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INVERSE RANDOM POTENTIAL SCATTERING FOR ELASTIC WAVES*

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Abstract. This paper is concerned with the inverse elastic scattering problem for a random potential in three dimensions. Interpreted as a distribution, the potential is assumed to be a microlocally isotropic Gaussian random field whose covariance operator is a classical pseudodifferential operator. Given the potential, the direct scattering problem is shown to be well-posed in the sense of distributions by studying the equivalent Lippmann–Schwinger integral equation. For the inverse scattering problem, we demonstrate that the microlocal strength of the random potential can be uniquely determined with probability one by a single realization of the high frequency limit of the averaged compressional or shear backscattered far-field pattern of the scattered wave. The analysis employs the integral operator theory, the Born approximation in the high frequency regime, the microlocal analysis for the Fourier integral operators, and the ergodicity of the wave field.

Key words. inverse scattering problem, elastic wave equation, generalized Gaussian random field, pseudodifferential operator, principal symbol, uniqueness

MSC codes. 35R30, 35R60, 60H15

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1. Introduction.

Inverse problems seek the causal factors which produce observations. Mathematically, they determine unknown parameters in partial differential equations via external measurements. The field of inverse problems has undergone tremendous growth in the last several decades. Motivated by diverse scientific and industrial applications such as radar and sonar, nondestructive testing, and medical imaging, inverse scattering has become an area of intense activity and is one of the most important research topics in scattering theory [11]. Recently, inverse scattering problems for elastic waves have received much attention due to significant applications in geophysics, seismology, and elastography [1]. We refer to [4, 5, 6, 7, 15, 16, 35, 36] and the references cited therein for mathematical and computational results on inverse problems in elasticity. A comprehensive account of elasticity can be found in the monograph [10].

Stochastic inverse problems refer to inverse problems that involve uncertainties, which become essential for mathematical models in order to take account of unpredictability of the environments, incomplete knowledge of the systems and measurements, and interference between different scales. In addition to the existing hurdles of

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nonlinearity and ill-posedness for deterministic counterparts, stochastic inverse problems have substantially more difficulties due to randomness and uncertainties. The area is wide open. It is highly desired to develop new models and methodologies. We refer to [2, 8, 17] and [14, 18, 19] on inverse scattering problems in random media and with random sources, respectively. Recent progress can be found in [3, 24, 25, 30, 31] on the inverse random source problems for the wave equations, where the goal is to determine the statistical properties of the random source from knowledge of the radiated random wave field.

The inverse random potential problem is more challenging than the inverse random source problem due to nonlinearity. There are only a few mathematical results on the inverse random potential problems, where the potential is assumed to be a microlocally isotropic Gaussian random field of order $-m$ as proposed in [22] (cf. Assumption 1.1). In [23], the authors considered the point source illumination and studied the inverse random potential problem for the two-dimensional Schrödinger equation with $m \in [2, 3)$ by examining the Born series and using the near-field data. It is shown that the microlocal strength of the random potential can be uniquely determined by the high frequency limit of the scattered wave averaged over the frequency band. The work was extended in [26] to the more complicated stochastic elastic wave equation of rougher potentials with $m \in (1, 2]$ in $\mathbb{R}^2$. The microlocal strength of the random potential is shown to be uniquely determined by the high frequency limit of two scattered fields averaged over the frequency band, where the two scattered fields are excited by two point sources with two unit orthonormal polarization vectors. It is unclear whether the near-field measurement used in the two-dimensional problems can be utilized to solve the three-dimensional problems. The difficulty arises from the fact that the fundamental solutions in three dimensions do not decay with respect to the frequency, which makes it challenging to estimate the higher order terms in the Born series. In [9], the authors investigated the inverse random potential problem for the three-dimensional Schrödinger equation with $m = 3$, where the plane wave was taken as the incidence and the far-field pattern was used as the data. They showed that the microlocal strength of the random potential can be uniquely recovered by the high frequency limit of the backscattered far-field pattern averaged over the frequency band. Motivated by [9], we study the inverse random potential problem for the three-dimensional elastic wave equation by using the far-field pattern, where the potential has a larger range of roughness with $m \in (2, 3]$. Since the compressional and shear components of the elastic wave have different wavenumbers and the displacement is a vector-valued function, the analysis is more sophisticated than the Schrödinger equation. We refer to [27, 28, 29] for closely related inverse problems of the three-dimensional Schrödinger equation. In [27], it was shown that the deterministic potential and statistics of the white noise perturbed source can be uniquely determined by using both the passive and the active measurement. The work was extended in [29] to the case that the source is a microlocally isotropic Gaussian random field. It was further extended in [28] to the case that both the potential and the source are microlocally isotropic Gaussian random fields. The estimates are technical for the problem with both the potential and the source being random. A priori information is needed about the relationship of the orders for the potential and source. These interesting results are summarized in the review paper [33]. An inverse random impedance problem for the acoustic wave equation can be found in [20].

This work is concerned with an inverse random potential scattering problem for the three-dimensional elastic wave equation. Specifically, we consider the stochastic elastic wave equation

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$$\mu\Delta\boldsymbol{u} + (\lambda+\mu)\nabla\nabla\cdot\boldsymbol{u} + \omega^2\boldsymbol{u} - \rho\boldsymbol{u} = 0 \quad \text{in } \mathbb{R}^3, \tag{1.1}$$

where $\omega > 0$ is the angular frequency, λ and μ are the Lamé constants satisfying $\mu > 0$ and $\lambda + 2\mu > 0$ such that the second order partial differential operator $\Delta^* := \mu\Delta + (\lambda+\mu)\nabla\nabla\cdot$ is strongly elliptic (cf. [34, section 10.4]), and the random potential ρ stands for a linear load inside a known homogeneous and isotropic elastic solid and is assumed to be a generalized Gaussian random field whose definition is given as follows.

Assumption 1.1. Assume that the random potential ρ is a centered microlocally isotropic Gaussian random field of order $-m$ supported in D with $m \in (2,3]$ and $D \subset \mathbb{R}^3$ being a bounded domain. More precisely, the covariance operator $\mathcal{C}_\rho$ of ρ, defined by

$$\langle \mathcal{C}_\rho \varphi, \psi\rangle := \mathbb{E}[\langle \rho, \varphi\rangle\langle \rho, \psi\rangle] \quad \forall \varphi, \psi \in \mathcal{D},$$

is a classical pseudodifferential operator. Its principal symbol has the form $\phi(x)|\xi|^{-m}$, where ϕ is called the microlocal strength of ρ and satisfies $\mathrm{supp}(\phi) \subset D$, $\phi \in C_0^\infty(D)$, and $\phi \geq 0$.

The total field $\boldsymbol{u}$ consists of the scattered field $\boldsymbol{u}^{sc}$ and the incident field $\boldsymbol{u}^{inc}$, which is assumed to be the elastic plane wave of the general form

$$\boldsymbol{u}^{inc}(x) = \alpha\boldsymbol{u}_{\mathrm{p}}^{inc}(x) + \beta\boldsymbol{u}_{\mathrm{s}}^{inc}(x), \quad \alpha, \beta \in \mathbb{C}. \tag{1.2}$$

Here, $\boldsymbol{u}_{\mathrm{p}}^{inc} := \theta e^{\mathrm{i}\kappa_{\mathrm{p}} x\cdot\theta}$ is the compressional plane wave and $\boldsymbol{u}_{\mathrm{s}}^{inc} := \theta^\perp e^{\mathrm{i}\kappa_{\mathrm{s}} x\cdot\theta}$ denotes the shear plane wave, where $\theta \in \mathbb{S}^2 := \{x \in \mathbb{R}^3 : |x| = 1\}$ represents the unit propagation direction, $\theta^\perp \in \mathbb{S}^2$ is a unit vector orthogonal to θ, and $\kappa_{\mathrm{p}} := c_{\mathrm{p}}\omega$ and $\kappa_{\mathrm{s}} := c_{\mathrm{s}}\omega$ with $c_{\mathrm{p}} := (\lambda+2\mu)^{-\frac{1}{2}}$ and $c_{\mathrm{s}} := \mu^{-\frac{1}{2}}$ denote the compressional and shear wavenumbers, respectively. In this paper, we consider separately these two types of incident plane waves: one is the compressional plane wave $\boldsymbol{u}^{inc} = \boldsymbol{u}_{\mathrm{p}}^{inc}$ with $\alpha = 1$ and $\beta = 0$; the other is the shear plane wave $\boldsymbol{u}^{inc} = \boldsymbol{u}_{\mathrm{s}}^{inc}$ with $\alpha = 0$ and $\beta = 1$. It can be verified that the incident field $\boldsymbol{u}^{inc}$ satisfies

$$\mu\Delta\boldsymbol{u}^{inc} + (\lambda+\mu)\nabla\nabla\cdot\boldsymbol{u}^{inc} + \omega^2\boldsymbol{u}^{inc} = 0 \quad \text{in } \mathbb{R}^3.$$

Since the problem is formulated in the whole space $\mathbb{R}^3$, an appropriate radiation condition is needed to ensure the uniqueness of the solution. As usual, the scattered field $\boldsymbol{u}^{sc}$ is required to satisfy the Kupradze–Sommerfeld radiation condition. Based on the Helmholtz decomposition (cf. [5, Appendix B]), the scattered field $\boldsymbol{u}^{sc}$ can be decomposed into the compressional wave component $\boldsymbol{u}_{\mathrm{p}}^{sc} := -\frac{1}{\kappa_{\mathrm{p}}^2}\nabla\nabla\cdot\boldsymbol{u}^{sc}$ and the shear wave component $\boldsymbol{u}_{\mathrm{s}}^{sc} := \frac{1}{\kappa_{\mathrm{s}}^2}\nabla\times(\nabla\times\boldsymbol{u}^{sc})$ in $\mathbb{R}^3\setminus\overline{D}$. The Kupradze–Sommerfeld radiation condition reads that $\boldsymbol{u}_{\mathrm{p}}^{sc}$ and $\boldsymbol{u}_{\mathrm{s}}^{sc}$ satisfy the Sommerfeld radiation condition

$$\lim_{|x|\to\infty} |x|\left(\partial_{|x|}\boldsymbol{u}_{\mathrm{p}}^{sc} - \mathrm{i}\kappa_{\mathrm{p}}\boldsymbol{u}_{\mathrm{p}}^{sc}\right) = 0, \quad \lim_{|x|\to\infty} |x|\left(\partial_{|x|}\boldsymbol{u}_{\mathrm{s}}^{sc} - \mathrm{i}\kappa_{\mathrm{s}}\boldsymbol{u}_{\mathrm{s}}^{sc}\right) = 0 \tag{1.3}$$

uniformly in all directions $\hat{x} := x/|x| \in \mathbb{S}^2$. The radiation condition (1.3) leads to the following asymptotic expansion of $\boldsymbol{u}^{sc}$:

$$\boldsymbol{u}^{sc}(x) = \frac{e^{\mathrm{i}\kappa_{\mathrm{p}}|x|}}{|x|}\boldsymbol{u}_{\mathrm{p}}^\infty(\hat{x}) + \frac{e^{\mathrm{i}\kappa_{\mathrm{s}}|x|}}{|x|}\boldsymbol{u}_{\mathrm{s}}^\infty(\hat{x}) + O(|x|^{-2}), \quad |x| \to \infty, \tag{1.4}$$

where $\boldsymbol{u}_{\mathrm{p}}^\infty(\hat{x})$ and $\boldsymbol{u}_{\mathrm{s}}^\infty(\hat{x})$ are known as the compressional and shear far-field patterns of the scattered field $\boldsymbol{u}^{sc}$, respectively.

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Note that the wave fields $\boldsymbol{u}, \boldsymbol{u}_{\rm p}^\infty, \boldsymbol{u}_{\rm s}^\infty$ also depend on the angular frequency ω and the propagation direction θ. For clarity, we write $\boldsymbol{v}(x)$ as $\boldsymbol{v}(x,\omega,\theta)$ when it is necessary to express explicitly the dependence of the wave field $\boldsymbol{v}$ on ω and θ.

Given the random potential ρ, the direct scattering problem is to investigate the well-posedness and regularity of the solution $\boldsymbol{u}$ to (1.1)–(1.3). The inverse scattering problem aims to determine the microlocal strength ϕ of the random potential from knowledge of the wave field $\boldsymbol{u}$. In this work, we consider both the direct and inverse scattering problems. The direct scattering problem is shown to be well-posed in the sense of distributions (cf. Theorem 2.2). Below, we present the main result on the uniqueness of the inverse scattering problem and outline the steps of its proof for readability.

