Electromagnetic field enhancement in a subwavelength rectangular open cavity

Yixian Gao[1] · Peijun Li[2] · Xiaokai Yuan[3]

Abstract

Consider the transverse magnetic polarization of the electromagnetic scattering of a plane wave by a perfectly conducting plane surface, which contains a two-dimensional subwavelength rectangular cavity. The enhancement is investigated fully for the electric and magnetic fields arising in such an interaction. The cavity wall is assumed to be a perfect electric conductor, while the cavity bottom is allowed to be either a perfect electric conductor or a perfect magnetic conductor. We show that the significant field enhancement may be achieved in both nonresonant and resonant regimes. The proofs are based on variational approaches, layer potential techniques, boundary integral equations, and asymptotic analysis. Numerical experiments are also presented to confirm the theoretical findings.

Keywords Cavity scattering problem · Electromagnetic field enhancement · Scattering resonances · Helmholtz equation · Variational formulation · Boundary integral equation · Asymptotic analysis

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1 Introduction

The electromagnetic scattering by open cavities has significant applications in many scientific areas. For instance, the radar cross section (RCS) is a quantity which measures the detectability of a target by a radar system. Clearly, it is of high importance in military and civil use for a deliberate control in the form of enhancement or reduction of the RCS of a target. Since the cavity RCS can dominate the total RCS, it is critical to have a thorough understanding of the electromagnetic scattering by a target, particularly a cavity, for successful implementation of any desired control of its RCS. The underlying scattering problems have received much attention by many researchers in both the engineering and mathematical communities. There has been a rapid development of the mathematical theories and computational methods in this area [3,8,9]. We refer to [27] for a survey of open cavity scattering problems.

Recently, the electromagnetic wave scattering problems for optical device surfaces containing subwavelength cavities have created a lot of interest in electromagnetic field enhancement and extraordinary optical transmission effects, due to the fact that the light can be localized and greatly enhanced near subwavelength cavities or apertures. Such features have important potential applications in diverse scientific areas, such as biological and chemical sensing, microscopy, spectroscopy and communication [6,10,13,18,19,25,33–35,37]. However, there are many mathematical and computational issues for the underlying problems. The complication arises from the multiscale nature of the structures and various enhancement behaviors that they induce. For instance, the enhancement can be attributed to surface plasmonic resonance [18,19,29], non-plasmonic resonances [35,37], or even without the resonant effect [28].

It is very helpful to provide a rigorous mathematical analysis for these phenomena, which turns out to be extremely challenging. The particular light patterns observed in the far-field or in the near-field may be the results of complex multi-interactions of surface waves, cavity resonances, resonant tunneling of plasmon waves, skin depth effects, etc. It requires the resolution of the full three-dimensional Maxwell equations in non-smooth geometries even for a quantitative description of the diffractive properties of electromagnetic waves in subwavelength structures. For most resonance phenomena, the localization or enhancement of light is very sensitive to geometrical parameters and the frequency. We may begin with simple geometries, where the analytical approaches are made possible to obtain an accurate description of the electromagnetic field. In [7,11], the field enhancement was considered for a single rectangle cavity and double rectangle cavities. Using asymptotic expansions of Green's function, they showed that the limiting Green function is a perfectly conducting plane with a dipole in place of the cavity when the width of the cavity shrinks to zero. In [14,22,23], an asymptotic expansion method was proposed to study the solution of the Helmholtz equation in a domain including a single thin slot. A quantitative analysis was made for the field enhancement of the Helmholtz equation for a single narrow open slit [28,30], or periodic slits [12,31,32]. We refer [4,5] for the study of closely related Helmholtz resonators and resonances in bubbly media.

Motivated by [28,30], we consider the electromagnetic scattering of a plane wave by a perfectly electrically conducting plane surface, which contains a subwavelength rectangular cavity. More specifically, the cavity is assumed to be invariant in the x_3-axis and only the transverse magnetic polarization (TM) is studied. Hence the three-dimensional Maxwell equations can be reduced into the two-dimensional Helmholtz equation. The cavity wall is assumed to be a perfect electric conductor (PEC), and the cavity bottom is assumed to be either a perfect magnetic conductor (PMC) or a perfect electric conductor (PEC). A cavity

Table 1 Electromagnetic field enhancement. ϵ: width of the cavity, d: depth of the cavity, λ: wavelength of the incident field

enhancement	PEC-PMC cavity		PEC-PEC cavity	
	Nonresonant	Resonant	Nonresonant	Resonant
Electric field	$O(1/\sin\kappa d)$	$O(1/\epsilon)$	$O(\lambda\sqrt{\epsilon}/d)$	$O(1/\epsilon)$
Magnetic field	no	$O(1/\epsilon)$	no	$O(1/\epsilon)$

is called the PEC-PMC type when it has a PEC wall and a PMC bottom, while a cavity is called the PEC-PEC type when it has a PEC wall and a PEC bottom. We investigate the electromagnetic field enhancement for both types of cavities. Using a combination of variational approaches, layer potential techniques, boundary integral equations, and asymptotic analysis, we demonstrate the field enhancement mechanisms for the underlying scattering problems different boundary conditions. Denote by λ, ϵ, d the wavelength of the incident field, the width of the cavity, and the depth of the cavity, respectively. For the PEC-PMC cavity, we prove in the nonresonant regime that the electric field enhancement has an order $O(1/\sin(\kappa d))$ and the magnetic field has no enhancement in the cavity provided that the length scales satisfy $\epsilon \ll d \ll \lambda$. Moreover, if $\epsilon \ll \lambda$, we show that the Fabry–Perot type resonance occurs. In the resonant regime, we derive an asymptotic expansion for the resonant frequencies and deduce an order $O(1/\epsilon)$ enhancement for both the electric and magnetic fields. For the PEC-PEC cavity, we carry the same analysis in both the nonresonant and resonant regimes. The electric field is shown to have a weak enhancement of order $O(\lambda\sqrt{\epsilon}/d)$ in the nonresonant regime with length scales satisfying $\epsilon \ll d \ll \lambda$. In the resonant regime, a similar quantity analysis is done for the resonant frequencies and the same order $O(1/\epsilon)$ is achieved for the electric and magnetic field enhancement. These field enhancement results are summarized in Table 1. Numerical experiments are presented and used to confirm our theoretical findings.

This work is closely related to [28,30], where a transmission problem was considered for small gaps. We take the similar approaches and some of the results are parallel. But the model problem is different and thus the Green's functions take different forms, which lead to the difference of the details for the analysis. In this work, we give a comprehensive account of the analysis for the enhancement of the fields and the resonant frequencies in both the PEC-PMC and PEC-PEC cavities. Specifically, below is a brief summary of the similarity and difference between our results and those presented in [11,28,30]. In the non-resonant region, for the PEC-PMC cavity problem, we obtain the same order of enhancement for the electric field as that of the PEC gap in [28]; for the PEC-PEC cavity problem, we achieve a weak enhancement for the electric field which was not obtained in [11,28]. In the resonant region, for both the PEC-PMC and PEC-PEC cavity problems, due to influence of the bottom boundary condition, we obtain the same enhancement order as that in [30], but the resonance sets are different.

The rest of the paper is organized as follows. The PEC-PMC cavity is examined in Sects. 2–3 and the PEC-PEC cavity is studied in Sects. 4–5. Sections 2 and 4 focus on the field enhancement for the non-resonant case. In each section, the model problems are introduced, the approximate problems are proposed, the error estimates are derived between the solutions of the exact and approximate problems, and the electric and magnetic field enhancement are discussed. Sections 3 and 5 address the field enhancement for the resonant case. In each section, the boundary integral equations are given, the asymptotic analysis is carried for the Fabry–Perot resonances, and the electric and magnetic field enhancement are presented under
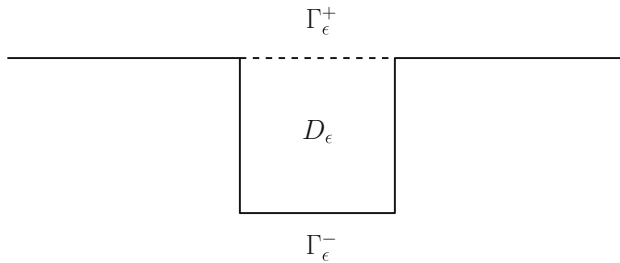
Fig. 1 The problem geometry of a rectangular open cavity

the resonant frequencies. The paper is concluded with some general remarks and directions for future work in Sect. 6.

