscr{T}^N \left(\boldsymbol{E}_{\boldsymbol{h}}^{\boldsymbol{N}}\right)_{\Gamma_R} + \boldsymbol{f}^N\right], \\ J_F^{(2)} &:= 2\left[\kappa^2 \boldsymbol{E}_h^N \cdot \boldsymbol{\nu} - \mathrm{i}\kappa \mathrm{div}_{\Gamma_R}\left(\mathscr{T}^N \left(\boldsymbol{E}_{\boldsymbol{h}}^{\boldsymbol{N}}\right)_{\Gamma_R}\right) - \mathrm{div}_{\Gamma_R} \boldsymbol{f}^N\right]. \end{aligned}$$

For any element $K \in \mathscr{M}_h$, we define the local error estimator η_K by

$$\eta_K^2 = h_K^2 \left(\|R_K^{(1)}\|_{L^2(K)}^2 + \|R_K^{(2)}\|_{L^2(K)}^2 \right) + h_K \sum_{F \in \partial K} \left(\|J_F^{(1)}\|_{L^2(F)}^2 + \|J_F^{(2)}\|_{L^2(F)}^2 \right).$$

The main result of the paper is stated as follows.

Theorem 3.1. *Let E and E_h^N be the solutions to the original scattering problem* (2.6) *and the discrete problem* (3.1)*, respectively. Then for sufficiently large integer N, the following a posteriori error estimate holds:*

$$\left\| E - E_h^N \right\|_{H(\mathrm{curl},\Omega)} \lesssim \left(\sum_{K \in \mathscr{M}_h} \eta_K^2 \right)^{1/2} + \left(\frac{R'}{R} \right)^N \|f\|_{TH^{-1/2}(\mathrm{div},\Gamma_R)}.$$

Hereafter, the notation $a \lesssim b$ means $a \leq Cb$, where C is a positive constant. It is clear to note from the theorem that there are two parts for the a posteriori error estimate. One comes from the finite element discretization error and another takes into account the DtN operator truncation error, which decays exponentially with respect to N due to $R' < R$.

4. Proof of the Main Theorem

This section is devoted to the proof of Theorem 3.1. Denote the error by $\xi = E - E_h^N$. It follows from (2.7) that

$$\begin{aligned} \|\xi\|_{H(\mathrm{curl},\Omega)}^2 = \Re \left\{ a(\xi,\xi) + \mathrm{i}\kappa \int_{\Gamma_R} \left(\mathscr{T} - \mathscr{T}^N \right) \xi_{\Gamma_R} \cdot \overline{\xi}_{\Gamma_R} \mathrm{d}s \right\} \\ - \kappa \Im \left\{ \int_{\Gamma_R} \mathscr{T}^N \xi_{\Gamma_R} \cdot \overline{\xi}_{\Gamma_R} \mathrm{d}s \right\} + \left(\kappa^2 + 1 \right) \int_\Omega \xi \cdot \overline{\xi} \mathrm{d}x. \end{aligned} \tag{4.1}$$

The goal is to estimate all the four terms on the right-hand side of (4.1). Below, Lemmas 4.1 and 4.3 concern the estimates of the first two terms; Lemma 4.4 is devoted to the estimate of the third term; Lemmas 4.6-4.10 address the error estimate of the last term.

Lemma 4.1. *Let the scattered field $E^s = E - E^{\mathrm{inc}}$ admit the Fourier series expansion in the domain $\mathbb{R}^3 \setminus \overline{B_{R'}}$*

$$E^s = \sum_{n \in \mathbb{N}} \sum_{|m| \leq n} E_{1n}^{sm}(\rho) U_n^m + E_{2n}^{sm}(\rho) V_n^m + E_{3n}^{sm}(\rho) X_n^m e_\rho.$$

Then for sufficiently large n, the following estimates hold:

$$\left| E_{jn}^{sm}(R) \right| \lesssim \left(\frac{R'}{R} \right)^n \left| E_{jn}^{sm}(R') \right|, \quad j = 1, 2.$$

Proof. By [13, Theorem 6.27] and the identities (C.1) and (C.3), the scattered field has the expansion

$$\begin{aligned}\boldsymbol{E}^s(\boldsymbol{x}) =& \sum_{n=1}^{\infty}\sum_{m=-n}^{n}\bigg[a_n^m \nabla\times\left\{\boldsymbol{x}h_n^{(1)}(\kappa|\boldsymbol{x}|)Y_n^m\left(\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\right)\right\}\\ &\qquad + b_n^m\nabla\times\nabla\times\left\{\boldsymbol{x}h_n^{(1)}(\kappa|\boldsymbol{x}|)Y_n^m\left(\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\right)\right\}\bigg]\\ =& \sum_{n=1}^{\infty}\sum_{m=-n}^{n}\bigg[-a_n^m\sqrt{n(n+1)}Rh_n^{(1)}(\kappa|\boldsymbol{x}|)\boldsymbol{V}_n^m\\ &\qquad + b_n^m\frac{\sqrt{n(n+1)}}{|\boldsymbol{x}|}\frac{\partial}{\partial\rho}\left(R|\boldsymbol{x}|h_n^{(1)}(\kappa|\boldsymbol{x}|)\right)\boldsymbol{U}_n^m\bigg]\\ &+\sum_{n=1}^{\infty}\sum_{m=-n}^{n} b_n^m\frac{n(n+1)}{|\boldsymbol{x}|}Rh_n^{(1)}(\kappa|\boldsymbol{x}|)X_n^m\boldsymbol{e}_\rho.\end{aligned}$$

Evaluating $\boldsymbol{E}^s(\boldsymbol{x})$ on Γ_R and $\Gamma_{R'}$, we get

$$\begin{aligned}E_{1n}^{sm}(R) &= b_n^m\frac{\sqrt{n(n+1)}}{R}\left[Rh_n^{(1)}(\kappa R)+\kappa R^2 h_n^{(1)'}(\kappa R)\right],\\ E_{2n}^{sm}(R) &= -a_n^m\sqrt{n(n+1)}Rh_n^{(1)}(\kappa R),\\ E_{1n}^{sm}(R') &= b_n^m\frac{\sqrt{n(n+1)}}{R'}\left[Rh_n^{(1)}(\kappa R')+\kappa RR' h_n^{(1)'}(\kappa R')\right],\\ E_{2n}^{sm}(R') &= -a_n^m\sqrt{n(n+1)}Rh_n^{(1)}(\kappa R'),\end{aligned}$$

which lead to

$$\frac{E_{1n}^{sm}(R)}{E_{1n}^{sm}(R')} = \frac{R'}{R}\frac{h_n^{(1)}(\kappa R)}{h_n^{(1)}(\kappa R')}\frac{1+z_n^{(1)}(\kappa R)}{1+z_n^{(1)}(\kappa R')},\qquad \frac{E_{2n}^{sm}(R)}{E_{2n}^{sm}(R')} = \frac{h_n^{(1)}(\kappa R)}{h_n^{(1)}(\kappa R')}. \tag{4.2}$$

By [13, Eq. (2.39)], for sufficiently large n, the spherical Hankel functions admit the asymptotic behavior

$$h_n^{(j)}(t)\sim(-1)^j\mathrm{i}\frac{(2n-1)!!}{t^{n+1}}.$$

For any integer n, it is shown in [28, Lemma 9.20] that

$$C_1 n \le \left|1+z_n^{(1)}(t)\right| \le C_2 n, \tag{4.3}$$

where C_1, C_2 are constants independent of n. Substituting the above estimates into (4.2) completes the proof. □

Since the obstacle D is assumed to be Lipschitz and the computational domain Ω can be non-convex, the classical Helmholtz decomposition may not hold. The following Birman-Solomyak decomposition is adopted in the a posteriori error analysis — cf. [7, 9].

Lemma 4.2. *For any $\boldsymbol{\phi} \in \boldsymbol{H}_{\partial D}(\mathrm{curl},\Omega)$, there exist $\boldsymbol{\Phi} \in \boldsymbol{H}^1(\Omega)$ satisfying $\nu \times \boldsymbol{\Phi} = 0$ on ∂D and $\varphi \in H^1_{\partial D}(\Omega) = \{u \in H^1(\Omega) : u = 0 \,\mathrm{on}\, \partial D\}$ such that $\boldsymbol{\phi} = \boldsymbol{\Phi} + \nabla\varphi$, and*

$$\|\varphi\|_{H^1(\Omega)} + \|\boldsymbol{\Phi}\|_{\boldsymbol{H}^1(\Omega)} \le C\|\boldsymbol{\phi}\|_{\boldsymbol{H}(\mathrm{curl},\Omega)},$$

where $C > 0$ is a constant depending on Ω.

Let U_h be the standard H^1-conforming piecewise linear isoparametric finite element space over $\mathscr{M}_h$. Lemma 4.3 is concerned with the bounds for the first two terms of (4.1). To prove it, we introduce the Clément interpolation operator $\Pi_h : H^1_{\partial D}(\Omega) \to U_h$ and the Beck-Hiptmair-Hoppe-Wohlmuth interpolation operator $\mathscr{P}_h : \boldsymbol{H}^1(\Omega) \cap \boldsymbol{H}_{\partial D}(\mathrm{curl},\Omega) \to \boldsymbol{V}_h$, which satisfy the following properties — cf. [5, 8]:

$$\|\phi - \Pi_h\phi\|_{L^2(K)} \le Ch_K\|\nabla\phi\|_{\boldsymbol{L}^2(\tilde{K})}, \quad \|\phi - \Pi_h\phi\|_{L^2(F)} \le Ch_F^{1/2}\|\nabla\phi\|_{\boldsymbol{L}^2(\tilde{F})}, \tag{4.4}$$

$$\|\boldsymbol{\Phi} - \mathscr{P}_h\boldsymbol{\Phi}\|_{\boldsymbol{L}^2(K)} \le Ch_K\|\nabla\boldsymbol{\Phi}\|_{\boldsymbol{L}^2(\tilde{K})}, \quad \|\boldsymbol{\Phi} - \mathscr{P}_h\boldsymbol{\Phi}\|_{\boldsymbol{L}^2(F)} \le Ch_F^{1/2}\|\nabla\boldsymbol{\Phi}\|_{\boldsymbol{L}^2(\tilde{F})}, \tag{4.5}$$

where C is a positive constant, $\tilde{K}$ or $\tilde{F}$ stands for the union of tetrahedra on $\mathscr{M}_h$ with non-empty intersection with K or F, respectively.

Lemma 4.3. *For sufficiently large N and for any $\boldsymbol{\psi} \in \boldsymbol{H}_{\partial D}(\mathrm{curl},\Omega)$, the following estimate holds:*

$$\begin{aligned}
&\left| a(\boldsymbol{\xi},\boldsymbol{\psi}) + \mathrm{i}\kappa\int_{\Gamma_R}\left(\mathscr{T} - \mathscr{T}^N\right)\boldsymbol{\xi}_{\Gamma_R}\cdot\overline{\boldsymbol{\psi}}_{\Gamma_R}\mathrm{d}s\right| \\
&\lesssim \left(\left(\sum_{K\in\mathscr{M}_h}\eta_K^2\right)^{1/2} + \left(\frac{R'}{R}\right)^N\|\boldsymbol{f}\|_{TH^{-1/2}(\mathrm{div},\Gamma_R)}\right)\|\boldsymbol{\psi}\|_{\boldsymbol{H}(\mathrm{curl},\Omega)}.
\end{aligned}$$

Proof. A simple calculation shows that

$$\begin{aligned}
&a(\boldsymbol{\xi},\boldsymbol{\psi}) + \mathrm{i}\kappa\int_{\Gamma_R}\left(\mathscr{T} - \mathscr{T}^N\right)\boldsymbol{\xi}_{\Gamma_R}\cdot\overline{\boldsymbol{\psi}}_{\Gamma_R}\mathrm{d}s \\
&= a(\boldsymbol{E},\boldsymbol{\psi}) - a^N\left(\boldsymbol{E}_h^N,\boldsymbol{\psi}_h\right) + a^N\left(\boldsymbol{E}_h^N,\boldsymbol{\psi}_h\right) - a\left(\boldsymbol{E}_h^N,\boldsymbol{\psi}\right) \\
&\quad + \mathrm{i}\kappa\int_{\Gamma_R}\left(\mathscr{T} - \mathscr{T}^N\right)\boldsymbol{E}_{\Gamma_R}\cdot\overline{\boldsymbol{\psi}}_{\Gamma_R}\mathrm{d}s - \mathrm{i}\kappa\int_{\Gamma_R}\left(\mathscr{T} - \mathscr{T}^N\right)\left(\boldsymbol{E}_h^N\right)_{\Gamma_R}\cdot\overline{\boldsymbol{\psi}}_{\Gamma_R}\mathrm{d}s \\
&= \left(\boldsymbol{f} - \boldsymbol{f}^N,\boldsymbol{\psi}\right) + \left(\boldsymbol{f}^N,\boldsymbol{\psi} - \boldsymbol{\psi}_h\right) - a^N\left(\boldsymbol{E}_h^N,\boldsymbol{\psi} - \boldsymbol{\psi}_h\right) + \mathrm{i}\kappa\int_{\Gamma_R}\left(\mathscr{T} - \mathscr{T}^N\right)\boldsymbol{E}_{\Gamma_R}\cdot\overline{\boldsymbol{\psi}}_{\Gamma_R}\mathrm{d}s \\
&= \left(\boldsymbol{f}^N,\boldsymbol{\psi} - \boldsymbol{\psi}_h\right) - a^N\left(\boldsymbol{E}_h^N,\boldsymbol{\psi} - \boldsymbol{\psi}_h\right) + \left(\boldsymbol{f} - \boldsymbol{f}^N,\boldsymbol{\psi}\right) + \mathrm{i}\kappa\int_{\Gamma_R}\left(\mathscr{T} - \mathscr{T}^N\right)\boldsymbol{E}_{\Gamma_R}\cdot\overline{\boldsymbol{\psi}}_{\Gamma_R}\mathrm{d}s \\
&=: J_1 + J_2.
\end{aligned}$$