THEOREM 1.2. *Let the random potential ρ satisfy Assumption 1.1 with $m \in (\frac{14}{5}, 3]$. Denote by $\boldsymbol{u}_{\rm p}^\infty$ and $\boldsymbol{u}_{\rm s}^\infty$ the compressional and shear far-field patterns of the scattered wave $\boldsymbol{u}^{sc}$ associated with $\boldsymbol{u}^{inc} = \boldsymbol{u}_{\rm p}^{inc}$ and $\boldsymbol{u}^{inc} = \boldsymbol{u}_{\rm s}^{inc}$, respectively. For any fixed $\theta \in \mathbb{S}^2$ and $\tau \geq 0$, it holds almost surely that*

$$\lim_{Q\to\infty} \frac{1}{Q}\int_Q^{2Q} \omega^m \boldsymbol{u}_{\rm p}^\infty(-\theta,\omega,\theta)\cdot \overline{\boldsymbol{u}_{\rm p}^\infty(-\theta,\omega+\tau,\theta)}d\omega = C_{\rm p}\hat{\phi}(2c_{\rm p}\tau\theta), \tag{1.5}$$

$$\lim_{Q\to\infty} \frac{1}{Q}\int_Q^{2Q} \omega^m \boldsymbol{u}_{\rm s}^\infty(-\theta,\omega,\theta)\cdot \overline{\boldsymbol{u}_{\rm s}^\infty(-\theta,\omega+\tau,\theta)}d\omega = C_{\rm s}\hat{\phi}(2c_{\rm s}\tau\theta), \tag{1.6}$$

where $C_{\rm p} = 2^{-m-4}\pi^{-2}c_{\rm p}^{4-m}$, $C_{\rm s} = 2^{-m-4}\pi^{-2}c_{\rm s}^{4-m}$, and

$$\hat{\phi}(\xi) = \mathcal{F}[\phi](\xi) = \int_{\mathbb{R}^3} \phi(x) e^{-{\rm i}x\cdot\xi}dx$$

is the Fourier transform of ϕ. Moreover, the microlocal strength ϕ is uniquely determined from (1.5) or (1.6) with $(\tau,\theta)\in\Theta$ and $\Theta\subset\mathbb{R}_+\times\mathbb{S}^2$ being any open domain.

Since the potential is random, the scattered wave and its far-field pattern are also random fields. In general, the scattering data used to recover the random coefficients involved in the stochastic inverse problems depend on the realizations of the random coefficients. Interestingly, the results in Theorem 1.2 demonstrate that the scattering data given on the left-hand side of (1.5)–(1.6) are statistically stable, i.e., they are independent of the realizations of the potential. The compressional or shear backscattered far-field pattern generated by any single realization of the random potential can determine with probability one the microlocal strength ϕ of the random potential.

To prove Theorem 1.2, we consider the equivalent Lippmann–Schwinger integral equation and show that the solution can be written as a Born series for sufficiently large frequency, i.e.,

$$\boldsymbol{u}^{sc}(x,\omega,\theta) = \sum_{j=1}^\infty \boldsymbol{u}_j(x,\omega,\theta).$$

Correspondingly, the far-field pattern $\boldsymbol{u}^\infty := \boldsymbol{u}_{\rm p}^\infty + \boldsymbol{u}_{\rm s}^\infty$ of the scattered field $\boldsymbol{u}^{sc}$ has the form

$$\boldsymbol{u}^\infty(\hat{x},\omega,\theta) = \boldsymbol{u}_1^\infty(\hat{x},\omega,\theta) + \boldsymbol{u}_2^\infty(\hat{x},\omega,\theta) + \boldsymbol{b}(\hat{x},\omega,\theta),$$

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where $\boldsymbol{b}(\hat{x},\omega,\theta) := \sum_{j=3}^{\infty} \boldsymbol{u}_j^{\infty}(\hat{x},\omega,\theta)$ and $\boldsymbol{u}_j^{\infty}$ denotes the far-field pattern of $\boldsymbol{u}_j$. For the first order far-field pattern $\boldsymbol{u}_1^{\infty}$, we show by using the Fourier analysis in section 3.1 that

$$\lim_{Q\to\infty} \frac{1}{Q}\int_Q^{2Q} \omega^m \boldsymbol{u}_{1,\mathrm{p}}^{\infty}(-\theta,\omega,\theta)\cdot \overline{\boldsymbol{u}_{1,\mathrm{p}}^{\infty}(-\theta,\omega+\tau,\theta)}d\omega = C_{\mathrm{p}}\hat{\phi}(2c_{\mathrm{p}}\tau\theta),$$
$$\lim_{Q\to\infty} \frac{1}{Q}\int_Q^{2Q} \omega^m \boldsymbol{u}_{1,\mathrm{s}}^{\infty}(-\theta,\omega,\theta)\cdot \overline{\boldsymbol{u}_{1,\mathrm{s}}^{\infty}(-\theta,\omega+\tau,\theta)}d\omega = C_{\mathrm{s}}\hat{\phi}(2c_{\mathrm{s}}\tau\theta),$$

where $\boldsymbol{u}_{j,\mathrm{p}}^{\infty}$ and $\boldsymbol{u}_{j,\mathrm{s}}^{\infty}$ are the compressional and shear far-field patterns of $\boldsymbol{u}_j^{\infty}$ for $j\in\mathbb{N}$. For the second order far-field pattern $\boldsymbol{u}_2^{\infty}$, the higher order far-field pattern $\boldsymbol{b}$, and their interactions to the first order far-field pattern, we employ microlocal analysis of Fourier integral operators and show that they are negligible in sections 3.2 and 3.3, i.e.,

$$\lim_{Q\to\infty} \frac{1}{Q}\int_Q^{2Q} \omega^m |\boldsymbol{u}_2^{\infty}(-\theta,\omega,\theta)|^2 d\omega = 0, \lim_{Q\to\infty} \frac{1}{Q}\int_Q^{2Q} \omega^m |\boldsymbol{b}(-\theta,\omega,\theta)|^2 d\omega = 0.$$

The paper is organized as follows. In section 2, the well-posedness is established for the direct scattering problem by studying the equivalent Lippmann–Schwinger integral equation; the convergence of the series solution is proved for the Lippmann–Schwinger integral equation for sufficiently large frequency. Section 3 is devoted to the inverse scattering problem, where a uniqueness result is obtained to determine the microlocal strength of the random potential. The paper concludes with some general remarks in section 4.

2. The direct scattering problem. This section is devoted to addressing the well-posedness of the scattering problem (1.1)–(1.3) and the regularity of the solution $\boldsymbol{u}$. The challenge arises from the roughness of the random potential ρ. By the following lemma, the potential ρ should be interpreted as a distribution in $W^{\frac{m-3}{2}-\epsilon,p}(D)$ almost surely for any $\epsilon>0$ and $p\in(1,\infty)$. The proof of Lemma 2.1 can be found in [23, 30].

LEMMA 2.1. *Let ρ be a microlocally isotropic Gaussian random field of order $-m$ in $D\subset\mathbb{R}^n$ with $m\in[0,n+2)$.*

(i) *If $m\in(n,n+2)$, then $\rho\in C^{0,\alpha}(D)$ almost surely for all $\alpha\in(0,\frac{m-n}{2})$.*
(ii) *If $m\in[0,n]$, then $\rho\in W^{\frac{m-n}{2}-\epsilon,p}(D)$ almost surely for any $\epsilon>0$ and $p\in(1,\infty)$.*

Since the potential ρ is a distribution, the well-posedness of the problem (1.1)–(1.3) is examined in the sense of distributions by studying the equivalent Lippmann–Schwinger integral equation

$$(\mathcal{I}+\mathcal{K}_\omega)\boldsymbol{u} = \boldsymbol{u}^{inc}, \tag{2.1}$$

where $\mathcal{I}$ is the identity operator and the operator $\mathcal{K}_\omega$ is defined by

$$(\mathcal{K}_\omega\boldsymbol{u})(x) := \int_{\mathbb{R}^3} \boldsymbol{G}(x,z,\omega)\rho(z)\boldsymbol{u}(z)dz.$$

Here, $\boldsymbol{G}\in\mathbb{C}^{3\times3}$ denotes the Green tensor for the elastic wave equation and is given by

$$\boldsymbol{G}(x,z,\omega) = \frac{1}{\mu}\Phi(x,z,\kappa_{\mathrm{s}})\boldsymbol{I} + \frac{1}{\omega^2}\nabla_x\nabla_x^{\top}\Big[\Phi(x,z,\kappa_{\mathrm{s}}) - \Phi(x,z,\kappa_{\mathrm{p}})\Big], \tag{2.2}$$

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where $\boldsymbol{I}$ is the 3×3 identity matrix and $\Phi(x,z,\kappa)=\frac{e^{\mathrm{i}\kappa|x-z|}}{4\pi|x-z|}$ is the fundamental solution of the three-dimensional Helmholtz equation.

In what follows, we denote by $\boldsymbol{V}=\{\boldsymbol{v}=(v_1,v_2,v_3)^\top : v_i\in V, i=1,2,3\}$ the Cartesian product vector space of the space V. For example, $\boldsymbol{W}^{\gamma,q}(D)$ denotes the Sobolev space of vector-valued functions $\boldsymbol{u}=(u_1,u_2,u_3)^\top$ with each component u_i, $i=1,2,3$, belonging to the scalar Sobolev space $W^{\gamma,q}(D)$. The notation $a\lesssim b$ stands for $a\le Cb$, where C is a positive constant whose value is not required and may change step by step in the proofs.

THEOREM 2.2. *Let ρ satisfy Assumption* 1.1. *Then the scattering problem* (1.1)–(1.3) *is well-defined in the sense of distributions and admits a unique solution $\boldsymbol{u}\in \boldsymbol{W}^{\gamma,q}_{\rm loc}(\mathbb{R}^3)$ almost surely with $q\in(2,\frac{6}{7-2m})$ and $\gamma\in(\frac{3-m}{2},\frac{3}{2q}-\frac14)$.*

Proof. To address the existence of the solution to the scattering problem (1.1)–(1.3), we first show that the Lippmann–Schwinger equation (2.1) admits a unique solution in $\boldsymbol{W}^{\gamma,q}_{\rm loc}(\mathbb{R}^3)$, and then we prove that the solution to (2.1) is also a solution to (1.1)–(1.3) in the sense of distributions.

By [32, Lemma 3.1], the operator $\mathcal{K}_\omega:\boldsymbol{W}^{\gamma,q}(U)\to\boldsymbol{W}^{\gamma,q}(U)$ is compact, where $\gamma\in(\frac{3-m}{2},\frac{3}{q}-\frac12)$ and $U\subset\mathbb{R}^3$ is any bounded open set with a locally Lipschitz boundary. Noting that the incident wave $\boldsymbol{u}^{inc}$ given in (1.2) is smooth in $\mathbb{R}^3$, we have $\boldsymbol{u}^{inc}\in\boldsymbol{W}^{\gamma,q}(U)$. It follows from the Fredholm alternative theorem that (2.1) admits a unique solution $\boldsymbol{u}\in\boldsymbol{W}^{\gamma,q}_{\rm loc}(\mathbb{R}^3)$ (cf. [25, 32]).

Next we show that the solution $\boldsymbol{u}$ obtained above is also a solution to the scattering problem (1.1)–(1.3). Denote by $\boldsymbol{\mathcal{D}}$ the space $\boldsymbol{C}_0^\infty(\mathbb{R}^3)$ equipped with a locally convex topology, which is also known as the space of test functions, and by $\langle\cdot,\cdot\rangle$ the following dual product between a pair of dual spaces $\boldsymbol{V}$ and $\boldsymbol{V}^*$:

$$\langle\boldsymbol{v},\boldsymbol{w}\rangle:=\int_{\mathbb{R}^3}\boldsymbol{v}(x)^\top\overline{\boldsymbol{w}(x)}dx\quad\forall\boldsymbol{v}\in\boldsymbol{V},\boldsymbol{w}\in\boldsymbol{V}^*.$$

Noting that $\boldsymbol{u}$ satisfies $\boldsymbol{u}=\boldsymbol{u}^{inc}-\mathcal{K}_\omega\boldsymbol{u}$, we have for any $\boldsymbol{\psi}\in\boldsymbol{\mathcal{D}}$ that

$$\begin{aligned}
&\langle\Delta^*\boldsymbol{u}+\omega^2\boldsymbol{u}-\rho\boldsymbol{u},\boldsymbol{\psi}\rangle\\
=&\langle\Delta^*\boldsymbol{u}^{inc}+\omega^2\boldsymbol{u}^{inc},\boldsymbol{\psi}\rangle-\left\langle\int_{\mathbb{R}^3}(\Delta^*+\omega^2)\boldsymbol{G}(\cdot,z,\omega)\rho(z)\boldsymbol{u}(z)dz,\boldsymbol{\psi}\right\rangle-\langle\rho\boldsymbol{u},\boldsymbol{\psi}\rangle\\
=&-\int_{\mathbb{R}^3}\rho(z)\boldsymbol{u}(z)^\top\langle(\Delta^*+\omega^2)\boldsymbol{G}(\cdot,z,\omega),\boldsymbol{\psi}\rangle dz-\langle\rho\boldsymbol{u},\boldsymbol{\psi}\rangle\\
=&\int_{\mathbb{R}^3}\rho(z)\boldsymbol{u}(z)^\top\overline{\boldsymbol{\psi}(z)}dz-\langle\rho\boldsymbol{u},\boldsymbol{\psi}\rangle=0,
\end{aligned}$$

where we use the facts that $\Delta^*\boldsymbol{u}^{inc}+\omega^2\boldsymbol{u}^{inc}=0$ and $(\Delta^*+\omega^2)\boldsymbol{G}(x,z,\omega)=-\delta(x-z)\boldsymbol{I}$ with δ being the Dirac delta function. Thus, $\boldsymbol{u}$ satisfies (1.1). Moreover, (2.1) implies that the scattered wave $\boldsymbol{u}^{sc}$ has the form

$$\boldsymbol{u}^{sc}(x)=-\int_{\mathbb{R}^3}\boldsymbol{G}(x,z,\omega)\rho(z)\boldsymbol{u}(z)dz,$$

which satisfies the Kupradze–Sommerfeld radiation condition (1.3) since $\boldsymbol{G}(\cdot,z,\omega)$ satisfies the Kupradze–Sommerfeld radiation condition (1.3). Hence, $\boldsymbol{u}$ is a solution of the scattering problem (1.1)–(1.3).