2 PEC-PMC cavity without resonance

In this section, we discuss the electromagnetic field enhancement for the PEC-PMC cavity problem when the wavenumber is sufficiently small, i.e., no resonance occurs.

2.1 Problem formulation

As is shown in Fig. 1, the problem geometry is assumed to be invariant in the x_3-axis. Let $\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2$. Consider a two-dimensional rectangular open cavity $D_\epsilon = \{\boldsymbol{x} \in \mathbb{R}^2 : 0 < x_1 < \epsilon, -d < x_2 < 0\}$, where $\epsilon > 0$ and $d > 0$ are constants. Denote by $\Gamma_\epsilon^+ = \{\boldsymbol{x} \in \mathbb{R}^2 : 0 < x_1 < \epsilon, x_2 = 0\}$ and $\Gamma_\epsilon^- = \{\boldsymbol{x} \in \mathbb{R}^2 : 0 < x_1 < \epsilon, x_2 = -d\}$ the open aperture and the bottom side of the cavity, respectively. Let $\mathbb{R}_+^2 = \{\boldsymbol{x} \in \mathbb{R}^2 : x_2 > 0\}$ and $\Gamma_0 = \{\boldsymbol{x} \in \mathbb{R}^2 : x_2 = 0\}$. Denote $\Omega = \mathbb{R}_+^2 \cup D_\epsilon$.

We assume that the cavity and the upper half-space are filled with the same homogeneous medium. The electromagnetic wave propagation is governed by the time-harmonic Maxwell equations (time dependence $e^{-i\omega t}$):

$$\nabla \times \boldsymbol{E} - i\omega\mu\boldsymbol{H} = 0, \quad \nabla \times \boldsymbol{H} + i\omega\varepsilon\boldsymbol{E} = 0 \quad \text{in } \Omega \times \mathbb{R}, \tag{2.1}$$

where $\omega > 0$ is the angular frequency, $\varepsilon > 0$ is the electric permittivity, $\mu > 0$ is the magnetic permeability, $\boldsymbol{E}$ and $\boldsymbol{H}$ are the electric field and the magnetic field, respectively. Furthermore, we assume that the boundary $\partial\Omega \setminus \Gamma_\epsilon^-$ is a perfect electrical conductor (PEC) while the boundary Γ_ϵ^- is a perfect magnetic conductor (PMC). Hence $(\boldsymbol{E}, \boldsymbol{H})$ satisfy the following boundary conditions:

$$\boldsymbol{\nu} \times \boldsymbol{E} = 0 \text{ on } (\partial\Omega \setminus \Gamma_\epsilon^-) \times \mathbb{R}, \quad \boldsymbol{\nu} \times \boldsymbol{H} = 0 \text{ on } \Gamma_\epsilon^- \times \mathbb{R}, \tag{2.2}$$

where $\boldsymbol{\nu} = (\nu, 0)$ is the unit outward normal vector on $\partial\Omega \times \mathbb{R}$, i.e., ν is the unit outward normal vector on $\partial\Omega$.

We consider a polarized time-harmonic electromagnetic wave $(\boldsymbol{E}^{\text{inc}}, \boldsymbol{H}^{\text{inc}})$ that impinges on the cavity from above. In the transverse magnetic polarization (TM), the magnetic field is parallel to the x_3-axis, which implies that the incident magnetic wave $\boldsymbol{H}^{\text{inc}} = (0, 0, u^{\text{inc}})$. Here $u^{\text{inc}}(\boldsymbol{x}) = e^{i\kappa \boldsymbol{x} \cdot \boldsymbol{d}}$ is a plane wave propagating in the direction $\boldsymbol{d} = (\sin\theta, -\cos\theta)$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ is the incident angle and $\kappa = \omega\sqrt{\varepsilon\mu}$ is the wavenumber. The corresponding

incident electric field $\boldsymbol{E}^{\text{inc}}$ is determined by Ampere's law $\boldsymbol{E}^{\text{inc}} = \mathrm{i}(\omega\varepsilon)^{-1}\nabla \times \boldsymbol{H}^{\text{inc}} = \sqrt{\mu/\varepsilon}(\cos\theta, \sin\theta, 0)e^{\mathrm{i}\kappa\boldsymbol{x}\cdot\boldsymbol{d}}$. Denote by $\lambda = 2\pi/\kappa$ the wavelength of the incident wave. In this section, we assume that the length scale of the underlying geometry satisfies $\epsilon \ll d \ll \lambda$.

Let $\boldsymbol{H} = (0, 0, u_\epsilon)$ be the total magnetic field. It can be verified from (2.1)–(2.2) that u_ϵ satisfies

$$\begin{cases} \Delta u_\epsilon + \kappa^2 u_\epsilon = 0 & \text{in } \Omega, \\ \partial_\nu u_\epsilon = 0 & \text{on } \partial\Omega \setminus \Gamma_\epsilon^-, \\ u_\epsilon = 0 & \text{on } \Gamma_\epsilon^-. \end{cases} \tag{2.3}$$

Denote by u^{ref} the reflected field due to the interaction between the incident field u^{inc} and the PEC plane Γ_0. It can be shown that $u^{\text{ref}}(\boldsymbol{x}) = e^{\mathrm{i}\kappa\boldsymbol{x}\cdot\boldsymbol{d}'}$, where $\boldsymbol{d}' = (\sin\theta, \cos\theta)$. In $\mathbb{R}_2^+$, the total field u_ϵ consists of the incident wave u^{inc}, the reflected wave u^{ref}, and the scattered field u_ϵ^{sc}. The scattered fields u_ϵ^{sc} is required to satisfy the Sommerfeld radiation condition:

$$\lim_{r\to\infty} \sqrt{r}\left(\partial_r u_\epsilon^{\text{sc}} - \mathrm{i}\kappa u_\epsilon^{\text{sc}}\right) = 0, \quad r = |\boldsymbol{x}|. \tag{2.4}$$

2.2 An approximate model

We introduce an approximate model for the problem (2.3)–(2.4) in order to estimate the field enhancement inside the cavity.

Let

$$\phi_0(x_1) = \frac{1}{\sqrt{\epsilon}}, \quad \phi_n(x_1) = \sqrt{\frac{2}{\epsilon}}\cos\left(\frac{n\pi x_1}{\epsilon}\right), \ n \geq 1$$

be an orthonormal basis on the interval $(0, \epsilon)$. It follows from the Neumann boundary condition in (2.3) that u_ϵ can be expanded as the series of waveguide modes:

$$u_\epsilon(\boldsymbol{x}) = \sum_{n=0}^{\infty}\left(\alpha_n^+ e^{-\mathrm{i}\beta_n x_2} + \alpha_n^- e^{\mathrm{i}\beta_n(x_2+d)}\right)\phi_n(x_1), \quad \boldsymbol{x} \in D_\epsilon, \tag{2.5}$$

where the coefficients β_n are defined as

$$\beta_n = \begin{cases} \kappa, & n = 0, \\ \mathrm{i}\sqrt{(n\pi/\epsilon)^2 - \kappa^2}, & n \geq 1. \end{cases} \tag{2.6}$$

If ϵ is small enough (in fact it only needs $\epsilon < \lambda/2$ here), it follows that β_n is are pure imaginary numbers for all $n \geq 1$. It is easy to note that for each n, if $n\pi/\epsilon \leq \kappa$, the series consists of two propagating wave modes traveling upward and downward respectively; if $n\pi/\epsilon > \kappa$ the series consists of two evanescent wave modes decaying exponentially away from the bottom side and the open aperture of the cavity, respectively.