By (2.11), we have

$$J_1 = -a^N\left(\boldsymbol{E}_h^N,\boldsymbol{\psi} - \boldsymbol{\psi}_h\right) + \left(\boldsymbol{f}^N,\boldsymbol{\psi} - \boldsymbol{\psi}_h\right)$$

$$
\begin{aligned}
&= -\int_\Omega \left(\nabla\times \boldsymbol{E}_h^N\right)\cdot\left(\nabla\times(\overline{\boldsymbol{\psi}-\boldsymbol{\psi}_h})\right)\mathrm{d}\boldsymbol{x} + \kappa^2\int_\Omega \boldsymbol{E}_h^N\cdot(\overline{\boldsymbol{\psi}-\boldsymbol{\psi}_h})\mathrm{d}\boldsymbol{x}\\
&\quad + \mathrm{i}\kappa\int_{\Gamma_R}\left(\mathscr{T}^N\boldsymbol{E}_h^N\right)_{\Gamma_R}\cdot\overline{\left(\boldsymbol{\psi}-\boldsymbol{\psi}_h\right)_{\Gamma_R}}\mathrm{d}s + \left(\boldsymbol{f}^N,\boldsymbol{\psi}-\boldsymbol{\psi}_h\right)\\
&=: J_1^1 + J_1^2 + J_1^3 + J_1^4.
\end{aligned}
$$

Denote the Birman-Solomyak decomposition of $\boldsymbol{\psi}$ by $\boldsymbol{\psi} = \boldsymbol{\Phi} + \nabla\phi$ and let $\boldsymbol{\Phi}_h, \phi_h$ be the Beck-Hiptmair-Hoppe-Wohlmuth interpolation of $\boldsymbol{\Phi}$ and the Clément interpolation of ϕ, respectively. Taking $\boldsymbol{\psi}_h = \boldsymbol{\Phi}_h + \nabla\phi_h$ and using Green's identity and $\nabla\times\nabla(\phi-\phi_h) = 0$, we get

$$
\begin{aligned}
J_1^1 &= -\int_\Omega \left(\nabla\times \boldsymbol{E}_h^N\right)\cdot\left(\nabla\times(\overline{\boldsymbol{\psi}-\boldsymbol{\psi}_h})\right)\mathrm{d}\boldsymbol{x} = -\int_\Omega \left(\nabla\times \boldsymbol{E}_h^N\right)\cdot\left(\nabla\times(\overline{\boldsymbol{\Phi}-\boldsymbol{\Phi}_h})\right)\mathrm{d}\boldsymbol{x}\\
&= -\sum_{K\in\mathcal{M}_h}\int_K \left(\nabla\times \boldsymbol{E}_h^N\right)\cdot\left(\nabla\times(\overline{\boldsymbol{\Phi}-\boldsymbol{\Phi}_h})\right)\mathrm{d}\boldsymbol{x}\\
&= -\sum_{K\in\mathcal{M}_h}\left[\int_K \nabla\times\left(\nabla\times \boldsymbol{E}_h^N\right)\cdot(\overline{\boldsymbol{\Phi}-\boldsymbol{\Phi}_h})\mathrm{d}\boldsymbol{x} + \int_{\partial K}\left(\nabla\times \boldsymbol{E}_h^N\right)\times\boldsymbol{\nu}\cdot(\overline{\boldsymbol{\Phi}-\boldsymbol{\Phi}_h})\mathrm{d}s\right]. \qquad (4.6)
\end{aligned}
$$

Similarly, we may deduce

$$
\begin{aligned}
J_1^2 &= \kappa^2\int_\Omega \boldsymbol{E}_h^N\cdot(\overline{\boldsymbol{\psi}-\boldsymbol{\psi}_h})\mathrm{d}\boldsymbol{x}\\
&= \kappa^2\int_\Omega \boldsymbol{E}_h^N\cdot(\overline{\boldsymbol{\Phi}-\boldsymbol{\Phi}_h})\mathrm{d}\boldsymbol{x} + \kappa^2\int_\Omega \boldsymbol{E}_h^N\cdot(\overline{\nabla\phi-\nabla\phi_h})\mathrm{d}\boldsymbol{x}\\
&= \sum_{K\in\mathcal{M}_h}\left[\kappa^2\int_K \boldsymbol{E}_h^N\cdot(\overline{\boldsymbol{\Phi}-\boldsymbol{\Phi}_h}) - \nabla\cdot\boldsymbol{E}_h^N(\overline{\phi-\phi_h})\mathrm{d}\boldsymbol{x} + \kappa^2\int_{\partial K}\boldsymbol{E}_h^N\cdot\boldsymbol{\nu}(\overline{\phi-\phi_h})\mathrm{d}s\right] \qquad (4.7)
\end{aligned}
$$

and

$$
\begin{aligned}
J_1^3 &= \mathrm{i}\kappa\int_{\Gamma_R}\mathscr{T}^N\left(\boldsymbol{E}_h^N\right)_{\Gamma_R}\cdot(\overline{\boldsymbol{\psi}-\boldsymbol{\psi}_h})_{\Gamma_R}\mathrm{d}s \qquad (4.8)\\
&= \mathrm{i}\kappa\int_{\Gamma_R}\mathscr{T}^N\left(\boldsymbol{E}_h^N\right)_{\Gamma_R}\cdot(\overline{\boldsymbol{\Phi}-\boldsymbol{\Phi}_h})_{\Gamma_R}\mathrm{d}s + \mathrm{i}\kappa\int_{\Gamma_R}\mathscr{T}^N\left(\boldsymbol{E}_h^N\right)_{\Gamma_R}\cdot(\overline{\nabla\phi-\nabla\phi_h})_{\Gamma_R}\mathrm{d}s\\
&= \sum_{K\in\mathcal{M}_h}\sum_{F\subset\Gamma_R\cap\partial K}\left[\int_{\Gamma_R}\mathrm{i}\kappa\mathscr{T}^N\left(\boldsymbol{E}_h^N\right)_{\Gamma_R}\cdot(\overline{\boldsymbol{\Phi}-\boldsymbol{\Phi}_h})_{\Gamma_R} - \mathrm{i}\kappa\int_{\Gamma_R}\mathrm{div}_{\Gamma_R}\left(\mathscr{T}^N\left(\boldsymbol{E}_h^N\right)_{\Gamma_R}\right)(\overline{\phi-\phi_h})\mathrm{d}s\right].
\end{aligned}
$$

A simple calculation shows that

$$
\begin{aligned}
J_1^4 = \left(\boldsymbol{f}^N,\boldsymbol{\psi}-\boldsymbol{\psi}_h\right) &= \int_{\Gamma_R}\boldsymbol{f}^N\cdot(\overline{\boldsymbol{\Phi}-\boldsymbol{\Phi}_h} + \overline{\nabla\phi-\nabla\phi_h})_{\Gamma_R}\mathrm{d}s\\
&= \int_{\Gamma_R}\boldsymbol{f}^N\cdot\overline{(\boldsymbol{\Phi}-\boldsymbol{\Phi}_h)_{\Gamma_R}}\mathrm{d}s - \int_{\Gamma_R}\mathrm{div}_{\Gamma_R}\boldsymbol{f}^N(\overline{\phi-\phi_h})\mathrm{d}s. \qquad (4.9)
\end{aligned}
$$

Substituting (4.6)-(4.9) into J_1 yields

$$
\begin{aligned}
J_1 &= J_1^1 + J_1^2 + J_1^3 + J_1^4 \\
&= \sum_{K\in\mathcal{M}_h} \int_K \left[\kappa^2 \boldsymbol{E}_h^N - \nabla\times\left(\nabla\times \boldsymbol{E}_h^N\right)\right]\cdot(\overline{\boldsymbol{\Phi}-\boldsymbol{\Phi}_h})\mathrm{d}\boldsymbol{x} \\
&\quad - \sum_{K\in\mathcal{M}_h} \int_K \kappa^2 \left(\nabla\cdot \boldsymbol{E}_h^N\right)(\overline{\phi-\phi_h})\mathrm{d}\boldsymbol{x} - \sum_{K\in\mathcal{M}_h}\int_{\partial K}\left(\nabla\times \boldsymbol{E}_h^N\right)\times\boldsymbol{\nu}\cdot(\overline{\boldsymbol{\Phi}-\boldsymbol{\Phi}_h})\mathrm{d}s \\
&\quad + \sum_{K\in\mathcal{M}_h}\kappa^2\int_{\partial K}\boldsymbol{E}_h^N\cdot\boldsymbol{\nu}(\overline{\phi-\phi_h})\mathrm{d}s + \sum_{K\in\mathcal{M}_h}\sum_{F\subset\Gamma_R\cap\partial K}\int_F \mathrm{i}\kappa\mathscr{T}^N\left(\boldsymbol{E}_h^N\right)_{\Gamma_R}\cdot(\overline{\boldsymbol{\Phi}-\boldsymbol{\Phi}_h})_{\Gamma_R}\mathrm{d}s \\
&\quad - \sum_{K\in\mathcal{M}_h}\sum_{F\subset\Gamma_R\cap\partial K}\int_F \mathrm{i}\kappa\,\mathrm{div}_{\Gamma_R}\left(\mathscr{T}^N\left(\boldsymbol{E}_h^N\right)_{\Gamma_R}\right)(\overline{\phi-\phi_h})\mathrm{d}s \\
&\quad + \sum_{K\in\mathcal{M}_h}\sum_{F\subset\Gamma_R\cap\partial K}\int_F \boldsymbol{f}^N\cdot\overline{(\boldsymbol{\Phi}-\boldsymbol{\Phi}_h)_{\Gamma_R}}\mathrm{d}s - \int_F \mathrm{div}_{\Gamma_R}\boldsymbol{f}^N(\overline{\phi-\phi_h})\mathrm{d}s \\
&= \sum_{K\in\mathcal{M}_h}\Bigg\{\int_K R_K^{(1)}\cdot(\overline{\boldsymbol{\Phi}-\boldsymbol{\Phi}_h})\mathrm{d}\boldsymbol{x} + \int_K R_K^{(2)}(\overline{\phi-\phi_h})\mathrm{d}\boldsymbol{x} \\
&\qquad + \sum_{F\subset\partial K}\left[\int_F \frac{1}{2}J_F^{(1)}\cdot(\overline{\boldsymbol{\Phi}-\boldsymbol{\Phi}_h})_{\Gamma_R}\mathrm{d}s + \int_F\frac{1}{2}J_F^{(2)}(\overline{\phi-\phi_h})\mathrm{d}s\right]\Bigg\}.
\end{aligned} \tag{4.10}
$$

It follows from the Clément interpolation (4.4) and the Beck-Hiptmair-Hoppe-Wohlmuth interpolation (4.5) that we get

$$
|J_1| \le C\Bigg(\sum_{K\in\mathcal{M}_h}\eta_K^2\Bigg)^{1/2}\|\boldsymbol{\psi}\|_{\boldsymbol{H}(\mathrm{curl},\Omega)}.
$$

On the other hand, we have from the definition of $\boldsymbol{f}$ and $\boldsymbol{f}^N$ that

$$
\begin{aligned}
\boldsymbol{f}-\boldsymbol{f}^N &= \left(\nabla\times\boldsymbol{E}^{\mathrm{inc}}\right)\times\boldsymbol{\nu} - \mathrm{i}\kappa\mathscr{T}\boldsymbol{E}_{\Gamma_R}^{\mathrm{inc}} - \left[\left(\nabla\times\boldsymbol{E}^{\mathrm{inc}}\right)\times\boldsymbol{\nu}-\mathrm{i}\kappa\mathscr{T}^N\boldsymbol{E}_{\Gamma_R}^{\mathrm{inc}}\right] \\
&= -\mathrm{i}\kappa\left(\mathscr{T}-\mathscr{T}^N\right)\boldsymbol{E}_{\Gamma_R}^{\mathrm{inc}}.
\end{aligned}
$$

Using (2.4) and (2.9) yields

$$
\begin{aligned}
J_2 &= \left(\boldsymbol{f}-\boldsymbol{f}^N,\boldsymbol{\psi}\right) + \mathrm{i}\kappa\int_{\Gamma_R}\left(\mathscr{T}-\mathscr{T}^N\right)\boldsymbol{E}_{\Gamma_R}\cdot\overline{\boldsymbol{\psi}}_{\Gamma_R}\mathrm{d}s \\
&= \mathrm{i}\kappa\int_{\Gamma_R}\left(\mathscr{T}-\mathscr{T}^N\right)\boldsymbol{E}_{\Gamma_R}^s\cdot\overline{\boldsymbol{\psi}}_{\Gamma_R}\mathrm{d}s \\
&= \mathrm{i}\kappa\sum_{n>N}\sum_{|m|\le n}\frac{\mathrm{i}\kappa R}{1+z_n^{(1)}(\kappa R)}E_{1n}^{sm}(R)\psi_{1n}^m(R) + \frac{1+z_n^{(1)}(\kappa R)}{\mathrm{i}\kappa R}E_{2n}^{sm}(R)\psi_{2n}^m(R)
\end{aligned}
$$

$$= \mathrm{i}\kappa \sum_{n>N}\sum_{|m|\le n} \frac{\mathrm{i}\kappa R}{1+z_n^{(1)}(\kappa R)} \frac{E_{1n}^{sm}(R)}{E_{1n}^{sm}(R')} E_{1n}^{sm}(R')\psi_{1n}^m(R) + \frac{1+z_n^{(1)}(\kappa R)}{\mathrm{i}\kappa R}\frac{E_{2n}^{sm}(R)}{E_{2n}^{sm}(R')}E_{2n}^{sm}(R')\psi_{2n}^m(R).$$

By Lemmas D.1 and 4.1, and the asymptotic property (4.3), we have

$$\begin{aligned}
|J_2| \le\ & \kappa \sum_{n>N}\sum_{|m|\le n} \left|\frac{\mathrm{i}\kappa R}{1+z_n^{(1)}(\kappa R)}\right| \left|\frac{E_{1n}^m(R)}{E_{1n}^m(R')}\right| \left|E_{1n}^m(R')\right| \left|\psi_{1n}^m(R)\right| \\
& + \left|\frac{1+z_n^{(1)}(\kappa R)}{\mathrm{i}\kappa R}\right| \left|\frac{E_{2n}^m(R)}{E_{2n}^m(R')}\right| \left|E_{2n}^m(R')\right| \left|\psi_{2n}^m(R)\right| \\
\lesssim\ & \kappa \sum_{n>N}\sum_{|m|\le n} \frac{1}{n}\left(\frac{R'}{R}\right)^n \left|E_{1n}^m(R')\right| \left|\psi_{1n}^m(R)\right| + n\left(\frac{R'}{R}\right)^n \left|E_{2n}^m(R')\right| \left|\psi_{2n}^m(R)\right| \\
\lesssim\ & \left(\frac{R'}{R}\right)^N \left(\sum_{n>N}\sum_{|m|\le n} \frac{1}{\sqrt{1+n(n+1)}}\left|E_{1n}^m(R')\right|^2 + \sqrt{1+n(n+1)}\left|E_{2n}^m(R')\right|^2\right)^{1/2} \\
& \times \left(\sum_{n>N}\sum_{|m|\le n} \frac{1}{\sqrt{1+n(n+1)}}\left|\psi_{1n}^m(R)\right|^2 + \sqrt{1+n(n+1)}\left|\psi_{2n}^m(R)\right|^2\right)^{1/2} \\
\lesssim\ & \left(\frac{R'}{R}\right)^N \|\boldsymbol{E}\|_{TH^{-1/2}(\mathrm{curl},\Gamma_{R'})}\|\boldsymbol{\psi}\|_{TH^{-1/2}(\mathrm{curl},\Gamma_R)} \\
\lesssim\ & \left(\frac{R'}{R}\right)^N \|\boldsymbol{E}\|_{\boldsymbol{H}(\mathrm{curl},\Omega)}\|\boldsymbol{\psi}\|_{\boldsymbol{H}(\mathrm{curl},\Omega)}.
\end{aligned}$$

Combining the estimates of J_1, J_2 and the condition (2.8), we complete the proof. □

Lemma 4.4. *For any $\boldsymbol{u} \in TH^{-1/2}(\mathrm{curl},\Gamma_R)$, the following estimate holds:*

$$-\Im\left\langle \mathscr{T}^N\boldsymbol{u},\boldsymbol{u}\right\rangle_{\Gamma_R} \lesssim \delta\|\nabla\times\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)} + C(\delta)\|\boldsymbol{u}\|^2_{\boldsymbol{L}^2(\Omega)},$$

where $\delta > 0$ is a constant and $C(\delta) > 0$ is also a constant depending on δ.

