THE SCATTERING RESONANCES FOR SCHRÖDINGER-TYPE OPERATORS WITH UNBOUNDED POTENTIALS

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Abstract. This paper addresses the meromorphic continuation of the outgoing resolvent associated with Schrödinger-type operators in three dimensions. The first part focuses on the classical Schrödinger-type operator involving unbounded potentials. The absence of nonzero real poles for the outgoing resolvent is investigated. The second part examines the fractional Schrödinger operator, including both bounded and unbounded potentials. The analysis relies on a resolvent identity that establishes a connection between the resolvents of the fractional Schrödinger operator and its classical counterpart.

1. Introduction and main results

This paper focuses on exploring the meromorphic continuation of outgoing resolvents for Schrödinger-type operators in three dimensions. Extensive literature exists on this subject, see e.g., [4, 6, 8–11, 23, 27, 31, 32, 34–37] and references therein. This area closely relates to the theory of scattering resonances, which are defined as poles in the meromorphic continuation. For a comprehensive study on the mathematical theory of scattering resonances, we refer to the monograph [12]. Scattering resonances phenomena have significant applications across various scientific and engineering research fields. For example, as highlighted in [12], the study of scattering resonances finds applications in determining the long-time behavior of wave equations, leading to resonance expansions of waves. However, existing research on the meromorphic continuation of resolvents primarily concentrates on bounded potentials. This work aims to extend some of these results to accommodate unbounded potentials for Schrödinger-type operators.

We introduce certain notations in this context. Throughout this discussion, $V(x)$ represents a nonnegative real-valued potential function with compact support, while $\Omega \subset \mathbb{R}^3$ stands for a bounded open set where $\mathrm{supp}V \subset \Omega$. Let $\rho(x)$ be a smooth cutoff function with compact support, and $T = \mathrm{diam}(\mathrm{supp}\rho)$ denotes the diameter of the support of ρ. For $a \in \mathbb{R}^+$, the notation $a+$ refers to a constant greater than a. The notation $a \lesssim b$ denotes $a \leq Cb$, where $C > 0$ serves as a generic constant that may vary throughout the proofs.

We begin by examining the Schrödinger equation

$$-\Delta u(x, \lambda) + V(x)u(x, \lambda) - \lambda^2 u(x, \lambda) = f(x), \quad x \in \mathbb{R}^3,$$

where $\lambda \in \mathbb{C}$, $f(x) \in L^2_{\mathrm{comp}}(\mathbb{R}^3)$, and $V \in L^p_{\mathrm{comp}}(\mathbb{R}^3)$ represents an unbounded potential with $p > 3/2$. The Schrödinger operator $-\Delta + V$ is self-adjoint; for further details regarding this

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property, we refer to Appendix B. Let $R_V(\lambda) = (-\Delta + V - \lambda^2)^{-1}$ be the outgoing resolvent of the Schrödinger operator.

The free resolvent $R_0(\lambda) : L^2(\mathbb{R}^3) \to H^2(\mathbb{R}^3)$ is holomorphic in $\Im\lambda > 0$ (cf. [12]). Thus, for $V \in L^p(\mathbb{R}^3)$ with $p > 3/2$ and $V \geq 0$, according to Lemma 2.1, the operator $I + V^{1/2}R_0(\lambda)V^{1/2} : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ is invertible for $\Im\lambda \gg 1$. This result, derived from the resolvent identity (2.7), establishes that $R_V(\lambda) : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ is holomorphic for $\Im\lambda \gg 1$. Moreover, due to the compactness of $V^{1/2}R_0(\lambda)V^{1/2} : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ as indicated by Lemma 2.2, the operator $R_V(\lambda) : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ is meromorphic in $\Im\lambda > 0$, a consequence of the analytic Fredholm theorem (cf. Theorem A.1). Further extension of this meromorphic behavior to the lower-half complex plane is demonstrated by Theorem 1.1 when multiplied by smooth cutoff functions.

Let us define the space $\mathcal{C}$ as follows:

$$\mathcal{C} = \{V : V \in L^p_{\text{comp}}(\mathbb{R}^3) \text{ with } p > 3/2 \text{ such that the resolvent identity (1.3) holds}\}.$$

From the discussions in Appendix B, it is evident that $\mathcal{C}$ contains all $V \in L^2_{\text{comp}}(\mathbb{R}^3)$.

The following resolvent estimate extends the findings presented in [12, Theorem 3.8] and [12, Theorem 3.10] by Dyatlov and Zworski, transitioning from considerations limited to bounded potentials to including unbounded L^p potentials.

Theorem 1.1. *Assume that $V \in \mathcal{C}$. Choose a cutoff function $\rho \in C_0^\infty(\mathbb{R}^3)$ such that $\rho = 1$ near supp$V$. Then the operator $\rho R_V(\lambda)\rho : L^2(\mathbb{R}^3) \to H^1(\mathbb{R}^3)$ is meromorphic in the complex plane $\mathbb{C}$. Moreover, there exist positive constants C_0 and M such that $\rho R_V(\lambda)\rho : L^2(\mathbb{R}^3) \to H^1(\mathbb{R}^3)$ is holomorphic in the domain*

$$\Omega_M = \{\lambda \in \mathbb{C} : \Im\lambda \geq -M \log|\lambda|, \, |\lambda| \geq C_0\},$$

and the following estimate holds for $\lambda \in \Omega_M$:

$$\|\rho R_V(\lambda)\rho f\|_{H^1(\mathbb{R}^3)} \lesssim \langle\lambda\rangle^2 e^{T(\mathrm{Im}\lambda)_-}\|f\|_{L^2(\mathbb{R}^3)}, \tag{1.1}$$

where $t_- := \max\{-t, 0\}$, $T = diam(supp\rho)$, and $\langle\lambda\rangle := (1 + |\lambda|^2)^{1/2}$.

Generally, the operator $-\Delta$ is defined in the Sobolev space $H^2(\mathbb{R}^3)$ and proves to be self-adjoint on $L^2(\mathbb{R}^3)$. It follows from the Fourier transform that

$$\widehat{-\Delta v}(\xi) = |\xi|^2\hat{v}(\xi), \quad v \in H^2(\mathbb{R}^3),$$

which immediately deduces the spectrum of $-\Delta$:

$$\sigma(-\Delta) = \{z = |\xi|^2 : \xi \in \mathbb{R}^3\} = [0, +\infty).$$

Hence, the resolvent $(-\Delta - z)^{-1}$ of $-\Delta$ is holomorphic for $z \in \mathbb{C}\backslash[0, +\infty)$ in the uniform operator topology of $\mathcal{B}(L^2)$, where $\mathcal{B}(L^2)$ denotes the set of all bounded operators acting on $L^2(\mathbb{R}^3)$.

Let $z = \lambda^2$, then the family of operators

$$R_0(\lambda) := (-\Delta - \lambda^2)^{-1} : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$$

are holomorphic in the upper-half complex plane $\mathbb{C}^+ := \{z \in \mathbb{C} : \Im z > 0\}$. Given that the kernel $\frac{e^{i\lambda|x-y|}}{|x-y|}$ of the resolvent $R_0(\lambda)$ exhibits exponential growth in the lower-half complex plane $\mathbb{C}^- := \{z \in \mathbb{C} : \Im z < 0\}$, it is typically assumed that $\lambda \in \mathbb{C}^+$. Furthermore, by the following resolvent identity which connects $R_V(\lambda)$ and $R_0(\lambda)$:

$$R_V(\lambda) = R_0(\lambda)(I + VR_0(\lambda))^{-1}, \tag{1.2}$$

it is also required that $\lambda \in \mathbb{C}^+$ for the investigation of the resolvent $R_V(\lambda) : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ in the presence of a potential function. In scattering theory, a crucial aspect is the study of the limiting absorption principle, which examines the mapping properties of $R_V(\lambda)$ as λ approaches the positive real axis of $\mathbb{C}$. In the seminal work by Agmon [1], the resolvent estimate of $R_V(\lambda)$ was established in weighted L^2 spaces for a class of short-range potentials. This result was further extended by Goldberg and Schlag [14] to L^p potentials, specifically $V \in L^p(\mathbb{R}^3) \cap L^{3/2}(\mathbb{R}^3)$ with $p > 3/2$, in three dimensions.

In [12], Dyatlov and Zworski studied the resolvent of the form $\rho R_V(\lambda)\rho$, where ρ is a fixed smooth function with compact support. Employing two cutoff functions, they facilitated the consideration of the meromorphic continuation of the resolvent $\rho R_V(\lambda)\rho : L^2 \to L^2$ from the upper-half complex plane to the lower-half complex plane. Their findings revealed that the free resolvent is holomorphic, characterized by the resolvent estimate $\|\rho R_0(\lambda)\rho\|_{L^2\to L^2} = \mathcal{O}(e^{T(\Im\lambda)_-}/\langle\lambda\rangle)$ for all $\lambda \in \mathbb{C}$. Notably, the decaying factor of $1/|\lambda|$ appearing in the resolvent estimate plays a crucial role in enabling the meromorphic continuation of the resolvent. This estimate was derived through the application of the Huygens principle for wave propagation in $\mathbb{R}^n$, where $n \geq 3$ is an odd number. Building upon the resolvent identity (1.2) and employing perturbation arguments, it was demonstrated in [12, Theorem 3.8] that $\rho R_V(\lambda)\rho : L^2 \to L^2$ exhibits meromorphic behavior across the entire complex plane $\mathbb{C}$. It is worth mentioning that $R_0(\lambda)$ in (1.2) contributes the crucial $1/|\lambda|$ decaying factor in the resolvent estimate, facilitating the application of the analytic Fredholm theorem (cf. Theorem A.1). Moreover, a resonance-free region was obtained in [12, Theorem 3.10]. However, this result assumed the potential V to be in L^∞_{comp}. Consequently, it prompts a natural inquiry: can the outcomes presented in [12] be extended from bounded potentials to include unbounded potentials?

The first part of this work presents an affirmative answer to the preceding query and extends the findings outlined in [12] to include unbounded L^p potentials with $p > 3/2$. Our focus lies specifically on exploring the meromorphic continuation of the resolvent into the lower-half complex plane. This extension allows for the utilization of the contour integral method to derive long-time asymptotics of the wave equation, consequently leading to resonance expansions of waves, as elaborated upon in [12]. The assumption $V \in L^{3/2}$ emerges as optimal for the well-posedness of the Dirichlet problem with L^p-type potentials in bounded domains. For instance, the work in [20] demonstrates instances where the Dirichlet problem lacks well-posedness for certain $V \in L^{3/2}_{\text{weak}}$. Additionally, this assumption remains optimal for the unique continuation property, as outlined in [18]. However, restricting V solely to $L^{3/2}_{\text{comp}}$ results in a loss of the crucial decaying factor of $1/|\lambda|$-type and a compact embedding result, both of which are vital for the meromorphic continuation of the resolvent into the lower-half complex plane. Detailed discussions concerning these aspects are provided in Lemmas 2.1 and 2.2. Therefore, the assumption $p > 3/2$ is both necessary and optimal for investigating the meromorphic continuation of the resolvent into the lower-half complex plane.