The uniqueness follows directly from the proof of [32, Theorem 4.3], which requires in addition $\gamma<\frac{3}{2q}-\frac14$ and concludes that the scattering problem (1.1)–(1.3) is equivalent to the Lippmann–Schwinger integral equation. ∎

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Due to the equivalence of the scattering problem (1.1)–(1.3) and the Lippmann–Schwinger integral equation (2.1), we only need to consider the Lippmann–Schwinger integral equation (2.1) in order to study the regularity of the solution.

Define the Born sequence

$$\boldsymbol{u}_j(x) = -(\mathcal{K}_\omega \boldsymbol{u}_{j-1})(x), \quad j \in \mathbb{N}, \tag{2.3}$$

where the leading term

$$\boldsymbol{u}_0(x) = \boldsymbol{u}^{inc}(x).$$

The rest of this section shows that, for sufficiently large frequency ω, the Born series $\sum_{j=0}^{\infty} \boldsymbol{u}_j$ converges to the solution $\boldsymbol{u}$ of the scattering problem (1.1)–(1.3).

Introduce the following weighted L^p space (cf. [26, 28]):

$$L^p_\zeta(\mathbb{R}^3) := \left\{ f \in L^1_{\rm loc}(\mathbb{R}^3) : \|f\|_{L^p_\zeta(\mathbb{R}^3)} < \infty \right\},$$

where

$$\|f\|_{L^p_\zeta(\mathbb{R}^3)} := \|(1+|\cdot|^2)^{\frac{\zeta}{2}} f\|_{L^p(\mathbb{R}^3)} = \left(\int_{\mathbb{R}^3} (1+|x|^2)^{\frac{\zeta p}{2}} |f(x)|^p dx \right)^{\frac{1}{p}}.$$

Let $\mathcal{S}$ be the set of all rapidly decreasing functions on $\mathbb{R}^3$ and let $\mathcal{S}'$ denote the dual space of $\mathcal{S}$. Define the space

$$H^{s,p}_\zeta(\mathbb{R}^3) := \left\{ f \in \mathcal{S}' : (\mathcal{I} - \Delta)^{\frac{s}{2}} f \in L^p_\zeta(\mathbb{R}^3) \right\},$$

which is equipped with the norm

$$\|f\|_{H^{s,p}_\zeta(\mathbb{R}^3)} = \|(\mathcal{I} - \Delta)^{\frac{s}{2}} f\|_{L^p_\zeta(\mathbb{R}^3)}.$$

We use the notation $H^s_\zeta(\mathbb{R}^3) := H^{s,2}_\zeta(\mathbb{R}^3)$ if in particular $p = 2$. Moreover, the space $H^{s,p}_0(\mathbb{R}^3)$ coincides with the classical Sobolev space $W^{s,p}(\mathbb{R}^3)$. These definitions enable us to present the following results which give the estimates for the operator $\mathcal{K}_\omega$.

Lemma 2.3. *Let ρ satisfy Assumption* 1.1. *Then for any $s \in (\frac{3-m}{2}, \frac{1}{2})$, it holds almost surely that*

$$\|\mathcal{K}_\omega\|_{\mathcal{L}(\boldsymbol{H}^s_{-1}(\mathbb{R}^3), \boldsymbol{H}^s_{-1}(\mathbb{R}^3))} \lesssim \omega^{-1+2s},$$
$$\|\mathcal{K}_\omega\|_{\mathcal{L}(\boldsymbol{H}^s_{-1}(\mathbb{R}^3), \boldsymbol{L}^\infty(\mathcal{V}))} \lesssim \omega^{s+\epsilon+\frac{1}{2}},$$

where $\mathcal{V} \subset \mathbb{R}^3$ is a bounded domain and $\epsilon > 0$ is an arbitrary constant.

Proof. Define the operator

$$(\mathcal{H}_\omega \boldsymbol{v})(x) := \int_{\mathbb{R}^3} \boldsymbol{G}(x, z, \omega) \boldsymbol{v}(z) dz.$$

Clearly, we have $\mathcal{K}_\omega \boldsymbol{u} = \mathcal{H}_\omega(\rho \boldsymbol{u})$. For any bounded set $\mathcal{V}$ and arbitrary constant $\epsilon > 0$, we may follow the same procedure as the one used in [26, Lemma 4.1] and obtain

$$\|\mathcal{H}_\omega\|_{\mathcal{L}(\boldsymbol{H}^{-s}_1(\mathbb{R}^3), \boldsymbol{H}^s_{-1}(\mathbb{R}^3))} \lesssim \omega^{-1+2s},$$
$$\|\mathcal{H}_\omega\|_{\mathcal{L}(\boldsymbol{H}^{-s}_1(\mathbb{R}^3), \boldsymbol{L}^\infty(\mathcal{V}))} \lesssim \omega^{s+\epsilon+\frac{1}{2}}.$$

Next we show that $\rho \boldsymbol{u} \in \boldsymbol{H}^{-s}_1(\mathbb{R}^3)$ for any $\boldsymbol{u} \in \boldsymbol{H}^s_{-1}(\mathbb{R}^3)$.

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For any $\boldsymbol{u},\boldsymbol{v}\in\boldsymbol{\mathcal{S}}$, it holds that $\langle\rho\boldsymbol{u},\boldsymbol{v}\rangle=\langle\rho,\boldsymbol{u}\cdot\boldsymbol{v}\rangle$. Define a cutoff function $\chi\in C_0^\infty(\mathbb{R}^3)$, which has a bounded support U with a locally Lipschitz boundary such that $D\subset U$ and $\chi(x)\equiv 1$ if $x\in D$. It is clear to note that

$$
\begin{aligned}
|\langle\rho\boldsymbol{u},\boldsymbol{v}\rangle| &= |\langle\rho,(\chi\boldsymbol{u})\cdot(\chi\boldsymbol{v})\rangle|\\
&= |\langle(\mathcal{I}-\Delta)^{-\gamma}\rho,(\mathcal{I}-\Delta)^{\gamma}[(\chi\boldsymbol{u})\cdot(\chi\boldsymbol{v})]\rangle|\\
&\lesssim \|\rho\|_{W^{-\gamma,p}(D)}\|(\mathcal{I}-\Delta)^{\gamma}[(\chi\boldsymbol{u})\cdot(\chi\boldsymbol{v})]\|_{L^{p'}(\mathbb{R}^3)},
\end{aligned}
$$

where $p,p'\in(1,\infty)$ are conjugate indices satisfying $\frac{1}{p}+\frac{1}{p'}=1$. Using the fractional Leibniz principle leads to

$$
\|(\mathcal{I}-\Delta)^{\gamma}[(\chi\boldsymbol{u})\cdot(\chi\boldsymbol{v})]\|_{L^{p'}(\mathbb{R}^3)}\le\|\chi\boldsymbol{u}\|_{\boldsymbol{L}^2(U)}\|\chi\boldsymbol{v}\|_{\boldsymbol{W}^{\gamma,q}(U)}+\|\chi\boldsymbol{v}\|_{\boldsymbol{L}^2(U)}\|\chi\boldsymbol{u}\|_{\boldsymbol{W}^{\gamma,q}(U)},
$$

where q satisfies $\frac{1}{p'}=\frac{1}{2}+\frac{1}{q}$. Since $s>\frac{3-m}{2}$, there exists $\gamma\in(\frac{3-m}{2},s)$ and $p'>1$ such that $\frac{1}{p'}-\frac{1}{2}=\frac{1}{q}>\frac{1}{2}-\frac{s-\gamma}{3}$, which implies from the Kondrachov compact embedding theorem that $\boldsymbol{H}^s(U)\hookrightarrow\boldsymbol{W}^{\gamma,q}(U)$. Hence,

$$
\begin{aligned}
|\langle\rho\boldsymbol{u},\boldsymbol{v}\rangle| &\lesssim \|\rho\|_{W^{-\gamma,p}(D)}\|\chi\boldsymbol{u}\|_{\boldsymbol{W}^{\gamma,q}(U)}\|\chi\boldsymbol{v}\|_{\boldsymbol{W}^{\gamma,q}(U)}\\
&\lesssim \|\rho\|_{W^{-\gamma,p}(D)}\|\chi\boldsymbol{u}\|_{\boldsymbol{H}^{s}(U)}\|\chi\boldsymbol{v}\|_{\boldsymbol{H}^{s}(U)}\\
&\lesssim \|\rho\|_{W^{-\gamma,p}(D)}\|\boldsymbol{u}\|_{\boldsymbol{H}^{s}_{-1}(\mathbb{R}^3)}\|\boldsymbol{v}\|_{\boldsymbol{H}^{s}_{-1}(\mathbb{R}^3)},
\end{aligned}
$$

where in the last step we use the fact $\|\chi\boldsymbol{u}\|_{\boldsymbol{H}^s(U)}\lesssim\|\boldsymbol{u}\|_{\boldsymbol{H}^s_{-2}(\mathbb{R}^3)}\le\|\boldsymbol{u}\|_{\boldsymbol{H}^s_{-1}(\mathbb{R}^3)}$ (cf. [9, 28]). The proof is completed by noting

$$
\|\rho\boldsymbol{u}\|_{\boldsymbol{H}^{-s}_{1}(\mathbb{R}^3)}:=\sup_{\boldsymbol{v}\in\boldsymbol{H}^s_{-1}(\mathbb{R}^3)}\frac{|\langle\rho\boldsymbol{u},\boldsymbol{v}\rangle|}{\|\boldsymbol{v}\|_{\boldsymbol{H}^s_{-1}(\mathbb{R}^3)}}\lesssim\|\rho\|_{W^{-\gamma,p}(D)}\|\boldsymbol{u}\|_{\boldsymbol{H}^s_{-1}(\mathbb{R}^3)}
$$

and $\rho\in W^{\frac{m-3}{2}-\epsilon,p}(D)\subset W^{-\gamma,p}(D)$ according to Lemma 2.1. ∎

By the definition of $\boldsymbol{u}_j$ given in (2.3), we have

$$
(\mathcal{I}+\mathcal{K}_\omega)\sum_{j=0}^{N}\boldsymbol{u}_j=\boldsymbol{u}_0+(-1)^N\mathcal{K}_\omega^{N+1}\boldsymbol{u}_0.
$$

For the leading term $\boldsymbol{u}_0=\boldsymbol{u}^{inc}$, a simple calculation yields

$$
\|\boldsymbol{u}_0\|_{\boldsymbol{L}^2(D)}\lesssim 1,\quad \|\boldsymbol{u}_0\|_{\boldsymbol{H}^1(D)}\lesssim\omega.
$$

Using the interpolation inequality [21] leads to

$$
\begin{aligned}
\|\boldsymbol{u}_0\|_{\boldsymbol{H}^s_{-1}(D)} &= \|(1+|\cdot|^2)^{-\frac{1}{2}}(\mathcal{I}-\Delta)^{\frac{s}{2}}\boldsymbol{u}_0\|_{\boldsymbol{L}^2(D)}\le\|(\mathcal{I}-\Delta)^{\frac{s}{2}}\boldsymbol{u}_0\|_{\boldsymbol{L}^2(D)}\\
&\lesssim \|\boldsymbol{u}_0\|_{\boldsymbol{H}^s(D)}\lesssim\|\boldsymbol{u}_0\|^{1-s}_{\boldsymbol{L}^2(D)}\|\boldsymbol{u}_0\|^{s}_{\boldsymbol{H}^1(D)}\lesssim\omega^s,
\end{aligned}
$$

which, together with Lemma 2.3, yields

$$
\begin{aligned}
\|\mathcal{K}_\omega^{N+1}\boldsymbol{u}_0\|_{\boldsymbol{H}^s_{-1}(\mathbb{R}^3)} &\lesssim \|\mathcal{K}_\omega\|^N_{\mathcal{L}(\boldsymbol{H}^s_{-1}(\mathbb{R}^3),\boldsymbol{H}^s_{-1}(\mathbb{R}^3))}\|\mathcal{K}_\omega\|_{\mathcal{L}(\boldsymbol{H}^s_{-1}(D),\boldsymbol{H}^s_{-1}(\mathbb{R}^3))}\|\boldsymbol{u}_0\|_{\boldsymbol{H}^s_{-1}(D)}\\
&\lesssim \omega^{(-1+2s)(N+1)}\omega^s\to 0\quad\text{as } N\to\infty.
\end{aligned}
$$