Proof. For any $\boldsymbol{u} \in TH^{-1/2}(\mathrm{curl},\Omega)$, it has the Fourier expansion in Ω'

$$\boldsymbol{u} = \sum_{n\in\mathbb{N}}\sum_{|m|\le n} u_{1n}^m \boldsymbol{U}_n^m + \sum_{n\in\mathbb{N}}\sum_{|m|\le n} u_{2n}^m \boldsymbol{V}_n^m.$$

Taking the imaginary part of (2.9) gives

$$\Im\left\langle \mathscr{T}^N\boldsymbol{u},\boldsymbol{u}\right\rangle_{\Gamma_R} = \sum_{n=0}^{N}\sum_{|m|\le n} \frac{\kappa R\big(1+\Re z_n^{(1)}(\kappa R)\big)}{\left|1+z_n^{(1)}(\kappa R)\right|^2}|u_{1n}^m|^2 - \sum_{n=0}^{N}\sum_{|m|\le n}\frac{1+\Re z_n^{(1)}(\kappa R)}{\kappa R}|u_{2n}^m|^2.$$

For fixed t, it is shown in [28, Lemma 9.20] that there exist positive constants c_1 and c_2 such that

$$c_1 n \le \left|z_n^{(1)}(t)\right| \le c_2 n, \quad \forall n\in\mathbb{N}.$$

Moreover, we have from [30, Theorem 2.6.1] that

$$1 \leq -\Re z_n^{(1)}(t) \leq n+1, \quad \forall n \in \mathbb{N}.$$

Hence, there exists a positive constant C independent of N such that

$$\frac{1+\Re z_n^{(1)}(\kappa R)}{\left|1+z_n^{(1)}(\kappa R)\right|^2} \geq -C\frac{1}{\sqrt{1+n(n+1)}},$$

which leads to

$$\begin{aligned}\Im\left\langle \mathscr{T}^N \boldsymbol{u}, \boldsymbol{u}\right\rangle_{\Gamma_R} &\geq \kappa R \sum_{n=0}^{N}\sum_{|m|\leq n}\frac{1+\Re z_n^{(1)}(\kappa R)}{\left|1+z_n^{(1)}(\kappa R)\right|^2}\left|u_{1n}^m\right|^2 \\ &\geq -C\sum_{|m|\leq 1}\frac{1}{\sqrt{1+n(n+1)}}\left|u_{1n}^m\right|^2 \geq -C\|\boldsymbol{u}\|^2_{TH^{-1/2}(\Gamma_R)}.\end{aligned}$$

The proof is completed by applying Lemma D.2. □

To estimate the last term of (4.1), we introduce a dual problem. Consider the Birman-Solomyak decomposition

$$\xi = \nabla q + \zeta, \quad q \in H_0^1(\Omega)$$

of ξ. The dual problem to (2.6) is to find $\boldsymbol{W} \in \boldsymbol{H}_{\partial D}(\mathrm{curl},\Omega)$ such that

$$a(\boldsymbol{\psi}, \boldsymbol{W}) = (\boldsymbol{\psi}, \boldsymbol{\zeta}), \quad \forall \boldsymbol{\psi} \in \boldsymbol{H}_{\partial D}(\mathrm{curl},\Omega). \tag{4.11}$$

By the Helmholtz decomposition, it is easy to note that

$$(\xi,\xi) = (\xi,\zeta) + (\xi,\nabla q). \tag{4.12}$$

Lemma 4.5. *The following estimate holds:*

$$\left|(\xi,\nabla q)\right| \lesssim \left(\sum_{K\in\mathcal{M}_n}\eta_K^2\right)\|\nabla q\|_{L^2(\Omega)}.$$

Proof. For any $q_h \in U_h \cap H_0^1(\Omega)$, it is easy to check that $\nabla q_h \in \boldsymbol{V}_h$. Then we have

$$\begin{aligned}&-\kappa^2\int_\Omega \xi\cdot\overline{\nabla q_h}\mathrm{d}\boldsymbol{x} = a\left(\boldsymbol{E},\nabla q_h\right) - a^N\left(\boldsymbol{E}_h^N,\nabla q_h\right) \\ &= \int_{\Gamma_R}\left(\boldsymbol{f}-\boldsymbol{f}^N\right)\cdot(\overline{\nabla q_h})_{\Gamma_R}\mathrm{d}s = -\int_{\Gamma_R}\mathrm{div}_{\Gamma_R}\left(\boldsymbol{f}-\boldsymbol{f}^N\right)q_h\mathrm{d}s = 0.\end{aligned}$$

Taking $q_h = \Pi_h q$ and using the integration by parts, we get

$$\begin{aligned}\left|(\xi,\nabla q)\right| = \left|\left(\xi,\nabla(q-q_h)\right)\right| &= \left|\sum_{K\in\mathcal{M}_h}\int_K \nabla\cdot\boldsymbol{E}_h^N(\overline{q-q_h})\mathrm{d}\boldsymbol{x} - \sum_{K\in\mathcal{M}_h}\int_{\partial K}\boldsymbol{E}_h^N\cdot\boldsymbol{\nu}(\overline{q-q_h})\mathrm{d}s\right| \\ &\lesssim \left(\sum_{K\in\mathcal{M}_n}\eta_K^2\right)\|\nabla q\|_{L^2(\Omega)},\end{aligned}$$

which completes the proof. □

It remains to estimate the first term of (4.12). It can be verified that the solution of dual problem (4.11) satisfies the boundary value problem

$$\begin{aligned} &\nabla\times(\nabla\times\boldsymbol{W})-\kappa^2\boldsymbol{W}=\boldsymbol{\zeta} && \text{in } \Omega,\\ &\boldsymbol{\nu}\times\boldsymbol{W}=0 && \text{on } \partial D,\\ &(\nabla\times\boldsymbol{W})\times\boldsymbol{\nu}=-\mathrm{i}\kappa\mathscr{T}^*\boldsymbol{W}_{\Gamma_R} && \text{on } \Gamma_R, \end{aligned} \tag{4.13}$$

where $\mathscr{T}^*$ is the adjoint operator to $\mathscr{T}$ and is defined as follows:

$$\mathscr{T}^*\boldsymbol{\phi}=\sum_{n\in\mathbb{N}}\sum_{|m|\le n}\frac{-\mathrm{i}\kappa R}{1+\overline{z_n^{(1)}(\kappa R)}}\phi_{1n}^m\boldsymbol{U}_n^m-\frac{1+\overline{z_n^{(1)}(\kappa R)}}{\mathrm{i}\kappa R}\phi_{2n}^m\boldsymbol{V}_n^m.$$

Let the solution of (4.13) admit the Fourier expansion in $B_R\setminus\overline{B_{R'}}$

$$\boldsymbol{W}=\sum_{n\in\mathbb{N}}\sum_{m=-n}^{n}w_{1n}^m(\rho)\boldsymbol{U}_n^m+w_{2n}^m(\rho)\boldsymbol{V}_n^m+w_{3n}^m(\rho)X_n^m\boldsymbol{e}_\rho. \tag{4.14}$$

When considering the problem (4.13) in $B_R\setminus\overline{B_{R'}}$, we can further assume $\nabla\cdot\boldsymbol{\zeta}=0$ since the domain $B_R\setminus\overline{B_{R'}}$ is smooth. In Lemmas 4.6 and 4.7, we present the ODE systems for the Fourier coefficients w_{2n}^m and w_{3n}^m. Lemmas 4.8 and 4.9 concern the solutions to the ODE systems. Finally, we deduce the estimate in Lemma 4.10.

Lemma 4.6. *If $\boldsymbol{W}$ satisfies* (4.13)*, then the coefficients w_{2n}^m of* (4.14) *satisfy the following ODE system:*

$$\begin{aligned} &w_{2n}^{m''}(\rho)+\frac{2}{\rho}w_{2n}^{m'}(\rho)+\left(\kappa^2-\frac{n(n+1)}{\rho^2}\right)w_{2n}^m(\rho)=-\zeta_{2n}^m(\rho), && \rho\in(R',R),\\ &w_{2n}^{m'}(R)-\frac{z_n^{(2)}(\kappa R)}{R}w_{2n}^m(R)=0, && \rho=R,\\ &w_{2n}^m(R')=w_{2n}^m(R'), && \rho=R'. \end{aligned}$$

Proof. It follows from (C.3) that

$$\left[-\frac{1}{\rho}\frac{\partial^2}{\partial\rho^2}\left(\rho w_{2n}^m(\rho)\right)+\frac{n(n+1)}{\rho^2}w_{2n}^m(\rho)-\kappa^2 w_{2n}^m(\rho)\right]V_n^m=\zeta_{2n}^m(\rho)V_n^m,$$

which gives

$$w_{2n}^{m''}(\rho)+\frac{2}{\rho}w_{2n}^{m'}(\rho)+\left(\kappa^2-\frac{n(n+1)}{\rho^2}\right)w_{2n}^m(\rho)=-\zeta_{2n}^m(\rho). \tag{4.15}$$

Substituting (C.2) into (2.4) leads to

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho w_{2n}^m(\rho)\right)\Big|_{\rho=R}V_n^m=(-\mathrm{i}\kappa)\,\mathrm{i}\,\frac{1+z_n^{(2)}(\kappa R)}{\kappa R}w_{2n}^m(R)V_n^m,$$

which shows

$$w_{2n}^{m'}(R)-\frac{z_n^{(2)}(\kappa R)}{R}w_{2n}^m(R)=0. \tag{4.16}$$

The proof is completed by combining (4.15) and (4.16). □

Lemma 4.7. *Let $v_n^m(\rho)=\rho w_{3n}^m(\rho)+C_n^m(R)$, where*

$$C_n^m(R)=\frac{1+z_n^{(2)}(\kappa R)}{z_n^{(2)}(\kappa R)}\frac{R}{\kappa^2}\zeta_{3n}^m(R),$$

then v_n^m satisfies the following ODE system:

$$\begin{aligned}
&v_n^{m''}(\rho)+\frac{2}{\rho}v_n^{m'}(\rho)+\left(\kappa^2-\frac{n(n+1)}{\rho^2}\right)v_n^m(\rho)=-\beta_n^m(\rho), && \rho\in(R',R),\\
&v_n^{m'}(R)-\frac{z_n^{(2)}(\kappa R)}{R}v_n^m(R)=0, && \rho=R,\\
&v_n^m(R')=v_n^m(R'), && \rho=R',
\end{aligned}$$

where

$$\beta_n^m(\rho)=\rho\zeta_{3n}^m(\rho)-\left(\kappa^2-\frac{n(n+1)}{\rho^2}\right)C_n^m(R).$$

Proof. Letting $u_n^m(\rho)=\rho w_{3n}^m(\rho)$, we have from (2.2) and (C.3) that

$$\begin{aligned}
&\left[-\frac{\sqrt{n(n+1)}}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho w_{1n}^m(\rho)\right)+\frac{n(n+1)}{\rho^2}w_{3n}^m(\rho)-\kappa^2 w_{3n}^m(\rho)\right]X_n^m\boldsymbol{e}_\rho\\
&=\zeta_{3n}^m(\rho)X_n^m\boldsymbol{e}_\rho.
\end{aligned} \tag{4.17}$$

Using Lemma C.1 and eliminating w_{1n}^m from (4.17) yield

$$u_n^{m''}(\rho)+\frac{2}{\rho}u_n^{m'}(\rho)+\left(\kappa^2-\frac{n(n+1)}{\rho^2}\right)u_n^m(\rho)=-\rho\zeta_{3n}^m(\rho). \tag{4.18}$$

Noting $v_n^m(\rho)=u_n^m(\rho)+C_n^m(R)$, we get

$$v_n^{m''}(\rho)+\frac{2}{\rho}v_n^{m'}(\rho)+\left(\kappa^2-\frac{n(n+1)}{\rho^2}\right)v_n^m(\rho)=-\rho\zeta_{3n}^m(\rho)+\left(\kappa^2-\frac{n(n+1)}{\rho^2}\right)C_n^m(R).$$

It follows from (2.4) and (C.2) that

$$\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho w_{1n}^m(\rho)\right)\Big|_{\rho=R}-\frac{\sqrt{n(n+1)}}{\rho}w_{3n}^m(\rho)\Big|_{\rho=R}\right]=(-\mathrm{i}\kappa)\frac{-\mathrm{i}\kappa R}{1+z_n^{(2)}(\kappa R)}w_{1n}^m(R),$$

which gives

$$w_{1n}^{m'}(R)+\frac{1+z_n^{(2)}(\kappa R)+\kappa^2R^2}{1+z_n^{(2)}(\kappa R)}\frac{1}{R}w_{1n}^m(R)=\frac{\sqrt{n(n+1)}}{R}w_{3n}^m(R). \tag{4.19}$$

Eliminating $w_{1n}^m(R)$ from (4.19) by using Lemma C.1, we obtain

$$
\begin{aligned}
&-\frac{1}{R}w_{1n}^m(R)+\frac{1}{\sqrt{n(n+1)}}\frac{1}{R}\frac{\partial^2}{\partial\rho^2}\big(\rho u_n^m(\rho)\big)\Big|_{\rho=R}+\frac{1+z_n^{(2)}(\kappa R)+\kappa^2R^2}{1+z_n^{(2)}(\kappa R)}\frac{1}{R}w_{1n}^m(R)\\
&=\frac{\sqrt{n(n+1)}}{R}w_{3n}^m(R).
\end{aligned}
$$