The motivation behind the proof of Theorem 1.1 originates from [12]. However, the proof presented in [12] for the L^2-based resolvent estimate of $\rho R_0(\lambda)\rho$ is limited to bounded potentials and cannot be extended to unbounded L^p-type potentials. To address unbounded L^p potentials, we utilize the resolvent identity (cf. Appendix B):

$$R_V(\lambda) = R_0(\lambda) - R_0(\lambda)V^{1/2}(I + V^{1/2}R_0(\lambda)V^{1/2})^{-1}V^{1/2}R_0(\lambda), \quad \Im\lambda > 0. \tag{1.3}$$

Subsequently, employing Fourier analysis techniques helps us handle singular L^p potentials, leading to an L^p-based resolvent estimate of $\rho R_0(\lambda)\rho$. We then show that the operator $V^{1/2}R_0(\lambda)V^{1/2} : L^2 \to L^2$ is compact. Crucially, we demonstrate that $\|V^{1/2}R_0(\lambda)V^{1/2}\|_{L^2\to L^2}$

exhibits a decaying factor of $1/|\lambda|$, requiring a slightly higher regularity assumption: $V \in L^p_{\text{comp}}, p > 3/2$. Finally, the proof concludes by applying the analytic Fredholm theorem and the regularity theory of elliptic equations.

Another important result in the first part of the work concerns the absence of nonzero real poles for the outgoing resolvent.

Theorem 1.2. *Assume that $V \in \mathcal{C}$. The outgoing resolvent $R_V(\lambda) : L^2_{\text{comp}}(\mathbb{R}^3) \to H^1_{\text{loc}}(\mathbb{R}^3)$ has no poles on $\mathbb{R} \setminus \{0\}$.*

The proof of the above theorem aligns with Goldberg–Schlag [14] and Ionescu–Jerison [17]. Utilizing the Tomas–Stein restriction theorem from [29, page 386] applicable to L^p potentials, we establish a result concerning the absence of embedded non-zero eigenvalues for $V \in L^p$. Importantly, it is noteworthy that the Tomas–Stein restriction theorem does not necessitate the potential function to have compact support.

In the second part of the paper, we study the fractional Schrödinger equation

$$(-\Delta)^\alpha u(x) - \lambda^{2\alpha} u(x) + V(x)u(x) = f(x), \quad x \in \mathbb{R}^3,$$

where $0 < \alpha < 1$ and $f(x) \in L^2_{\text{comp}}(\mathbb{R}^3)$. The fractional Laplacian is defined via the Fourier transform

$$(-\Delta)^\alpha u = \mathcal{F}^{-1}\{|\xi|^{2\alpha}\hat{u}(\xi)\}, \quad u \in H^{2\alpha}(\mathbb{R}^3).$$

Alternatively, the fractional Laplacian can also be defined pointwisely through the principle value integral

$$(-\Delta)^\alpha u = C(\alpha)\text{p.v.} \int_{\mathbb{R}^3} \frac{u(x) - u(y)}{|x - y|^{\alpha+3}} \mathrm{d}y,$$

where $C(\alpha)$ is a normalizing constant [15]. Regarding the self-adjointness of the fractional Schrödinger operator $(-\Delta)^\alpha - \lambda^{2\alpha} + V$, further details are provided in Appendix B.

Denote the outgoing resolvent of the fractional Schrödinger operator by

$$R_{\alpha,V}(\lambda) = ((-\Delta)^\alpha - \lambda^{2\alpha} + V)^{-1}.$$

Let $\mathbb{R}^+ = [0, +\infty)$. Given that the spectrum of the fractional Schrödinger operator satisfies $\sigma((-\Delta)^\alpha) = \mathbb{R}^+$ for $0 < \alpha < 1$, the resolvent $R_{\alpha,0}(\lambda) = ((-\Delta)^\alpha - \lambda^{2\alpha})^{-1} : L^2(\mathbb{R}^3) \to H^{2\alpha}(\mathbb{R}^3)$ is holomorphic in $0 < \arg\lambda < \frac{\pi}{\alpha}$. Furthermore, for $V \in L^\infty_{\text{comp}}(\mathbb{R}^3)$, $R_{\alpha,V}(\lambda) = ((-\Delta)^\alpha - \lambda^{2\alpha} + V)^{-1} : L^2(\mathbb{R}^3) \to H^{2\alpha}(\mathbb{R}^3)$ is also holomorphic in $0 < \arg\lambda < \frac{\pi}{\alpha}$ (cf. [22]).

For any fixed $\theta_0 \in (0, \frac{\pi}{2})$, we denote the sectorial domain

$$S_{\theta_0} = \{\lambda \in \mathbb{C} : \arg\lambda \in [-\theta_0, \theta_0] \cup [\pi - \theta_0, \pi + \theta_0], \lambda \neq 0\}.$$

The following theorem concerns the meromorphic continuation of the resolvent for the fractional Schrödinger operator involving bounded potentials. Let $\mathbb{R}^- = (-\infty, 0]$. We select the branch $\mathbb{C} \setminus \mathrm{i}\mathbb{R}^-$ such that $z^{2\alpha}$ is analytic.

Theorem 1.3. *Assuming $V \in L^\infty_{\text{comp}}(\mathbb{R}^3)$ with supp$V \subset \Omega$, where Ω is a bounded open set and $1/2 < \alpha < 1$. For any fixed $\rho \in C_0^\infty(\mathbb{R}^3)$ such that $\rho = 1$ on supp$V$ and supp$\rho \subset \Omega$, the outgoing resolvent $\rho R_{\alpha,V}(\lambda)\rho$ is meromorphic in S_{θ_0}. Moreover, $\rho R_{\alpha,V}(\lambda)\rho$ is holomorphic and satisfies the following resolvent estimate for $\lambda \in \Omega_M \cap S_{\theta_0}$:*

$$\|\rho R_{\alpha,V}(\lambda)\rho\|_{L^2(\Omega)\to H^s(\Omega)} \lesssim \langle\lambda\rangle^{1+s-2\alpha} e^{T(\Im\lambda)_-}, \tag{1.4}$$

where $0 \le s < 2\alpha - 1$ and

$$\Omega_M := \{\lambda : \Im\lambda \ge -M\log(|\lambda|),\ |\lambda| > C_0\} \setminus \mathrm{i}\mathbb{R}.$$

Here, $M < (2\alpha - 1)/T$, $T =$ diam(suppρ), and C_0 is a sufficiently large constant.

Assuming $1/2 < \alpha < 1$ and $0 < \beta < \frac{2\alpha-1}{3}$, let us introduce the space

$$\mathcal{F} = \{V : V \in L^p_{\text{comp}}(\mathbb{R}^3) \text{ with } p > \frac{3}{2\beta} \text{ such that the resolvent identity (3.6) holds}\}.$$

For further details regarding the assumption, we direct the reader to the discussions in Appendix B.

The following theorem addresses the meromorphic continuation of the resolvent for the fractional Schrödinger operator with unbounded potentials.

From Theorem 3.1 and Lemma 3.5, we deduce that $I + V^{1/2}R_{\alpha,0}(\lambda)V^{1/2} : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ is invertible for $|\lambda| \gg 1$ in $\{\lambda : \Im\lambda > 0, \lambda \notin \mathrm{i}\mathbb{R}^+\}$. This result, combined with the resolvent identity

$$R_{\alpha,V}(\lambda) = R_{\alpha,0}(\lambda) - R_{\alpha,0}(\lambda)V^{1/2}(I + V^{1/2}R_{\alpha,0}(\lambda)V^{1/2})^{-1}V^{1/2}R_{\alpha,0}(\lambda),$$

implies that $R_{\alpha,V}(\lambda) : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ is holomorphic for $|\lambda| \gg 1$ in $\{\lambda : \Im\lambda > 0, \lambda \notin \mathrm{i}\mathbb{R}^+\}$. Additionally, given that $V^{1/2}R_{0,\alpha}(\lambda)V^{1/2} : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ is compact by Lemma 3.4, we can assert that $R_{\alpha,V}(\lambda) : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ is meromorphic in $\{\lambda : \Im\lambda > 0, \lambda \in S_{\theta_0}\}$ by utilizing the analytic Fredholm theorem (cf. Theorem A.1). When multiplied by the smooth cutoff functions, the following theorem demonstrates its meromorphic extension to S_{θ_0}.

Theorem 1.4. *Assuming $V \in \mathcal{F}$, let $\rho \in C_0^\infty(\mathbb{R}^3)$ be such that $\rho = 1$ on supp$V$ and supp$\rho \subset \Omega$. Then the outgoing resolvent $\rho R_{\alpha,V}(\lambda)\rho : L^2(\Omega) \to H^\beta(\Omega)$ is meromorphic in S_{θ_0}. Moreover, $\rho R_{\alpha,V}(\lambda)\rho$ is holomorphic and satisfies the following resolvent estimate for $\lambda \in \Omega_M \cap S_{\theta_0}$:*

$$\|\rho R_{\alpha,V}(\lambda)\rho\|_{L^2(\Omega)\to H^\beta(\Omega)} \lesssim |\lambda|^{-(4\alpha-4\beta-2)}e^{T(\Im\lambda)_-}, \tag{1.5}$$

where

$$\Omega_M := \{\lambda : \Im\lambda \geq -M\log(|\lambda|),\, |\lambda| > C_0\} \setminus \mathrm{i}\mathbb{R}.$$

Here, $T = diam(\text{supp}\rho)$, $M < (2\alpha - 3\beta - 1)/T$, and C_0 is a sufficiently large constant.

The motivations behind the proofs of Theorems 1.3 and 1.4 stem from [12] and Theorem 1.1. However, due to our focus on nonlocal fractional operators, the strategies employed in [12] and Theorem 1.1 for standard elliptic operators cannot be directly applied to fractional operators. Notably, the standard elliptic interior regularity might not hold in the context of fractional operators. Recently, [5] established an interior regularity for solutions of the fractional Laplacian, showing that for a fractional Laplacian with $0 < \alpha < 1$, the solution exhibits interior $H^{2\alpha-\varepsilon}$ regularity, where ε is any small positive number. However, it remains unknown whether this result represents an optimal characterization. Furthermore, the interior estimate in [5] requires that the interior $H^{2\alpha-\varepsilon}$ norm should be bounded by the global $L^2_\sigma(\mathbb{R}^n)$ norm of the solution. Here, $L^2_\sigma(\mathbb{R}^n)$ denotes the standard weighted L^2 space. Yet, it is important to note that when $\lambda \in \mathbb{C}^-$, the solution exhibits exponential growth and does not fall into the $L^2_\sigma(\mathbb{R}^n)$ space. There exist several studies on the resolvents of fractional Schrödinger operators, such as [16, 24, 25]. These works concern the family of resolvents in the upper-half complex plane and explore their limiting absorption principles. However, as of now, there is no known research addressing the meromorphic continuation of the resolvent of the fractional Schrödinger operator from the upper-half complex plane to the lower-half complex plane.