Hence, we conclude

$$
(\mathcal{I}+\mathcal{K}_\omega)\sum_{j=0}^{N}\boldsymbol{u}_j\to\boldsymbol{u}_0=(\mathcal{I}+\mathcal{K}_\omega)\boldsymbol{u}\quad\text{as } N\to\infty.
$$

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Noting the invertibility of the operator $\mathcal{I}+\mathcal{K}_\omega$, we have

$$\boldsymbol{u}=\sum_{j=0}^{\infty}\boldsymbol{u}_j \quad \text{in} \quad \boldsymbol{H}^s_{-1}(\mathbb{R}^3). \tag{2.4}$$

Moreover, for any bounded domain $U\subset\mathbb{R}^3$, it holds that

$$\begin{aligned}
\|\boldsymbol{u}-\sum_{j=0}^{N}\boldsymbol{u}_j\|_{\boldsymbol{L}^\infty(U)} &\lesssim \sum_{j=N+1}^{\infty}\|\mathcal{K}_\omega^j\boldsymbol{u}_0\|_{\boldsymbol{L}^\infty(U)}\\
&\lesssim \sum_{j=N+1}^{\infty}\|\mathcal{K}_\omega\|_{\mathcal{L}(\boldsymbol{H}^s_{-1}(\mathbb{R}^3),\boldsymbol{L}^\infty(U))}\|\mathcal{K}_\omega\|^{j-2}_{\mathcal{L}(\boldsymbol{H}^s_{-1}(\mathbb{R}^3),\boldsymbol{H}^s_{-1}(\mathbb{R}^3))}\\
&\qquad\times\|\mathcal{K}_\omega\|_{\mathcal{L}(\boldsymbol{H}^s_{-1}(D),\boldsymbol{H}^s_{-1}(\mathbb{R}^3))}\|\boldsymbol{u}_0\|_{\boldsymbol{H}^s_{-1}(D)}\\
&\lesssim \sum_{j=N+1}^{\infty}\omega^{s+\epsilon+\frac12+(j-1)(-1+2s)+s}\to 0 \quad \text{as } N\to\infty,
\end{aligned}$$

which implies that the convergence (2.4) also holds in $\boldsymbol{L}^\infty(U)$.

3. The inverse scattering problem. This section studies the inverse problem, which aims to determine the microlocal strength ϕ of the random potential ρ from the backscattered far-field pattern of the scattered wave.

By (2.4), we rewrite the scattered wave as

$$\boldsymbol{u}^{sc}(x)=\boldsymbol{u}_1(x)+\boldsymbol{u}_2(x)+\boldsymbol{b}(x), \tag{3.1}$$

where the residual $\boldsymbol{b}(x):=\sum_{j=3}^\infty\boldsymbol{u}_j(x)$. Note that

$$\boldsymbol{u}_j(x)=-(\mathcal{K}_\omega\boldsymbol{u}_{j-1})(x)=-\int_{\mathbb{R}^3}\boldsymbol{G}(x,z,\omega)\rho(z)\boldsymbol{u}_{j-1}(z)dz, \tag{3.2}$$

where the Green tensor $\boldsymbol{G}$ has the asymptotic behavior (cf. [12, section 2.2])

$$\begin{aligned}
\boldsymbol{G}(x,z,\omega)=&\frac{c_{\rm p}^2}{4\pi}\hat{x}\otimes\hat{x}\frac{e^{{\rm i}\kappa_{\rm p}|x|}}{|x|}e^{-{\rm i}\kappa_{\rm p}\hat{x}\cdot z}\\
&+\frac{c_{\rm s}^2}{4\pi}(\boldsymbol{I}-\hat{x}\otimes\hat{x})\frac{e^{{\rm i}\kappa_{\rm s}|x|}}{|x|}e^{-{\rm i}\kappa_{\rm s}\hat{x}\cdot z}+O(|x|^{-2}).
\end{aligned}\tag{3.3}$$

Here, the symbol $\hat{x}\otimes\hat{x}:=\hat{x}^\top\hat{x}\in\mathbb{R}^{3\times3}$ is the tensor product. Substituting (3.3) into (3.2) leads to

$$\boldsymbol{u}_j(x)=\frac{e^{{\rm i}\kappa_{\rm p}|x|}}{|x|}\boldsymbol{u}^\infty_{j,{\rm p}}(\hat{x})+\frac{e^{{\rm i}\kappa_{\rm s}|x|}}{|x|}\boldsymbol{u}^\infty_{j,{\rm s}}(\hat{x})+O(|x|^{-2}), \tag{3.4}$$

where $\boldsymbol{u}^\infty_{j,{\rm p}}$ and $\boldsymbol{u}^\infty_{j,{\rm s}}$ are the compressional and shear far-field patterns of $\boldsymbol{u}_j$, respectively. A simple calculation from (3.2) and (3.4) gives

$$\begin{aligned}
\boldsymbol{u}^\infty_{j,{\rm p}}(\hat{x})&:=-\frac{c_{\rm p}^2}{4\pi}\hat{x}\otimes\hat{x}\int_{\mathbb{R}^3}e^{-{\rm i}\kappa_{\rm p}\hat{x}\cdot z}\rho(z)\boldsymbol{u}_{j-1}(z)dz,\\
\boldsymbol{u}^\infty_{j,{\rm s}}(\hat{x})&:=-\frac{c_{\rm s}^2}{4\pi}(\boldsymbol{I}-\hat{x}\otimes\hat{x})\int_{\mathbb{R}^3}e^{-{\rm i}\kappa_{\rm s}\hat{x}\cdot z}\rho(z)\boldsymbol{u}_{j-1}(z)dz.
\end{aligned}\tag{3.5}$$

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Combining (1.4), (3.1), and (3.4), we get the following compressional and shear far-field patterns $\boldsymbol{u}_{\rm p}^\infty$ and $\boldsymbol{u}_{\rm s}^\infty$ of the scattered wave $\boldsymbol{u}^{sc}$:

$$
\begin{aligned}
&\boldsymbol{u}_{\rm p}^\infty(\hat{x}) = \boldsymbol{u}_{1,{\rm p}}^\infty(\hat{x}) + \boldsymbol{u}_{2,{\rm p}}^\infty(\hat{x}) + \boldsymbol{b}_{\rm p}^\infty(\hat{x}), \quad \boldsymbol{b}_{\rm p}^\infty(\hat{x}) := \sum_{j=3}^{\infty} \boldsymbol{u}_{j,{\rm p}}^\infty(\hat{x}),\\
&\boldsymbol{u}_{\rm s}^\infty(\hat{x}) = \boldsymbol{u}_{1,{\rm s}}^\infty(\hat{x}) + \boldsymbol{u}_{2,{\rm s}}^\infty(\hat{x}) + \boldsymbol{b}_{\rm s}^\infty(\hat{x}), \quad \boldsymbol{b}_{\rm s}^\infty(\hat{x}) := \sum_{j=3}^{\infty} \boldsymbol{u}_{j,{\rm s}}^\infty(\hat{x}).
\end{aligned}
\tag{3.6}
$$

As mentioned in the introduction, two types of incident plane waves are used as the illumination and two corresponding backscattered far-field patterns are measured as the data to reconstruct the strength ϕ: one is the compressional plane wave $\boldsymbol{u}_0(x) = \boldsymbol{u}_{\rm p}^{inc}(x) = \theta e^{{\rm i}\kappa_{\rm p} x\cdot\theta}$ and the compressional far-field pattern $\boldsymbol{u}_{\rm p}^\infty(\hat{x})$ is measured; the other is the shear plane wave $\boldsymbol{u}_0(x) = \boldsymbol{u}_{\rm s}^{inc}(x) = \theta^\perp e^{{\rm i}\kappa_{\rm s} x\cdot\theta}$ and the shear far-field pattern $\boldsymbol{u}_{\rm s}^\infty(\hat{x})$ is measured.

To prove Theorem 1.2, we analyze separately the three terms in the far-field patterns (3.6): the first order far-field patterns $\boldsymbol{u}_{1,{\rm p}}^\infty$ and $\boldsymbol{u}_{1,{\rm s}}^\infty$, the second order far-field patterns $\boldsymbol{u}_{2,{\rm p}}^\infty$ and $\boldsymbol{u}_{2,{\rm s}}^\infty$, and the higher order far-field patterns $\boldsymbol{b}_{\rm p}^\infty$ and $\boldsymbol{b}_{\rm s}^\infty$.

3.1. The first order far-field patterns. We begin with analyzing the first order backscattered far-field patterns by employing the Fourier analysis and ergodicity arguments. Below is the main result of this subsection.

THEOREM 3.1. *Let the random potential ρ satisfy Assumption* 1.1*, and let $\boldsymbol{u}_{1,{\rm p}}^\infty$ and $\boldsymbol{u}_{1,{\rm s}}^\infty$ be given by* (3.5) *with $\boldsymbol{u}_0 = \boldsymbol{u}_{\rm p}^{inc}$ and $\boldsymbol{u}_0 = \boldsymbol{u}_{\rm s}^{inc}$, respectively. For any fixed $\theta \in \mathbb{S}^2$ and $\tau \ge 0$, it holds almost surely that*

$$
\lim_{Q\to\infty} \frac{1}{Q} \int_Q^{2Q} \omega^m \boldsymbol{u}_{1,{\rm p}}^\infty(-\theta,\omega,\theta) \cdot \overline{\boldsymbol{u}_{1,{\rm p}}^\infty(-\theta,\omega+\tau,\theta)} d\omega = C_{\rm p} \hat{\phi}(2c_{\rm p}\tau\theta),
\tag{3.7}
$$

$$
\lim_{Q\to\infty} \frac{1}{Q} \int_Q^{2Q} \omega^m \boldsymbol{u}_{1,{\rm s}}^\infty(-\theta,\omega,\theta) \cdot \overline{\boldsymbol{u}_{1,{\rm s}}^\infty(-\theta,\omega+\tau,\theta)} d\omega = C_{\rm s} \hat{\phi}(2c_{\rm s}\tau\theta),
\tag{3.8}
$$

where $C_{\rm p}$ and $C_{\rm s}$ are constants defined in Theorem 1.2.

The proof of Theorem 3.1 is left to the end of this subsection. The following lemmas are useful for the proof of Theorem 3.1.

LEMMA 3.2. *Under assumptions in Theorem* 3.1*, for any fixed $\theta \in \mathbb{S}^2$ and $\tau \ge 0$, it holds that*

$$
\lim_{Q\to\infty} \frac{1}{Q} \int_Q^{2Q} \omega^m \mathbb{E}\left[\boldsymbol{u}_{1,{\rm p}}^\infty(-\theta,\omega,\theta) \cdot \overline{\boldsymbol{u}_{1,{\rm p}}^\infty(-\theta,\omega+\tau,\theta)}\right] d\omega = C_{\rm p} \hat{\phi}(2c_{\rm p}\tau\theta),
\tag{3.9}
$$

$$
\lim_{Q\to\infty} \frac{1}{Q} \int_Q^{2Q} \omega^m \mathbb{E}\left[\boldsymbol{u}_{1,{\rm s}}^\infty(-\theta,\omega,\theta) \cdot \overline{\boldsymbol{u}_{1,{\rm s}}^\infty(-\theta,\omega+\tau,\theta)}\right] d\omega = C_{\rm s} \hat{\phi}(2c_{\rm s}\tau\theta),
\tag{3.10}
$$

where $C_{\rm p}$ and $C_{\rm s}$ are constants defined in Theorem 1.2.