A simple calculation yields

$$
\frac{1}{\sqrt{n(n+1)}}\frac{\partial^2}{\partial\rho^2}\big(\rho u_n^m(\rho)\big)\Big|_{\rho=R}+\frac{\kappa^2 R}{1+z_n^{(2)}(\kappa R)}Rw_{1n}^m(R)=\sqrt{n(n+1)}w_{3n}^m(R),
$$

which gives

$$
\begin{aligned}
&\frac{1}{\sqrt{n(n+1)}}\frac{\partial^2}{\partial\rho^2}\big(\rho u_n^m(\rho)\big)\Big|_{\rho=R}+\frac{\kappa^2 R}{1+z_n^{(2)}(\kappa R)}\frac{1}{\sqrt{n(n+1)}}\frac{\partial}{\partial\rho}\big(\rho u_n^m(\rho)\big)\Big|_{\rho=R}\\
&=\sqrt{n(n+1)}w_{3n}^m(R).
\end{aligned}
$$

Substituting (4.18) into the above equation leads to

$$
\begin{aligned}
&\frac{1}{\sqrt{n(n+1)}}\left[R\left(\frac{n(n+1)}{R^2}-\kappa^2\right)u_n^m(R)-R^2\zeta_{3n}^m(R)\right]\\
&\quad+\frac{\kappa^2R}{1+z_n^{(2)}(\kappa R)}\frac{1}{\sqrt{n(n+1)}}\left[u_n^m(R)+Ru_n^{m'}(R)\right]\\
&=\sqrt{n(n+1)}\frac{1}{R}u_n^m(R).
\end{aligned}
$$

It is easy to verify

$$
\frac{\kappa^2R^2}{1+z_n^{(2)}(\kappa R)}u_n^{m'}(R)+\left[\frac{\kappa^2R}{1+z_n^{(2)}(\kappa R)}-\kappa^2R\right]u_n^m(R)=R^2\zeta_{3n}^m(R),
$$

which yields

$$
u_n^{m'}(R)-\frac{z_n^{(2)}(\kappa R)}{R}u_n^m(R)=\frac{1+z_n^{(2)}(\kappa R)}{\kappa^2}\zeta_{3n}^m(R).
$$

Noting $v_n^m(\rho)=u_n^m(\rho)+C_n^m(R)$ again, we get

$$
v_n^{m'}(R)-\frac{z_n^{(2)}(\kappa R)}{R}v_n^m(R)=0,
$$

which completes the proof. □

Once v_n^m is solved, w_{1n}^m can be computed directly from Lemma C.1 as follows:

$$\begin{aligned}w_{1n}^m(\rho)&=\frac{1}{\sqrt{n(n+1)}}\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho^2 w_{3n}^m(\rho)\right)\\&=\frac{1}{\sqrt{n(n+1)}}\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\left(v_n^m(\rho)-C_n^m(R)\right)\right)\\&=\frac{1}{\sqrt{n(n+1)}}\frac{1}{\rho}\left[v_n^m(\rho)+\rho v_n^{m'}(\rho)-C_n^m(R)\right].\end{aligned}\tag{4.20}$$

Moreover, evaluating (4.20) at $\rho=R$ yields

$$\begin{aligned}w_{1n}^m(R)&=\frac{1}{\sqrt{n(n+1)}}\left[\frac{1}{R}v_n^m(R)+v_n^{m'}(R)-\frac{1}{R}C_n^m(R)\right]\\&=\frac{1}{\sqrt{n(n+1)}}\left[\frac{1}{R}v_n^m(R)+\frac{z_n^{(2)}(\kappa R)}{R}v_n^m(R)-\frac{1}{R}C_n^m(R)\right]\\&=\frac{1}{\sqrt{n(n+1)}}\frac{1}{R}\left[1+z_n^{(2)}(\kappa R)\right]v_n^m(R)-\frac{1}{\sqrt{n(n+1)}}\frac{1}{R}C_n^m(R).\end{aligned}\tag{4.21}$$

The following results are concerned with the solutions to the ODE systems in Lemmas 4.6 and 4.7. The proof can be found in [4].

Lemma 4.8. *Let $v(\rho)$ satisfy the ODE system*

$$\begin{aligned}&v''(\rho)+\frac{2}{\rho}v'(\rho)+\left(\kappa^2-\frac{n(n+1)}{\rho^2}\right)v(\rho)=-\xi(\rho), && \rho\in(R',R),\\&v'(R)-\frac{z_n^{(2)}(\kappa R)}{R}v(R)=0, && \rho=R,\\&v(R')=v(R'), && \rho=R',\end{aligned}$$

which has a unique solution given by

$$v(\rho)=S_n(\rho)v(R')+\frac{\mathrm{i}\kappa}{2}\int_{R'}^{\rho}t^2W_n(\rho,t)\xi(t)\mathrm{d}t+\frac{\mathrm{i}\kappa}{2}\int_{R'}^{R}t^2S_n(t)W_n(R',\rho)\xi(t)\mathrm{d}t,$$

where

$$S_n(\rho)=\frac{h_n^{(2)}(\kappa\rho)}{h_n^{(2)}(\kappa R')},\quad W_n(\rho,t)=\det\begin{bmatrix}h_n^{(1)}(\kappa\rho) & h_n^{(2)}(\kappa\rho)\\ h_n^{(1)}(\kappa t) & h_n^{(2)}(\kappa t)\end{bmatrix}.$$

Taking $\rho=R$ in the solution yields

$$v(R)=S_n(R)v(R')+\frac{\mathrm{i}\kappa}{2}\int_{R'}^{R}t^2S_n(R)W_n(R',t)\xi(t)\mathrm{d}t.$$

Moreover, it follows from the asymptotic property of $h_n^{(2)}(t)$ that

$$|S_n(R)|\lesssim\left(\frac{R'}{R}\right)^n,\quad |W_n(R',t)|\lesssim\frac{1}{n}\left(\frac{t}{R'}\right)^n,\quad n\to\infty.$$

The estimates of w_{1n}^m and w_{2n}^m at $\rho=R$ are given in the following lemma.

Lemma 4.9. *Let* w_{1n}^m *and* w_{2n}^m *be the Fourier coefficients of* $\boldsymbol{W}$. *They satisfy the estimates*

$$\begin{aligned}\left|w_{2n}^m(R)\right| &\lesssim \left(\frac{R'}{R}\right)^n \left|w_{2n}^m(R')\right| + \frac{1}{n^2}\left\|\zeta_{2n}^m(t)\right\|_{L^\infty([R',R])},\\ \left|w_{1n}^m(R)\right| &\lesssim \left(\frac{R'}{R}\right)^n \left|w_{3n}^m(R')\right| + \frac{1}{n^2}\left\|\zeta_{3n}^m\right\|_{L^\infty([R',R])} + \left|\zeta_{3n}^m(R)\right|.\end{aligned}$$

Proof. It follows from Lemmas 4.6 and 4.8 that

$$\left|w_{2n}^m(R)\right| \lesssim \left(\frac{R'}{R}\right)^n \left|w_{2n}^m(R')\right| + \frac{1}{n^2}\left\|\zeta_{2n}^m(t)\right\|_{L^\infty([R',R])}.$$

In addition, we have from Lemma 4.9 that

$$v_n^m(R) = S_n(R)v_n^m(R') + \frac{\mathrm{i}\kappa}{2}\int_{R'}^R t^2 S_n(R)W_n(R',t)\beta_n^m(t)\mathrm{d}t.$$

Substituting the above equation into (4.21), we obtain

$$\begin{aligned}w_{1n}^m(R) =& \frac{1}{\sqrt{n(n+1)}}\frac{1}{R}\left[1+z_n^{(2)}(\kappa R)\right]v_n^m(R) - \frac{1}{\sqrt{n(n+1)}}\frac{1}{R}C_n^m(R)\\ =& \frac{1}{\sqrt{n(n+1)}}\frac{1}{R}\left[1+z_n^{(2)}(\kappa R)\right]S_n(R)v_n^m(R') - \frac{1}{\sqrt{n(n+1)}}\frac{1}{R}C_n^m(R)\\ &+ \frac{1}{\sqrt{n(n+1)}}\frac{1}{R}\left[1+z_n^{(2)}(\kappa R)\right]\frac{\mathrm{i}\kappa}{2}\int_{R'}^R t^2 S_n(R)W_n(R',t)\beta_n^m(t)\,\mathrm{d}t\\ =& \frac{1}{\sqrt{n(n+1)}}\frac{1}{R}\left\{\left[1+z_n^{(2)}(\kappa R)\right]S_n(R)\left[R'w_{3n}^m(R')+C_n^m(R)\right] - C_n^m(R)\right\}\\ &+ \frac{\mathrm{i}\kappa}{2}\frac{1}{\sqrt{n(n+1)}}\frac{1}{R}\left[1+z_n^{(2)}(\kappa R)\right]\int_{R'}^R t^3 S_n(R)W_n(R',t)\zeta_{3n}^m(t)\mathrm{d}t\\ &- \frac{\mathrm{i}\kappa}{2}\frac{1}{\sqrt{n(n+1)}}\frac{1}{R}\left[1+z_n^{(2)}(\kappa R)\right]\int_{R'}^R t^2 S_n(R)W_n(R',t)\left(\kappa^2 - \frac{n(n+1)}{t^2}\right)C_n^m(R)\mathrm{d}t,\end{aligned}$$

which can be simplified to

$$\begin{aligned}w_{1n}^m(R) =& \frac{1}{\sqrt{n(n+1)}}\frac{R'}{R}\left[1+z_n^{(2)}(\kappa R)\right]S_n(R)w_{3n}^m(R')\\ &+ \frac{\mathrm{i}\kappa}{2}\frac{1}{\sqrt{n(n+1)}}\frac{1}{R}\left[1+z_n^{(2)}(\kappa R)\right]S_n(R)\int_{R'}^R t^3 W_n(R',t)\zeta_{3n}^m(t)\mathrm{d}t\\ &- \frac{1}{\sqrt{n(n+1)}}\frac{1}{R}C_n^m(R) + \frac{1}{\sqrt{n(n+1)}}\frac{1}{R}\left[1+z_n^{(2)}(\kappa R)\right]S_n(R)C_n^m(R)\end{aligned}$$

$$\times\left\{1-\frac{i\kappa^3}{2}\int_{R'}^{R}t^2W_n(R',t)\mathrm{d}t+\frac{i\kappa}{2}n(n+1)\int_{R'}^{R}W_n(R',t)\mathrm{d}t\right\}.$$

Using the asymptotic expansion (cf. [4, p. 12])

$$W_n(R',t)\sim-\frac{2i}{(2n+1)\kappa R'}\left(\frac{t}{R'}\right)^n,\quad n\to\infty,$$

we get from straightforward calculations that

$$\begin{aligned}\int_{R'}^{R}W_n(R',t)\mathrm{d}t&\sim-\frac{2i}{(2n+1)(n+1)}\frac{1}{\kappa}\left(\frac{R}{R'}\right)^{n+1},\\ \int_{R'}^{R}t^2W_n(R',t)\mathrm{d}t&\sim-\frac{2i}{(2n+1)(n+3)}\frac{R^2}{\kappa}\left(\frac{R}{R'}\right)^{n+1}.\end{aligned}$$

Substituting the above equations into $w_{1n}^m(R)$ yields

$$\begin{aligned}&1-\frac{i\kappa^3}{2}\int_{R'}^{R}t^2W_n(R',t)\mathrm{d}t+\frac{i\kappa}{2}(n+1)n\int_{R'}^{R}W_n(R',t)\mathrm{d}t\\ &\sim 1-\frac{\kappa^2R^2}{(2n+1)(n+3)}\left(\frac{R}{R'}\right)^{n+1}+\frac{n}{2n+1}\left(\frac{R}{R'}\right)^{n+1},\end{aligned}$$

which gives

$$\begin{aligned}&\Bigg|\frac{1}{\sqrt{n(n+1)}}\frac{1}{R}\left[1+z_n^{(2)}(\kappa R)\right]S_n(R)C_n^m(R)\\ &\qquad\times\left\{1-\frac{i\kappa^3}{2}\int_{R'}^{R}t^2W_n(R',t)\mathrm{d}t+\frac{i\kappa}{2}\sqrt{(n+1)n}\int_{R'}^{R}W_n(R',t)\mathrm{d}t\right\}\\ &\qquad-\frac{1}{\sqrt{n(n+1)}}\frac{1}{R}C_n^m(R)\Bigg|\\ &\lesssim\left|\frac{1+z_n^{(2)}(\kappa R)}{2n+1}\frac{1}{R'}C_n^m(R)\right|.\end{aligned}$$

It is shown in [24, Lemma 3.1] that

$$z_n(t)=-(n+1)+\frac{t^4}{16n}+\frac{t^2}{2n}+\mathcal{O}\left(\frac{1}{n^2}\right).$$

Hence,

$$\left|\frac{1+z_n^{(2)}(\kappa R)}{2n+1}\frac{1}{R'}C_n^m(R)\right|\lesssim\frac{1}{2R'}\left|C_n^m(R)\right|.$$

Plugging the above equation into to $w_{1n}^m(R)$, we obtain

$$\left|w_{1n}^m(R)\right| \lesssim \left(\frac{R'}{R}\right)^n \left|w_{3n}^m(R')\right| + \frac{1}{n^2}\left\|\zeta_{3n}^m\right\|_{L^\infty([R',R])} + \left|C_n^m(R)\right|.$$

Noting

$$\left|C_n^m(R)\right| = \left|\frac{1+z_n^{(2)}(\kappa R)}{z_n^{(2)}(\kappa R)}\frac{R}{\kappa^2}\zeta_{3n}^m(R)\right| \sim \frac{R}{\kappa^2}\left|\zeta_{3n}^m(R)\right|,$$

we have

$$\left|w_{1n}^m(R)\right| \lesssim \left(\frac{R'}{R}\right)^n \left|w_{3n}^m(R')\right| + \frac{1}{n^2}\left\|\zeta_{3n}^m\right\|_{L^\infty([R',R])} + \left|\zeta_{3n}^m(R)\right|.$$

The estimate for $w_{2n}^m(R)$ can be obtained by following the same steps. □

The following result is crucial to prove Theorem 3.1.