Using the series (2.5), we may reformulate (2.3)–(2.4) into the following coupled problem:

$$\begin{cases} \Delta u_\epsilon + \kappa^2 u_\epsilon = 0 & \text{in } \mathbb{R}_2^+, \\ u_\epsilon = \sum_{n=0}^{\infty}\left(\alpha_n^+ e^{-\mathrm{i}\beta_n x_2} + \alpha_n^- e^{\mathrm{i}\beta_n(x_2+d)}\right)\phi_n(x_1) & \text{in } D_\epsilon, \\ \partial_\nu u_\epsilon = 0 & \text{on } \partial\Omega \setminus \Gamma_\epsilon^-, \\ u_\epsilon = 0 & \text{on } \Gamma_\epsilon^-, \\ u_\epsilon(x_1, 0+) = u_\epsilon(x_1, 0-), \ \partial_{x_2}u_\epsilon(x_1, 0+) = \partial_{x_2}u_\epsilon(x_1, 0-) & \text{on } \Gamma_\epsilon^+, \\ \lim_{r\to\infty} \sqrt{r}\left(\partial_r u_\epsilon^{\text{sc}} - \mathrm{i}\kappa u_\epsilon^{\text{sc}}\right) = 0 & \text{in } \mathbb{R}_2^+, \end{cases} \tag{2.7}$$

where $u_\epsilon^{\text{sc}} = u_\epsilon - (u^{\text{inc}} + u^{\text{ref}})$ in $\mathbb{R}_2^+$. The continuity conditions are imposed along the open aperture Γ_ϵ^+, where $0+$ and $0-$ indicate the limits taken from above and below Γ_ϵ^+, respectively.

For simplicity of notation, we define $u_\epsilon^+(x_1) = u_\epsilon(x_1, 0)$ and $u_\epsilon^-(x_1) = u_\epsilon(x_1, -d)$. Let $u_{\epsilon,n}^\pm = \langle u_\epsilon^\pm, \phi_n \rangle_{\Gamma_\epsilon^\pm}$ be the Fourier coefficients for u_ϵ^+ and u_ϵ^-, respectively, where the inner product $\langle \cdot, \cdot \rangle_{\Gamma_\epsilon^\pm}$ is defined by

$$\langle u_\epsilon^\pm, \phi_n \rangle_{\Gamma_\epsilon^\pm} := \int_0^\epsilon u_\epsilon^\pm(x_1) \bar{\phi}_n(x_1) dx_1.$$

For any $s \in \mathbb{R}$, denote by $H^s(\Gamma_\epsilon^+)$ the trace functional space on Γ_ϵ^+ with the norm given by

$$\|u_\epsilon^+\|_{H^s(\Gamma_\epsilon^+)}^2 = \sum_{n=0}^\infty \left(1 + \left(\frac{n\pi}{\epsilon} \right)^2 \right)^s |u_{\epsilon,n}^+|^2. \tag{2.8}$$

Matching the series (2.5) on Γ_ϵ^+, Γ_ϵ^- and using the continuity conditions in (2.7), we get

$$\begin{cases} \alpha_n^+ + \alpha_n^- e^{i\beta_n d} = u_{\epsilon,n}^+, \\ \alpha_n^+ e^{i\beta_n d} + \alpha_n^- = u_{\epsilon,n}^-. \end{cases}$$

Solving the above equations by Cramer's rule yields that

$$\alpha_n^+ = \frac{e^{i\beta_n d} u_{\epsilon,n}^- - u_{\epsilon,n}^+}{e^{i2\beta_n d} - 1}, \quad \alpha_n^- = \frac{e^{i\beta_n d} u_{\epsilon,n}^+ - u_{\epsilon,n}^-}{e^{i2\beta_n d} - 1}. \tag{2.9}$$

Therefore, in the cavity D_ϵ, the total field

$$u_\epsilon(x) = \sum_{n=0}^\infty \left(\frac{e^{i\beta_n d} u_{\epsilon,n}^- - u_{\epsilon,n}^+}{e^{i2\beta_n d} - 1} e^{-i\beta_n x_2} + \frac{e^{i\beta_n d} u_{\epsilon,n}^+ - u_{\epsilon,n}^-}{e^{i2\beta_n d} - 1} e^{i\beta_n (x_2 + d)} \right) \phi_n(x_1). \tag{2.10}$$

Since u_ϵ satisfies the homogeneous Dirichlet boundary condition on Γ_ϵ^-, we have

$$\partial_{x_2} u_\epsilon(x_1, 0-) = \sum_{n=0}^\infty i\beta_n \left(\frac{e^{i\beta_n d} u_{\epsilon,n}^+ - u_{\epsilon,n}^-}{e^{i2\beta_n d} - 1} e^{i\beta_n d} - \frac{e^{i\beta_n d} u_{\epsilon,n}^- - u_{\epsilon,n}^+}{e^{i2\beta_n d} - 1} \right) \phi_n(x_1) \quad \text{on } \Gamma_\epsilon^+, \tag{2.11}$$

$$u_\epsilon(x_1, -d) = \sum_{n=0}^\infty \left(\frac{e^{i\beta_n d} u_{\epsilon,n}^- - u_{\epsilon,n}^+}{e^{i2\beta_n d} - 1} e^{i\beta_n d} + \frac{e^{i\beta_n d} u_{\epsilon,n}^+ - u_{\epsilon,n}^-}{e^{i2\beta_n d} - 1} \right) \phi_n(x_1) = 0 \quad \text{on } \Gamma_\epsilon^-. \tag{2.12}$$

It follows from (2.12) that

$$u_{\epsilon,n}^- = 0. \tag{2.13}$$

Substituting (2.13) into (2.11) yields

$$\partial_{x_2} u_\epsilon(x_1, 0-) = \sum_{n=0}^\infty i\beta_n \frac{e^{i2\beta_n d} + 1}{e^{i2\beta_n d} - 1} u_{\epsilon,n}^+ \phi_n(x_1) \quad \text{on } \Gamma_\epsilon^+. \tag{2.14}$$

Given a function $u \in H^1(\mathbb{R}_2^+)$ and let $u^+ = u(x_1, 0)$, we define a Dirichlet-to-Neumann (DtN) operator:

$$\mathscr{B}^{\text{PMC}}[u] = \sum_{n=0}^\infty i\beta_n \frac{e^{i2\beta_n d} + 1}{e^{i2\beta_n d} - 1} u_n^+ \phi_n(x_1) \quad \text{on } \Gamma_\epsilon^+, \tag{2.15}$$

where the Fourier coefficients $u_n^+ = \langle u^+, \phi_n \rangle_{\Gamma_\epsilon^+}$.

Lemma 2.1 *The DtN operator $\mathscr{B}^{\mathrm{PMC}}$ is bounded from $H^{1/2}(\Gamma_\epsilon^+) \to H^{-1/2}(\Gamma_\epsilon^+)$, i.e.,*

$$\|\mathscr{B}^{\mathrm{PMC}}[u]\|_{H^{-1/2}(\Gamma_\epsilon^+)} \le C(\lambda, d)\|u\|_{H^{1/2}(\Gamma_\epsilon^+)} \quad \forall u \in H^{1/2}(\Gamma_\epsilon^+),$$

where the constant $C(\lambda, d) = \max\left\{4, \frac{\kappa}{|\tan \kappa d|}\right\}$.

Proof For any $u, w \in H^{1/2}(\Gamma_\epsilon^+)$, they have the Fourier series expansions

$$u = \sum_{n=0}^{\infty} u_n^+ \phi_n(x_1), \quad w = \sum_{n=0}^{\infty} w_n^+ \phi_n(x_1),$$

where $u_n^+ = \langle u, \phi_n \rangle_{\Gamma_\epsilon^+}$ and $w_n^+ = \langle w, \phi_n \rangle_{\Gamma_\epsilon^+}$. It follows from (2.15) that

$$\langle \mathscr{B}^{\mathrm{PMC}}[u], w \rangle_{\Gamma_\epsilon^+} = \sum_{n=0}^{\infty} \mathrm{i}\beta_n \frac{e^{\mathrm{i}2\beta_n d} + 1}{e^{\mathrm{i}2\beta_n d} - 1} u_n^+ \bar{w}_n^+.$$