Proof. Using (1.2), (3.5), and noting $(\theta\otimes\theta)\theta = \theta$ and $(\theta\otimes\theta)\theta^\perp = 0$, we obtain

$$
\begin{aligned}
\boldsymbol{u}_{1,{\rm p}}^\infty(-\theta,\omega,\theta) &= -\frac{c_{\rm p}^2}{4\pi} \theta\otimes\theta \int_{\mathbb{R}^3} e^{{\rm i}\kappa_{\rm p}\theta\cdot z} \rho(z) \theta e^{{\rm i}\kappa_{\rm p}\theta\cdot z} dz\\
&= -\frac{c_{\rm p}^2}{4\pi} \theta \int_{\mathbb{R}^3} e^{2{\rm i}\kappa_{\rm p}\theta\cdot z} \rho(z) dz
\end{aligned}
$$

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and

$$\boldsymbol{u}_{1,\mathrm{s}}^{\infty}(-\theta,\omega,\theta) = -\frac{c_{\mathrm{s}}^2}{4\pi}(\boldsymbol{I}-\theta\otimes\theta)\int_{\mathbb{R}^3} e^{\mathrm{i}\kappa_{\mathrm{s}}\theta\cdot z}\rho(z)\theta^{\perp}e^{\mathrm{i}\kappa_{\mathrm{s}}\theta\cdot z}dz$$
$$= -\frac{c_{\mathrm{s}}^2}{4\pi}\theta^{\perp}\int_{\mathbb{R}^3} e^{2\mathrm{i}\kappa_{\mathrm{s}}\theta\cdot z}\rho(z)dz.$$

It suffices to show (3.9) since the proof is similar for (3.10).

We have for $\omega_1,\omega_2\geq 1$ that

$$\mathbb{E}\left[\boldsymbol{u}_{1,\mathrm{p}}^{\infty}(-\theta,\omega_1,\theta)\cdot\overline{\boldsymbol{u}_{1,\mathrm{p}}^{\infty}(-\theta,\omega_2,\theta)}\right]$$
$$= \frac{c_{\mathrm{p}}^4}{16\pi^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{2\mathrm{i}c_{\mathrm{p}}\omega_1\theta\cdot y}e^{-2\mathrm{i}c_{\mathrm{p}}\omega_2\theta\cdot z}\mathbb{E}\left(\rho(y)\rho(z)\right)dydz$$
$$= \frac{c_{\mathrm{p}}^4}{16\pi^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{2\mathrm{i}c_{\mathrm{p}}\theta\cdot(\omega_1 y-\omega_2 z)}K_\rho(y,z)dydz, \tag{3.11}$$

where $K_\rho\in\mathcal{D}'(\mathbb{R}^3\times\mathbb{R}^3;\mathbb{R})$ is the symmetric covariance kernel of ρ satisfying

$$\langle\mathcal{C}_\rho\varphi,\psi\rangle = \mathbb{E}[\langle\rho,\varphi\rangle\langle\rho,\psi\rangle] = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} K_\rho(y,z)\varphi(y)\overline{\psi(z)}dydz \quad \forall\varphi,\psi\in\mathcal{D}.$$

Let $s_\rho\in\mathcal{S}^{-m}(\mathbb{R}^3\times\mathbb{R}^3)$ be the symbol of the covariance operator $\mathcal{C}_\rho$ satisfying

$$(\mathcal{C}_\rho\varphi)(x) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} e^{\mathrm{i}x\cdot\xi}s_\rho(x,\xi)\hat{\varphi}(\xi)d\xi \quad \forall\varphi\in\mathcal{D},$$

where $\mathcal{S}^{-m}(\mathbb{R}^3\times\mathbb{R}^3)$ is defined by

$$\mathcal{S}^{-m}(\mathbb{R}^3\times\mathbb{R}^3) := \Big\{s(x,\xi)\in C^\infty(\mathbb{R}^3\times\mathbb{R}^3):$$
$$|\partial_\xi^{\gamma_1}\partial_x^{\gamma_2}s(x,\xi)|\leq C(\gamma_1,\gamma_2)(1+|\xi|)^{-m-|\gamma_1|}\Big\}$$

with γ_1 and γ_2 being any multiple indices and $|\gamma_1|$ denoting the sum of its components. A simple calculation gives the oscillatory integral form of K_ρ (cf. [30]):

$$K_\rho(y,z) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} e^{\mathrm{i}(z-y)\cdot\xi}s_\rho(z,\xi)d\xi. \tag{3.12}$$

According to Assumption 1.1, we have $s_\rho(x,\xi)=\phi(x)|\xi|^{-m}+a(x,\xi)$, where $a\in\mathcal{S}^{-m-1}(\mathbb{R}^3\times\mathbb{R}^3)$, and $\mathrm{supp}K_\rho\subset D\times D$. Substituting (3.12) into (3.11) yields

$$\mathbb{E}\left[\boldsymbol{u}_{1,\mathrm{p}}^{\infty}(-\theta,\omega_1,\theta)\cdot\overline{\boldsymbol{u}_{1,\mathrm{p}}^{\infty}(-\theta,\omega_2,\theta)}\right]$$
$$= \frac{c_{\mathrm{p}}^4}{16\pi^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{2\mathrm{i}c_{\mathrm{p}}\theta\cdot(\omega_1 y-\omega_2 z)}\left[\frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} e^{\mathrm{i}(z-y)\cdot\xi}s_\rho(z,\xi)d\xi\right]dydz$$
$$= \frac{c_{\mathrm{p}}^4}{16\pi^2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{-2\mathrm{i}c_{\mathrm{p}}\omega_2\theta\cdot z+\mathrm{i}z\cdot\xi}s_\rho(z,\xi)\delta(\xi-2c_{\mathrm{p}}\omega_1\theta)d\xi dz$$
$$= \frac{c_{\mathrm{p}}^4}{16\pi^2}\int_D s_\rho(z,2c_{\mathrm{p}}\omega_1\theta)e^{2\mathrm{i}c_{\mathrm{p}}(\omega_1-\omega_2)\theta\cdot z}dz$$
$$= \frac{c_{\mathrm{p}}^4}{16\pi^2}\left[\int_D \phi(z)|2c_{\mathrm{p}}\omega_1\theta|^{-m}e^{2\mathrm{i}c_{\mathrm{p}}(\omega_1-\omega_2)\theta\cdot z}dz + \int_D a(z,2c_{\mathrm{p}}\omega_1\theta)e^{2\mathrm{i}c_{\mathrm{p}}(\omega_1-\omega_2)\theta\cdot z}dz\right]$$
$$= C_{\mathrm{p}}\hat{\phi}(2c_{\mathrm{p}}(\omega_2-\omega_1)\theta)\omega_1^{-m}+O(\omega_1^{-m-1}). \tag{3.13}$$

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Letting $\omega_1 = \omega$ and $\omega_2 = \omega + \tau$ in (3.13) gives

$$\mathbb{E}\left[\boldsymbol{u}^{\infty}_{1,\mathrm{p}}(-\theta,\omega,\theta)\cdot\overline{\boldsymbol{u}^{\infty}_{1,\mathrm{p}}(-\theta,\omega+\tau,\theta)}\right] = C_{\mathrm{p}}\hat{\phi}(2c_{\mathrm{p}}\tau\theta)\omega^{-m} + O(\omega^{-m-1}),$$

which implies (3.9) and completes the proof. ∎

LEMMA 3.3. *Under assumptions in Theorem* 3.1*, it holds for all* $\theta\in\mathbb{S}^2$*,* $\omega_1,\omega_2\ge 1$*, and* $N\in\mathbb{N}$ *that*

$$\left|\mathbb{E}\left[\boldsymbol{u}^{\infty}_{1,\mathrm{p}}(-\theta,\omega_1,\theta)\cdot\overline{\boldsymbol{u}^{\infty}_{1,\mathrm{p}}(-\theta,\omega_2,\theta)}\right]\right| \lesssim \omega_1^{-m}(1+|\omega_1-\omega_2|)^{-N}, \tag{3.14}$$

$$\left|\mathbb{E}\left[\boldsymbol{u}^{\infty}_{1,\mathrm{s}}(-\theta,\omega_1,\theta)\cdot\overline{\boldsymbol{u}^{\infty}_{1,\mathrm{s}}(-\theta,\omega_2,\theta)}\right]\right| \lesssim \omega_1^{-m}(1+|\omega_1-\omega_2|)^{-N}, \tag{3.15}$$

$$\left|\mathbb{E}\left[\boldsymbol{u}^{\infty}_{1,\mathrm{p}}(-\theta,\omega_1,\theta)\cdot\boldsymbol{u}^{\infty}_{1,\mathrm{p}}(-\theta,\omega_2,\theta)\right]\right| \lesssim \omega_1^{-m}(1+\omega_1+\omega_2)^{-N}, \tag{3.16}$$

$$\left|\mathbb{E}\left[\boldsymbol{u}^{\infty}_{1,\mathrm{s}}(-\theta,\omega_1,\theta)\cdot\boldsymbol{u}^{\infty}_{1,\mathrm{s}}(-\theta,\omega_2,\theta)\right]\right| \lesssim \omega_1^{-m}(1+\omega_1+\omega_2)^{-N}. \tag{3.17}$$

Proof. For the case $|\omega_1-\omega_2|<1$, it follows from (3.13) that

$$\begin{aligned}
&\left|\mathbb{E}\left[\boldsymbol{u}^{\infty}_{1,\mathrm{p}}(-\theta,\omega_1,\theta)\cdot\overline{\boldsymbol{u}^{\infty}_{1,\mathrm{p}}(-\theta,\omega_2,\theta)}\right]\right|\\
&\le \frac{c_{\mathrm{p}}^4}{16\pi^2}\int_D\left|s_\rho(z,2c_{\mathrm{p}}\omega_1\theta)e^{2\mathrm{i}c_{\mathrm{p}}(\omega_1-\omega_2)\theta\cdot z}\right|dz\\
&\lesssim (1+\omega_1)^{-m}\lesssim 2^N(1+\omega_1)^{-m}(1+|\omega_1-\omega_2|)^{-N}\\
&\lesssim \omega_1^{-m}(1+|\omega_1-\omega_2|)^{-N},
\end{aligned}$$

where we use the fact $s_\rho\in\mathcal{S}^{-m}$ and hence $|s_\rho(z,2c_{\mathrm{p}}\omega_1\theta)|\lesssim(1+|\omega_1|)^{-m}$.

For the case $|\omega_1-\omega_2|\ge 1$, denoting $z=(z_1,z_2,z_3)^\top$ and $\theta=(\theta_1,\theta_2,\theta_3)^\top$, we obtain from (3.13) and the integration by parts that

$$\begin{aligned}
&\mathbb{E}\left[\boldsymbol{u}^{\infty}_{1,\mathrm{p}}(-\theta,\omega_1,\theta)\cdot\overline{\boldsymbol{u}^{\infty}_{1,\mathrm{p}}(-\theta,\omega_2,\theta)}\right]\\
&=\frac{c_{\mathrm{p}}^4}{16\pi^2}\int_D s_\rho(z,2c_{\mathrm{p}}\omega_1\theta)e^{2\mathrm{i}c_{\mathrm{p}}(\omega_1-\omega_2)\theta\cdot z}dz\\
&=\frac{c_{\mathrm{p}}^4}{16\pi^2}\frac{1}{2\mathrm{i}c_{\mathrm{p}}(\omega_1-\omega_2)\theta_1}\int_D s_\rho(z,2c_{\mathrm{p}}\omega_1\theta)\\
&\qquad\times e^{2\mathrm{i}c_{\mathrm{p}}(\omega_1-\omega_2)(\theta_2z_2+\theta_3z_3)}de^{2\mathrm{i}c_{\mathrm{p}}(\omega_1-\omega_2)\theta_1z_1}dz_2dz_3\\
&=-\frac{c_{\mathrm{p}}^4}{16\pi^2}\frac{1}{2\mathrm{i}c_{\mathrm{p}}(\omega_1-\omega_2)\theta_1}\int_D\partial_{z_1}s_\rho(z,2c_{\mathrm{p}}\omega_1\theta)e^{2\mathrm{i}c_{\mathrm{p}}(\omega_1-\omega_2)\theta\cdot z}dz
\end{aligned}$$

$$=(-1)^N\frac{c_{\mathrm{p}}^4}{16\pi^2}\frac{1}{(2\mathrm{i}c_{\mathrm{p}}(\omega_1-\omega_2)\theta_1)^N}\int_D\partial^N_{z_1}s_\rho(z,2c_{\mathrm{p}}\omega_1\theta)e^{2\mathrm{i}c_{\mathrm{p}}(\omega_1-\omega_2)\theta\cdot z}dz. \tag{3.18}$$

Since $s_\rho\in\mathcal{S}^{-m}$, we have

$$\left|\partial^N_{z_1}s_\rho(z,2c_{\mathrm{p}}\omega_1\theta)\right|\lesssim(1+\omega_1)^{-m}.$$

Combining the above estimates leads to

$$\begin{aligned}
&\left|\mathbb{E}\left[\boldsymbol{u}^{\infty}_{1,\mathrm{p}}(-\theta,\omega_1,\theta)\cdot\overline{\boldsymbol{u}^{\infty}_{1,\mathrm{p}}(-\theta,\omega_2,\theta)}\right]\right|\\
&\lesssim(1+\omega_1)^{-m}\,|\omega_1-\omega_2|^{-N}
\end{aligned}$$

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$$\begin{aligned}
&\lesssim \left(1+\frac{1}{|\omega_1-\omega_2|}\right)^N \omega_1^{-m}(1+|\omega_1-\omega_2|)^{-N}\\
&\lesssim 2^N\omega_1^{-m}(1+|\omega_1-\omega_2|)^{-N}\\
&\lesssim \omega_1^{-m}(1+|\omega_1-\omega_2|)^{-N}, \qquad (3.19)
\end{aligned}$$

which shows (3.14).