Lemma 4.10. *Let* $\boldsymbol{W}$ *be the solution to the dual problem* (4.11). *Then the following estimate holds:*

$$\kappa\left|\int_{\Gamma_R}\left(\mathscr{T}-\mathscr{T}^N\right)\xi_{\Gamma_R}\cdot\overline{\boldsymbol{W}}_{\Gamma_R}\right| \lesssim \frac{1}{N}\|\xi\|^2_{H(\mathrm{curl}\,\Omega)}.$$

Proof. Using (2.4) and (2.9), we have

$$\begin{aligned}\kappa\left|\int_{\Gamma_R}\left(\mathscr{T}-\mathscr{T}^N\right)\xi_{\Gamma_R}\cdot\overline{\boldsymbol{W}}_{\Gamma_R}\right| \le{}& \kappa\left|\sum_{n=N+1}\sum_{|m|\le n}\frac{\mathrm{i}\kappa R}{1+z_n^{(1)}(\kappa R)}\xi_{1n}^m(R)\overline{w}_{1n}^m(R)\right| \\ &+\kappa\left|\sum_{n=N+1}\sum_{|m|\le n}\frac{1+z_n^{(1)}(\kappa R)}{\mathrm{i}\kappa R}\xi_{2n}^m(R)\overline{w}_{2n}^m(R)\right|.\end{aligned}$$

It follows from the Cauchy-Schwarz inequality that

$$\begin{aligned}&\sum_{n=N+1}\sum_{|m|\le n}\left|\frac{1+z_n^{(1)}(\kappa R)}{\kappa R}\right|\left|\xi_{2n}^m(R)\right|\left|w_{2n}^m(R)\right| \\ \le{}& \sum_{n=N+1}\sum_{|m|\le n}\left|\frac{1+z_n^{(1)}(\kappa R)}{\kappa R}\right|\left(1+n(n+1)\right)^{1/4}\left|\xi_{2n}^m(R)\right|\left(1+n(n+1)\right)^{-1/4}\left|w_{2n}^m(R)\right| \\ \le{}& \frac{1}{N^2}\left[\sum_{n=N+1}\sum_{|m|\le n}\sqrt{1+n(n+1)}\left|\xi_{2n}^m(R)\right|^2\right]^{1/2} \\ &\times\left[\sum_{n=N+1}\sum_{|m|\le n}\left|\frac{1+z_n^{(1)}(\kappa R)}{\kappa R}\right|^2\frac{N^4}{\sqrt{1+n(n+1)}}\left|w_{2n}^m(R)\right|^2\right]^{1/2} \\ \le{}& \frac{1}{N^2}\|\xi\|_{TH^{-\frac{1}{2}}(\mathrm{curl},\Gamma_R)}\left[\sum_{n=N+1}\sum_{|m|\le n}n^5\left|w_{2n}^m(R)\right|^2\right]^{1/2}.\end{aligned}$$

By Lemma 4.9, we have

$$\begin{aligned}\sum_{n=N+1}\sum_{|m|\le n} n^5\left|w_{2n}^m(R)\right|^2 &= \sum_{n=N+1}\sum_{|m|\le n} n^5\left[\left(\frac{R'}{R}\right)^{2n}\left|w_{2n}^m(R')\right|^2+\frac{1}{n^4}\left\|\zeta_{2n}^m(t)\right\|^2_{L^\infty([R',R])}\right]\\ &= \sum_{n=N+1}\sum_{|m|\le n} n^5\left(\frac{R'}{R}\right)^{2n}\left|w_{2n}^m(R')\right|^2+n\left\|\zeta_{2n}^m(t)\right\|^2_{L^\infty([R',R])}. \end{aligned}\tag{4.22}$$

We have from Lemma D.1 that

$$\begin{aligned}&\sum_{n=N+1}\sum_{|m|\le n} n^5\left(\frac{R'}{R}\right)^{2n}\left|w_{2n}^m(R')\right|^2 \lesssim \max\left(n^4\left(\frac{R'}{R}\right)^{2n}\right)\|\boldsymbol{W}\|^2_{TH^{-1/2}(\mathrm{curl},\Gamma_R)}\\ \lesssim& \max\left(n^4\left(\frac{R'}{R}\right)^{2n}\right)\|\boldsymbol{W}\|^2_{\boldsymbol{H}(\mathrm{curl},\Omega)} \lesssim \max\left(n^4\left(\frac{R'}{R}\right)^{2n}\right)\|\boldsymbol{\zeta}\|^2_{\boldsymbol{H}^1(\Omega)}. \end{aligned}\tag{4.23}$$

It is shown in [17] that

$$\|\zeta(t)\|^2_{L^\infty([R',R])}\le\left(\frac{2}{R-R'}+n\right)\|\zeta(t)\|^2_{L^2([R',R])}+\frac{1}{n}\|\zeta'(t)\|^2_{L^2([R',R])}. \tag{4.24}$$

Moreover,

$$\begin{aligned}&\left\|\zeta_{2n}^{m'}(t)\right\|^2_{L^2(R',R)}\le\left(\frac{1}{R'}\right)^2\left\|t\zeta_{2n}^{m'}(t)\right\|^2_{L^2(R',R)}\\ \le&\left(\frac{1}{R'}\right)^2\left\|t\zeta_{2n}^{m'}(t)+\zeta_{2n}^m(t)\right\|^2_{L^2(R',R)}+\left(\frac{1}{R'}\right)^2\left\|\zeta_{2n}^m(t)\right\|^2_{L^2(R',R)}. \end{aligned}$$

Combining (D.8) and (4.24) leads to

$$\begin{aligned}&\sum_{n=N+1}\sum_{|m|\le n} n\left\|\zeta_{2n}^m(t)\right\|^2_{L^\infty([R',R])}\\ \le&\sum_{n=N+1}\sum_{|m|\le n}\left[\left(\frac{2}{R-R'}+n\right)n+\left(\frac{1}{R'}\right)^2\right]\left\|\zeta_{2n}^m(t)\right\|^2_{L^2([R',R])}\\ &+\left(\frac{1}{R'}\right)^2\left\|t\zeta_{2n}^{m'}(t)+\zeta_{2n}^m(t)\right\|^2_{L^2([R',R])}\\ \lesssim&\|\nabla\times\boldsymbol{\zeta}\|^2_{L^2(\Omega)}. \end{aligned}\tag{4.25}$$

Since $\max(n^4(R'/R)^{2n})$ is bounded, we have from Lemma D.1, (4.22)-(4.23) and (4.25) that

$$\sum_{n=N+1}\sum_{|m|\le n}\left|\frac{1+z_n^{(1)}(\kappa R)}{\kappa R}\right|\left|\xi_{2n}^m(R)\right|\left|w_{2n}^m(R)\right|\lesssim\frac{1}{N^2}\|\boldsymbol{\xi}\|^2_{H(\mathrm{curl},\Omega)}. \tag{4.26}$$

Next is to estimate $\xi_{1n}^m(R)\overline{w}_{1n}^m(R)$. It follows from Lemma 4.9 that

$$\left|\sum_{n=N+1}\sum_{|m|\le n}\frac{\mathrm{i}\kappa R}{1+z_n^{(1)}(\kappa R)}\xi_{1n}^m(R)\overline{w}_{1n}^m(R)\right|\lesssim\sum_{n=N+1}\sum_{|m|\le n}\frac{1}{n}\left|\xi_{1n}^m(R)\right|\left|w_{1n}^m(R)\right|$$

$$
\begin{aligned}
&\lesssim \sum_{n=N+1}\sum_{|m|\le n}\frac{1}{n}\left|\xi_{1n}^m(R)\right|\left[\left(\frac{R'}{R}\right)^n\left|w_{3n}^m(R')\right|+\frac{1}{n^2}\left\|\zeta_{3n}^m\right\|_{L^\infty([R',R])}+\left|\zeta_{3n}^m(R)\right|\right]\\
&=: I_1+I_2+I_3.
\end{aligned}
$$

A straightforward calculation yields

$$
\begin{aligned}
I_1 &= \sum_{n=N+1}\sum_{|m|\le n}\frac{1}{n}\left(\frac{R'}{R}\right)^n\left|\xi_{1n}^m(R)\right|\left|w_{3n}^m(R')\right|\\
&= \sum_{n=N+1}\sum_{|m|\le n}\frac{1}{n^2}\frac{1}{\sqrt{n}}\left|\xi_{1n}^m(R)\right|\left(\frac{R'}{R}\right)^n n^{3/2}\left|w_{3n}^m(R')\right|\\
&\lesssim \frac{1}{N^2}\left(\sum_{n=N+1}\sum_{|m|\le n}\frac{1}{\sqrt{1+n(n+1)}}\left|\xi_{1n}^m(R)\right|^2\right)^{1/2}\left(\sum_{n=N+1}\sum_{|m|\le n}\left(\frac{R'}{R}\right)^{2n}n^3|w_{3n}^m(R')|^2\right)^{1/2}\\
&\lesssim \frac{1}{N^2}\|\boldsymbol{\xi}\|_{TH^{-1/2}(\mathrm{curl},\Gamma_R)}\max\left(n^2\left(\frac{R'}{R}\right)^n\right)\left(\sum_{n=N+1}\sum_{|m|\le n}\frac{1}{\sqrt{n(n+1)}}\left|w_{3n}^m(R')\right|^2\right)^{1/2}\\
&\lesssim \frac{1}{N^2}\|\boldsymbol{\xi}\|_{\boldsymbol{H}(\mathrm{curl},\Omega)}\|\boldsymbol{W}\|_{\boldsymbol{H}^{-1/2}(\Gamma_{R'})}\lesssim \frac{1}{N^2}\|\boldsymbol{\xi}\|^2_{H(\mathrm{curl},\Omega)}.
\end{aligned}
$$

Since $\nabla\cdot\boldsymbol{\zeta}=0$, it can be obtained from Lemma C.1 that

$$
\zeta_{3n}^{m'}(\rho)+\frac{2}{\rho}\zeta_{3n}^m(\rho)=\sqrt{n(n+1)}\frac{1}{\rho}\zeta_{1n}^m(\rho).
$$

Then we have

$$
\begin{aligned}
\int_{R'}^R\left|\zeta_{3n}^{m'}(t)\right|^2\mathrm{d}t &= \int_{R'}^R\left|\sqrt{n(n+1)}\frac{1}{\rho}\zeta_{1n}^m(\rho)-\frac{2}{\rho}\zeta_{3n}^m(\rho)\right|^2\mathrm{d}t\\
&\lesssim n(n+1)\left\|\zeta_{1n}^m\right\|^2_{L^2([R',R])}+\left\|\zeta_{3n}^m\right\|^2_{L^2([R',R])}.
\end{aligned}
$$

Substituting the above equation into (4.24) gives

$$
\begin{aligned}
\left\|\zeta_{3n}^m(t)\right\|^2_{L^\infty([R',R])} &\le \left(\frac{2}{R-R'}+n\right)\left\|\zeta_{3n}^m(t)\right\|^2_{L^2([R',R])}+\frac{1}{n}\left\|\zeta_{3n}^{m'}(t)\right\|^2_{L^2([R',R])}\\
&\lesssim \left(\frac{2}{R-R'}+n\right)\left\|\zeta_{3n}^m(t)\right\|^2_{L^2([R',R])}+n\left\|\zeta_{1n}^m\right\|^2_{L^2([R',R])}.
\end{aligned}
$$

For I_2, we have

$$
\begin{aligned}
I_2 &= \sum_{n=N+1}\sum_{|m|\le n}\frac{1}{n^3}\left|\xi_{1n}^m(R)\right|\left\|\zeta_{3n}^m\right\|_{L^\infty([R',R])}\\
&\lesssim \frac{1}{N^2}\left(\sum_{n=N+1}\sum_{|m|\le n}\frac{1}{\sqrt{n(n+1)}}\left|\xi_{1n}^m(R)\right|^2\right)^{1/2}\left(\sum_{n=N+1}\sum_{|m|\le n}\frac{1}{n}\left\|\zeta_{3n}^m\right\|^2_{L^\infty([R',R])}\right)^{1/2}\\
&\lesssim \frac{1}{N^2}\|\boldsymbol{\xi}\|_{\boldsymbol{H}(\mathrm{curl},\Omega)}\|\boldsymbol{\zeta}\|_{L^2(\Omega)}\le \frac{1}{N^2}\|\boldsymbol{\xi}\|^2_{H(\mathrm{curl},\Omega)}.
\end{aligned}
$$

Similarly, it can be shown that

$$
\begin{aligned}
I_3 &= \sum_{n=N+1} \sum_{|m|\le n} \frac{1}{n} \left|\xi_{1n}^m(R)\right| \left|\zeta_{3n}^m(R)\right| \\
&= \sum_{n=N+1} \sum_{|m|\le n} \frac{1}{n} \big(1+n(n+1)\big)^{-1/4} \left|\xi_{1n}^m(R)\right| \big(1+n(n+1)\big)^{1/4} \left|\zeta_{3n}^m(R)\right| \\
&\lesssim \frac{1}{N}\bigg(\sum_{n=N+1} \sum_{|m|\le n} \frac{1}{\sqrt{1+n(n+1)}} \left|\xi_{1n}^m(R)\right|^2 \bigg)^{1/2} \bigg(\sum_{n=N+1} \sum_{|m|\le n} \sqrt{1+n(n+1)} \left|\zeta_{3n}^m(R)\right|^2 \bigg)^{1/2} \\
&\lesssim \frac{1}{N} \|\xi\|_{H^{-1/2}(\Gamma_R)} \|\zeta\|_{H^{1/2}(\Gamma_R)} \lesssim \frac{1}{N} \|\xi\|^2_{H(\mathrm{curl},\Omega)}.
\end{aligned}
$$

Combining the estimates of I_1, I_2, I_3 and Lemma 4.2, we obtain

$$
\left| \sum_{n=N+1} \sum_{|m|\le n} \frac{\mathrm{i}\kappa R}{1+z_n^{(1)}(\kappa R)} \xi_{1n}^m(R) \overline{w}_{1n}^m(R) \right| \lesssim \frac{1}{N} \|\xi\|^2_{\boldsymbol{H}(\mathrm{curl},\Omega)}. \tag{4.27}
$$

It follows from (4.26) and (4.27) that

$$
\kappa \left| \int_{\Gamma_R} \left(\mathscr{T} - \mathscr{T}^N\right) \xi_{\Gamma_R} \cdot \overline{\boldsymbol{W}_{\Gamma_R}} \right| \le \frac{1}{N} \|\xi\|^2_{H(\mathrm{curl},\Omega)},
$$

which completes the proof. ☐

Combining (4.11)-(4.12), (4.5), and Lemmas 4.5 and 4.10, we obtain

$$
\begin{aligned}
\|\xi\|^2_{L^2(\Omega)} &= \left|\big(\xi,\zeta\big) + (\xi,\nabla q)\right| \\
&\le \left| a(\xi,\boldsymbol{W}) + \mathrm{i}\kappa \int_{\Gamma_R} \left(\mathscr{T} - \mathscr{T}^N\right) \xi_{\Gamma_R} \cdot \overline{\boldsymbol{W}_{\Gamma_R}} \mathrm{d}s \right| \\
&\quad + \left| \mathrm{i}\kappa \int_{\Gamma_R} \left(\mathscr{T} - \mathscr{T}^N\right) \xi_{\Gamma_R} \cdot \overline{\boldsymbol{W}_{\Gamma_R}} \mathrm{d}s \right| + \left|(\xi,\nabla q)\right| \\
&\lesssim \frac{1}{N} \|\xi\|^2_{\boldsymbol{H}(\mathrm{curl},\Omega)} + \Bigg(\bigg(\sum_{K\in\mathscr{M}_h} \eta_K^2 \bigg)^{1/2} + \left(\frac{R'}{R}\right)^N \|\boldsymbol{f}\|_{TH^{-1/2}(\mathrm{div},\Gamma_R)} \Bigg) \|\xi\|_{\boldsymbol{H}(\mathrm{curl},\Omega)}.
\end{aligned} \tag{4.28}
$$

Now we are ready to prove Theorem 3.1.