The analysis of the meromorphic continuation of the resolvent for the fractional Schrödinger operator relies on the following resolvent formula of the free fractional resolvent (cf. [22, (5.28)]) for $\lambda \in \{\lambda : \Im\lambda > 0, \lambda \notin \mathrm{i}\mathbb{R}^+\}$:

$$((-\Delta)^\alpha - \lambda^{2\alpha})^{-1} = \frac{\lambda^{2-2\alpha}}{\alpha}(-\Delta - \lambda^2)^{-1} + \frac{\sin\alpha\pi}{\pi}\int_0^{+\infty} \frac{\gamma^\alpha(\gamma-\Delta)^{-1}}{\gamma^{2\alpha} - 2\gamma^\alpha\lambda^{2\alpha}\cos\alpha\pi + \lambda^{4\alpha}}\mathrm{d}\gamma, \quad (1.6)$$

which establishes a connection between the free resolvent for the fractional Schrödinger operator and the classical one. It is crucial to note that the requirement $\lambda \notin \mathrm{i}\mathbb{R}^+$ yielding $\lambda^2 \in \mathbb{C} \setminus \mathbb{R}^-$ is necessary for the integral term in (1.6) to be well-defined. For detailed insights, refer to [22]. An important observation is the necessity of a $1/|\gamma|$-type decay for the resolvent estimate of $\rho(\gamma-\Delta)^{-1}\rho$ for the convergence of the integral in (1.6). According to [12], this is viable due to the resolvent estimate $\|\rho(\gamma-\Delta)^{-1}\rho\|_{L^2\to L^2} = \mathcal{O}(e^{T(\Im\sqrt{\gamma})_-}/|\gamma|^{1/2})$ for the classical free resolvent. Consequently, the meromorphic continuation of the free fractional resolvent and the related resolvent estimates are derived using interpolation inequalities. The meromorphic continuation of the fractional resolvent with a bounded potential follows from a combination of (1.2) and the perturbation argument in [12]. For an unbounded potential, a similar argument to the proof of Theorem 1.1 is employed, utilizing the resolvent identity (1.3) and interpolation inequalities.

The paper's structure is as follows. In Section 2, we investigate the meromorphic continuation of the resolvent of the classical Schrödinger operator featuring an unbounded L^p potential. We obtain a region free of resonances and derive associated resolvent estimates. Additionally, we provide the L^p-based resolvent estimate of $\rho R_0(\lambda)\rho$. Section 3 is dedicated to exploring the meromorphic continuation of the resolvents for fractional Schrödinger operators, including both bounded and unbounded potentials.

2. The Schrödinger operator

This section is to address the outgoing resolvent of the Schrödinger operator defined by $R_V(\lambda) := (-\Delta + V - \lambda^2)$. We begin with the free outgoing resolvent $R_0(\lambda) := (-\Delta - \lambda^2)$.

Building upon [7, Lemma 3.3], originally rooted in [21], the following lemma plays an important role in achieving the meromorphic continuation of the resolvent. In contrast to the assumption $V \in L^{3/2}$ in [7], we impose slightly greater regularity on the potential function V by requiring $V \in L^p$ with $p > 3/2$. This choice arises from two reasons: Firstly, the resolvent introduced in [7] primarily serves the construction of complex geometric solutions, divergent from our objectives. Secondly, we are primarily concerned with the extension of the resolvent into the lower-half complex plane and the establishment of a region free from resonances. In contrast, the outcomes presented in [7] only hold outside an indeterminate countable set, precluding the derivation of an explicit resonance-free region.

Lemma 2.1. *Assuming $V \in L^p_{\mathrm{comp}}(\mathbb{R}^3)$ with $p > 3/2$, there exist constants $D > 0$ and $F > 0$ such that for $\lambda \in \Omega_D$ where*

$$\Omega_D := \{\lambda \in \mathbb{C} : \Im\lambda \geq -D\log|\lambda|,\ |\lambda| \geq F\},$$

the following inequality holds:

$$\|V^{1/2}R_0(\lambda)V^{1/2}\|_{L^2(\mathbb{R}^3)\to L^2(\mathbb{R}^3)} \leq \frac{1}{2}. \quad (2.1)$$

Proof. Given the kernel of the outgoing free resolvent $R_0(\lambda)$ as

$$R_0(\lambda, x, y) := \frac{1}{4\pi}\frac{e^{\mathrm{i}\lambda|x-y|}}{|x-y|}$$

such that

$$R_0(\lambda,x,y)f=\int_{\mathbb{R}^3}R_0(\lambda,x,y)f(y)\mathrm{d}y,$$

we find that $R_0(\lambda)$ defines a bounded operator $R_0(\lambda):L^2(\mathbb{R}^3)\to L^2(\mathbb{R}^3)$ for $\lambda\in\mathbb{C}$ with $\Im\lambda>0$. On the other hand, for $\lambda\in\mathbb{C}$ with $\Im\lambda\le 0$, the operator $R_0(\lambda):L^2(\mathbb{R}^3)\to L^2(\mathbb{R}^3)$ is unbounded. However, considering $R_0(\lambda)$ as an operator mapping $L^2_{\mathrm{comp}}(\mathbb{R}^3)$ onto $L^2_{\mathrm{loc}}(\mathbb{R}^3)$, where for any fixed $\rho\in C_0^\infty(\mathbb{R}^3)$, the operator $\rho R_0(\lambda)\rho:L^2(\mathbb{R}^3)\to L^2(\mathbb{R}^3)$ is bounded, allows the extension of the operator $R_0(\lambda)$ into a holomorphic family of operators for all $\lambda\in\mathbb{C}$. For any fixed $\rho\in C_0^\infty(\mathbb{R}^3)$, $\rho R_0(\lambda)\rho:L^2(\mathbb{R}^3)\to L^2(\mathbb{R}^3)$ is a weak holomorphic family of operators if the function $I(\lambda):=\langle\rho R_0(\lambda)\rho g_1,g_2\rangle_{L^2(\mathbb{R}^3)}$ is holomorphic in $\mathbb{C}$ for any given $g_1,g_2\in L^2(\mathbb{R}^3)$. Furthermore, the fact that $\rho R_0(\lambda)\rho$ is weakly holomorphic implies strong holomorphicity, a concept detailed in [33].

Let $\rho\in C_0^\infty(\mathbb{R}^3)$ be a fixed cutoff function with $\rho=1$ on $\mathrm{supp}V$. Given

$$|\rho(x)R_0(\lambda,x,y)\rho(y)|\le\frac{e^{T(\Im\lambda)_-}}{|x-y|},\quad \lambda\in\mathbb{C},$$

where $T=\mathrm{diam}(\mathrm{supp}\rho)$, utilizing the Hardy–Littlewood–Sobolev inequality (cf. [28, Theorem 0.3.2]), for $\lambda\in\mathbb{C}$, we obtain

$$\|\rho R_0(\lambda)\rho\|_{L^{\frac65}(\mathbb{R}^3)\to L^6(\mathbb{R}^3)}\lesssim e^{T(\Im\lambda)_-}. \tag{2.2}$$

Denote $m=V^{1/2}$ for simplicity. Considering $mR_0(\lambda)m=m\rho R_0(\lambda)\rho m$, from (2.2) and the Hölder inequality, we derive

$$\begin{aligned}\|mR_0(\lambda)mg\|_{L^2}&\le\|m\|_{L^3}\|\rho R_0(\lambda)\rho mg\|_{L^6}\lesssim e^{T(\Im\lambda)_-}\|m\|_{L^3}\|mg\|_{L^{\frac65}}\\&\lesssim e^{T(\Im\lambda)_-}\|m\|_{L^3}\|m\|_{L^3}\|g\|_{L^2}.\end{aligned}\tag{2.3}$$

Let $\tau=p-3/2$. We decompose $m=m_1+m_2$ such that $m_1=m\chi_{\{|m|\le\theta\}}$ and $m_2=m\chi_{\{|m|\ge\theta\}}$, where $\theta=Me^{\tilde T(\Im\lambda)_-}$ with $\tilde T$ and M being two positive constants to be determined later. Thus, $m_1\in L^\infty(\mathbb{R}^3)$ satisfies

$$\|m_1\|_{L^\infty}\le Me^{\tilde T(\Im\lambda)_-},\quad \|m_1\|_{L^3}\le\|m\|_{L^3}.$$

Let $\varepsilon>0$ be a positive constant such that $\varepsilon<\frac12$. Through a straightforward calculation, we can find a sufficiently large $M>0$ such that

$$\begin{aligned}\|m_2\|_{L^3}&=\Big(\int_{\chi_{\{|m|\ge\theta\}}}|V|^{3/2}\mathrm{d}x\Big)^{1/3}\le\Big(\int_{\chi_{\{|m|\ge\theta\}}}\frac{m^{2\tau}}{\theta^{2\tau}}|V|^{3/2}\mathrm{d}x\Big)^{1/3}\\&\le\theta^{-\frac{2\tau}{3}}\Big(\int_{\chi_{\{|m|\ge\theta\}}}|V|^{3/2+\tau}\mathrm{d}x\Big)^{1/3}\lesssim\frac{\varepsilon}{3}e^{\frac{-2\tau\tilde T}{3}(\Im\lambda)_-}.\end{aligned}$$

Now, we fix the constant M. For a given $g\in L^2$, using the L^2 estimate $\|\rho R_0(\lambda)\rho\|_{L^2\to L^2}=\mathcal{O}(\frac{1}{|\lambda|})$ (cf. [12, Theorem 3.1]) and (2.3), we derive

$$\begin{aligned}\|m\rho R_0(\lambda)\rho mg\|_{L^2}&\le\|m_1\rho R_0(\lambda)\rho m_1g\|_{L^2}+\|m_1\rho R_0(\lambda)\rho m_2g\|_{L^2}+\|m_2\rho R_0(\lambda)\rho mg\|_{L^2}\\&\lesssim\|m_1\|_{L^\infty}\|\rho R_0(\lambda)\rho\|_{L^2\to L^2}\|m_1\|_{L^\infty}\|g\|_{L^2}\\&\quad+e^{T(\Im\lambda)_-}\|m_1\|_{L^3}\|m_2\|_{L^3}\|g\|_{L^2}+e^{T(\Im\lambda)_-}\|m_2\|_{L^3}\|m\|_{L^3}\|g\|_{L^2}\\&\lesssim\Big(\frac{e^{(T+2\tilde T)(\Im\lambda)_-}}{|\lambda|}+\frac{2\varepsilon}{3}e^{(T-\frac{2\tau\tilde T}{3})(\Im\lambda)_-}\Big)\|g\|_{L^2}.\end{aligned}\tag{2.4}$$

The implicit constants in (2.4) are independent of M.

First, select $\tilde{T}$ to be sufficiently large such that $(T-\frac{2\tau\tilde{T}}{3})<0$. Subsequently, given $\lambda\in\Omega_D$, we have

$$e^{(T+2\tilde{T})(\Im\lambda)_-}\le|\lambda|^{D(T+2\tilde{T})}.$$

Next, we can choose a small enough value for D so that $D(T+2\tilde{T})<1$. Consequently, for $|\lambda|\ge F$, where F is sufficiently large, we get

$$\frac{e^{(T+2\tilde{T})(\Im\lambda)_-}}{|\lambda|}\le|\lambda|^{D(T+2\tilde{T})-1}<\frac{\varepsilon}{3},$$

which shows that (2.4) is bounded by $\varepsilon\|g\|_{L^2}$. We arrive at (2.1) due to the condition $\varepsilon<\frac{1}{2}$, thus concluding the proof. □

The following lemma concerns the compactness of the operator $V^{1/2}R_0(\lambda)V^{1/2}:L^2(\mathbb{R}^3)\to L^2(\mathbb{R}^3)$. The proof draws from the insights presented in [12, Theorem 2.1].