The inequality (3.15) can be obtained by following the same procedure; the inequalities (3.16) and (3.17) can be proved similarly by replacing ω_2 with $-\omega_2$ in (3.18) and (3.19), respectively. ∎

The following two lemmas help to replace the results in the expectation sense stated in Lemma 3.2 with the ones in the almost surely sense given in Theorem 3.1. Lemma 3.5 gives a sufficient condition for the convergence to its expectation of the time average of a stochastic process. The proof of Lemma 3.4 can be found in [9]. The proof of Lemma 3.5 is motivated by [13] and is given below for the reader's convenience.

LEMMA 3.4. *Let X and Y be two random variables such that the pair (X,Y) is a Gaussian random vector. If $\mathbb{E}[X]=\mathbb{E}[Y]=0$, then*

$$\mathbb{E}\left[(X^2-\mathbb{E}X^2)(Y^2-\mathbb{E}Y^2)\right]=2\left(\mathbb{E}[XY]\right)^2.$$

LEMMA 3.5. *Let $\{X_t\}_{t\geq 0}$ be a real-valued centered stochastic process with continuous paths and $\mathbb{E}[X_t]=0$. Assume that for some constants $\eta\geq 0$ and $\sigma>0$, it holds that*

$$|\mathbb{E}[X_tX_{t+r}]|\lesssim(1+|r-\eta|)^{-\sigma}\quad \forall t,r\geq 0.$$

Then

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T X_t dt=\lim_{T\to\infty}\frac{1}{T}\int_T^{2T}X_t dt=0\quad a.s.$$

Proof. Without loss of generality, we assume that $\sigma\in(0,1)$. If the condition in Lemma 3.5 holds for $\sigma'\geq 1$, then we can always find some $\sigma\in(0,1)$ such that

$$|\mathbb{E}[X_tX_{t+r}]|\lesssim(1+|r-\eta|)^{-\sigma'}<(1+|r-\eta|)^{-\sigma}.$$

For T being large enough such that $T>\eta$, we have

$$\begin{aligned}
&\mathbb{E}\left|\frac{1}{T}\int_0^T X_t dt\right|^2=\frac{1}{T^2}\int_0^T\int_0^T\mathbb{E}[X_tX_u]dtdu\\
&\lesssim\frac{1}{T^2}\int_0^T\int_0^T(1+||t-u|-\eta|)^{-\sigma}dtdu\\
&=\frac{1}{T^2}\int_0^T\bigg[\int_0^{(u-\eta)\vee 0}(1+u-\eta-t)^{-\sigma}dt+\int_{(u-\eta)\vee 0}^{u}(1+t-u+\eta)^{-\sigma}dt\\
&\quad+\int_u^{(u+\eta)\wedge T}(1+u+\eta-t)^{-\sigma}dt+\int_{(u+\eta)\wedge T}^{T}(1+t-u-\eta)^{-\sigma}dt\bigg]du\\
&=\frac{2(1+\eta)^{1-\sigma}}{(1-\sigma)T}+\frac{2\left[(1+T-\eta)^{2-\sigma}-(1-\eta)^{2-\sigma}\right]}{(2-\sigma)(1-\sigma)T^2}
\end{aligned}$$

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$$
\begin{aligned}
&- \frac{2}{(1-\sigma)T^2}\left[\frac{(1+\eta)^{2-\sigma}-(1-\eta)^{2-\sigma}}{2-\sigma}+2(T-\eta)\right]\\
&\lesssim \frac{1}{T^\sigma},
\end{aligned}
$$

where we use the notation $a\vee b := \max\{a,b\}$ and $a\wedge b := \min\{a,b\}$.

Based on the estimate above, the rest of the proof follows directly from [13, p. 95]. More precisely, we choose some constant $k>0$ sufficiently large such that $k\sigma>1$ and $2^k>\eta$. Taking $T_n := (2n)^k>\eta$ for $n\in\mathbb{N}_+$, then the time averages $\xi_n := \frac{1}{T_n}\int_0^{T_n} X_t dt$, $n\in\mathbb{N}_+$, satisfy

$$
\sum_{n=1}^\infty \mathbb{E}|\xi_n|^2 \lesssim \sum_{n=1}^\infty \frac{1}{(2n)^{k\sigma}} < \infty,
$$

which implies $\lim_{n\to\infty}\xi_n = 0$ almost surely according to Markov's inequality and the Borel–Cantelli lemma. Define the nonnegative random variable

$$
Y_n := \sup_{T_n\le T<T_{n+1}}\left|\frac{1}{T}\int_0^T X_t dt - \xi_n\right|.
$$

We next investigate the convergence of Y_n. Note that

$$
\begin{aligned}
\mathbb{E}|Y_n|^2 &= \mathbb{E}\left(\sup_{T_n\le T<T_{n+1}}\left|\frac{1}{T}\int_0^T X_t dt - \frac{1}{T_n}\int_0^{T_n} X_t dt\right|\right)^2\\
&= \mathbb{E}\left(\sup_{T_n\le T<T_{n+1}}\left|\left(\frac{1}{T}-\frac{1}{T_n}\right)\int_0^{T_n} X_t dt + \frac{1}{T}\int_{T_n}^T X_t dt\right|\right)^2\\
&\le \mathbb{E}\left(\frac{T_{n+1}-T_n}{T_n^2}\int_0^{T_n}|X_t|dt + \frac{1}{T_n}\int_{T_n}^{T_{n+1}}|X_t|dt\right)^2\\
&\le 2\left(\frac{T_{n+1}-T_n}{T_n^2}\right)^2\int_0^{T_n}\int_0^{T_n}\mathbb{E}|X_tX_u|dtdu\\
&\quad+\frac{2}{T_n^2}\int_{T_n}^{T_{n+1}}\int_{T_n}^{T_{n+1}}\mathbb{E}|X_tX_u|dtdu\\
&\le 4\frac{(T_{n+1}-T_n)^2}{T_n^2} = 4\left(\frac{(n+1)^k-n^k}{n^k}\right)^2 \lesssim \frac{1}{n^2},
\end{aligned}
$$

and hence

$$
\sum_{n=1}^\infty \mathbb{E}|Y_n|^2 = \sum_{n=1}^\infty \frac{1}{n^2} < \infty.
$$

We then get $\lim_{n\to\infty} Y_n = 0$ almost surely based on the same procedure as the proof of the almost sure convergence for ξ_n. The almost sure convergence of both Y_n and ξ_n leads to

$$
\lim_{T\to\infty}\frac{1}{T}\int_0^T X_t dt = 0 \quad a.s.
$$

and

$$
\lim_{T\to\infty}\frac{1}{T}\int_T^{2T} X_t dt = \lim_{T\to\infty}\left[\frac{1}{T}\int_0^{2T} X_t dt - \frac{1}{T}\int_0^T X_t dt\right] = 0 \quad a.s.,
$$

which complete the proof. ∎

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Now we are in position to prove Theorem 3.1.

Proof of Theorem 3.1. By Lemma 3.2, to prove (3.7), it suffices to show that

$$\begin{aligned}&\lim_{Q\to\infty}\frac{1}{Q}\int_Q^{2Q}\omega^m \boldsymbol{u}_{1,\mathrm{p}}^{\infty}(-\theta,\omega,\theta)\cdot\overline{\boldsymbol{u}_{1,\mathrm{p}}^{\infty}(-\theta,\omega+\tau,\theta)}d\omega\\ &=\lim_{Q\to\infty}\frac{1}{Q}\int_Q^{2Q}\omega^m\mathbb{E}\left[\boldsymbol{u}_{1,\mathrm{p}}^{\infty}(-\theta,\omega,\theta)\cdot\overline{\boldsymbol{u}_{1,\mathrm{p}}^{\infty}(-\theta,\omega+\tau,\theta)}\right]d\omega,\end{aligned}$$

or equivalently,

$$\sum_{j=1,2,3}\lim_{Q\to\infty}\frac{1}{Q}\int_Q^{2Q}\omega^m\left(u_j(\omega)\overline{u_j(\omega+\tau)}-\mathbb{E}\left[u_j(\omega)\overline{u_j(\omega+\tau)}\right]\right)d\omega=0, \tag{3.20}$$

where $\boldsymbol{u}_{1,\mathrm{p}}^{\infty}(-\theta,\omega,\theta)=(u_1(\omega),u_2(\omega),u_3(\omega))^{\top}$.

Denote by $U_j(\omega)$ and $V_j(\omega)$ the real and imaginary parts of $u_j(\omega)$, respectively, which read

$$u_j(\omega)=U_j(\omega)+\mathrm{i}V_j(\omega),\quad j=1,2,3. \tag{3.21}$$

It then leads to

$$\begin{aligned}2u_j(\omega)\overline{u_j(\omega+\tau)}&=2(U_j(\omega)+\mathrm{i}V_j(\omega))(U_j(\omega+\tau)-\mathrm{i}V_j(\omega+\tau))\\ &=(1+\mathrm{i})\left[U_j^2(\omega)+U_j^2(\omega+\tau)+V_j^2(\omega)+V_j^2(\omega+\tau)\right]\\ &\quad-(U_j(\omega)-U_j(\omega+\tau))^2-(V_j(\omega)-V_j(\omega+\tau))^2\\ &\quad-\mathrm{i}(U_j(\omega)+V_j(\omega+\tau))^2-\mathrm{i}(V_j(\omega)-U_j(\omega+\tau))^2.\end{aligned}$$

For simplicity, let W_ω be any random variable in the set $\Gamma:=\{U_j(\omega),U_j(\omega+\tau),V_j(\omega),V_j(\omega+\tau),U_j(\omega)-U_j(\omega+\tau),V_j(\omega)-V_j(\omega+\tau),U_j(\omega)+V_j(\omega+\tau),V_j(\omega)-U_j(\omega+\tau)\}_{j=1,2,3}$.

Then it is only required to show

$$\lim_{Q\to\infty}\frac{1}{Q}\int_Q^{2Q}\omega^m\left(W_\omega^2-\mathbb{E}W_\omega^2\right)d\omega=0, \tag{3.22}$$

which indicates (3.20). Using Lemmas 3.5 and 3.4 and noting that W_ω is Gaussian since ρ is Gaussian, to get (3.22), we need to show that for any $W_\omega\in\Gamma$, there exist positive constants η and σ such that

$$\begin{aligned}&\left|\mathbb{E}\left[\omega^m\left(W_\omega^2-\mathbb{E}W_\omega^2\right)(\omega+r)^m\left(W_{\omega+r}^2-\mathbb{E}W_{\omega+r}^2\right)\right]\right|\\ &=2\left(\mathbb{E}\left[\omega^{\frac{m}{2}}(\omega+r)^{\frac{m}{2}}W_\omega W_{\omega+r}\right]\right)^2\lesssim(1+|r-\eta|)^{-\sigma}\quad\forall\omega\geq1,\,r\geq0.\end{aligned} \tag{3.23}$$

It follows from (3.21) that

$$U_j(\omega)=\frac{1}{2}\left[u_j(\omega)+\overline{u_j(\omega)}\right],\quad V_j(\omega)=\frac{1}{2\mathrm{i}}\left[u_j(\omega)-\overline{u_j(\omega)}\right],$$

which give

$$\begin{aligned}U_j(\omega_1)U_j(\omega_2)&=\frac{1}{4}\left[u_j(\omega_1)+\overline{u_j(\omega_1)}\right]\left[u_j(\omega_2)+\overline{u_j(\omega_2)}\right],\\ V_j(\omega_1)V_j(\omega_2)&=-\frac{1}{4}\left[u_j(\omega_1)-\overline{u_j(\omega_1)}\right]\left[u_j(\omega_2)-\overline{u_j(\omega_2)}\right],\\ U_j(\omega_1)V_j(\omega_2)&=\frac{1}{4\mathrm{i}}\left[u_j(\omega_1)+\overline{u_j(\omega_1)}\right]\left[u_j(\omega_2)-\overline{u_j(\omega_2)}\right].\end{aligned}$$

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Using the same procedure as that in Lemma 3.3 yields

$$\begin{aligned}|\mathbb{E}[U_j(\omega_1)U_j(\omega_2)]| &\lesssim \omega_1^{-m}(1+|\omega_1-\omega_2|)^{-N},\\ |\mathbb{E}[V_j(\omega_1)V_j(\omega_2)]| &\lesssim \omega_1^{-m}(1+|\omega_1-\omega_2|)^{-N},\\ |\mathbb{E}[U_j(\omega_1)V_j(\omega_2)]| &\lesssim \omega_1^{-m}(1+|\omega_1-\omega_2|)^{-N},\end{aligned}$$

which lead to

$$\begin{aligned}\left|\mathbb{E}\left[\omega^{\frac{m}{2}}(\omega+r)^{\frac{m}{2}}U_j(\omega)U_j(\omega+r)\right]\right| &\lesssim \left(\frac{1+r/\omega}{1+r}\right)^{\frac{m}{2}}(1+r)^{\frac{m}{2}-N}\\ &\lesssim (1+r)^{\frac{m}{2}-N}\end{aligned}$$

for any $\omega\ge 1$ and $N\in\mathbb{N}$. Similarly, we may conclude that

$$\left|\mathbb{E}\left[\omega^{\frac{m}{2}}(\omega+r)^{\frac{m}{2}}W_\omega W_{\omega+r}\right]\right| \lesssim (1+r)^{\frac{m}{2}-N} \tag{3.24}$$

holds for $W_\omega\in\{U_j(\omega),V_j(\omega),U_j(\omega+\tau),V_j(\omega+\tau)\}_{j=1,2,3}$.