Proof. It follows from the error representation formula (4.1) and Lemmas 4.3 and 4.4 that

$$
\begin{aligned}
\|\xi\|^2_{H(\mathrm{curl},\Omega)} \le C\Bigg(\bigg(\sum_{T\in\mathscr{M}_h} \eta_T^2 \bigg)^{1/2} + \left(\frac{R'}{R}\right)^N \|\boldsymbol{f}\|_{TH^{-1/2}(\mathrm{div},\Gamma_R)} \Bigg) \|\xi\|_{\boldsymbol{H}(\mathrm{curl},\Omega)} \\
+ \delta \|\nabla\times\xi\|^2_{L^2(\Omega)} + C(\delta)\|\xi\|^2_{L^2(\Omega)} + C\|\xi\|^2_{L^2(\Omega)},
\end{aligned}
$$

which gives after taking $\delta = 1/2$ that

$$\|\xi\|^2_{H(\mathrm{curl},\Omega)} \lesssim \left(\left(\sum_{T\in\mathcal{M}_h} \eta_T^2 \right)^{1/2} + \left(\frac{R'}{R}\right)^N \|\boldsymbol{f}\|_{TH^{-1/2}(\mathrm{div},\Gamma_R)} \right) \|\xi\|_{\boldsymbol{H}(\mathrm{curl},\Omega)} + \|\xi\|^2_{\boldsymbol{L}^2(\Omega)}. \tag{4.29}$$

The proof is completed by substituting (4.28) into (4.29). □

5. Numerical Experiments

In this section, we present two numerical examples to demonstrate the efficiency of the proposed method. By Theorem 3.1, the a posteriori error estimator contains both the finite element approximation error ϵ_h and the DtN operator truncation error ϵ_N. Explicitly, we have

$$\epsilon_h = \left(\sum_{T\in\mathcal{M}_h} \eta_T^2 \right)^{1/2}, \quad \epsilon_N = \left(\frac{R'}{R}\right)^N \|\boldsymbol{f}\|_{TH^{-1/2}(\mathrm{div},\Gamma_R)}.$$

The algorithm of the adaptive finite element method is summarized in Algorithm 1, which is similar to that in [19] for solving the Maxwell equation in biperiodic structures.

Algorithm 1: The adaptive finite element method with the truncated DtN operator.

Input: Tolerance $\epsilon > 0$, ratio $\theta \in (0,1)$, and radius R.
Output: Refined mesh $\mathcal{M}_h$ and solution $\boldsymbol{E}$.
Fix the computational domain $\Omega = B_R \setminus \overline{D}$.
Choose N and $\hat{R}$ and such that $\epsilon_N \le 10^{-8}$.
Generate a mesh $\mathcal{M}_h$ over Ω and compute local error estimators.
while $\epsilon_h > \epsilon$ **do**
    Refine the mesh $\mathcal{M}_h$ by the strategy:
    if $\eta_{\hat{T}} > \theta \max_{T\in\mathcal{M}_h} \eta_T$ **then**
        refine the element $\hat{T} \in \mathcal{M}_h$.
    end
    Solve the discrete problem (3.1) on the new mesh $\mathcal{M}_h$.
    Compute the local error estimators.
end

In the implementation, we employ the toolbox of parallel hierarchical grid (PHG) [31] for the adaptive mesh generation and apply the direct solver of MUMPS [29] for the resulting linear systems.

Example 5.1. Let the obstacle $D = B_{0.1}$ be a ball centered at the origin with radius 0.1. An analytical solution can be constructed for such an obstacle. Specifically, the Dirichlet boundary condition on Γ_D can be set by taking the exact solution

$$\boldsymbol{E}(\boldsymbol{x}) = \boldsymbol{G}(\boldsymbol{x}) + k^{-2}\nabla\nabla\cdot\boldsymbol{G}(\boldsymbol{x}),$$

where the wavenumber $k = 2$ and

$$G = (0,0,\Phi)^\top, \quad \Phi(x,y) = \frac{e^{ik|x-y|}}{4\pi|x-y|}, \quad y = (0,0,0)^\top,$$

i.e. the point source is located at $y = (0,0,0)^\top$. The truncated computational domain is defined by $B_{0.5}$, which is a ball centered at the origin with radius 0.5.

The surface plots of the amplitude of the field E_h^N are shown in Fig. 1. Fig. 2 shows the curves of $\log\|E - E_h^N\|$ versus $\log N_k$ for both the a priori and the a posteriori error estimates, where N_k is the total number of unknowns or degrees of freedom (DoFs). It can be seen that the meshes and the corresponding numerical error are quasi-optimal, i.e., the estimate $\log\|E - E_h^N\| = \mathcal{O}(N_k^{-1/3})$ holds for sufficiently large N_k.

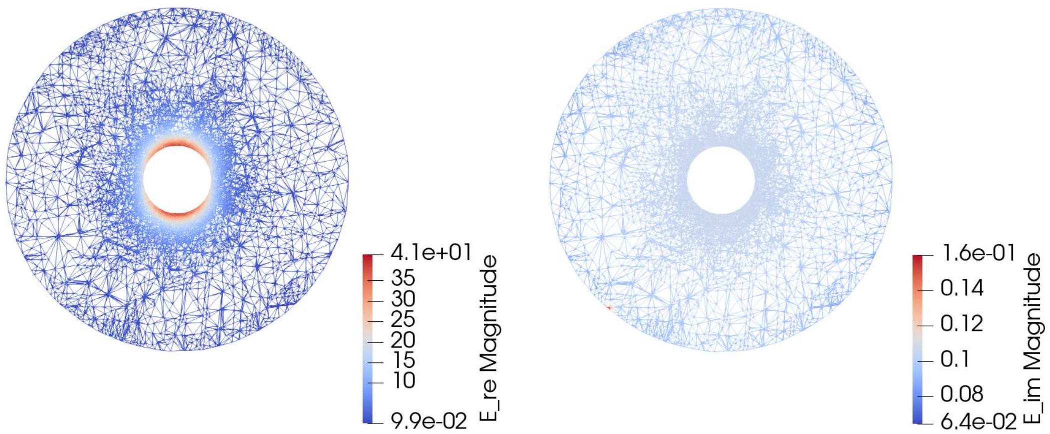

Figure 1: Example 5.1: The amplitude of the real part and the imaginary part of the solution E_h^N on the plane $\{x \in \mathbb{R}^3 : x_1 = 0\}$.

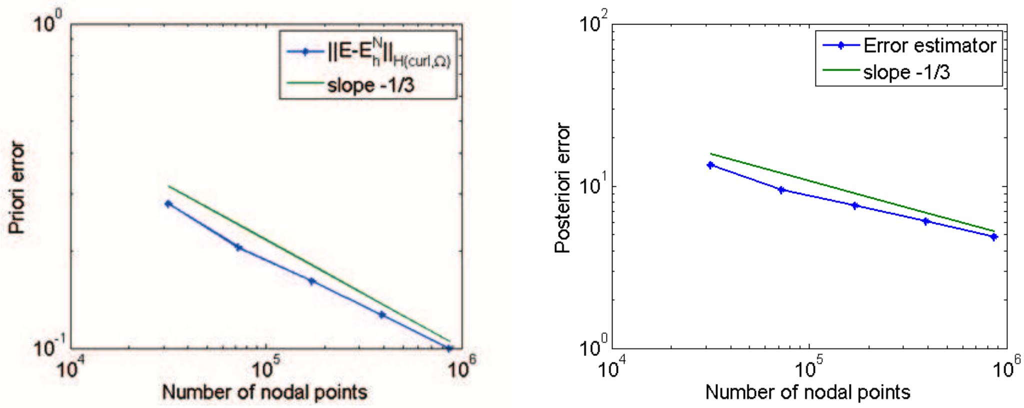

Figure 2: Example 5.1: Quasi-optimality of the a priori error (left) and the a posteriori error (right) estimates.

Example 5.2. In this example, we consider the scattering of a time-harmonic plane wave by a U-shaped obstacle, where the incident plane wave is

$$\boldsymbol{E}^{\rm inc} = \boldsymbol{p} e^{ik\boldsymbol{q}\cdot\boldsymbol{x}} = e^{-ikz}(1,0,0)^\top,$$

and the U-shaped obstacle D is shown in Fig. 3. We assume that the obstacle is a perfect electric conductor so that the Dirichlet boundary condition on Γ_D can be set by $\boldsymbol{\nu}\times\boldsymbol{E}^s = -\boldsymbol{\nu}\times\boldsymbol{E}^{\rm inc}$, i.e., the tangential trace of the total electric field vanishes on the surface of the obstacle.

The surface plots of the amplitude of the field $\boldsymbol{E}_h^N$ and the corresponding mesh are shown in Fig. 3. The solution has local singularities around the corners of the U-shaped obstacle. It is clear that the method captures the solution behavior by generating finer meshes around corners. Fig. 4 shows the curves of $\log\|\boldsymbol{E}-\boldsymbol{E}_h^N\|$ versus $\log N_k$ for the a posteriori error estimate. Again, it can be observed that the meshes and the corresponding numerical error are quasi-optimal, i.e., the estimate $\log\|\boldsymbol{E}-\boldsymbol{E}_h^N\| = \mathscr{O}(N_k^{-1/3})$ holds for sufficiently large N_k.

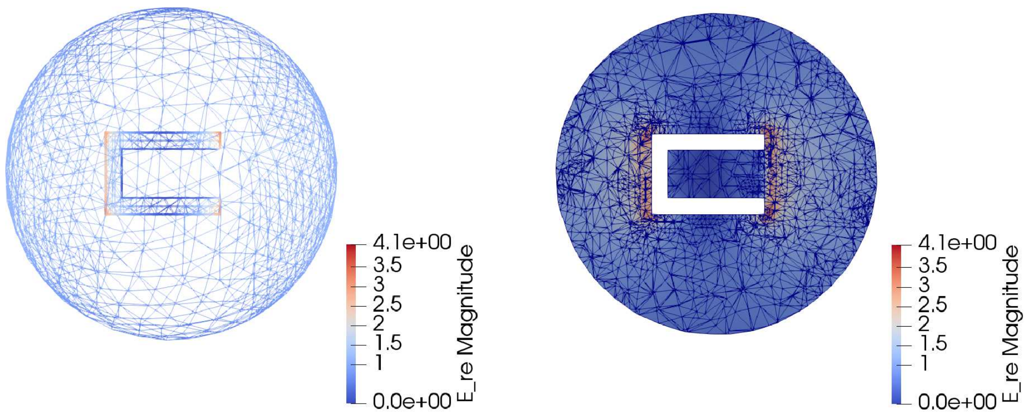

Figure 3: Example 5.2: The mesh of the computational domain (left). The amplitude of the real part of the solution $\boldsymbol{E}_h^N$ on the plane $\{\boldsymbol{x}\in\mathbb{R}^3 : x_2 = 0.04\}$ (right).

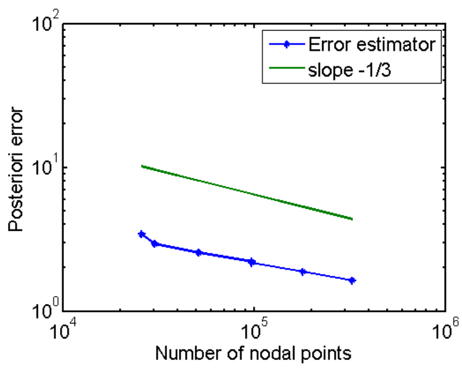

Figure 4: Example 5.2: Quasi-optimality of the a posteriori error estimates.

6. Conclusion

We have presented an adaptive finite element method with the truncated DtN operator for solving the three-dimensional electromagnetic scattering problem. The a posteriori error estimate is deduced for the discrete problem. The estimate contains both the finite element approximation error and the DtN operator truncation error. Moreover, we show that the latter decays exponentially with respect to the number of truncation terms. The a posteriori error estimate based adaptive finite element method is developed for the discrete problem. Numerical results show that the proposed method is effective to solve the electromagnetic scattering problem in three dimensions.

Appendix A. Spherical Harmonic Functions

Let $\boldsymbol{x} = (x_1, x_2, x_3)$ and (ρ, θ, φ) be the Cartesian coordinates and the spherical coordinates, respectively. We have $x_1 = \rho \sin\theta \cos\varphi$, $x_2 = \rho \sin\theta \sin\varphi$, $x_3 = \rho\cos\theta$, where $\theta \in [0, \pi]$, $\varphi \in [0, 2\pi]$ are the Euler angles of $\boldsymbol{x}$ and $\rho = |\boldsymbol{x}|$. The local orthonormal basis $\{\boldsymbol{e}_\rho, \boldsymbol{e}_\theta, \boldsymbol{e}_\varphi\}$ is given by

$$
\begin{aligned}
\boldsymbol{e}_\rho &= (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta),\\
\boldsymbol{e}_\theta &= (\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta),\\
\boldsymbol{e}_\varphi &= (-\sin\varphi, \cos\varphi, 0).
\end{aligned}
$$

Denote by $\Gamma = \{\boldsymbol{x} \in \mathbb{R}^3 : |\boldsymbol{x}| = 1\}$ and $\Gamma_R = \{\boldsymbol{x} \in \mathbb{R}^3 : |\boldsymbol{x}| = R\}$ the unit sphere and the sphere with radius R, respectively. Let $\{Y_n^m(\theta, \varphi) : |m| \le n, n = 0, 1, 2, \dots\}$ be the orthonormal sequence of spherical harmonics of order n on the unit sphere Γ. Explicitly, we have

$$
Y_n^m(\theta,\varphi) = \sqrt{\frac{(2n+1)(n-|m|)!}{4\pi(n+|m|)!}} P_n^{|m|}(\cos\theta) e^{\mathrm{i}m\varphi},
$$

where $P_n^m(t), 0 \le m \le n, -1 \le t \le 1$ are the associated Legendre functions and are defined by

$$
P_n^m(t) = (1-t^2)^{m/2} \frac{\mathrm{d}^m}{\mathrm{d}t^m} P_n(t)
$$

with P_n being the n-th order Legendre polynomial. Define rescaled harmonics of order n

$$
X_n^m(\theta,\varphi) = \frac{1}{R} Y_n^m(\theta,\varphi).
$$

It can be easily verified that $\{X_n^m(\theta,\varphi) : |m| \le n, n = 0, 1, 2, \dots\}$ form a complete orthonormal system in $L^2(\Gamma_R)$, which is the functional space of complex square integrable functions on the sphere Γ_R.