Note that the coefficients β_n are given by (2.6). For $n = 0$, a straightforward calculation yields that

$$\left| \mathrm{i}\kappa \frac{e^{\mathrm{i}2\kappa d} + 1}{e^{\mathrm{i}2\kappa d} - 1} u_0^+ \bar{w}_0^+ \right| = \frac{\kappa |\cos \kappa d|}{|\sin \kappa d|} |u_0^+||\bar{w}_0^+| \le \frac{\kappa}{|\tan \kappa d|} |u_0^+||\bar{w}_0^+|, \tag{2.16}$$

where we use the fact that $d \ll \lambda$, which implies $\kappa d = \frac{2\pi d}{\lambda}$ tends to zero and $\kappa / \tan(\kappa d) \to 1/d$. For $n \ge 1$, using the fact that $\epsilon \ll \lambda$, we get

$$\left| \mathrm{i}\beta_n \frac{e^{\mathrm{i}2\beta_n d} + 1}{e^{\mathrm{i}2\beta_n d} - 1} u_n^+ \bar{w}_n^+ \right| \le 4 \frac{n\pi}{\epsilon} |u_n^+||\bar{w}_n^+| \le 4 \left(1 + \left(\frac{n\pi}{\epsilon}\right)^2\right)^{1/2} |u_n^+| \left(1 + \left(\frac{n\pi}{\epsilon}\right)^2\right)^{1/2} |\bar{w}_n^+|, \tag{2.17}$$

where we use the estimates

$$|\mathrm{i}\beta_n| = \left| -\sqrt{\left(\frac{n\pi}{\epsilon}\right)^2 - \kappa^2} \right| \le \frac{n\pi}{\epsilon}, \quad \left| \frac{e^{\mathrm{i}2\beta_n d} + 1}{e^{\mathrm{i}2\beta_n d} - 1} \right| = \left| \frac{e^{-2\sqrt{(n\pi/\epsilon)^2 - \kappa^2}d} + 1}{e^{-2\sqrt{(n\pi/\epsilon)^2 - \kappa^2}d} - 1} \right| \le 4.$$

Combining (2.16)–(2.17) and using (2.8), we obtain from the Cauchy–Schwarz inequality that

$$\left| \langle \mathscr{B}^{\mathrm{PMC}}[u], w \rangle_{\Gamma_\epsilon^+} \right| \le C(\lambda, d)\|u\|_{H^{1/2}(\Gamma_\epsilon^+)}\|w\|_{H^{1/2}(\Gamma_\epsilon^+)},$$

which completes the proof. □

Using (2.14)–(2.15) and the continuity of $\partial_{x_2} u_\epsilon$ on Γ_ϵ^+, we obtain the transparent boundary condition (TBC):

$$\partial_{x_2} u_\epsilon = \mathscr{B}^{\mathrm{PMC}}[u_\epsilon] \quad \text{on } \Gamma_\epsilon^+.$$

Hence the problem (2.3)–(2.4) can be reduced to the following boundary value problem:

$$\begin{cases} \Delta u_\epsilon + \kappa^2 u_\epsilon = 0 & \text{in } \mathbb{R}_+^2, \\ \partial_\nu u_\epsilon = 0 & \text{on } \Gamma_0 \setminus \Gamma_\epsilon^+, \\ \partial_{x_2} u_\epsilon = \mathscr{B}^{\mathrm{PMC}}[u_\epsilon] & \text{on } \Gamma_\epsilon^+, \\ \lim_{r \to \infty} \sqrt{r} \left(\partial_r u_\epsilon^{\mathrm{sc}} - \mathrm{i}\kappa u_\epsilon^{\mathrm{sc}} \right) = 0 & \text{in } \mathbb{R}_+^2. \end{cases} \tag{2.18}$$

By calculating the Fourier coefficients $u_{\epsilon,n}^+$, which give the coefficients $u_{\epsilon,n}^-$ by (2.13), we may obtain the solution in the cavity D_ϵ from the formula (2.10).

To find an approximate model to (2.18), we examine the series (2.5) more closely. Note that if $\epsilon \ll \lambda$, the wave modes $e^{-i\beta_n x_2}\phi_n(x_1)$ and $e^{i\beta_n(x_2+d)}\phi_n(x_1)$ decay exponentially in the cavity, with a decaying rate of $O(e^{-n/\epsilon})$ for all $n \geq 1$. Only the leading wave modes $e^{-i\kappa x_2}\phi_0(x_1)$ and $e^{i\kappa(x_2+d)}\phi_0(x_1)$ propagate in the cavity. The observation motivates us to approximate the DtN map (2.15) by dropping the high order modes and define an approximate DtN map by

$$\mathscr{B}_0^{\text{PMC}}[v] = i\kappa \frac{e^{i2\kappa d} + 1}{e^{i2\kappa d} - 1} v_0^+ \phi_0(x_1). \tag{2.19}$$

Now we arrive at an approximate model problem:

$$\begin{cases} \Delta v_\epsilon + \kappa^2 v_\epsilon = 0 & \text{in } \mathbb{R}_+^2, \\ \partial_\nu v_\epsilon = 0 & \text{on } \Gamma_0 \setminus \Gamma_\epsilon^+, \\ \partial_{x_2} v_\epsilon = \mathscr{B}_0^{\text{PMC}}[v_\epsilon] & \text{on } \Gamma_\epsilon^+, \\ \lim_{r \to \infty} \sqrt{r}\left(\partial_r v_\epsilon^{\text{sc}} - i\kappa v_\epsilon^{\text{sc}}\right) = 0 & \text{in } \mathbb{R}_+^2, \end{cases} \tag{2.20}$$

where $v_\epsilon = 0$ on Γ_ϵ^- and $v_\epsilon^{\text{sc}} = v_\epsilon - (u^{\text{inc}} + u^{\text{ref}})$ in $\mathbb{R}_+^2$. Accordingly, we may approximate u_ϵ by one single mode:

$$v_\epsilon(\boldsymbol{x}) = \left(\tilde{\alpha}_0^+ e^{-i\kappa x_2} + \tilde{\alpha}_0^- e^{i\kappa(x_2+d)}\right)\phi_0(x_1), \quad \boldsymbol{x} \in D_\epsilon, \tag{2.21}$$

where the coefficients $\tilde{\alpha}_0^+$ and $\tilde{\alpha}_0^-$ are given by

$$\tilde{\alpha}_0^+ = \frac{e^{i\kappa d} v_{\epsilon,0}^- - v_{\epsilon,0}^+}{e^{i2\kappa d} - 1}, \quad \tilde{\alpha}_0^- = \frac{e^{i\kappa d} v_{\epsilon,0}^+ - v_{\epsilon,0}^-}{e^{i2\kappa d} - 1}. \tag{2.22}$$

It follows from $v_\epsilon = 0$ on Γ_ϵ^- that $\tilde{\alpha}_0^- = -\tilde{\alpha}_0^+ e^{i\kappa d}$, which yields

$$v_{\epsilon,0}^- = 0. \tag{2.23}$$

2.3 Enhancement of the approximated field

This section introduces the estimates for the solution of the approximate model problem.

Theorem 2.2 *Let v_ϵ be the solution of the approximate model problem* (2.20) *and be given by* (2.21) *in the cavity, then the following estimate holds:*

$$\|\nabla v_\epsilon\|_{L^2(D_\epsilon)} = \frac{\kappa\sqrt{\epsilon d}}{\sin \kappa d}(1 + O(\epsilon \ln \epsilon)).$$

Proof In $\mathbb{R}_+^2$, since the incident field u^{inc} and the reflected field u^{ref} satisfy the Helmholtz equation in (2.20), the scattered field v_ϵ^{sc} also satisfies

$$\Delta v_\epsilon^{\text{sc}} + \kappa^2 v_\epsilon^{\text{sc}} = 0 \quad \text{in } \mathbb{R}_+^2.$$

Noting that $\partial_{x_2} u^{\text{inc}} + \partial_{x_2} u^{\text{ref}} = 0$ on Γ_0, we have

$$\partial_{x_2} v_\epsilon^{\text{sc}} = 0 \quad \text{on } \Gamma_0 \setminus \Gamma_\epsilon^+.$$

Let G be the half-space Green function of the Helmholtz equation with Neumann boundary condition, i.e., it satisfies

$$\begin{cases} \Delta G(\boldsymbol{x}, \boldsymbol{y}) + \kappa^2 G(\boldsymbol{x}, \boldsymbol{y}) = \delta(\boldsymbol{x}, \boldsymbol{y}), & \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}_2^+, \\ \frac{\partial G(\boldsymbol{x}, \boldsymbol{y})}{\partial \nu_y} = 0, & \text{on } \Gamma_0. \end{cases}$$

It is easy to note that

$$G(\boldsymbol{x}, \boldsymbol{y}) = -\frac{\mathrm{i}}{4} \left(H_0^{(1)}(\kappa|\boldsymbol{x} - \boldsymbol{y}|) + H_0^{(1)}(\kappa|\boldsymbol{x}' - \boldsymbol{y}|) \right), \tag{2.24}$$

where $\boldsymbol{y} = (y_1, y_2)$, $H_0^{(1)}$ is the Hankel function of the first kind with order 0, and $\boldsymbol{x}'$ is the reflection of the point $\boldsymbol{x}$ with respect to Γ_0, i.e., $\boldsymbol{x}' = (x_1, -x_2)$.