Lemma 2.2. *Assume that $V\in L^p(\mathbb{R}^3)$ with $p>3/2$ and has compact support. Then the operator $V^{1/2}R_0(\lambda)V^{1/2}:L^2(\mathbb{R}^3)\to L^2(\mathbb{R}^3)$ is compact.*

Proof. Select a cutoff function $\rho\in C_0^\infty(\mathbb{R}^3)$ such that $\rho=1$ on supp$V$. We have

$$\int_{\mathbb{R}^3}|\rho(x)\rho(y)R_0(\lambda,x,y)|\mathrm{d}x\lesssim\int_{\mathbb{R}^3}|\rho(x)\rho(y)|\frac{1}{|x-y|}e^{T(\Im\lambda)_-}\mathrm{d}x\lesssim e^{T(\Im\lambda)_-}$$

and

$$\int_{\mathbb{R}^3}|\rho(x)\rho(y)R_0(\lambda,x,y)|\mathrm{d}y\lesssim\int_{\mathbb{R}^3}|\rho(x)\rho(y)|e^{(\Im\lambda)_-|x-y|}\mathrm{d}y\lesssim e^{T(\Im\lambda)_-},$$

where $R_0(\lambda,x,y)$ is the Schwartz kernel of the free resolvent $(-\Delta-\lambda^2)^{-1}$ and $T=\mathrm{diam}(\mathrm{supp}\rho)$. By applying Schur's test (cf. Theorem A.2), it follows that for any $1\le q\le\infty$

$$\|\rho R_0(\lambda)\rho\|_{L^q(\mathbb{R}^3)\to L^q(\mathbb{R}^3)}\lesssim e^{T(\Im\lambda)_-}. \tag{2.5}$$

We proceed to show

$$\|\rho R_0(\lambda)\rho\|_{L^q(\mathbb{R}^3)\to W^{2,q}(\mathbb{R}^3)}\lesssim\langle\lambda\rangle^2e^{T(\Im\lambda)_-}. \tag{2.6}$$

To establish this, we utilize the following elliptic regularity estimate: Given $\tilde{\rho}\in C_0^\infty(\mathbb{R}^3)$ such that $\tilde{\rho}=1$ near the support of ρ, we derive from [29, (7.13)] that

$$\|\rho u\|_{W^{2,q}(\mathbb{R}^3)}\le C\big(\|\tilde{\rho}u\|_{L^q(\mathbb{R}^3)}+\|\tilde{\rho}\Delta u\|_{L^q(\mathbb{R}^3)}\big).$$

Consequently, we obtain

$$\|\rho R_0(\lambda)\rho f\|_{W^{2,q}(\mathbb{R}^3)}\lesssim\|\tilde{\rho}R_0(\lambda)\rho f\|_{L^q(\mathbb{R}^3)}+\|\tilde{\rho}\Delta R_0(\lambda)\rho f\|_{L^q(\mathbb{R}^3)}.$$

By (2.5), we deduce $\tilde{\rho}\Delta R_0(\lambda)\rho f=\rho f+\tilde{\rho}\lambda^2R_0(\lambda)\rho f$, satisfying

$$\|\tilde{\rho}\Delta R_0(\lambda)\rho f\|_{L^q}\lesssim\langle\lambda\rangle^2e^{T(\Im\lambda)_-}\|f\|_{L^q}.$$

Thus, we establish the estimate (2.6).

We continue to prove the compactness. Given $V\in L^p$ where $p>3/2$, employing Hölder's inequality yields $V^{1/2}f\in L^{\frac{6}{5}+}$. Consequently, $\rho R_0(\lambda)\rho V^{1/2}f\in W^{2,\frac{6}{5}+}(\Omega)$, which is compactly embedded in $L^6(\Omega)$ by Proposition A.4. Applying Hölder's inequality once more, we obtain that $V^{1/2}R_0(\lambda)V^{1/2}f=V^{1/2}\rho R_0(\lambda)\rho V^{1/2}f$ is also compactly embedded in $L^2(\Omega)$, which finalizes the proof. □

We are prepared to present the proof of Theorem 1.1.

Proof. By Appendix B, the outgoing resolvent formula for $V \geq 0$ is given by

$$R_V(\lambda) = R_0(\lambda) - R_0(\lambda)V^{1/2}(I + V^{1/2}R_0(\lambda)V^{1/2})^{-1}V^{1/2}R_0(\lambda), \quad \Im\lambda > 0. \tag{2.7}$$

Denote the domain

$$\Omega_M := \{\lambda \in \mathbb{C} : \Im\lambda \geq -M \log|\lambda|, \, |\lambda| \geq C_0\}.$$

By Lemma 2.1 and the Neumann series argument, it follows that there exist $M > 0$ and $C_0 > 0$ such that for $\lambda \in \Omega_M$, the operator $I + V^{1/2}R_0(\lambda)V^{1/2} : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ is invertible. Furthermore, it satisfies

$$\|(I + V^{1/2}R_0(\lambda)V^{1/2})^{-1}\|_{L^2(\mathbb{R}^3)\to L^2(\mathbb{R}^3)} \leq 2, \quad \lambda \in \Omega_M.$$

Next we prove that $\rho R_V(\lambda)\rho$ has the mapping property $\rho R_V(\lambda)\rho : L^2(\mathbb{R}^3) \to H^1(\mathbb{R}^3)$, where $\|\rho R_V(\lambda)\rho\|_{L^2\to H^1} = \mathcal{O}(\langle\lambda\rangle e^{T(\mathrm{Im}\lambda)_-})$ with $T = \mathrm{diam}(\mathrm{supp}\rho)$.

Based on the resolvent estimate [12, Theorem 3.1], it follows from Proposition A.4 that $\rho R_0(\lambda)\rho f \in H^1(\Omega) \subset L^6(\Omega)$, where

$$\|\rho R_0(\lambda)\rho\|_{L^2(\mathbb{R}^3)\to H^1(\mathbb{R}^3)} = \mathcal{O}(e^{T(\Im\lambda)_-}).$$

Thus, for $f \in L^2(\mathbb{R}^3)$, Hölder's inequality implies $V^{1/2}\rho R_0(\lambda)\rho f \in L^2(\mathbb{R}^3)$, which gives

$$V^{1/2}(I + V^{1/2}R_0(\lambda)V^{1/2})^{-1}V^{1/2}\rho R_0(\lambda)\rho f \in L^{\frac{6}{5}}(\mathbb{R}^3)$$

and

$$\|V^{1/2}(I + V^{1/2}R_0(\lambda)V^{1/2})^{-1}V^{1/2}\rho R_0(\lambda)\rho\|_{L^2(\mathbb{R}^3)\to L^{6/5}(\mathbb{R}^3)} = \mathcal{O}(e^{T(\Im\lambda)_-}).$$

Using (2.7) and the resolvent estimate in (2.2) yields

$$\rho R_V(\lambda)\rho : L^2(\mathbb{R}^3) \to L^6(\mathbb{R}^3)$$

and

$$\|\rho R_V(\lambda)\rho\|_{L^2(\mathbb{R}^3)\to L^6(\mathbb{R}^3)} = \mathcal{O}(e^{T(\Im\lambda)_-}). \tag{2.8}$$

Choose $\tilde{\rho} \in C_0^\infty(\mathbb{R}^3)$ such that $\tilde{\rho} = 1$ near $\mathrm{supp}\rho$. It follows from the standard elliptic regularity theory [29, (7.13)] that

$$\|\rho R_V(\lambda)\rho f\|_{H^1(\mathbb{R}^3)} \lesssim \|\tilde{\rho}\Delta R_V(\lambda)\rho f\|_{H^{-1}(\mathbb{R}^3)} + \|\tilde{\rho} R_V(\lambda)\rho f\|_{L^2(\mathbb{R}^3)}.$$

Since $\tilde{\rho}\Delta R_V(\lambda)\rho f = -\rho f + V R_V(\lambda)\rho f - \tilde{\rho}\lambda^2 R_V(\lambda)\rho f$ and $V R_V(\lambda)\rho f \in L^{6/5} \subset H^{-1}$ by Proposition A.4, we have

$$\|\tilde{\rho}\Delta R_V(\lambda)\rho f\|_{H^{-1}(\mathbb{R}^3)} \lesssim \langle\lambda\rangle^2 e^{T(\Im\lambda)_-}\|f\|_{L^2(\mathbb{R}^3)}.$$

Moreover, as stated in (2.8), it holds that

$$\|\tilde{\rho} R_V(\lambda)\rho f\|_{L^2(\mathbb{R}^3)} \lesssim e^{T(\Im\lambda)_-}\|f\|_{L^2(\mathbb{R}^3)}$$

for $\lambda \in \Omega_M$. This concludes the proof of the estimate (1.1).

By Lemma 2.2, the operator $V^{1/2}R_0(\lambda)V^{1/2} : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ is compact. Using the analytic Fredholm theory stated in Theorem A.1, we obtain that $\rho R_V(\lambda)\rho : L^2(\mathbb{R}^3) \to H^1(\mathbb{R}^3)$ is meromorphic in the complex plane $\mathbb{C}$, which completes the proof. □

Remark 2.3. *If $V \in L^{n/2}$ where $n \geq 3$ is odd, by replacing the Hardy–Littlewood–Sobolev inequality* (2.2) *with the usual uniform Sobolev inequality, as established in Bourgain–Shao–Sogge–Yao [3] and Kenig–Ruiz–Sogge [19], where $\|R_0(\lambda)\|_{L^{\frac{2n}{n+2}}(\mathbb{R}^n)\to L^{\frac{2n}{n-2}}(\mathbb{R}^n)} = \mathcal{O}(1)$ for $\Im\lambda \geq 0$, it is observed that the operator*

$$\rho R_V(\lambda)\rho : L^2(\mathbb{R}^3) \to H^1(\mathbb{R}^3)$$

is holomorphic in $\tilde{\Omega}_M := \{\lambda : |\lambda| \geq \tilde{C}_0, \Im\lambda > 0\}$ with $\|\rho R_V(\lambda)\rho\|_{L^2\to H^1} = \mathcal{O}(1)$ and is continuous up to the real axis, where $\tilde{C}_0$ is a sufficiently large constant. However, the domain of holomorphy is limited to the upper-half complex plane and cannot be extended to the lower-half complex plane. Consequently, the proof of Theorem 1.1 relies on the Hardy–Littlewood–Sobolev inequality (2.2) in $\mathbb{R}^3$ for all $\lambda \in \mathbb{C}$.

Next, we prove the absence of nonzero real poles for the outgoing resolvent $R_V(\lambda)$. Denote by $L^2(\mathbb{S}^2, \mathrm{d}\mu)$ the L^2 space on the unit sphere $\mathbb{S}^2$ with surface measure $\mathrm{d}\mu$. The following proposition and theorem are crucial in our proof, which can be found [14, Proposition 2.4] and [17] or [14, Theorem 1.2], respectively.

Proposition 2.4. *Let $1 \leq p < 4/3$. For any $\delta < \frac{1}{2} - \frac{2}{p'}$, where $p' = \frac{p}{p-1}$, and for any $f \in L^p(\mathbb{R}^3)$ satisfying $\hat{f} = 0$ on $\mathbb{S}^2$ in the $L^2(\mathbb{S}^2, \mathrm{d}\mu)$ sense, it holds that*

$$\sup_{\varepsilon>0} \|(1+|x|)^{\delta-1/2} R_0(1 \pm \mathrm{i}\varepsilon) f\|_{L^2} \lesssim \|f\|_{L^p}.$$

Theorem 2.5. *Consider $V \in L^{3/2}(\mathbb{R}^3)$, real-valued, and nonnegative. Assume that $u \in H^1_{\mathrm{loc}}(\mathbb{R}^3)$ satisfies $(-\Delta + V)u = \lambda^2 u$, where $\lambda \in \mathbb{R} \setminus \{0\}$ in the distributional sense. Furthermore, if $\|(1+|x|)^{\delta-1/2} u\|_{L^2} < \infty$ for some $\delta > 0$, then $u \equiv 0$.*

We proceed to prove Theorem 1.2, employing the argument presented in [14, Lemma 3.2].