Appendix B. Surface Differential Operators and Basis Functions

For a smooth scalar function ϕ defined on Γ_R, let

$$\nabla_\Gamma \phi = \frac{\partial \phi}{\partial \theta}\boldsymbol{e}_\theta + \frac{1}{\sin\theta}\frac{\partial \phi}{\partial \varphi}\boldsymbol{e}_\varphi$$

be the surface gradient on Γ_R. The surface vector curl is defined by

$$\mathbf{curl}_\Gamma \phi = \nabla_\Gamma \phi \times \boldsymbol{e}_\rho.$$

Given a tangent vector function $\boldsymbol{\phi}$ to Γ_R, it can be expressed in the local orthonormal basis

$$\boldsymbol{\phi} = \phi_\theta \boldsymbol{e}_\theta + \phi_\varphi \boldsymbol{e}_\varphi,$$

where

$$\phi_\theta = \boldsymbol{\phi}\cdot\boldsymbol{e}_\theta, \quad \phi_\varphi = \boldsymbol{\phi}\cdot\boldsymbol{e}_\varphi.$$

Using ϕ_θ and ϕ_φ, we can define the surface divergence and the surface scalar curl

$$\mathrm{div}_\Gamma \boldsymbol{\phi} = \frac{1}{\sin\theta}\left[\frac{\partial}{\partial\theta}(\phi_\theta \sin\theta) + \frac{\partial \phi_\varphi}{\partial\varphi}\right], \qquad \mathrm{curl}_\Gamma \boldsymbol{\phi} = \frac{1}{\sin\theta}\left[\frac{\partial}{\partial\theta}(\phi_\varphi \sin\theta) - \frac{\partial \phi_\theta}{\partial\varphi}\right].$$

Following [13, Theorem 6.23], we introduce an orthonormal basis for $TL(\Gamma_R)$

$$\boldsymbol{U}_n^m(\theta,\varphi) = \frac{1}{\sqrt{n(n+1)}}\nabla_\Gamma X_n^m(\theta,\varphi) \tag{B.1}$$

and

$$\boldsymbol{V}_n^m(\theta,\varphi) = \boldsymbol{e}_\rho \times \boldsymbol{U}_n^m = -\frac{1}{\sqrt{n(n+1)}}\mathbf{curl}_\Gamma X_n^m(\theta,\varphi) \tag{B.2}$$

for $|m| \le n, n = 0,1,2,\ldots$.

Appendix C. Identities of Differential Operators

Let f be a smooth function. It can be verified that the curl operator satisfies

$$\begin{aligned}
&\nabla \times \big(f(\rho)\boldsymbol{U}_n^m\big) = \frac{1}{\rho}\frac{\partial}{\partial\rho}\big(\rho f(\rho)\big)\boldsymbol{V}_n^m,\\
&\nabla \times \big(f(\rho)\boldsymbol{V}_n^m\big) = -\frac{1}{\rho}\frac{\partial}{\partial\rho}\big(\rho f(\rho)\big)\boldsymbol{U}_n^m - \frac{\sqrt{n(n+1)}}{\rho}f(\rho)X_n^m\boldsymbol{e}_\rho,\\
&\nabla \times \big(f(\rho)X_n^m\boldsymbol{e}_\rho\big) = -\frac{\sqrt{n(n+1)}}{\rho}f(\rho)\boldsymbol{V}_n^m.
\end{aligned} \tag{C.1}$$

Moreover, we may show from (C.1) that

$$\begin{aligned}
\big(\nabla \times \big(f(\rho)\boldsymbol{U}_n^m\big)\big) \times \boldsymbol{e}_\rho &= \frac{1}{\rho}\frac{\partial}{\partial \rho}\big(\rho f(\rho)\big)\boldsymbol{U}_n^m, \\
\big(\nabla \times \big(f(\rho)\boldsymbol{V}_n^m\big)\big) \times \boldsymbol{e}_\rho &= \frac{1}{\rho}\frac{\partial}{\partial \rho}\big(\rho f(\rho)\big)\boldsymbol{V}_n^m, \\
\big(\nabla \times \big(f(\rho)X_n^m\boldsymbol{e}_\rho\big)\big) \times \boldsymbol{e}_\rho &= -\frac{\sqrt{n(n+1)}}{\rho} f(\rho)\boldsymbol{U}_n^m.
\end{aligned} \tag{C.2}$$

Taking the curl on both sides of (C.1), we have

$$\begin{aligned}
\nabla \times \big(\nabla \times \big(f(\rho)\boldsymbol{U}_n^m\big)\big) &= -\frac{1}{\rho}\frac{\partial^2}{\partial \rho^2}\big(\rho f(\rho)\big)\boldsymbol{U}_n^m - \frac{\sqrt{n(n+1)}}{\rho^2}\frac{\partial}{\partial \rho}\big(\rho f(\rho)\big)X_n^m\boldsymbol{e}_\rho, \\
\nabla \times \big(\nabla \times \big(f(\rho)\boldsymbol{V}_n^m\big)\big) &= \left[-\frac{1}{\rho}\frac{\partial^2}{\partial \rho^2}\big(\rho f(\rho)\big) + \frac{n(n+1)}{\rho^2} f(\rho)\right]\boldsymbol{V}_n^m, \\
\nabla \times \big(\nabla \times \big(f(\rho)X_n^m\boldsymbol{e}_\rho\big)\big) &= \frac{\sqrt{n(n+1)}}{\rho}\frac{\partial}{\partial \rho} f(\rho)\boldsymbol{U}_n^m + \frac{n(n+1)}{\rho^2} f(\rho)X_n^m\boldsymbol{e}_\rho.
\end{aligned} \tag{C.3}$$

The divergence operator satisfies

$$\begin{aligned}
&\nabla \cdot \big(f(\rho)\boldsymbol{U}_n^m\big) = -f(\rho)\sqrt{(n+1)n}\,\frac{1}{\rho}X_n^m, \\
&\nabla \cdot \big(f(\rho)\boldsymbol{V}_n^m\big) = 0, \\
&\nabla \cdot \big(f(\rho)X_n^m\boldsymbol{e}_\rho\big) = \frac{1}{\rho^2}\frac{\partial}{\partial \rho}\big(\rho^2 f(\rho)\big)X_n^m.
\end{aligned} \tag{C.4}$$

The following result can be easily obtained from (C.4).

Lemma C.1. *Given any smooth vector function*

$$\boldsymbol{v}_n^m = v_{1n}^m(\rho)\boldsymbol{U}_n^m + v_{2n}^m(\rho)\boldsymbol{V}_n^m + v_{3n}^m(\rho)X_n^m\boldsymbol{e}_\rho,$$

if $\nabla \cdot \boldsymbol{v}_n^m = 0$*, then its coefficients satisfy the following equation:*

$$\frac{\partial}{\partial \rho}\big(\rho^2 v_{3n}^m(\rho)\big) = \sqrt{n(n+1)}\rho v_{1n}^m(\rho).$$

Appendix D. Function Spaces and Trace Theorems

Denote by $L^2(\Omega)$ and $\boldsymbol{L}^2(\Omega) = L^2(\Omega)^3$ the standard Hilbert space of complex square integrable functions in Ω and the corresponding Cartesian product space, respectively. Let

$$\boldsymbol{H}(\mathrm{curl}, \Omega) := \left\{\boldsymbol{\phi} \in \boldsymbol{L}^2(\Omega) : \nabla \times \boldsymbol{\phi} \in \boldsymbol{L}^2(\Omega)\right\},$$

which has the norm

$$\|\boldsymbol{\phi}\|_{\boldsymbol{H}(\mathrm{curl},\Omega)} = \left(\|\boldsymbol{\phi}\|^2_{\boldsymbol{L}^2(\Omega)} + \|\nabla\times\boldsymbol{\phi}\|^2_{\boldsymbol{L}^2(\Omega)}\right)^{1/2}. \tag{D.1}$$

To describe the Calderón operator and the TBC, it is necessary to introduce some trace function spaces defined on Γ_R. Let $H^s(\Gamma_R), s\in\mathbb{R}$ be the standard trace Sobolev space and $\boldsymbol{H}^s(\Gamma_R) = H^s(\Gamma_R)^3$ be the corresponding Cartesian product space. Define the tangential function spaces

$$TL(\Gamma_R) := \left\{\boldsymbol{\phi}\in\boldsymbol{L}^2(\Gamma_R) : \boldsymbol{\phi}\cdot\boldsymbol{e}_\rho = 0\right\}, \quad TH^s(\Gamma_R) = \left\{\boldsymbol{\phi}\in\boldsymbol{H}^s(\Gamma_R) : \boldsymbol{\phi}\cdot\boldsymbol{e}_\rho = 0\right\},$$

where $\boldsymbol{e}_\rho$ is the unit normal vector to Γ_R. It is shown in [13, Theorem 6.23] that for any $\boldsymbol{\phi}\in TL(\Gamma_R)$, it has the Fourier series expansion

$$\boldsymbol{\phi} = \sum_{n\in\mathbb{N}}\sum_{|m|\le n}\phi^m_{1n}\boldsymbol{U}^m_n + \phi^m_{2n}\boldsymbol{V}^m_n,$$

where $\{(\boldsymbol{U}^m_n, \boldsymbol{V}^m_n) : |m|\le n, n=0,1,\dots\}$ is an orthonormal basis for $TL(\Gamma_R)$. The norm for functions in $TL(\Gamma_R)$ and $TH^s(\Gamma_R)$ can be characterized by

$$\|\boldsymbol{\phi}\|_{TL(\Gamma_R)} = \left(\sum_{n\in\mathbb{N}}\sum_{|m|\le n}\left|\phi^m_{1n}\right|^2 + \left|\phi^m_{2n}\right|^2\right)^{1/2}$$

and

$$\|\boldsymbol{\phi}\|_{TH^s(\Gamma_R)} = \left[\sum_{n\in\mathbb{N}}\sum_{|m|\le n}\big(1+n(n+1)\big)^s\left(\left|\phi^m_{1n}\right|^2 + \left|\phi^m_{2n}\right|^2\right)\right]^{1/2}.$$

Let

$$\begin{aligned} TH^{-1/2}(\mathrm{curl},\Gamma_R) &= \left\{\boldsymbol{\phi}\in TH^{-1/2}(\Gamma_R) : \mathrm{curl}_{\Gamma_R}\boldsymbol{\phi}\in H^{-1/2}(\Gamma_R)\right\}, \\ TH^{-1/2}(\mathrm{div},\Gamma_R) &= \left\{\boldsymbol{\phi}\in TH^{-1/2}(\Gamma_R) : \mathrm{div}_{\Gamma_R}\boldsymbol{\phi}\in H^{-1/2}(\Gamma_R)\right\}, \end{aligned}$$

which are equipped with the norms

$$\|\boldsymbol{\phi}\|_{TH^{-1/2}(\mathrm{curl},\Gamma_R)} = \left[\sum_{n\in\mathbb{N}}\sum_{|m|\le n}\left(\frac{1}{\sqrt{1+n(n+1)}}\left|\phi^m_{1n}\right|^2 + \sqrt{1+n(n+1)}\left|\phi^m_{2n}\right|^2\right)\right]^{1/2}, \tag{D.2}$$

$$\|\boldsymbol{\phi}\|_{TH^{-1/2}(\mathrm{div},\Gamma_R)} = \left[\sum_{n\in\mathbb{N}}\sum_{|m|\le n}\left(\sqrt{1+n(n+1)}\left|\phi^m_{1n}\right|^2 + \frac{1}{\sqrt{1+n(n+1)}}\left|\phi^m_{2n}\right|^2\right)\right]^{1/2}.$$

Let us introduce two trace regularity results. Similar results can be found in [23, Lemmas 3.3 and 3.4] for the overfilled cavity problem of Maxwell's equations.

Lemma D.1. *For any $\boldsymbol{\phi} \in \boldsymbol{H}(\mathrm{curl}, \Omega)$, the following estimate holds:*

$$\|\boldsymbol{\phi}_{\Gamma_R}\|_{TH^{-1/2}(\mathrm{curl},\Gamma_R)} \le C\|\boldsymbol{\phi}\|_{\boldsymbol{H}(\mathrm{curl},\Omega)},$$

where $C > 0$ only depends on the domain Ω.