It follows from Green's identity that

$$v_\epsilon^{\mathrm{sc}}(\boldsymbol{x}) = \int_{\Gamma_\epsilon^+} G(\boldsymbol{x}, \boldsymbol{y}) \frac{\partial v_\epsilon^{\mathrm{sc}}(\boldsymbol{y})}{\partial \nu_y} \mathrm{d}s_y, \quad \boldsymbol{x} \in \mathbb{R}_+^2,$$

which gives

$$v_\epsilon(\boldsymbol{x}) = u^{\mathrm{inc}}(\boldsymbol{x}) + u^{\mathrm{ref}}(\boldsymbol{x}) + \int_{\Gamma_\epsilon^+} G(\boldsymbol{x}, \boldsymbol{y}) \frac{\partial v_\epsilon^{\mathrm{sc}}(\boldsymbol{y})}{\partial \nu_y} \mathrm{d}s_y, \quad \boldsymbol{x} \in \mathbb{R}_+^2.$$

By the fact that $\partial_{x_2} u^{\mathrm{inc}} + \partial_{x_2} u^{\mathrm{ref}} = 0$ on Γ_0, especially on Γ_ϵ^+, we have

$$\frac{\partial v_\epsilon^{\mathrm{sc}}}{\partial \nu_y} = \frac{\partial v_\epsilon}{\partial \nu_y} - \frac{\partial (u^{\mathrm{inc}} + u^{\mathrm{ref}})}{\partial \nu_y} = \frac{\partial v_\epsilon}{\partial \nu_y} \quad \text{on } \Gamma_\epsilon^+.$$

It follows from the continuity of single layer potentials [16,17] and the above equality that

$$v_\epsilon(\boldsymbol{x}) = u^{\mathrm{inc}}(\boldsymbol{x}) + u^{\mathrm{ref}}(\boldsymbol{x}) - \frac{\mathrm{i}}{2} \int_{\Gamma_\epsilon^+} H_0^{(1)}(\kappa|\boldsymbol{x} - \boldsymbol{y}|) \frac{\partial v_\epsilon(\boldsymbol{y})}{\partial \nu_y} \mathrm{d}s_y, \quad \boldsymbol{x} \in \Gamma_\epsilon^+. \tag{2.25}$$

Using (2.21)–(2.23) yields that

$$\partial_{x_2} v_\epsilon = \mathrm{i}\kappa \left(-\tilde{\alpha}_0^+ + \tilde{\alpha}_0^- e^{\mathrm{i}\kappa d} \right) \phi_0(x_1) \quad \text{on } \Gamma_\epsilon^+.$$

Substituting the above equality into (2.25) and using the fact that $\phi_0(x_1) = \frac{1}{\sqrt{\epsilon}}$, we get

$$v_\epsilon(x_1, 0) = u^{\mathrm{inc}}(x_1, 0) + u^{\mathrm{ref}}(x_1, 0) + \frac{\kappa}{2} \left(-\tilde{\alpha}_0^+ + \tilde{\alpha}_0^- e^{\mathrm{i}\kappa d} \right) \frac{1}{\sqrt{\epsilon}} h_1(x_1), \quad x_1 \in (0, \epsilon),$$

where

$$h_1(x_1) = \int_0^\epsilon H_0^{(1)}(\kappa|x_1 - y_1|) \mathrm{d}y_1. \tag{2.26}$$

Therefore, the Fourier coefficients $v_{\epsilon,0}^+$ may be expressed as

$$v_{\epsilon,0}^+ = \langle u^{\mathrm{inc}}, \phi_0 \rangle_{\Gamma_\epsilon^+} + \langle u^{\mathrm{ref}}, \phi_0 \rangle_{\Gamma_\epsilon^+} + \frac{\kappa}{2} \left(-\tilde{\alpha}_0^+ + \tilde{\alpha}_0^- e^{\mathrm{i}\kappa d} \right) \frac{1}{\sqrt{\epsilon}} \langle h_1, \phi_0 \rangle_{\Gamma_\epsilon^+}.$$

It follows from the fact $u^{\mathrm{inc}}(x_1, 0) = u^{\mathrm{ref}}(x_1, 0)$ and (2.21) that

$$\tilde{\alpha}_0^+ + \tilde{\alpha}_0^- e^{\mathrm{i}\kappa d} = 2\langle u^{\mathrm{inc}}, \phi_0 \rangle_{\Gamma_\epsilon^+} + \frac{\kappa}{2\sqrt{\epsilon}} \langle h_1, \phi_0 \rangle_{\Gamma_\epsilon^+} \left(-\tilde{\alpha}_0^+ + \tilde{\alpha}_0^- e^{\mathrm{i}\kappa d} \right).$$

By $\tilde{\alpha}_0^- = -\tilde{\alpha}_0^+ e^{i\kappa d}$ and the above equation, we obtain

$$\tilde{\alpha}_0^+ = \frac{2\langle u^{\mathrm{inc}}, \phi_0 \rangle_{\Gamma_\epsilon^+}}{(1+c_0)-(1-c_0)e^{i2\kappa d}}, \quad \tilde{\alpha}_0^- = -\frac{2e^{i\kappa d}\langle u^{\mathrm{inc}}, \phi_0 \rangle_{\Gamma_\epsilon^+}}{(1+c_0)-(1-c_0)e^{i2\kappa d}}, \tag{2.27}$$

where $c_0 := \frac{\kappa}{2\sqrt{\epsilon}}\langle h_1, \phi_0 \rangle_{\Gamma_\epsilon^+}$.

Note that $x_1, y_1 \in \Gamma_\epsilon^+$ for small $|x_1 - y_1|$, asymptotically, it holds that (cf. [36])

$$H_0^{(1)}(\kappa|x_1 - y_1|) = \frac{2i}{\pi}\ln|x_1 - y_1| + \frac{2i}{\pi}\ln\frac{\kappa}{2} + \gamma_0 + O(|x_1 - y_1|^2 \ln|x_1 - y_1|),$$

where γ_0 is the Euler constant defined by $\gamma_0 = \lim\limits_{\tau \to \infty} \sum\limits_{m=1}^{\tau} \frac{1}{m} - \ln \tau$. A direct calculation yields

$$\begin{aligned} c_0 &= \frac{\kappa}{2\sqrt{\epsilon}}\langle h_1, \phi_0 \rangle_{\Gamma_\epsilon^+} = \frac{\kappa}{2\epsilon}\int_0^\epsilon \int_0^\epsilon H_0^{(1)}(\kappa|x_1 - y_1|)\mathrm{d}y_1\mathrm{d}x_1 \\ &= \frac{i\kappa}{\pi}\left(\epsilon\ln\epsilon - \frac{3}{2}\epsilon\right) + \frac{\kappa}{2}\left(\frac{2i}{\pi}\ln\frac{\kappa}{2} + \gamma_0\right)\epsilon + O(\epsilon^3 \ln\epsilon), \end{aligned} \tag{2.28}$$

which gives

$$\begin{aligned} &|(1+c_0)-(1-c_0)e^{i2\kappa d}| \\ &= \left|\kappa\gamma_0\epsilon\cos\kappa d + i2\left(\frac{\kappa}{\pi}\left(\epsilon\ln\epsilon - \frac{3}{2}\epsilon + \epsilon\ln\frac{\kappa}{2}\right)\cos\kappa d - \sin\kappa d\right) + O(\epsilon^3 \ln\epsilon)\right|. \end{aligned}$$

It follows from the assumption $\epsilon \ll d \ll \lambda$ that

$$|(1+c_0)-(1-c_0)e^{i2\kappa d}| = 2\sin(\kappa d) + O(\epsilon\ln\epsilon\kappa d). \tag{2.29}$$

By the definition of the incident wave, we get

$$|\langle u^{\mathrm{inc}}, \phi_0 \rangle_{\Gamma_\epsilon^+}| = \left|\int_0^\epsilon e^{i\kappa\sin\theta x_1}\frac{1}{\sqrt{\epsilon}}\mathrm{d}x_1\right| = \frac{1}{\sqrt{\epsilon}}\frac{2\sin(\frac{\kappa\sin\theta\epsilon}{2})}{\kappa\sin\theta} = \sqrt{\epsilon}(1+O(\kappa^2\epsilon^2)). \tag{2.30}$$