Proof. It suffices to prove that

$$I + V^{1/2} R_0(\lambda) V^{1/2} : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$$

is invertible for every $\lambda \in \mathbb{R} \setminus \{0\}$. By Lemma 2.2, the operator $V^{1/2} R_0(\lambda) V^{1/2} : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ is compact. By employing the Fredholm alternative principle, our objective is to demonstrate that for each $\lambda \in \mathbb{R} \setminus \{0\}$, the kernel of the operator $I + V^{1/2} R_0(\lambda) V^{1/2}$ is trivial.

Assume that there exists a function $f \in L^2(\mathbb{R}^3)$ such that for some $\lambda \in \mathbb{R} \setminus \{0\}$,

$$(I + V^{1/2} R_0(\lambda) V^{1/2}) f = 0. \tag{2.9}$$

Thus, letting $w = V^{1/2} f$, we have

$$w + V R_0(\lambda) w = 0. \tag{2.10}$$

Next, letting $g = R_0(\lambda) w$ and substituting it into (2.10) gives

$$(-\Delta + V - \lambda^2) g = 0. \tag{2.11}$$

Let $R_0(\lambda) = (-\Delta - (\lambda + \mathrm{i}0)^2)^{-1}$. From (2.10), we deduce that $(R_0(\lambda)w, w) = -(R_0(\lambda)w, V R_0(\lambda) w) = -(g, Vg)$, implying $\Im(R_0(\lambda)w, w) = 0$ since V is real-valued. According to the Stein-Tomas theorem (cf. [29]):

$$\Im(R_0(\lambda)w, w) = \lim_{\varepsilon\to 0} \Im(R_0(\lambda + \mathrm{i}\varepsilon)w, w) = c\lambda \int_{\mathbb{S}^2} |\hat{w}(\lambda\xi)|^2 \mathrm{d}s(\xi),$$

where c is some nonzero constant, one has that $\hat{w}(\lambda\xi) = 0$ on $|\lambda|\mathbb{S}^2$. Given $w \in L^{\frac{6}{5}}(\mathbb{R}^3)$ and $\frac{6}{5} < \frac{4}{3}$, we derive from Proposition 2.4 that $(1+|x|)^{\delta-1/2} R_0(\lambda) w \in L^2(\mathbb{R}^3)$ for some $\delta > 0$. Consequently, we have $(1+|x|)^{\delta-1/2} g \in L^2(\mathbb{R}^3)$. As $w \in L^{\frac{6}{5}}(\mathbb{R}^3)$, the Hardy–Littlewood–Sobolev inequality infers $g = R_0(\lambda) w \in L^6(\mathbb{R}^3)$, which, combined with Hölder's inequality, gives $Vg \in L^{\frac{6}{5}}(\mathbb{R}^3)$. Then, it follows from (2.11) and the standard elliptic regularity theory that we obtain $g \in W^{2,6/5}_{\mathrm{loc}}(\mathbb{R}^3)$. By the Sobolev embedding theorem in Proposition A.4, we have $g \in H^1_{\mathrm{loc}}(\mathbb{R}^3)$. Thus, an application of Theorem 2.5 gives $g = 0$. Consequently, we arrive at $f = 0$ by (2.9), which shows that the kernel of the operator $I + V^{1/2}R_0(\lambda)V^{1/2}$ is trivial. Therefore, for each $\lambda \in \mathbb{R} \setminus \{0\}$, the operator $I + V^{1/2}R_0(\lambda)V^{1/2}$ is invertible, thereby completing the proof. $\square$

3. The fractional Schrödinger operator

In this section, we explore the meromorphic continuation of the resolvents for the fractional Schrödinger operators involving bounded and unbounded potentials, respectively.

3.1. **Bounded potentials.** First, we consider the fractional Schrödinger equation without a potential function

$$(-\Delta)^\alpha u(x) - \lambda^{2\alpha} u(x) = f(x), \quad x \in \mathbb{R}^3.$$

Denote the free fractional resolvent by $R_{\alpha,0}(\lambda) = ((-\Delta)^\alpha - \lambda^{2\alpha})^{-1}$. Let $\mathbb{R}^- = (-\infty, 0]$. The branch $\mathbb{C} \setminus \mathrm{i}\mathbb{R}^-$ is chosen such that $z^{2\alpha}$ is holomorphic, i.e., $-\frac{\pi}{2} < \arg z < \frac{3\pi}{2}$.

Theorem 3.1. *For a fixed $\rho \in C_0^\infty(\mathbb{R}^3)$, the resolvent $\rho R_{0,\alpha}(\lambda)\rho : L^2(\Omega) \to H^s(\Omega)$ is holomorphic in S_{θ_0} provided $\frac{1}{2} < \alpha < 1$. Moreover, for $\lambda \in S_{\theta_0}$, it holds that*

$$\|\rho R_{\alpha,0}(\lambda)\rho\|_{L^2(\mathbb{R}^3)\to H^s(\mathbb{R}^3)} \lesssim |\lambda|^{1+s-2\alpha} e^{T(\Im\lambda)_-}, \tag{3.1}$$

where $T = diam(supp\rho)$ and $0 \le s < 2\alpha - 1$.

Proof. We adopt the resolvent formula (cf. [22, (5.28)]) given by

$$((-\Delta)^\alpha - \lambda^{2\alpha})^{-1} = \frac{\lambda^{2-2\alpha}}{\alpha}(-\Delta - \lambda^2)^{-1} + \frac{\sin\alpha\pi}{\pi}\int_0^{+\infty} \frac{\gamma^\alpha(\gamma - \Delta)^{-1}}{\gamma^{2\alpha} - 2\gamma^\alpha\lambda^{2\alpha}\cos\alpha\pi + \lambda^{4\alpha}}\mathrm{d}\gamma, \tag{3.2}$$

which is well-defined and holomorphic for $\lambda \in S_{\theta_0}$. Consequently,

$$\rho R_{\alpha,0}(\lambda)\rho = \frac{\lambda^{2-2\alpha}}{\alpha}\rho(-\Delta - \lambda^2)^{-1}\rho + \frac{\sin\alpha\pi}{\pi}\int_0^{+\infty} \frac{\gamma^\alpha\rho(\gamma - \Delta)^{-1}\rho}{\gamma^{2\alpha} - 2\gamma^\alpha\lambda^{2\alpha}\cos\alpha\pi + \lambda^{4\alpha}}\mathrm{d}\gamma.$$

For any fixed $\rho \in C_0^\infty(\mathbb{R}^3)$, we define $\rho R_{\alpha,0}(\lambda)\rho : L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ as a weak holomorphic family of operators at λ if the function

$$\begin{aligned} I(\lambda) :&= \langle \rho R_{\alpha,0}(\lambda)\rho\, g_1, g_2\rangle_{L^2(\mathbb{R}^3)} \\ &= \frac{\lambda^{2-2\alpha}}{\alpha}\langle \rho(-\Delta - \lambda^2)^{-1}\rho g_1, g_2\rangle_{L^2(\mathbb{R}^3)} \\ &\quad + \frac{\sin\alpha\pi}{\pi}\int_0^{+\infty} \frac{\gamma^\alpha\langle\rho(\gamma - \Delta)^{-1}\rho g_1, g_2\rangle_{L^2(\mathbb{R}^3)}}{\gamma^{2\alpha} - 2\gamma^\alpha\lambda^{2\alpha}\cos\alpha\pi + \lambda^{4\alpha}}\mathrm{d}\gamma \end{aligned}$$

is holomorphic at λ for any given $g_1, g_2 \in L^2(\mathbb{R}^3)$. Furthermore, the weak holomorphicity of $\rho R_{\alpha,0}(\lambda)\rho$ implies its strong holomorphicity. For further insights into the integral of operators, we refer the reader to [33].

In the following discussion, we demonstrate the extension of $\rho R_{0,\alpha}(\lambda)\rho$ to the lower-half complex plane through multiplication with two smooth cutoff functions. It is clear to note that the resolvent $\rho(\gamma - \Delta)^{-1}\rho$ requires a $1/|\gamma|$-type decaying factor for the integral in (3.2) to converge. To achieve this, we recall the resolvent estimate (cf. [12, Theorem 3.1]):

$$\|\rho(-\Delta - \lambda^2)^{-1}\rho\|_{L^2\to H^j} \lesssim \langle\lambda\rangle^{j-1} e^{T(\Im\lambda)_-}, \quad j = 0, 1, 2,$$

and apply the interpolation inequality

$$\|v\|_{H^s(\Omega)} \le C\|v\|_{L^2(\Omega)}^{1-s}\|v\|_{H^1(\Omega)}^s, \quad 0 \le s < 1,$$

which lead us to derive

$$\|\rho(-\Delta-\lambda^2)^{-1}\rho\|_{L^2\to H^s}\lesssim\langle\lambda\rangle^{-(1-s)}e^{T(\Im\lambda)_-},\quad 0\le s\le 1. \tag{3.3}$$

We analyze the denominator $\gamma^{2\alpha}-2\gamma^\alpha\lambda^{2\alpha}\cos\alpha\pi+\lambda^{4\alpha}$. If the branch $\mathbb{C}\setminus\mathrm{i}\mathbb{R}^-$ is chosen such that $z^{2\alpha}$ is holomorphic, a direct calculation yields that for $\lambda\notin\mathrm{i}\mathbb{R}$, the denominator cannot be zero for $\gamma\in[0,\infty)$. To illustrate, consider

$$\gamma^{2\alpha}-2\gamma^\alpha\lambda^{2\alpha}\cos\alpha\pi+\lambda^{4\alpha}=0.$$

By letting $\gamma^\alpha=t$, the above equation reduces to a quadratic form

$$t^2-2t\lambda^{2\alpha}\cos\alpha\pi+\lambda^{4\alpha}=0.$$

Applying the quadratic root formula yields

$$\lambda^{2\alpha}=\gamma^\alpha e^{\pm\mathrm{i}\alpha\pi},$$

resulting in $\lambda=\pm\mathrm{i}\sqrt{\gamma}$. Therefore, for $\lambda\in\mathbb{C}\setminus\mathrm{i}\mathbb{R}$ in the complex plane, the denominator cannot be zero for $\gamma\ge 0$.