Proof. Let $\Omega' := B_R \setminus \overline{B_{R'}} \subset \Omega$. For any $\boldsymbol{\phi} \in \boldsymbol{H}(\mathrm{curl}, \Omega)$, it has the following Fourier series expansion in Ω':

$$\boldsymbol{\phi} = \sum_{n\in\mathbb{N}}\sum_{|m|\le n} \phi_{1n}^m(\rho)\boldsymbol{U}_n^m + \phi_{2n}^m(\rho)\boldsymbol{V}_n^m + \phi_{3n}^m(\rho)X_n^m\boldsymbol{e}_\rho.$$

It is easy to verify from (C.1) that

$$\begin{aligned}\nabla\times\boldsymbol{\phi} = & -\frac{1}{\rho}\sum_{n\in\mathbb{N}}\sum_{|m|\le n}\sqrt{n(n+1)}\phi_{2n}^m(\rho)X_n^m\boldsymbol{e}_\rho - \frac{1}{\rho}\sum_{n\in\mathbb{N}}\sum_{|m|\le n}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\phi_{2n}^m(\rho)\right)\boldsymbol{U}_n^m \\ & -\frac{1}{\rho}\sum_{n\in\mathbb{N}}\sum_{|m|\le n}\left(\sqrt{n(n+1)}\phi_{3n}^m(\rho) - \frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\phi_{1n}^m(\rho)\right)\right)\boldsymbol{V}_n^m. \end{aligned} \tag{D.3}$$

Substituting (D.3) into (D.1), we obtain

$$\begin{aligned} & \|\boldsymbol{\phi}\|^2_{\boldsymbol{H}(\mathrm{curl},B_R\setminus\overline{B'_R})} \\ = & \frac{1}{R^2}\sum_{n\in\mathbb{N}}\sum_{|m|\le n}\left\{\int_{R'}^R\left[\rho^2+n(n+1)\right]\left|\phi_{2n}^m(\rho)\right|^2 + \left|\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\phi_{2n}^m(\rho)\right)\right|^2\mathrm{d}\rho\right. \\ & + \int_{R'}^R\left|\sqrt{n(n+1)}\phi_{3n}^m(\rho) - \frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\phi_{1n}^m(\rho)\right)\right|^2\mathrm{d}\rho \\ & \left. + \int_{R'}^R\rho^2\left(\left|\phi_{1n}^m\right|^2 + \left|\phi_{3n}^m\right|^2\right)\mathrm{d}\rho\right\} \\ = & \frac{1}{R^2}\sum_{n\in\mathbb{N}}\sum_{|m|\le n}\left\{\int_{R'}^R\left[\rho^2+n(n+1)\right]\left|\phi_{2n}^m(\rho)\right|^2 + \left|\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\phi_{2n}^m(\rho)\right)\right|^2\mathrm{d}\rho\right. \\ & + \int_{R'}^R\rho^2\left|\phi_{1n}^m\right|^2 + \frac{\rho^2}{\rho^2+n(n+1)}\left|\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\phi_{1n}^m(\rho)\right)\right|^2\mathrm{d}\rho \\ & \left. + \int_{R'}^R\left|\sqrt{\frac{n(n+1)}{\rho^2+n(n+1)}}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\phi_{1n}^m(\rho)\right) - \sqrt{\rho^2+n(n+1)}\phi_{3n}^m(\rho)\right|^2\mathrm{d}\rho\right\}, \end{aligned}$$

which gives

$$\begin{aligned} & \|\boldsymbol{\phi}\|^2_{\boldsymbol{H}(\mathrm{curl},\Omega)} \ge \|\boldsymbol{\phi}\|^2_{\boldsymbol{H}(\mathrm{curl},B_R\setminus\overline{B'_R})} \\ \ge & \frac{1}{R^2}\sum_{n\in\mathbb{N}}\sum_{|m|\le n}\left\{\int_{R'}^R\left[\rho^2+n(n+1)\right]\left|\phi_{2n}^m(\rho)\right|^2 + \left|\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\phi_{2n}^m(\rho)\right)\right|^2\mathrm{d}\rho\right. \\ & \left. + \int_{R'}^R\rho^2\left|\phi_{1n}^m\right|^2 + \frac{\rho^2}{\rho^2+n(n+1)}\left|\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\phi_{1n}^m(\rho)\right)\right|^2\mathrm{d}\rho\right\}. \end{aligned} \tag{D.4}$$

A simple calculation shows that

$$(R-R')R^2\,|\xi(R)|^2 = \int_{R'}^{R} |\rho\xi(\rho)|^2\,\mathrm{d}\rho + \int_{R'}^{R}\int_{\rho}^{R} \frac{\mathrm{d}}{\mathrm{d}\tau}|\tau\xi(\tau)|^2\,\mathrm{d}\tau\mathrm{d}\rho$$
$$\le \int_{R'}^{R} |\rho\xi(\rho)|^2\,\mathrm{d}\rho + 2(R-R')\int_{R'}^{R} |\rho\xi(\rho)|\left|\frac{\mathrm{d}}{\mathrm{d}\rho}\big(\rho\xi(\rho)\big)\right|\mathrm{d}\rho. \tag{D.5}$$

It follows from (D.5) that

$$\frac{1}{\sqrt{1+n(n+1)}}\left|\phi_{1n}^m(R)\right|^2$$
$$\le \frac{1}{(R-R')R^2}\frac{1}{\sqrt{1+n(n+1)}}\left[\int_{R'}^{R}\left|\rho\phi_{1n}^m(\rho)\right|^2 + (R-R')\sqrt{1+n(n+1)}\left|\rho\phi_{1n}^m(\rho)\right|^2\mathrm{d}\rho\right.$$
$$\left. + \frac{(R-R')}{\sqrt{1+n(n+1)}}\int_{R'}^{R}\left|\frac{\mathrm{d}}{\mathrm{d}\rho}\big(\rho\phi_{1n}^m(\rho)\big)\right|^2\mathrm{d}\rho\right]$$
$$\le \frac{1}{R^2}\left(1+\frac{1}{R-R'}\right)\int_{R'}^{R}\left|\rho\phi_{1n}^m(\rho)\right|^2\mathrm{d}\rho + \frac{1}{1+n(n+1)}\frac{1}{R^2}\int_{R'}^{R}\left|\frac{\mathrm{d}}{\mathrm{d}\rho}\big(\rho\phi_{1n}^m(\rho)\big)\right|^2\mathrm{d}\rho. \tag{D.6}$$

Similarly, we have

$$\sqrt{1+n(n+1)}R^2\left|\phi_{2n}^m(R)\right|^2 \le \left[1+n(n+1)+\frac{\sqrt{1+n(n+1)}}{R-R'}\right]\int_{R'}^{R}\left|\rho\phi_{2n}^m(\rho)\right|^2\mathrm{d}\rho$$
$$+\int_{R'}^{R}\left|\frac{\mathrm{d}}{\mathrm{d}\rho}\big(\rho\phi_{2n}^m(\rho)\big)\right|^2\mathrm{d}\rho. \tag{D.7}$$

Taking the summation of (D.6) and (D.7) over $n\in\mathbb{N}$, and using the definition (D.2) and (D.4), we complete the proof. ☐

Lemma D.2. *For any $\delta>0$, there is a positive constant $C(\delta)$ such that the following estimate holds:*

$$\|\boldsymbol{\phi}_{\Gamma_R}\|^2_{TH^{-1/2}(\Gamma_R)} \lesssim \delta\|\nabla\times\boldsymbol{\phi}\|^2_{\boldsymbol{L}^2(\Omega)} + C(\delta)\|\boldsymbol{\phi}\|^2_{\boldsymbol{L}^2(\Omega)}, \quad \forall\boldsymbol{\phi}\in\boldsymbol{H}(\mathrm{curl},\Omega).$$

Proof. For any $\boldsymbol{\phi}\in\boldsymbol{H}(\mathrm{curl},\Omega)$, it has the Fourier series expansion in $\Omega'=B_R\setminus\overline{B_{R'}}$

$$\boldsymbol{\phi}=\sum_{n\in\mathbb{N}}\sum_{|m|\le n}\phi_{1n}^m(\rho)\boldsymbol{U}_n^m+\phi_{2n}^m(\rho)\boldsymbol{V}_n^m+\phi_{3n}^m(\rho)X_n^m\boldsymbol{e}_\rho.$$

By (D.3), we have

$$\|\nabla\times\boldsymbol{\phi}\|^2_{\boldsymbol{L}^2(B_R\setminus\overline{B_{R'}})} = \frac{1}{R^2}\sum_{n\in\mathbb{N}}\sum_{|m|\le n}\left\{\int_{R'}^{R} n(n+1)\left|\phi_{2n}^m(\rho)\right|^2+\left|\frac{\mathrm{d}}{\mathrm{d}\rho}\big(\rho\phi_{2n}^m(\rho)\big)\right|^2\mathrm{d}\rho\right. \tag{D.8}$$

$$
\begin{aligned}
&\quad + \int_{R'}^{R} \left| \sqrt{n(n+1)}\phi_{3n}^m(\rho) - \frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\phi_{1n}^m(\rho)\right) \right|^2 \mathrm{d}\rho \Bigg\} \\
&= \frac{1}{R^2}\sum_{n\in\mathbb{N}}\sum_{|m|\le n} \Bigg\{ \int_{R'}^{R} n(n+1)\left|\phi_{2n}^m(\rho)\right|^2 + \left|\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\phi_{2n}^m(\rho)\right)\right|^2 \mathrm{d}\rho \\
&\quad + \int_{R'}^{R} \frac{\rho^2}{\rho^2+n(n+1)} \left|\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\phi_{1n}^m(\rho)\right)\right|^2 - \rho^2\left|\phi_{3n}^m\right|^2 \mathrm{d}\rho \\
&\quad + \int_{R'}^{R} \left| \sqrt{\frac{n(n+1)}{\rho^2+n(n+1)}}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\phi_{1n}^m(\rho)\right) - \sqrt{\rho^2+n(n+1)}\phi_{3n}^m(\rho) \right|^2 \mathrm{d}\rho \Bigg\},
\end{aligned}
$$

which gives

$$
\begin{aligned}
&\|\nabla\times\boldsymbol{\phi}\|_{L^2(\Omega)}^2 \ge \|\nabla\times\boldsymbol{\phi}\|_{L^2(B_R\setminus\overline{B_{R'}})}^2 \\
&\ge \frac{1}{R^2}\sum_{n\in\mathbb{N}}\sum_{|m|\le n} \Bigg\{ \int_{R'}^{R} n(n+1)\left|\phi_{2n}^m(\rho)\right|^2 + \left|\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\phi_{2n}^m(\rho)\right)\right|^2 \mathrm{d}\rho \\
&\quad + \int_{R'}^{R} \frac{\rho^2}{\rho^2+n(n+1)} \left|\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\phi_{1n}^m(\rho)\right)\right|^2 - \rho^2\left|\phi_{3n}^m\right|^2 \mathrm{d}\rho \Bigg\}. \qquad \text{(D.9)}
\end{aligned}
$$

It follows from (D.5) that

$$
\begin{aligned}
\frac{1}{\sqrt{1+n(n+1)}}\left|\phi_{1n}^m(R)\right|^2 &\le \frac{1}{(R-R')R^2}\frac{1}{\sqrt{1+n(n+1)}} \\
&\quad \times \Bigg[\int_{R'}^{R} \left|\rho\phi_{1n}^m(\rho)\right|^2 + \frac{1}{\hat{\delta}}(R-R')\sqrt{1+n(n+1)}\left|\rho\phi_{1n}^m(\rho)\right|^2 \mathrm{d}\rho \\
&\quad + \frac{(R-R')}{\sqrt{1+n(n+1)}}\hat{\delta}\int_{R'}^{R}\left|\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\phi_{1n}^m(\rho)\right)\right|^2 \mathrm{d}\rho \Bigg] \\
&\le \frac{1}{R^2}\left(\frac{1}{\hat{\delta}}+\frac{1}{R-R'}\right)\int_{R'}^{R}\left|\rho\phi_{1n}^m(\rho)\right|^2 \mathrm{d}\rho \\
&\quad + \frac{1}{1+n(n+1)}\frac{\hat{\delta}}{R^2}\int_{R'}^{R}\left|\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho\phi_{1n}^m(\rho)\right)\right|^2 \mathrm{d}\rho. \qquad \text{(D.10)}
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
\frac{1}{\sqrt{1+n(n+1)}}\left|\phi_{2n}^m(R)\right|^2 &\le \frac{1}{R-R'}\frac{1}{R^2}\frac{1}{\sqrt{1+n(n+1)}} \\
&\quad \times \Bigg[\int_{R'}^{R}\left|\rho\phi_{2n}^m(\rho)\right|^2 \mathrm{d}\rho + \frac{1}{\hat{\delta}}(R-R')\frac{1}{\sqrt{1+n(n+1)}} \\
&\quad \times \int_{R'}^{R}\left|\rho\phi_{2n}^m(\rho)\right|^2 \mathrm{d}\rho + \hat{\delta}(R-R')\sqrt{1+n(n+1)}
\end{aligned}
$$

$$
\begin{aligned}
&\times \int_{R'}^{R} \left| \frac{\mathrm{d}}{\mathrm{d}\rho} \big(\rho \phi_{2n}^{m}(\rho)\big) \right|^{2} \mathrm{d}\rho \Bigg] \\
\le\, & \frac{\hat{\delta}}{R^2} \int_{R'}^{R} n(n+1) \left|\phi_{2n}^{m}(\rho)\right|^{2} + \left| \frac{\mathrm{d}}{\mathrm{d}\rho} \big(\rho \phi_{2n}^{m}(\rho)\big) \right|^{2} \mathrm{d}\rho \\
&+ \frac{1}{R-R'} \left[\frac{1}{R^2} + \frac{(R-R')^2}{\hat{\delta}} \right] \int_{R'}^{R} \left|\rho \phi_{2n}^{m}(\rho)\right|^{2} \mathrm{d}\rho. \qquad \text{(D.11)}
\end{aligned}
$$

Taking $\hat{\delta} = \delta R^2$ and summing over $n \in \mathbb{N}$, we obtain from (D.9)-(D.11) that

$$
\begin{aligned}
\|\boldsymbol{\phi}_{\Gamma_R}\|^2_{TH^{-1/2}(\Gamma_R)} \le\, & \delta \sum_{n\in\mathbb{N}} \sum_{|m|\le n} \int_{R'}^{R} \Bigg[\frac{1}{1+n(n+1)} \left| \frac{\mathrm{d}}{\mathrm{d}\rho} \big(\rho \phi_{1n}^{m}(\rho)\big) \right|^{2} + n(n+1) \left|\phi_{2n}^{m}(\rho)\right|^{2} \\
&+ \left| \frac{\mathrm{d}}{\mathrm{d}\rho} \big(\rho \phi_{2n}^{m}(\rho)\big) \right|^{2} - \left|\rho \phi_{3n}^{m}(\rho)\right|^{2} \Bigg] \mathrm{d}\rho \\
&+ \delta \sum_{n\in\mathbb{N}} \sum_{|m|\le n} \int_{R'}^{R} \left|\rho \phi_{3n}^{m}(\rho)\right|^{2} \mathrm{d}\rho \\
&+ C(\delta) \sum_{n\in\mathbb{N}} \sum_{|m|\le n} \int_{R'}^{R} \Big[\left|\rho \phi_{1n}^{m}(\rho)\right|^{2} + \left|\rho \phi_{2n}^{m}(\rho)\right|^{2} + \left|\rho \phi_{3n}^{m}(\rho)\right|^{2} \Big] \mathrm{d}\rho \\
\lesssim\, & \delta \|\nabla \times \boldsymbol{\phi}\|^2_{\boldsymbol{L}^2(\Omega)} + C(\delta) \|\boldsymbol{\phi}\|^2_{\boldsymbol{L}^2(\Omega)},
\end{aligned}
$$

which completes the proof. □

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