Using (2.27)–(2.30) gives

$$|\tilde{\alpha}_0^+| = |\tilde{\alpha}_0^-| = \frac{\sqrt{\epsilon} + O(\kappa^2\epsilon^{3/2})}{2\sin\kappa d + O(\epsilon\ln\epsilon\kappa d)} = \frac{\sqrt{\epsilon}}{2\sin\kappa d}(1+O(\epsilon\ln\epsilon)).$$

It follows from (2.21) that

$$\|\nabla v_\epsilon\|_{L^2(D_\epsilon)}^2 \le \int_0^\epsilon \int_{-d}^0 \left(\kappa(|\tilde{\alpha}_0^+| + |\tilde{\alpha}_0^-|)\phi_0(x_1)\right)^2\mathrm{d}x_2\mathrm{d}x_1 \le \frac{\epsilon\kappa^2 d}{\sin^2(\kappa d)}(1+O(\epsilon\ln\epsilon)).$$

Substituting (2.27) into (2.21) yields

$$v_\epsilon(x) = \left(\frac{2\langle u^{\mathrm{inc}}, \phi_0 \rangle_{\Gamma_\epsilon^+}}{(1+c_0)-(1-c_0)e^{i2\kappa d}}e^{-i\kappa x_2} - \frac{2e^{i\kappa d}\langle u^{\mathrm{inc}}, \phi_0 \rangle_{\Gamma_\epsilon^+}}{(1+c_0)-(1-c_0)e^{i2\kappa d}}e^{i\kappa(x_2+d)}\right)\phi_0(x_1). \tag{2.31}$$

We have from a simple calculation that

$$\begin{aligned} \frac{\partial v_\epsilon}{\partial x_2} &= -i\kappa e^{i\kappa d}\frac{2\langle u^{\mathrm{inc}}, \phi_0 \rangle_{\Gamma_\epsilon^+}}{(1+c_0)-(1-c_0)e^{i2\kappa d}}\phi_0(x_1)\left(e^{-i\kappa(x_2+d)} + e^{i\kappa(x_2+d)}\right) \\ &= -i\kappa e^{i\kappa d}\frac{2\langle u^{\mathrm{inc}}, \phi_0 \rangle_{\Gamma_\epsilon^+}}{(1+c_0)-(1-c_0)e^{i2\kappa d}}2\cos\kappa(x_2+d)\phi_0(x_1). \end{aligned} \tag{2.32}$$

Noting that ϵ is small enough and $\kappa(x_2 + d) \ll 1$, we have from (2.29)–(2.30) and (2.32) that there exists a positive constant C_1 such that

$$\left| \frac{\partial v_\epsilon}{\partial x_2} \right| \geq \frac{\kappa}{\sin \kappa d} (1 + O(\epsilon \ln \epsilon)).$$

Combining the above estimates yields

$$\|\nabla v_\epsilon\|_{L^2(D_\epsilon)}^2 \geq \int_0^\epsilon \int_{-d}^0 \left| \frac{\partial v_\epsilon}{\partial x_2} \right|^2 dx_2 dx_1 \geq \frac{\epsilon \kappa^2 d}{\sin^2 \kappa d} (1 + O(\epsilon \ln \epsilon)),$$

which completes the proof. □

Theorem 2.3 *Let v_ϵ be the solution of the approximate model problem* (2.20) *and be given by* (2.21), *then the following estimates holds:*

$$\|v_\epsilon\|_{L^2(D_\epsilon)} = \frac{1}{\sin(\kappa d)} \sqrt{\frac{\epsilon(2\kappa d - \sin(2\kappa d))}{\kappa}} (1 + O(\epsilon \ln \epsilon)).$$

Proof It follows from (2.31) that

$$\int_0^\epsilon \int_{-d}^0 |v_\epsilon(x_1, x_2)|^2 dx_1 dx_2 = \frac{\left| 2\langle u^{\text{inc}}, \phi_0 \rangle_{\Gamma_\epsilon^+} e^{i\kappa d} \right|^2}{\left| (1 + c_0) - (1 - c_0)^{i2\kappa d} \right|^2} \int_{-d}^0 |e^{-i\kappa(x_2+d)} - e^{i\kappa(x_2+d)}|^2 dx_2$$

$$= \frac{\left| 2\langle u^{\text{inc}}, \phi_0 \rangle_{\Gamma_\epsilon^+} \right|^2}{\left| (1 + c_0) - (1 - c_0)^{i2\kappa d} \right|^2} \int_{-d}^0 4 \sin^2 \kappa(x_2 + d) dx_2.$$

Noting that $d \ll \lambda$, we have

$$\int_{-d}^0 4 \sin^2 \kappa(x_2 + d) dx_2 = \frac{2\kappa d - \sin(2\kappa d)}{\kappa}. \tag{2.33}$$

It follows from (2.29)–(2.30) that

$$\frac{\left| 2\langle u^{\text{inc}}, \phi_0 \rangle_{\Gamma_\epsilon^+} e^{i\kappa d} \right|^2}{\left| (1 + c_0) - (1 - c_0)^{i2\kappa d} \right|^2} = \frac{\epsilon}{\sin^2(\kappa d)} (1 + O(\epsilon \ln \epsilon)).$$

Combining the above estimates, we obtain

$$\|v_\epsilon\|_{L^2(D_\epsilon)}^2 = \frac{\epsilon(2\kappa d - \sin(2\kappa d))}{\kappa \sin^2(\kappa d)} (1 + O(\epsilon \ln \epsilon)),$$

which completes the proof. □

2.4 Accuracy of the approximate model

To show the accuracy of the approximate model, we introduce a TBC in $\mathbb{R}_+^2$ and reformulate the problems (2.18) and (2.20) in a bounded domain. Let $B_R^+ = \{x \in \mathbb{R}^2 : |x - x_c| < R, x_2 > 0\}$ be the upper half disc with radius R centered at $x_c = (\epsilon/2, 0)$, where R is a sufficiently large positive real number. Denote by $\partial B_R^+ = \{x \in \mathbb{R}^2 : |x - x_c| = R, x_2 > 0\}$ be the upper half circle. Let $\Gamma_R^+ = \{x \in \mathbb{R}^2 : |x_1 - \epsilon/2| < R, x_2 = 0\}$ be the line segment on Γ_0 with length $2R$. The problem geometry is shown in Fig. 2.
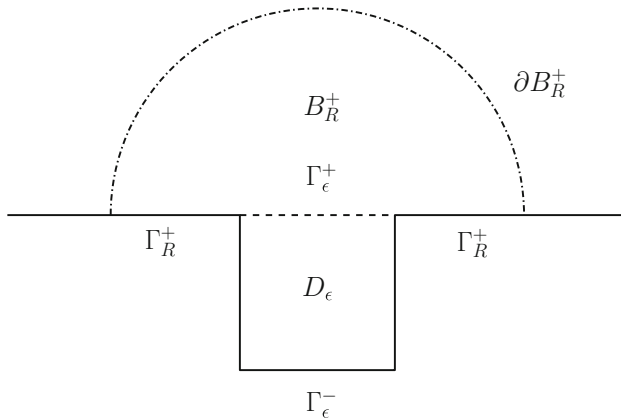
Fig. 2 The problem geometry with a transparent boundary

In the exterior domain $\mathbb{R}_+^2 \setminus \bar{B}_R^+$, by noting the PEC condition on $\Gamma_0 \setminus \Gamma_R^+$ and the radiation condition (2.4), the scattered field can be expressed as the Fourier series:

$$u_\epsilon^{\text{sc}}(r, \eta) = \sum_{n=0}^{\infty} \frac{H_n^{(1)}(\kappa r)}{H_n^{(1)}(\kappa R)} u_{\epsilon,n}^{\text{sc}} \cos(n\eta), \quad 0 < \eta < \pi,$$

where $u_{\epsilon,n}^{\text{sc}} = \frac{2}{\pi} \int_0^\pi u_\epsilon^{\text{sc}}(R, \eta) \cos(n\eta) \mathrm{d}\eta$, $H_n^{(1)}$ is the Hankel function of the first kind with order n, and the polar coordinate $r = \sqrt{(x_1 - \epsilon/2)^2 + x_2^2}$. Taking the normal derivative of u_ϵ^{sc} on ∂B_R^+ yields

$$\frac{\partial u_\epsilon^{\text{sc}}(R, \eta)}{\partial r} = \sum_{n=0}^{\infty} \frac{\kappa (H_n^{(1)})'(\kappa R)}{H_n^{(1)}(\kappa R)} u_{\epsilon,n}^{\text{sc}} \cos(n\eta). \tag{2.34}$$