For the case $W_\omega=U_j(\omega)-U_j(\omega+\tau)$, we have from Lemma 3.3 that

$$\begin{aligned}&|\mathbb{E}\left[(U_j(\omega)-U_j(\omega+\tau))(U_j(\omega+r)-U_j(\omega+r+\tau))\right]|\\ &\lesssim |\mathbb{E}[U_j(\omega)U_j(\omega+r)]|+|\mathbb{E}[U_j(\omega)U_j(\omega+r+\tau)]|\\ &\quad +|\mathbb{E}[U_j(\omega+\tau)U_j(\omega+r)]|+|\mathbb{E}[U_j(\omega+\tau)U_j(\omega+r+\tau)]|\\ &\lesssim \omega^{-m}(1+r)^{-N}+\omega^{-m}(1+r+\tau)^{-N}+(\omega+\tau)^{-m}(1+|r-\tau|)^{-N}\\ &\quad +(\omega+\tau)^{-m}(1+r)^{-N}\\ &\lesssim \omega^{-m}(1+r)^{-N}+(\omega+\tau)^{-m}(1+|r-\tau|)^{-N}+(\omega+\tau)^{-m}(1+r)^{-N}.\end{aligned}$$

Hence,

$$\begin{aligned}&\left|\mathbb{E}\left[\omega^{\frac{m}{2}}(\omega+r)^{\frac{m}{2}}(U_j(\omega)-U_j(\omega+\tau))(U_j(\omega+r)-U_j(\omega+r+\tau))\right]\right|\\ &\lesssim (1+r)^{\frac{m}{2}-N}+\left(\frac{\omega+r}{\omega+\tau}\right)^{\frac{m}{2}}(1+|r-\tau|)^{-N}+\left(\frac{\omega+r}{\omega+\tau}\right)^{\frac{m}{2}}(1+r)^{-N}\\ &\lesssim (1+r)^{\frac{m}{2}-N}+(1+|r-\tau|)^{\frac{m}{2}-N}.\end{aligned}$$

Similarly, we may show that the inequality

$$\left|\mathbb{E}\left[\omega^{\frac{m}{2}}(\omega+r)^{\frac{m}{2}}W_\omega W_{\omega+r}\right]\right| \lesssim (1+r)^{\frac{m}{2}-N}+(1+|r-\tau|)^{\frac{m}{2}-N} \tag{3.25}$$

holds for $W_\omega\in\{U_j(\omega)-U_j(\omega+\tau),V_j(\omega)-V_j(\omega+\tau),U_j(\omega)+V_j(\omega+\tau),V_j(\omega)-U_j(\omega+\tau)\}_{j=1,2,3}$.

Combining (3.24) and (3.25), we get that (3.23) holds for all $W_\omega\in\Gamma$, which completes the proof of (3.7). The proof of (3.8) is analogous to the proof of (3.7) and is omitted here. ∎

3.2. The second order far-field patterns. In this subsection, we show that the contribution of the second order backscattered far-field pattern can be ignored. According to (3.2) and (3.5), the far-field patterns $\boldsymbol{u}_{2,\mathrm{p}}^\infty$ and $\boldsymbol{u}_{2,\mathrm{s}}^\infty$ associated with incident waves $\boldsymbol{u}_0=\boldsymbol{u}_{\mathrm{p}}^{inc}$ and $\boldsymbol{u}_0=\boldsymbol{u}_{\mathrm{s}}^{inc}$, respectively, admit the following forms:

$$\begin{aligned}\boldsymbol{u}_{2,\mathrm{p}}^\infty(-\theta,\omega,\theta) &= \frac{c_{\mathrm{p}}^2}{4\pi}\theta\otimes\theta\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\rho(y)\rho(z)\boldsymbol{G}(y,z,\omega)\theta e^{\mathrm{i}\kappa_{\mathrm{p}}\theta\cdot(y+z)}dzdy,\\ \boldsymbol{u}_{2,\mathrm{s}}^\infty(-\theta,\omega,\theta) &= \frac{c_{\mathrm{s}}^2}{4\pi}(\boldsymbol{I}-\theta\otimes\theta)\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\rho(y)\rho(z)\boldsymbol{G}(y,z,\omega)\theta^\perp e^{\mathrm{i}\kappa_{\mathrm{s}}\theta\cdot(y+z)}dzdy.\end{aligned} \tag{3.26}$$

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The main result of this subsection is stated in the following theorem.

THEOREM 3.6. *Let the random potential ρ satisfy Assumption* 1.1*, and let $\boldsymbol{u}_{2,\mathrm{p}}^\infty$ and $\boldsymbol{u}_{2,\mathrm{s}}^\infty$ be given by* (3.26)*. For all $\theta\in\mathbb{S}^2$, it holds almost surely that*

$$\lim_{Q\to\infty}\frac{1}{Q}\int_Q^{2Q}\omega^m\left|\boldsymbol{u}_{2,\mathrm{p}}^\infty(-\theta,\omega,\theta)\right|^2 d\omega=0,$$
$$\lim_{Q\to\infty}\frac{1}{Q}\int_Q^{2Q}\omega^m\left|\boldsymbol{u}_{2,\mathrm{s}}^\infty(-\theta,\omega,\theta)\right|^2 d\omega=0.$$

The Green tensor $\boldsymbol{G}$ given in (2.2) can be split into three parts

$$\boldsymbol{G}(y,z,\omega)=\boldsymbol{G}_1(y,z,\omega)+\boldsymbol{G}_2(y,z,\omega)+\boldsymbol{G}_3(y,z,\omega), \tag{3.27}$$

where

$$\boldsymbol{G}_1(y,z,\omega)=\frac{c_\mathrm{s}^2}{4\pi}\frac{e^{\mathrm{i}\kappa_\mathrm{s}|y-z|}}{|y-z|}\boldsymbol{I},$$
$$\boldsymbol{G}_2(y,z,\omega)=\frac{c_\mathrm{p}^2e^{\mathrm{i}\kappa_\mathrm{p}|y-z|}-c_\mathrm{s}^2e^{\mathrm{i}\kappa_\mathrm{s}|y-z|}}{4\pi|y-z|^3}(y-z)\otimes(y-z),$$
$$\boldsymbol{G}_3(y,z,\omega)=\omega^{-2}\frac{\beta(y,z,\omega)}{4\pi|y-z|^5}\left[|y-z|^2\boldsymbol{I}-3(y-z)\otimes(y-z)\right].$$

Here

$$\beta(y,z,\omega):=e^{\mathrm{i}\kappa_\mathrm{s}|y-z|}(\mathrm{i}\kappa_\mathrm{s}|y-z|-1)-e^{\mathrm{i}\kappa_\mathrm{p}|y-z|}(\mathrm{i}\kappa_\mathrm{p}|y-z|-1).$$

Substituting (3.27) into (3.26), we can see that $\boldsymbol{u}_{2,\mathrm{p}}^\infty$ and $\boldsymbol{u}_{2,\mathrm{s}}^\infty$ also consist of three parts corresponding to $\boldsymbol{G}_1$, $\boldsymbol{G}_2$, and $\boldsymbol{G}_3$. The components in the first and second parts are linear combinations of

$$\mathbb{I}(\omega,\theta):=\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\rho(y)\rho(z)e^{\mathrm{i}c_1\omega\theta\cdot(y+z)}e^{\mathrm{i}c_2\omega|y-z|}\mathbb{K}(y,z)dydz \tag{3.28}$$

and the components in the third part are linear combinations of

$$\mathbb{J}(\omega,\theta):=\omega^{-2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\rho(y)\rho(z)e^{\mathrm{i}c_1\omega\theta\cdot(y+z)}\beta(y,z,\omega)\mathbb{K}(y,z)dydz, \tag{3.29}$$

where $c_1,c_2\in\{c_\mathrm{s},c_\mathrm{p}\}$ and

$$\mathbb{K}(y,z)=\frac{(y_1-z_1)^{p_1}(y_2-z_2)^{p_2}(y_3-z_3)^{p_3}}{|y-z|^{p_4}}.$$

Here, $(p_1,p_2,p_3,p_4)\in S_\mathbb{I}$ for $\mathbb{I}(\omega,\theta)$ and $(p_1,p_2,p_3,p_4)\in S_\mathbb{J}$ for $\mathbb{J}(\omega,\theta)$ with

$$S_\mathbb{I}:=\{(0,0,0,1),(2,0,0,3),(0,2,0,3),(0,0,2,3),(1,1,0,3),(1,0,1,3),(0,1,1,3)\},$$
$$S_\mathbb{J}:=\{(0,0,0,3),(2,0,0,5),(0,2,0,5),(0,0,2,5),(1,1,0,5),(1,0,1,5),(0,1,1,5)\}$$

such that $p_1+p_2+p_3-p_4=-1$ for $S_\mathbb{I}$ and $p_1+p_2+p_3-p_4=-3$ for $S_\mathbb{J}$.

Since the components of $\boldsymbol{u}_{2,\mathrm{p}}^\infty$ and $\boldsymbol{u}_{2,\mathrm{s}}^\infty$ are linear combinations of $\mathbb{I}(\omega,\theta)$ and $\mathbb{J}(\omega,\theta)$, Theorem 3.6 can be obtained directly from the following lemma, whose proof is technical and is given in supplementary materials (supplement.pdf [local/web 297KB]) to avoid a possible distraction from the presentation of the main results.

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LEMMA 3.7. *Let the random potential ρ satisfy Assumption* 1.1, *and let $\mathbb{I}(\omega,\theta)$ and $\mathbb{J}(\omega,\theta)$ be given by* (3.28) *and* (3.29), *respectively. For all $\theta\in\mathbb{S}^2$, it holds almost surely that*

$$\lim_{Q\to\infty}\frac{1}{Q}\int_Q^{2Q}\omega^m|\mathbb{I}(\omega,\theta)|^2d\omega=0, \tag{3.30}$$

$$\lim_{Q\to\infty}\frac{1}{Q}\int_Q^{2Q}\omega^m|\mathbb{J}(\omega,\theta)|^2d\omega=0. \tag{3.31}$$

3.3. The higher order far-field patterns. It follows from (3.6) that the higher order backscattered far-field patterns can be expressed by

$$\boldsymbol{b}_{\rm p}^\infty(-\theta,\omega,\theta)=\sum_{j=3}^\infty\boldsymbol{u}_{j,{\rm p}}^\infty(-\theta,\omega,\theta), \tag{3.32}$$

$$\boldsymbol{b}_{\rm s}^\infty(-\theta,\omega,\theta)=\sum_{j=3}^\infty\boldsymbol{u}_{j,{\rm s}}^\infty(-\theta,\omega,\theta), \tag{3.33}$$

where

$$\boldsymbol{u}_{j,{\rm p}}^\infty(-\theta,\omega,\theta)=-\frac{c_{\rm p}^2}{4\pi}\theta\otimes\theta\int_{\mathbb{R}^3}e^{{\rm i}c_{\rm p}\omega\theta\cdot z}\rho(z)\boldsymbol{u}_{j-1}(z)dz, \tag{3.34}$$

$$\boldsymbol{u}_{j,{\rm s}}^\infty(-\theta,\omega,\theta)=-\frac{c_{\rm s}^2}{4\pi}(\boldsymbol{I}-\theta\otimes\theta)\int_{\mathbb{R}^3}e^{{\rm i}c_{\rm s}\omega\theta\cdot z}\rho(z)\boldsymbol{u}_{j-1}(z)dz. \tag{3.35}$$

The goal is to estimate the order of $\boldsymbol{b}_{\rm p}^\infty$ and $\boldsymbol{b}_{\rm s}^\infty$ with respect to the frequency ω and to show that the contribution of the higher order far-field patterns can be ignored as well.