A straightforward calculation yields

$$\begin{aligned}
&\Big\|\frac{\sin\alpha\pi}{\pi}\int_0^{+\infty}\frac{\gamma^\alpha\rho(\gamma-\Delta)^{-1}\rho}{\gamma^{2\alpha}-2\gamma^\alpha\lambda^{2\alpha}\cos\alpha\pi+\lambda^{4\alpha}}\mathrm{d}\gamma\Big\|_{L^2\to H^s}\\
&\lesssim\int_0^{+\infty}\frac{|\gamma|^\alpha\|\rho(-\Delta-(\mathrm{i}\sqrt{\gamma})^2)^{-1}\rho\|_{L^2\to H^s}}{|\gamma^{2\alpha}-2\gamma^\alpha\lambda^{2\alpha}\cos\alpha\pi+\lambda^{4\alpha}|}\mathrm{d}\gamma\\
&\lesssim\int_0^{+\infty}\frac{|\gamma|^{\alpha-(1-s)/2}}{|\gamma^{2\alpha}-2\gamma^\alpha\lambda^{2\alpha}\cos\alpha\pi+\lambda^{4\alpha}|}\mathrm{d}\gamma,\quad\lambda\in\mathbb{C}\setminus\mathrm{i}\mathbb{R}.
\end{aligned}\tag{3.4}$$

Letting $\frac{\gamma}{|\lambda|^2}=t$ and using a change of variables, we obtain

$$\begin{aligned}
(3.4)&\lesssim\int_0^\infty\frac{|\lambda|^{2\alpha-(1-s)}|t|^{\alpha-(1-s)/2}}{||\lambda|^{4\alpha}t^{2\alpha}-2|\lambda|^{2\alpha}\lambda^{2\alpha}t^\alpha\cos\alpha\pi+\lambda^{4\alpha}|}|\lambda|^2\mathrm{d}t\\
&\lesssim|\lambda|^{1+s-2\alpha}\int_0^\infty\frac{|t|^{\alpha-(1-s)/2}}{|t^{2\alpha}-2t^\alpha\frac{\lambda^{2\alpha}}{|\lambda|^{2\alpha}}\cos\alpha\pi+\frac{\lambda^{4\alpha}}{|\lambda|^{4\alpha}}|}\mathrm{d}t\\
&\lesssim|\lambda|^{1+s-2\alpha},
\end{aligned}$$

where the last integral is well-defined for $\alpha+(1-s)/2>1$, equivalent to $0\le s<2\alpha-1$. This also implies $\alpha>1/2$. From (3.3), we have

$$\|\frac{\lambda^{2-2\alpha}}{\alpha}\rho(-\Delta-\lambda^2)^{-1}\rho\|_{L^2\to H^s}\lesssim\langle\lambda\rangle^{1-2\alpha+s}e^{T(\Im\lambda)_-}.$$

It remains to demonstrate that there exists a positive constant C independent of $\lambda\in S_{\theta_0}$ such that

$$\int_0^\infty\frac{|t|^{\alpha-(1-s)/2}}{|t^{2\alpha}-2t^\alpha\frac{\lambda^{2\alpha}}{|\lambda|^{2\alpha}}\cos\alpha\pi+\frac{\lambda^{4\alpha}}{|\lambda|^{4\alpha}}|}\mathrm{d}t\le C.$$

First, we choose M to be sufficiently large and independent of λ such that for $t\ge M$,

$$|t^{2\alpha}-2t^\alpha\frac{\lambda^{2\alpha}}{|\lambda|^{2\alpha}}\cos\alpha\pi+\frac{\lambda^{4\alpha}}{|\lambda|^{4\alpha}}|\ge\frac{1}{2}t^{2\alpha},$$

which, given $\alpha + (1-s)/2 > 1$, leads to

$$\int_M^\infty \frac{|t|^{\alpha-(1-s)/2}}{|t^{2\alpha} - 2t^\alpha \frac{\lambda^{2\alpha}}{|\lambda|^{2\alpha}} \cos\alpha\pi + \frac{\lambda^{4\alpha}}{|\lambda|^{4\alpha}}|} \mathrm{d}t \le \int_M^\infty \frac{|t|^{\alpha-(1-s)/2}}{\frac{1}{2}t^{2\alpha}} \mathrm{d}t \le 2\int_M^\infty \frac{1}{t^{\alpha+(1-s)/2}} \le C_1$$

where C_1 is a constant independent of λ.

Moving on to the integral

$$\int_0^M \frac{|t|^{\alpha-(1-s)/2}}{|t^{2\alpha} - 2t^\alpha \frac{\lambda^{2\alpha}}{|\lambda|^{2\alpha}} \cos\alpha\pi + \frac{\lambda^{4\alpha}}{|\lambda|^{4\alpha}}|} \mathrm{d}t,$$

we aim to demonstrate that for $t \in [0, M]$ and $\lambda \in S_{\theta_0}$, there exists a constant $c_0 > 0$ independent of λ such that

$$|t^{2\alpha} - 2t^\alpha \frac{\lambda^{2\alpha}}{|\lambda|^{2\alpha}} \cos\alpha\pi + \frac{\lambda^{4\alpha}}{|\lambda|^{4\alpha}}| \ge c_0.$$

To establish this, let $\frac{\lambda}{|\lambda|} = e^{\mathrm{i}\theta}$, where $-\frac{\pi}{2} < \theta < \frac{3\pi}{2}$. Define

$$\widetilde{S}_{\theta_0} = \{\theta : \theta \in [-\theta_0, \theta_0] \cup [\pi - \theta_0, \pi + \theta_0]\}$$

with $\theta \in \widetilde{S}_{\theta_0}$. The denominator then becomes

$$t^{2\alpha} - 2t^\alpha e^{\mathrm{i}2\alpha\theta} \cos\alpha\pi + e^{\mathrm{i}4\alpha\theta} = (e^{\mathrm{i}2\alpha\theta} - t^\alpha e^{\mathrm{i}\alpha\pi})(e^{\mathrm{i}2\alpha\theta} - t^\alpha e^{-\mathrm{i}\alpha\pi}).$$

Since

$$e^{\mathrm{i}2\alpha\theta} - t^\alpha e^{\mathrm{i}\alpha\pi} = \cos 2\alpha\theta - t^\alpha \cos\alpha\pi + \mathrm{i}(\sin 2\alpha\theta - t^\alpha \sin\alpha\pi),$$

we have

$$|e^{\mathrm{i}2\alpha\theta} - t^\alpha e^{\mathrm{i}\alpha\pi}|^2 = 1 + t^{2\alpha} - 2t^\alpha \cos(2\alpha\theta - \alpha\pi).$$

Similarly,

$$|e^{\mathrm{i}2\alpha\theta} - t^\alpha e^{-\mathrm{i}\alpha\pi}|^2 = 1 + t^{2\alpha} - 2t^\alpha \cos(2\alpha\theta + \alpha\pi).$$

This yields

$$\begin{aligned} &|t^{2\alpha} - 2t^\alpha e^{\mathrm{i}2\alpha\theta} \cos\alpha\pi + e^{\mathrm{i}4\alpha\theta}|^2 \\ &= (1 + t^{2\alpha} - 2t^\alpha \cos(2\alpha\theta + \alpha\pi))(1 + t^{2\alpha} - 2t^\alpha \cos(2\alpha\theta - \alpha\pi)) := F(t, \theta). \end{aligned}$$

According to the previous analysis, the denominator could only be zero when $\lambda \in \mathrm{i}\mathbb{R}$. On the other hand, for $\theta \in \widetilde{S}_{\theta_0}$ we have $\mathrm{dist}(\widetilde{S}_{\theta_0}, \{-\frac{\pi}{2}, \frac{3\pi}{2}\}) > 0$. Consequently, given the continuity of the function $F(t, \theta)$ for $t \in [0, M]$ and $\theta \in \widetilde{S}_{\theta_0}$, there exists $c_0 > 0$ independent of λ such that $F(t, \theta) \ge c_0^2$. Therefore, there exists $C_2 > 0$ independent of λ such that

$$\int_0^M \frac{|t|^{\alpha-(1-s)/2}}{|t^{2\alpha} - 2t^\alpha \frac{\lambda^{2\alpha}}{|\lambda|^{2\alpha}} \cos\alpha\pi + \frac{\lambda^{4\alpha}}{|\lambda|^{4\alpha}}|} \mathrm{d}t \le C_2.$$

Thus, the proof is completed. ☐

Remark 3.2. *The necessity of $u \in H^s(\Omega)$ with $0 \le s < 2\alpha - 1$ becomes evident for ensuring convergence of the integral of operators in* (3.2).

The following theorem deals with the meromorphic continuation of the fractional Schrödinger operator. The proof aligns with the argument presented in [12, Theorem 3.8].

Lemma 3.3. *Assume that* $V \in L^\infty_{\mathrm{comp}}(\mathbb{R}^3)$. *For a fixed* $\rho \in C_0^\infty(\mathbb{R}^3)$ *with* $\rho = 1$ *on* $\mathrm{supp} V$, *the resolvent*

$$\rho R_{\alpha,V}(\lambda)\rho : L^2(\Omega) \to L^2(\Omega)$$

is a meromorphic family of operators with respect to $\lambda \in S_{\theta_0}$.

Proof. A direct calculation gives

$$I + VR_{\alpha,0}(\lambda) = (I + VR_{\alpha,0}(\lambda)(1-\rho))(I + VR_{\alpha,0}(\lambda)\rho).$$

Fix $\rho \in C_0^\infty(\mathbb{R}^3)$ such that $\rho = 1$ on $\mathrm{supp}V$. For $\Im\lambda \gg 1$ and $\lambda \in S_{\theta_0}$, utilizing the resolvent estimate (3.1), we find

$$\|VR_{\alpha,0}(\lambda)\rho\|_{L^2(\Omega)\to L^2(\Omega)} \lesssim |\lambda|^{1-2\alpha}\|V\|_{L^\infty(\mathbb{R}^3)} < \frac{1}{2},$$

which implies that the operator $I + VR_{\alpha,0}(\lambda)\rho$ is invertible through a Neumann series argument. Moreover, since the operator $VR_{\alpha,0}(\lambda)\rho$ is compact on $L^2(\mathbb{R}^3)$ due to the resolvent estimate (3.1) (as $H^s(\Omega)$ with $0 < s < 1$ is compactly embedded in $L^2(\Omega)$ [26, Theorem 7.1]), we have from Theorem A.1 that the operator $(I + VR_{\alpha,0}(\lambda)\rho)^{-1} : L^2(\Omega) \to L^2(\Omega)$ is meromorphic for $\lambda \in S_{\theta_0}$. On the other hand, it is easy to verify that

$$(I + VR_{\alpha,0}(\lambda)(1-\rho))^{-1} = I - VR_{\alpha,0}(\lambda)(1-\rho).$$

Hence, the operator $I + VR_{\alpha,0}(\lambda) : L^2(\Omega) \to L^2(\Omega)$ is invertible for $\Im\lambda \gg 1$ and its inverse is given by

$$(I + VR_{\alpha,0}(\lambda))^{-1} = (I + VR_{\alpha,0}(\lambda)\rho)^{-1}(I - VR_{\alpha,0}(\lambda)(1-\rho)).$$

Furthermore, $(I + VR_{\alpha,0}(\lambda))^{-1}$ is meromorphic for $\lambda \in S_{\theta_0}$.

By the resolvent identity

$$R_{\alpha,V}(\lambda) = R_{\alpha,0}(\lambda)(I + VR_{\alpha,0}(\lambda))^{-1},$$

it can be verified that

$$\rho R_{\alpha,V}(\lambda)\rho = \rho R_{\alpha,0}(\lambda)(I + VR_{\alpha,0}(\lambda)\rho)^{-1}(I - VR_{\alpha,0}(\lambda)(1-\rho))\rho. \tag{3.5}$$

Considering

$$(I - VR_{\alpha,0}(\lambda)(1-\rho))\rho : L^2(\Omega) \to L^2(\Omega)$$

and

$$(I + VR_{\alpha,0}(\lambda)\rho)^{-1} : L^2(\Omega) \to L^2(\Omega),$$

with $\eta, \tilde{\eta} \in C_0^\infty(\mathbb{R}^3)$ such that $\eta\rho = \rho, \tilde{\eta}\eta = \eta$, we have

$$(1-\tilde{\eta})(I + VR_{\alpha,0}(\lambda)\rho)^{-1}\eta = 0, \quad \Im\lambda \gg 1.$$

Therefore, by analytic continuation, the above identity also holds for all $\lambda \in \mathbb{C}$ at which $(I + VR_{\alpha,0}(\lambda)\rho)^{-1}$ is analytic.