For any $v \in H^{1/2}(\partial B_R^+)$ with the Fourier expansion

$$v = \sum_{n=0}^{\infty} v_n \cos(n\eta), \quad v_n = \frac{2}{\pi} \int_0^\pi v(\eta) \cos(n\eta) \mathrm{d}\eta,$$

we define the DtN operator on ∂B_R^+:

$$(\mathscr{T}v)(\eta) = \sum_{n=0}^{\infty} \frac{\kappa (H_n^{(1)})'(\kappa R)}{H_n^{(1)}(\kappa R)} v_n \cos(n\eta), \quad 0 < \eta < \pi. \tag{2.35}$$

It follows from (2.34) and (2.35) that the normal derivative of total field on ∂B_R^+ can be written as

$$\partial_r u_\epsilon = \partial_r u_\epsilon^{\text{sc}} + \partial_r (u^{\text{inc}} + u^{\text{ref}}) = \mathscr{T}(u_\epsilon) + g, \tag{2.36}$$

where $g = \partial_r (u^{\text{inc}} + u^{\text{ref}}) - \mathscr{T}(u^{\text{inc}} + u^{\text{ref}})$.

Lemma 2.4 *The DtN operator $\mathscr{T} : H^{1/2}(\partial B_R^+) \to H^{-1/2}(\partial B_R^+)$ is bounded.*

Proof Let $A_n = \frac{(H_n^{(1)})'(\kappa R)}{H_n^{(1)}(\kappa R)}$. By the relation of the Hankel function and the modified Bessel function (cf. [1]), it yields that

$$|A_n| = \left| \frac{K_n'(i\kappa R)}{K_n(i\kappa R)} \right|,$$

where K_n is the modified Bessel function of the second kind. Using the formula $zK_n'(z) = -nK_n(z) - zK_{n-1}(z)$, we obtain

$$|A_n| = \left| \frac{K_n'(i\kappa R)}{K_n(i\kappa R)} \right| = \left| \frac{n}{i\kappa R} + \frac{K_{n-1}(i\kappa R)}{K_n(i\kappa R)} \right| \le \frac{n}{\kappa R} + 1 \le c\sqrt{1+n^2},$$

where c is a positive constant.

For any $v, w \in H^{1/2}(\partial B_R^+)$, we have from the definition of (2.35) that

$$
\begin{aligned}
\left| \langle \mathscr{T} v, w \rangle_{\partial B_R^+} \right| &= \left| \kappa R \int_0^\pi \sum_{n=0}^\infty A_n v_n \cos(n\eta) \bar{w} \mathrm{d}\eta \right| = \left| \frac{\kappa \pi R}{2} \sum_{n=0}^\infty A_n v_n \bar{w}_n \right| \\
&\le \frac{c\kappa \pi R}{2} \sum_{n=0}^\infty \sqrt{1+n^2} |v_n \bar{w}_n| \\
&\le \frac{c\kappa \pi R}{2} \left(\sum_{n=0}^\infty \sqrt{1+n^2} |v_n|^2 \right)^{\frac{1}{2}} \left(\sum_{n=0}^\infty \sqrt{1+n^2} |w_n|^2 \right)^{\frac{1}{2}} \\
&= C(R)\kappa \|v\|_{H^{1/2}(\partial B_R^+)} \|w\|_{H^{1/2}(\partial B_R^+)}.
\end{aligned}
$$

Thus we have

$$\|\mathscr{T} v\|_{H^{-1/2}(\partial B_R^+)} = \sup_{w \in H^{1/2}(\partial B_R^+)} \frac{\left| \langle \mathscr{T} v, w \rangle_{\partial B_R^+} \right|}{\|w\|_{H^{1/2}(\partial B_R^+)}} \le C(R)\kappa \|v\|_{H^{1/2}(\partial B_R^+)},$$

which completes the proof. $\square$

Lemma 2.5 *We have*

$$\mathrm{Re}\langle \mathscr{T} v, v \rangle_{\partial B_R^+} \le 0, \quad \mathrm{Im}\langle \mathscr{T} v, v \rangle_{\partial B_R^+} > 0, \quad \forall v \in H^{1/2}(\partial B_R^+).$$

Proof It follows from the definition (2.35) that

$$\langle \mathscr{T} v, v \rangle_{\partial B_R^+} = \frac{\kappa \pi R}{2} \sum_{n=0}^\infty A_n |v_n|^2. \tag{2.37}$$

Note that $H_n^{(1)} = J_n + iY_n$, where J_n and Y_n are the Bessel functions of the first and second kind, respectively, and the modulus of each Hankel function is decreasing function [1], then

$$\mathrm{Re}A_n = \frac{J_n(kR)J_n'(kR) + Y_n(kR)Y_n'(kR)}{J_n^2(kR) + Y_n^2(kR)} = \frac{1}{2} \frac{(J_n^2)'(kR) + (Y_n^2)'(kR)}{J_n^2(kR) + Y_n^2(kR)} \le 0. \tag{2.38}$$

It follows from the Wronskian formula [1],

$$
\begin{aligned}
\mathrm{Im}A_n &= \frac{J_n(\kappa R)Y_n'(\kappa R) - Y_n(\kappa R)J_n'(\kappa R)}{J_n^2(\kappa R) + Y_n^2(\kappa R)} \\
&= \frac{W(J_n(\kappa R), Y_n(\kappa R))}{J_n^2(\kappa R) + Y_n^2(\kappa R)} = \frac{2}{\kappa \pi R} \frac{1}{J_n^2(\kappa R) + Y_n^2(\kappa R)} > 0, \tag{2.39}
\end{aligned}
$$

where $W(\cdot,\cdot)$ denotes the Wronskian determinant of J_n and Y_n. The proof is completed by combining (2.37)–(2.39).   □

By the TBC operator (2.36), the model problem (2.18) can be reformulated as follows:

$$
\begin{cases}
\Delta u_\epsilon + \kappa^2 u_\epsilon = 0 & \text{in } B_R^+, \\
\partial_\nu u_\epsilon = 0 & \text{on } \Gamma_R^+ \setminus \Gamma_\epsilon^+, \\
\partial_{x_2} u_\epsilon = \mathscr{B}^{\text{PMC}}[u_\epsilon] & \text{on } \Gamma_\epsilon^+, \\
\partial_r u_\epsilon = \mathscr{T}[u_\epsilon] + g & \text{on } \partial B_R^+.
\end{cases}
\tag{2.40}
$$

The variational formulation of (2.40) is to find $u_\epsilon \in H^1(B_R^+)$ such that

$$
a(u_\epsilon, w) = \langle g, w \rangle_{\partial B_R^+} \quad \forall w \in H^1(B_R^+),
\tag{2.41}
$$

where the sesquilinear form

$$
a(u_\epsilon, w) = \int_{B_R^+} \left(\nabla u_\epsilon \cdot \nabla \bar{w} - \kappa^2 u_\epsilon \bar{w} \right) \mathrm{d}\boldsymbol{x} - \int_{\partial B_R^+} \mathscr{T}[u_\epsilon] \bar{w} \mathrm{d}\gamma + \int_{\Gamma_\epsilon^+} \mathscr{B}^{\text{PMC}}[u_\epsilon] \bar{w} \mathrm{d}\gamma.
\tag{2.42}
$$

Theorem 2.6 *The variational problem* (2.41) *has a unique solution* $u_\epsilon \in H^1(B_R^+)$.