THEOREM 3.8. *For any $s\in(\frac{3-m}{2},\frac{1}{2})$, it holds almost surely that*

$$\sup_{\theta\in\mathbb{S}^2}\left|\boldsymbol{b}_{\rm p}^\infty(-\theta,\omega,\theta)\right|\lesssim\omega^{-2+6s}, \tag{3.36}$$

$$\sup_{\theta\in\mathbb{S}^2}|\boldsymbol{b}_{\rm s}^\infty(-\theta,\omega,\theta)|\lesssim\omega^{-2+6s}. \tag{3.37}$$

Proof. Define a cutoff function $\chi\in C_0^\infty(\mathbb{R}^3)$ supported in bounded domain U such that $D\subset U$ and $\chi(z)=1$ if $z\in D$. For any $s\in(\frac{3-m}{2},\frac{1}{2})$, $p\geq 3/s$, and p' satisfying $1/p+1/p'=1$, it follows from (3.32) and (3.34) that

$$\begin{aligned}\left|\boldsymbol{b}_{\rm p}^\infty(-\theta,\omega,\theta)\right|&\lesssim\left|\int_{\mathbb{R}^3}e^{{\rm i}c_{\rm p}\omega\theta\cdot z}\chi(z)\rho(z)\sum_{j=3}^\infty\boldsymbol{u}_{j-1}(z)dz\right|\\&\lesssim\|\rho\|_{W^{-s,p}(\mathbb{R}^3)}\Big\|\chi e^{{\rm i}c_{\rm p}\omega\theta\cdot(\cdot)}\chi\sum_{j=3}^\infty\boldsymbol{u}_{j-1}\Big\|_{\boldsymbol{W}^{s,p'}(\mathbb{R}^3)}\\&\lesssim\|\rho\|_{W^{-s,p}(\mathbb{R}^3)}\big\|\chi e^{{\rm i}c_{\rm p}\omega\theta\cdot(\cdot)}\big\|_{H^s(\mathbb{R}^3)}\Big\|\chi\sum_{j=3}^\infty\boldsymbol{u}_{j-1}\Big\|_{\boldsymbol{H}^s(\mathbb{R}^3)}\\&\lesssim\|\rho\|_{W^{-s,p}(\mathbb{R}^3)}\big\|\chi e^{{\rm i}c_{\rm p}\omega\theta\cdot(\cdot)}\big\|_{H^s(\mathbb{R}^3)}\Big\|\sum_{j=3}^\infty\boldsymbol{u}_{j-1}\Big\|_{\boldsymbol{H}^s_{-1}(\mathbb{R}^3)},\end{aligned} \tag{3.38}$$

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where we use the facts (cf. [9, 26])

$$\|fg\|_{W^{s,p'}(\mathbb{R}^3)} \lesssim \|f\|_{H^s(\mathbb{R}^3)} \|g\|_{H^s(\mathbb{R}^3)} \quad \forall f, g \in \mathcal{S}$$

for $p \geq \frac{3}{s}$ and p' satisfying $\frac{1}{p} + \frac{1}{p'} = 1$, and

$$\|\chi \boldsymbol{u}\|_{\boldsymbol{H}^s(\mathbb{R}^3)} \lesssim \|\boldsymbol{u}\|_{\boldsymbol{H}^s_{-2}(\mathbb{R}^3)} \lesssim \|\boldsymbol{u}\|_{\boldsymbol{H}^s_{-1}(\mathbb{R}^3)} \quad \forall \boldsymbol{u} \in \boldsymbol{\mathcal{S}}$$

with $\boldsymbol{\mathcal{S}}$ being dense in $\boldsymbol{H}^s_{-1}(\mathbb{R}^3)$. It is easy to check that

$$\left\|\chi e^{\mathrm{i}c_{\mathrm{p}}\omega\theta\cdot(\cdot)}\right\|_{L^2(\mathbb{R}^3)} \lesssim 1, \quad \left\|\chi e^{\mathrm{i}c_{\mathrm{p}}\omega\theta\cdot(\cdot)}\right\|_{H^1(\mathbb{R}^3)} \lesssim \omega.$$

Using the interpolation between spaces $L^2(\mathbb{R}^3)$ and $H^1(\mathbb{R}^3)$ yields

$$\left\|\chi e^{\mathrm{i}c_{\mathrm{p}}\omega\theta\cdot(\cdot)}\right\|_{H^s(\mathbb{R}^3)} \lesssim \left\|\chi e^{\mathrm{i}c_{\mathrm{p}}\omega\theta\cdot(\cdot)}\right\|^{1-s}_{L^2(\mathbb{R}^3)} \left\|\chi e^{\mathrm{i}c_{\mathrm{p}}\omega\theta\cdot(\cdot)}\right\|^{s}_{H^1(\mathbb{R}^3)} \lesssim \omega^s. \tag{3.39}$$

Note also that

$$\begin{aligned}
&\Big\|\sum_{j=3}^{\infty} \boldsymbol{u}_{j-1}\Big\|_{\boldsymbol{H}^s_{-1}(\mathbb{R}^3)} = \Big\|\sum_{j=2}^{\infty} \boldsymbol{u}_{j}\Big\|_{H^s_{-1}(\mathbb{R}^3)} \\
&\lesssim \sum_{j=2}^{\infty} \|\mathcal{K}_\omega^j \boldsymbol{u}_0\|_{H^s_{-1}(\mathbb{R}^3)} \\
&\lesssim \sum_{j=2}^{\infty} \|\mathcal{K}_\omega\|^{j-1}_{\mathcal{L}(H^s_{-1}(\mathbb{R}^3), H^s_{-1}(\mathbb{R}^3))} \|\mathcal{K}_\omega\|_{\mathcal{L}(H^s_{-1}(D), H^s_{-1}(\mathbb{R}^3))} \|\boldsymbol{u}_0\|_{H^s_{-1}(D)} \\
&\lesssim \sum_{j=2}^{\infty} \|\mathcal{K}_\omega\|^{j}_{\mathcal{L}(H^s_{-1}(\mathbb{R}^3), H^s_{-1}(\mathbb{R}^3))} \|\boldsymbol{u}_0\|_{H^s(D)} \\
&\lesssim \sum_{j=2}^{\infty} \omega^{j(-1+2s)} \omega^s \lesssim \omega^{-2+5s},
\end{aligned} \tag{3.40}$$

where we use Lemma 2.3 and the inequality $\|\cdot\|_{H^s_{-1}(\mathbb{R}^3)} \leq \|\cdot\|_{H^s(\mathbb{R}^3)}$ which can be easily checked by the definition. Combining (3.38)–(3.40), we get

$$\sup_{\theta \in \mathbb{S}^2} \left|\boldsymbol{b}^\infty_{\mathrm{p}}(-\theta, \omega, \theta)\right| \lesssim \omega^{-2+6s},$$

which completes the proof of (3.36). The estimate (3.37) can be obtained similarly by using (3.33) and (3.35). ∎

Now we are in position to prove the main result of the work.

Proof of Theorem 1.2. Recall from (3.6) that the compressional far-field pattern $\boldsymbol{u}^\infty_{\mathrm{p}}$ has the form

$$\boldsymbol{u}^\infty_{\mathrm{p}}(-\theta, \omega, \theta) = \boldsymbol{u}^\infty_{1,\mathrm{p}}(-\theta, \omega, \theta) + \boldsymbol{u}^\infty_{2,\mathrm{p}}(-\theta, \omega, \theta) + \boldsymbol{b}^\infty_{\mathrm{p}}(-\theta, \omega, \theta).$$

A simple calculation gives

$$\frac{1}{Q}\int_Q^{2Q} \omega^m \boldsymbol{u}^\infty_{\mathrm{p}}(-\theta, \omega, \theta) \cdot \overline{\boldsymbol{u}^\infty_{\mathrm{p}}(-\theta, \omega + \tau, \theta)} d\omega = \sum_{i,j=1}^{3} I_{i,j},$$

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where

$$I_{i,j} := \frac{1}{Q}\int_Q^{2Q} \omega^m \boldsymbol{v}_i(-\theta,\omega,\theta)\cdot\overline{\boldsymbol{v}_j(-\theta,\omega+\tau,\theta)}d\omega, \quad i,j=1,2,3,$$

with $\boldsymbol{v}_1=\boldsymbol{u}_{1,\mathrm{p}}^\infty$, $\boldsymbol{v}_2=\boldsymbol{u}_{2,\mathrm{p}}^\infty$, and $\boldsymbol{v}_3=\boldsymbol{b}_{\mathrm{p}}^\infty$. It then follows from Theorems 3.1, 3.6, and 3.8 that

$$\lim_{Q\to\infty} I_{1,1} = C_{\mathrm{p}}\hat{\phi}(2c_{\mathrm{p}}\tau\theta),$$

$$\begin{aligned}\left|\lim_{Q\to\infty} I_{1,2}\right| \le \lim_{Q\to\infty}&\left[\frac{1}{Q}\int_Q^{2Q}\omega^m|\boldsymbol{u}_{1,\mathrm{p}}^\infty(-\theta,\omega,\theta)|^2d\omega\right]^{\frac12}\\ &\times\left[\frac{1}{Q}\int_Q^{2Q}\omega^m|\boldsymbol{u}_{2,\mathrm{p}}^\infty(-\theta,\omega+\tau,\theta)|^2d\omega\right]^{\frac12}=0\end{aligned}$$

and

$$\begin{aligned}&\left|\lim_{Q\to\infty} I_{1,3}\right|\\ &\le \lim_{Q\to\infty}\left[\frac{1}{Q}\int_Q^{2Q}\omega^m|\boldsymbol{u}_{1,\mathrm{p}}^\infty(-\theta,\omega,\theta)|^2d\omega\right]^{\frac12}\left[\frac{1}{Q}\int_Q^{2Q}\omega^m|\boldsymbol{b}_{\mathrm{p}}^\infty(-\theta,\omega+\tau,\theta)|^2d\omega\right]^{\frac12}\\ &\lesssim \lim_{Q\to\infty}\left[\frac{1}{Q}\int_Q^{2Q}\omega^m|\boldsymbol{u}_{1,\mathrm{p}}^\infty(-\theta,\omega,\theta)|^2d\omega\right]^{\frac12}\left[\frac{1}{Q}\int_Q^{2Q}\omega^m(\omega+\tau)^{-4+12s}d\omega\right]^{\frac12}\\ &\le \lim_{Q\to\infty}\left[\frac{1}{Q}\int_Q^{2Q}\omega^m|\boldsymbol{u}_{1,\mathrm{p}}^\infty(-\theta,\omega,\theta)|^2d\omega\right]^{\frac12}\left[\frac{1}{Q}\int_Q^{2Q}\omega^{m-4+12s}d\omega\right]^{\frac12}\\ &=\left[C_{\mathrm{p}}\hat{\phi}(0)\right]^{\frac12}\lim_{Q\to\infty}\left[\frac{(2Q)^{m-3+12s}-Q^{m-3+12s}}{(m-3+12s)Q}\right]^{\frac12}=0\end{aligned}$$

for any $s\in(\frac{3-m}{2},\frac13-\frac{m}{12})$, where the domain is nonempty since $m>\frac{14}{5}$ and hence such an s exists.

Based on the same procedure used for the estimates of $\{I_{1,j}\}_{j=1,2,3}$, we may also show that elements in $\{I_{i,j}\}_{i=2,3,j=1,2,3}$ have limits of zero when $Q\to\infty$ and conclude that (1.5) holds. The result (1.6) for the shear far-field pattern can be obtained similarly. The details are omitted.

Due to continuation of a dense set and the fact that ϕ is analytic, the microlocal strength ϕ can be uniquely determined by $\{\hat{\phi}(2c_{\mathrm{p}}\tau\theta)\}_{(\tau,\theta)\in\Theta}$ or $\{\hat{\phi}(2c_{\mathrm{s}}\tau\theta)\}_{(\tau,\theta)\in\Theta}$ with Θ being any open domain of $\mathbb{R}_+\times\mathbb{S}^2$. ∎

4. Conclusion. In this paper, we have studied the inverse scattering problem for the three-dimensional time-harmonic elastic wave equation with a random potential. The potential is assumed to be a microlocally isotropic Gaussian random field such that its covariance operator is a classical pseudodifferential operator and should be interpreted as a distribution. For the direct problem, we prove that it is well-posed in the sense of distributions by examining the equivalent Lippmann–Schwinger integral equation. For the inverse scattering problem, we show that the strength of the random potential can be uniquely determined by a single realization of the high frequency

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limit of the averaged compressional (resp., shear) backscattered far-field pattern of the scattered wave associated to the compressional (resp., shear) plane incident wave.

This paper is concerned with the three-dimensional problem in a homogeneous medium, where the Green tensor has an explicit form which makes it possible to get the reconstruction formula of the strength. It is open for the two-dimensional inverse random potential scattering problem with the far-field data due to the complexity of the Hankel functions involved in the Green tensor. The problem is even more challenging if the medium is inhomogeneous where the explicit Green tensors are not available any more. Another interesting problem is to simultaneously reconstruct both the order m and the strength ϕ of the random potential ρ. The present approach seems not to work since the measurement depends on the given order m. We hope to be able to report the progress on these problems elsewhere in the future.

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