Finally, since $\rho R_{\alpha,0}(\lambda)\rho$ is analytic for $\lambda \in S_{\theta_0}$, by (3.5), we conclude that $\rho R_{\alpha,V}(\lambda)\rho : L^2(\Omega) \to L^2(\Omega)$ is meromorphic for $\lambda \in S_{\theta_0}$, thereby completing the proof. □

Now, we present the proof of Theorem 1.3.

Proof. The fact that $\rho R_{\alpha,V}(\lambda)\rho$ is meromorphic in S_{θ_0} follows directly from Lemma 3.3. Consider $\rho \in C_0^\infty(\mathbb{R}^3)$ such that $\rho = 1$ on supp$V$. For $\lambda \in \Omega_M$ and $|\lambda|$ large enough, we obtain

$$\begin{aligned}\|VR_{\alpha,0}(\lambda)\rho\|_{L^2(\mathbb{R}^3)\to L^2(\mathbb{R}^3)} &\lesssim \|V\|_{L^\infty(\mathbb{R}^3)}|\lambda|^{1-2\alpha}e^{T(\Im\lambda)_-}\\ &\lesssim e^{T(M\log(|\lambda|))}/|\lambda|^{2\alpha-1}\\ &\leq \frac{1}{2}.\end{aligned}$$

Combining (3.5) and the resolvent estimate (3.1) of $R_{\alpha,0}(\lambda)$ leads to (1.4). □

3.2. **Unbounded potentials.** In this section, we analyze the meromorphic continuation of the fractional resolvent operator denoted as $\rho R_{\alpha,V}(\lambda)\rho : L^2 \to L^2$, specifically concerning unbounded potentials. Referring to (1.3), we have the resolvent identity

$$R_{\alpha,V}(\lambda) = R_{\alpha,0}(\lambda) - R_{\alpha,0}(\lambda)V^{1/2}(I + V^{1/2}R_{\alpha,0}(\lambda)V^{1/2})^{-1}V^{1/2}R_{\alpha,0}(\lambda). \tag{3.6}$$

The following lemma provides an L^p-based mapping property of the free resolvent $\rho R_{\alpha,0}(\lambda)\rho$, complementing the one established in Theorem 3.1, which deals with L^2.

Lemma 3.4. *Fix a cutoff function $\rho \in C_0^\infty(\mathbb{R}^3)$ such that $\rho = 1$ on supp$V$. Assume that $0 < \beta < \frac{2\alpha-1}{3}$ and $0 < \beta_1 < \beta$. Then the operator*

$$\rho R_{\alpha,0}(\lambda)\rho : L^{\frac{6}{3+2\beta_1}}(\Omega) \to W^{\beta,2+}(\Omega)$$

is holomorphic in S_{θ_0}. Moreover, the following resolvent estimate holds for $\lambda \in S_{\theta_0}$:

$$\|\rho R_{\alpha,0}(\lambda)\rho\|_{L^{\frac{6}{3+2\beta_1}}(\Omega)\to W^{\beta,2+}(\Omega)} \lesssim |\lambda|^{-(2\alpha-3\beta-1)}e^{T(\Im\lambda)_-}. \tag{3.7}$$

Consequently, the operator $\rho R_{\alpha,0}(\lambda)\rho : L^{\frac{6}{3+\beta_1}}(\Omega) \to L^{\frac{6}{3-2\beta}}(\Omega)$ is compact.

Proof. Using the resolvent estimate (cf. [12, Theorem 3.1]) for the classical Laplacian operator $\|\rho(-\Delta-\lambda^2)^{-1}\rho\|_{L^2\to L^2} = \mathcal{O}(e^{T(\Im\lambda)_-}/|\lambda|)$ and the Hardy–Littlewood–Sobolev inequality (2.2), and noting for $0 \leq t \leq 1$

$$\frac{5}{6}t + \frac{1}{2}(1-t) = \frac{3+2t}{6}, \quad \frac{1-t}{2} + \frac{t}{6} = \frac{3-2t}{6},$$

we have from the Riesz–Thorin interpolation theorem (cf. Theorem A.3) that

$$\begin{aligned}&\|\rho(-\Delta-\lambda^2)^{-1}\rho\|_{L^{\frac{6}{3+2t}}\to L^{\frac{6}{3-2t}}}\\ &\quad\lesssim \|\rho(-\Delta-\lambda^2)^{-1}\rho\|_{L^2\to L^2}^{1-t}\|\rho(-\Delta-\lambda^2)^{-1}\rho\|_{L^{6/5}\to L^6}^{t}\\ &\quad\lesssim |\lambda|^{-(1-t)}e^{T(\Im\lambda)_-}, \quad 0 \leq t \leq 1.\end{aligned}$$

Subsequently, by setting $t = \beta_1$, it follows that

$$\|\rho(-\Delta-\lambda^2)^{-1}\rho\|_{L^{\frac{6}{3+2\beta_1}}\to L^{\frac{6}{3+2\beta_1}}} = \mathcal{O}(e^{T(\Im\lambda)_-}/|\lambda|^{1-\beta_1}).$$

Reiterating the proof for (2.6) yields

$$\|\rho(-\Delta-\lambda^2)^{-1}\rho\|_{L^{\frac{6}{3+2\beta_1}}\to W^{2,\frac{6}{3+2\beta_1}}} = \mathcal{O}(|\lambda|^{1+\beta_1}e^{T(\Im\lambda)_-}),$$

which implies after using the interpolation that

$$\|\rho(-\Delta-\lambda^2)^{-1}\rho\|_{L^{\frac{6}{3+2\beta_1}}\to W^{1,\frac{6}{3+2\beta_1}}} = \mathcal{O}(|\lambda|^{\beta_1}e^{T(\Im\lambda)_-}).$$

Using the interpolation inequality

$$\|v\|_{W^{s,q}} \leq C\|v\|_{L^q}^{1-s}\|v\|_{W^{1,q}}^{s}, \quad 0 \leq s \leq 1$$

and letting $s = 2\beta$, $q = \frac{6}{3+2\beta_1}$, we obtain

$$\|\rho(-\Delta-\lambda^2)^{-1}\rho\|_{L^{\frac{6}{3+2\beta_1}}\to W^{2\beta,\frac{6}{3+2\beta_1}}} = \mathcal{O}(e^{T(\Im\lambda)_-}/|\lambda|^{1-2\beta-\beta_1}).$$

At this point, a decay factor of $\frac{1}{|\lambda|}$ emerges when $1-2\beta-\beta_1>0$. This condition holds true for $\beta<\frac{1}{3}$.

Given that $\beta_1<\beta$, we derive from the Sobolev embedding in Proposition A.4 that $W^{2\beta,\frac{6}{3+2\beta_1}}(\Omega)\subset W^{\beta,2+}(\Omega)$, resulting in

$$\|\rho(-\Delta-\lambda^2)^{-1}\rho\|_{L^{\frac{6}{3+2\beta_1}}\to W^{\beta,2+}} = \mathcal{O}(e^{T(\Im\lambda)_-}/|\lambda|^{1-2\beta-\beta_1}),\quad 0<\beta<\frac{1}{3}.$$

Furthermore, by following the argument from the proof of Theorem 3.1, we obtain that the integral in the resolvent formula (3.2) converges provided $\beta<\frac{2\alpha-1}{3}<1/3$, which further gives (3.7). Additionally, the compactness follows from the fact that $W^{\beta,2+}(\Omega)$ is compactly embedded in $L^{\frac{6}{3-2\beta}}(\Omega)$ according to Proposition A.4. □

The following result is a direct consequence of the above lemma.

Lemma 3.5. *Assume that $V\in L^p_{\mathrm{comp}}(\mathbb{R}^3)$ where $p>\frac{3}{2\beta}$ with $0<\beta<\frac{2\alpha-1}{3}$. Then the operator $V^{1/2}R_{\alpha,0}(\lambda)V^{1/2}:L^2\to L^2$ is compact for $\lambda\in S_{\theta_0}$ and satisfies*

$$\|V^{1/2}R_{\alpha,0}(\lambda)V^{1/2}\|_{L^2\to L^2}\lesssim |\lambda|^{-(2\alpha-3\beta-1)}e^{T(\Im\lambda)_-},\quad \lambda\in S_{\theta_0}. \tag{3.8}$$

Proof. For a given $f\in L^2(\mathbb{R}^3)$, considering $V^{1/2}\in L^{\frac{3}{\beta}+}_{\mathrm{comp}}(\mathbb{R}^3)$, the application of the Hölder inequality yields $V^{1/2}f\in L^{\frac{6}{3+2\beta_1}}$ for $\beta_1<\beta$. Furthermore, with the selection of a cutoff function $\rho\in C_0^\infty(\mathbb{R}^3)$ satisfying $\rho=1$ on supp$V$, Lemma 3.4 implies that the operator $\rho R_{\alpha,0}(\lambda)\rho : L^{\frac{6}{3+2\beta_1}}(\Omega)\to L^{\frac{6}{3-2\beta}}(\Omega)$ is compact. Its norm satisfies (3.7), leading to (3.8) and establishing the compactness of the operator $V^{1/2}R_{\alpha,0}(\lambda)V^{1/2}:L^2\to L^2$ through Hölder's inequality. □

The following lemma addresses the meromorphic continuation of the fractional Schrödinger operator.

Lemma 3.6. *Assuming $V\in L^p_{\mathrm{comp}}(\mathbb{R}^3)$, where $p>\frac{3}{2\beta}$ with $0<\beta<\frac{2\alpha-1}{3}$, and considering a fixed $\rho\in C_0^\infty(\mathbb{R}^3)$ such that $\rho=1$ on* supp$V$*, the resolvent opertor*

$$\rho R_{\alpha,V}(\lambda)\rho : L^2(\Omega)\to L^2(\Omega)$$

constitutes a meromorphic family of operators with respect to $\lambda\in S_{\theta_0}$.

Proof. Given $f\in L^2(\Omega)$, we have from Theorem 3.1 that $R_{\alpha,0}(\lambda)f\in H^\beta(\Omega)\subset L^{\frac{6}{3-2\beta}}(\Omega)$ by Proposition A.4. Noting $V^{1/2}\in L^{3/\beta}(\Omega)$ and using Hölder's inequality, we deduce $V^{1/2}R_{\alpha,0}(\lambda)f\in L^2(\Omega)$. Moreover, for sufficiently large $|\lambda|$ with $\lambda\in\{\lambda:\Im\lambda>0,\lambda\in S_{\theta_0}\}$, we obtain from Lemma 3.5 that

$$\|V^{1/2}R_{\alpha,0}(\lambda)V^{1/2}\|_{L^2\to L^2}\le\frac{1}{2},$$

which indicates that the operator $I+V^{1/2}R_{\alpha,0}(\lambda)V^{1/2}:L^2(\Omega)\to L^2(\Omega)$ is invertible via the Neumann series argument. Considering the compactness of the operator $V^{1/2}R_{\alpha,0}(\lambda)V^{1/2}$: $L^2(\Omega)\to L^2(\Omega)$ from Lemma 3.5, we have from the analytic Fredholm theorem that $(I+V^{1/2}R_{\alpha,0}(\lambda)V^{1/2})^{-1}:L^2\to L^2$ is meromorphic. The proof is completed by noting the holomorphic nature of $R_{\alpha,0}(\lambda)$ in S_{θ_0}, $V^{1/2}R_{\alpha,0}(\lambda)f\in L^2(\Omega)$, and the resolvent identity (3.6). □

Now, we present the proof of Theorem 1.4.

Proof. The fact that $\rho R_{\alpha,V}(\lambda)\rho$ is meromorphic in S_{θ_0} comes from Lemma 3.6. For $\lambda \in \Omega_M \cap S_{\theta_0}$ and a sufficiently large constant C_0, we have

$$\begin{aligned}
\|V^{1/2}R_{\alpha,0}(\lambda)V^{1/2}\|_{L^2(\mathbb{R}^3)\to L^2(\mathbb{R}^3)} &\|V\|_{L^{\frac{3}{2\alpha}}}\|\rho R_{\alpha,0}(\lambda)\rho\|_{L^{\frac{6}{3+2\alpha}}\to L^{\frac{6}{3-2\alpha}}}\|V\|_{L^{\frac{3}{2\alpha}}} \\
&\lesssim |\lambda|^{-(2\alpha-3\beta-1)}e^{T(\Im\lambda)_-} \\
&\lesssim e^{T(M\log(|\lambda|))}/|\lambda|^{2\alpha-3\beta-1} \\
&\le \frac{1}{2},
\end{aligned}$$

which implies that $I + V^{1/2}R_{\alpha,0}(\lambda)V^{1/2}$ is invertible by the Neumann series argument.

From (3.1), it follows that

$$\|\rho R_{\alpha,0}(\lambda)\rho\|_{L^2(\Omega)\to H^\beta(\Omega)} \lesssim |\lambda|^{1+\beta-2\alpha}e^{T(\Im\lambda)_-}.$$

Using the resolvent identity (3.6) and the resolvent estimate (3.7) of $\rho R_{\alpha,0}(\lambda)\rho$, we derive (1.5). $\square$

Appendix A. Useful Lemmas

The theorem presented below concerns the analytic Fredholm theory, detailed in [12, Theorem C.8].

Theorem A.1. *Assume that $\Omega \subset \mathbb{C}$ is a connected open set and $\{A(z)\}_{z\in\Omega}$ represents a holomorphic family of Fredholm operators. If $A(z_0)^{-1}$ exists at a certain point $z_0 \in \Omega$, then the family $z \to A(z)^{-1}, z \in \Omega$, constitutes a meromorphic family of operators with poles of finite rank.*

The following theorem, known as the generalized Young's inequality or Schur's test, is referenced in [2, Corollary 1.3].

Theorem A.2. *Suppose that $K(x,y)$ is measurable on $\mathbb{R}^n \times \mathbb{R}^n$ and*

$$\sup_x \Big(\int |K(x,y)|^r \mathrm{d}y\Big)^{\frac{1}{r}}, \quad \sup_y \Big(\int |K(x,y)|^r \mathrm{d}x\Big)^{\frac{1}{r}} \le C$$

for some $1 \le r \le \infty$. Define

$$Tf(x) = \int K(x,y)\mathrm{d}y.$$

If $1 \le p \le q \le \infty$ satisfy $\frac{1}{r} = 1 - (\frac{1}{p} - \frac{1}{q})$, then the following estimate holds:

$$\|Tf\|_{L^q} \le C\|f\|_{L^p}.$$

The following result is referred to as the Riesz–Thorin interpolation theorem, as documented in [2, Theorem 1.1].

Theorem A.3. *Let T be a linear map from from $L^{p_0} \cap L^{p_1}$ to $L^{q_0} \cap L^{q_1}$ such that*

$$\|Tf\|_{L^{q_j}} \le M_j\|f\|_{L^{p_j}} \quad j = 0, 1$$

with $1 \le p_j, q_j \le \infty$. If

$$\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1}, \quad \frac{1}{q_t} = \frac{1-t}{q_0} + \frac{t}{q_1}$$

for some $t \in (0,1)$, then

$$\|Tf\|_{L^{q_t}} \le M_0^{1-t}M_1^t\|f\|_{L^{p_t}}.$$

The following proposition concerns results regarding compact embedding and Sobolev embedding theorems, as detailed in [13, Chapter 7].

Proposition A.4. *Let Ω be a bounded convex domain with a smooth boundary in $\mathbb{R}^n$. The following compact embedding holds*

$$W^{k,p}(\Omega) \subset\subset L^q(\Omega) \quad \text{for } kp < n, q < \frac{np}{n-kp},$$
$$W^{k,p}(\Omega) \subset L^{q^*}(\Omega) \quad \text{for } kp < n, q^* = \frac{np}{n-kp}.$$

Appendix B. Self-adjointness and Resolvent identity

In this section, we prove the self-adjointness of the Schrödinger operators $-\Delta + V$ and $(-\Delta)^\alpha + V$ with the unbounded potentials, and the resolvent identities (1.3) and (3.6). We show that $\mathcal{C}$ contains all $V \in L^2_{\mathrm{comp}}(\mathbb{R}^3)$ and $\mathcal{F}$ contains all $V \in L^{3/s}_{\mathrm{comp}}(\mathbb{R}^3)$, with $0 \leq s < 2\alpha - 1$ as specified in Theorem 3.1.

First, for $V \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$, $V \geq 0$, we demonstrate the self-adjointness of the the Schrödinger operator $-\Delta + V$ utilizing the quadratic form. The operator $-\Delta$ is self-adjoint on $H^2(\mathbb{R}^3)$ associated with the quadratic form $q_{-\Delta}(f) = \int_{\mathbb{R}^3} |\nabla f|^2 \mathrm{d}x$. The multiplier operator $V : f \to V(x)f$ is self-adjoint on the set $\{f \in L^2(\mathbb{R}^3) : Vf \in L^2(\mathbb{R}^3)\}$ associated with the quadratic form $q_V(f) = \int_{\mathbb{R}^3} V|f|^2 \mathrm{d}x$. Since both $q_{-\Delta}$ and q_V are closed, thus by [30, Theorem 7.5.11], their sum $q_{-\Delta} + q_V$ is also closed. Consequently, following [30, Proposition 7.5.6], the Schrödinger operator $H := -\Delta + V$ can be defined as a self-adjoint operator associated with the quadratic form $q_{-\Delta} + q_V$. For the domain of H obtained via the quadratic form, we refer to [30, Proposition 7.5.6], and for further details, we direct attention to Example 7.5.12 in [30].

Secondly, we prove the resolvent identity (1.3) provided that $I + V^{1/2}R_0(\lambda)V^{1/2}, \Im\lambda > 0$ is invertible. The invertibility of the operator $I + V^{1/2}R_0(\lambda)V^{1/2}$ is validated by referencing Lemma 2.1 and employing the Neumann series argument.

For two nonnegative self-adjoint operators A and B, let $Q(A)$ and $Q(B)$ be the form domains and q_A and q_B be the associated quadratic forms. Denote the resolvent $R_P(z) = (P-z)^{-1}$, where P is an operator. The following lemma (cf. [33, Lemma 6.30]) is useful in the subsequent analysis.

Lemma B.1. *Assuming $A - \gamma \geq 0$, where γ is a constant and B is self-adjoint. If $Q(A) \subset Q(B)$ and $q_A + q_B$ is a closed semi-bounded form, then*

$$R_{A+B}(z) = R_A(z) - (|B|^{1/2}R_{A+B}(z^*))^* sign(B)|B|^{1/2}R_A(z)$$

for $z \in \rho(A) \cap \rho(A+B)$. Here, $A + B$ represents the self-adjoint operator associated with $q_A + q_B$.

In our scenario, let $z = \lambda^2$, $A = -\Delta$, and $B = T_V$, where $T_V : f \to V(x)f$ is the multiplier operator. Thus, $-\Delta + V = A + B$. We have $Q(A) = H^1(\mathbb{R}^3)$ and

$$Q(B) = \{f \in L^2(\mathbb{R}^3) : V^{1/2}f \in L^2(\mathbb{R}^3)\}.$$

For specific details, we refer to page 77 of [33]. We confirm that $Q(A) \subset Q(B)$. To elaborate, considering $f \in H^1(\mathbb{R}^3) \subset L^6_{\mathrm{loc}}(\mathbb{R}^3)$ implies that $|f|^2 \in L^3_{\mathrm{loc}}(\mathbb{R}^3)$. Since $V \in L^p_{\mathrm{comp}}(\mathbb{R}^3)$ with $p > 3/2$, by Hölder's inequality, we deduce that $V^{1/2}|f| \in L^2(\mathbb{R}^3)$.

Let $R_V(\lambda) = (-\Delta + V - \lambda^2)^{-1}$ and $R_0(\lambda) = (-\Delta - \lambda^2)^{-1}$, where $\Im\lambda > 0$. Considering the range of $V^{1/2}R_0(\lambda)$ contained in $Q(B)$ for $V \in L^2_{\mathrm{comp}}(\mathbb{R}^3)$, we employ Lemma B.1 to derive

$$R_V(\lambda) = R_0(\lambda) - R_V(\lambda)V^{1/2}V^{1/2}R_0(\lambda). \tag{B.1}$$

Multiplying both sides of the above identity by $V^{1/2}$, we obtain

$$R_V(\lambda)V^{1/2} = R_0(\lambda)V^{1/2} - R_V(\lambda)V^{1/2}V^{1/2}R_0(\lambda)V^{1/2}.$$

As $I + V^{1/2}R_0(\lambda)V^{1/2}$ is invertible, we get

$$R_V(\lambda)V^{1/2} = R_0(\lambda)V^{1/2}(I + V^{1/2}R_0(\lambda)V^{1/2})^{-1}. \tag{B.2}$$

Substituting (B.2) into the second term on the right-hand side of (B.1) yields the desired identity.

In a similar manner, we investigate the fractional Schrödinger operator. First, in this case, we have $A = (-\Delta)^\alpha$ and $B = T_V$, where $T_V : f \to V(x)f$ is the multiplier operator. The self-adjointness of A is evident on $H^\alpha(\mathbb{R}^3)$, while V is self-adjoint on the set $\{f \in L^2(\mathbb{R}^3) : Vf \in L^2(\mathbb{R}^3)\}$. The quadratic forms associated with A and B are denoted as $q_A(f) = \int_{\mathbb{R}^3} |(-\Delta)^{\alpha/2}f|^2 \mathrm{d}x$ and $q_B(f) = \int_{\mathbb{R}^3} V|f|^2\mathrm{d}x$, respectively. Hence, the Schrödinger operator $(-\Delta)^\alpha + V$ is self-adjoint associated with $q_A + q_B$.

Second, we prove the resolvent identity (3.6) for the fractional Schrödinger operator under the condition that the operator

$$I + V^{1/2}R_{\alpha,0}(\lambda)V^{1/2}, \quad \Im\lambda > 0,$$

is invertible. For the invertibility of the above operator, we refer to Lemma 3.5 and the Neumann series argument. Notably, we have $Q(A) = H^\alpha(\mathbb{R}^3)$ and

$$Q(B) = \{f \in L^2(\mathbb{R}^3) : V^{1/2}f \in L^2(\mathbb{R}^3)\}.$$

Furthermore, $Q(A) \subset Q(B)$ due to $V^{1/2}|f| \in L^2(\mathbb{R}^3)$ for $f \in H^\alpha(\mathbb{R}^3)$ and $V \in L^p_{\mathrm{loc}}(\mathbb{R}^3)$ with $p > \frac{3}{2\beta}$ and $0 < \beta < \frac{2\alpha-1}{3}$. Additionally, it is clear to note that the range of $V^{1/2}R_{\alpha,0}(\lambda)$ is contained in $Q(B)$ for $V \in L^{3/s}_{\mathrm{comp}}(\mathbb{R}^3)$, where $0 \le s < 2\alpha - 1$ is specified in Theorem 3.1. Subsequently, the remainder of the proof aligns with the preceding arguments by using Lemma B.1.

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