Proof By the definition (2.15), we have for sufficiently small ϵ that

$$
\langle \mathscr{B}^{\text{PMC}}[u_\epsilon], u_\epsilon \rangle_{\Gamma_\epsilon^+} = \sum_{n=0}^\infty \mathrm{i}\beta_n \frac{e^{\mathrm{i}2\beta_n d} + 1}{e^{\mathrm{i}2\beta_n d} - 1} |u_{\epsilon,n}|^2
$$
$$
= \frac{\kappa \cos(\kappa d)}{\sin(\kappa d)} |u_{\epsilon,0}|^2 + \sum_{n=1}^\infty \sqrt{(n\pi/\epsilon)^2 - \kappa^2} \frac{e^{-\sqrt{(n\pi/\epsilon)^2 - \kappa^2}} + 1}{1 - e^{-\sqrt{(n\pi/\epsilon)^2 - \kappa^2}}} |u_{\epsilon,n}|^2 > 0.
$$

It follows from Lemma 2.5 and the above inequality that

$$
\mathrm{Re}\{a(u_\epsilon, u_\epsilon)\} = \|\nabla u_\epsilon\|_{L^2(B_R^+)}^2 - \kappa^2 \|u_\epsilon\|_{L^2(B_R^+)}^2 - \mathrm{Re}\langle \mathscr{T}[u_\epsilon], u_\epsilon \rangle_{\partial B_R^+} + \langle \mathscr{B}^{\text{PMC}}[u_\epsilon], u_\epsilon \rangle_{\Gamma_\epsilon^+}
$$
$$
\geq \|\nabla u_\epsilon\|_{L^2(B_R^+)}^2 - \kappa^2 \|u_\epsilon\|_{L^2(B_R^+)}^2,
$$

which shows that the sesquilinear form $a(\cdot,\cdot)$ satisfies a Gårding type inequality.

Next is show the uniqueness. It suffices to show that $u_\epsilon = 0$ if $g = 0$. A simple calculation yields

$$
\mathrm{Im}\, a(u_\epsilon, u_\epsilon) = -\mathrm{Im}\langle \mathscr{T}[u_\epsilon], u_\epsilon \rangle_{\partial B_R^+} = -\frac{\kappa \pi R}{2} \sum_{n=0}^\infty \mathrm{Im}\, A_n |u_{\epsilon,n}|^2 = 0.
$$

It follows from Lemma 2.5 that $u_{\epsilon,n} = 0$ for all n. Therefore $u_\epsilon = 0$ on ∂B_R^+. By the definition (2.35), we have $\mathscr{T} u_\epsilon = 0$, which implies $\partial_r u_\epsilon = \mathscr{T} u_\epsilon = 0$ on ∂B_R^+ by (2.36). Hence we have $u_\epsilon = 0$ in $\mathbb{R}_+^2 \setminus \bar{B}_R^+$. It follows from the unique continuation [21] that $u_\epsilon = 0$ in B_R^+. The proof is completed by applying the Fredholm alternative theorem.   □

Correspondingly, the approximate model problem (2.20) can be written as follows:

$$
\begin{cases}
\Delta v_\epsilon + \kappa^2 v_\epsilon = 0 & \text{in } B_R^+, \\
\partial_\nu v_\epsilon = 0 & \text{on } \Gamma_R^+ \setminus \Gamma_\epsilon^+, \\
\partial_{x_2} v_\epsilon = \mathscr{B}_0^{\text{PMC}}[v_\epsilon] & \text{on } \Gamma_\epsilon^+, \\
\partial_r v_\epsilon = \mathscr{T}[v_\epsilon] + g & \text{on } \partial B_R^+.
\end{cases}
\tag{2.43}
$$

The variational formulation of (2.43) is to find $v_\epsilon \in H^1(B_R^+)$ such that

$$a_0(v_\epsilon, w) = \langle g, w \rangle_{\partial B_R^+} \quad \forall w \in H^1(B_R^+), \tag{2.44}$$

where the sesquilinear form

$$a_0(v_\epsilon, w) = \int_{B_R^+} \left(\nabla v_\epsilon \cdot \nabla \bar{\epsilon} - \kappa^2 v_\epsilon \bar{w} \right) \mathrm{d}\boldsymbol{x} - \int_{\partial B_R^+} \mathscr{T}[v_\epsilon]\bar{w}\mathrm{d}\gamma + \int_{\Gamma_\epsilon^+} \mathscr{B}_0^{\mathrm{PMC}}[v_\epsilon]\bar{w}\mathrm{d}\gamma. \tag{2.45}$$

The well-posedness of the variational problem (2.44) can be shown similarly to Theorem 2.6. The proof is omitted for brevity.

In the cavity D_ϵ, the wave fields u_ϵ and v_ϵ are given by (2.10) and (2.21)–(2.22), respectively. Their Fourier coefficients can be calculated from the solutions obtained in (2.40) and (2.43). The following lemma gives the accuracy of approximate model (2.43) and (2.21)–(2.22).

Lemma 2.7 *Let u_ϵ be the expansion of (2.10) inside the cavity with the Fourier coefficients given by the model (2.40), and let v_ϵ be the expansion of (2.21)–(2.22) with the Fourier coefficients given by the approximate model (2.43), then the following estimates hold:*

$$\|\nabla u_\epsilon - \nabla v_\epsilon\|_{L^2(D_\epsilon)} \le C(R) \frac{\kappa^2 \sqrt{d}}{\cos(\kappa d)}\epsilon, \quad \|u_\epsilon - v_\epsilon\|_{L^2(D_\epsilon)} \le C(R) \frac{\kappa \sqrt{d}}{\cos(\kappa d)}\epsilon\sqrt{\epsilon |\ln \epsilon|},$$

where $C(R)$ is a positive constant independent of ϵ.

Proof The proof consists of five steps in order to show the estimates between u_ϵ and v_ϵ.

Step 1: we first give the estimate of $\|u_\epsilon - u_0\|_{H^1(B_R^+)}$ and $\|v_\epsilon - u_0\|_{H^1(B_R^+)}$, where u_0 is the solution of (2.40) when there is no cavity, i.e., u_0 satisfies the variational problem

$$b(u_0, w) = \langle g, w \rangle_{\partial B_R^+} \quad \forall w \in H^1(B_R^+), \tag{2.46}$$

where the sesquilinear form

$$b(u_0, w) = \int_{B_R^+} \left(\nabla u_0 \cdot \nabla \bar{w} - \kappa^2 u_0 \bar{w} \right) \mathrm{d}\boldsymbol{x} - \int_{\partial B_R^+} \mathscr{T}[u_0]\bar{w}\mathrm{d}\gamma.$$

It follows from (2.41) and (2.46) that

$$a(u_\epsilon - u_0, w) = \langle g, w \rangle_{\partial B_R^+} - a(u_0, w) = b(u_0, w) - a(u_0, w) = -\int_{\Gamma_\epsilon^+} \mathscr{B}^{\mathrm{PMC}}[u_0]\bar{w}\mathrm{d}\gamma. \tag{2.47}$$

Using (2.15) and (2.19), we have from Lemma 2.1 that

$$|b(u_0, w) - a(u_0, w)| \le \left| \int_{\Gamma_\epsilon^+} \mathscr{B}_0^{\mathrm{PMC}}[u_0]\bar{w}\mathrm{d}\gamma \right| + \left| \int_{\Gamma_\epsilon^+} (\mathscr{B}^{\mathrm{PMC}} - \mathscr{B}_0^{\mathrm{PMC}})[u_0]\bar{w}\mathrm{d}\gamma \right|$$

$$\le \frac{\kappa}{|\cot(\kappa d)|} |u_{0,0}^+||\bar{w}_0^+| + C\|u_0\|_{H^{1/2}(\Gamma_\epsilon^+)}\|w\|_{H^{1/2}(\Gamma_\epsilon^+)}, \tag{2.48}$$

where $u_{0,0}^+ = \langle u_0, \phi_0 \rangle_{\Gamma_\epsilon^+}$ and $w_0^+ = \langle w, \phi_0 \rangle_{\Gamma_\epsilon^+}$.

Using the estimates [28, Lemma A.1, Lemma A.2]

$$|u_{0,0}^+| \le \tilde{C}(R)\sqrt{\epsilon}\|u_0\|_{H^2(B_R^+)} \quad \text{if } u_0 \in H^2(B_R^+),$$

$$|w_0^+| \le \tilde{C}(R)\sqrt{\epsilon |\ln \epsilon|}\|w\|_{H^1(B_R^+)} \quad \text{if } w \in H^1(B_R^+),$$

$$\|u_0\|_{H^{1/2}(\Gamma_\epsilon^+)} \leq \tilde{C}(R)\epsilon\|u_0\|_{H^3(B_R^+)} \quad \text{if } u_0 \in H^3(B_R^+),$$

$$\|w\|_{H^{1/2}(\Gamma_\epsilon^+)} \leq \tilde{C}(R)\|w\|_{H^1(B_R^+)} \quad \text{if } w \in H^1(B_R^+),$$

we obtain from (2.48) that

$$|b(u_0, w) - a(u_0, w)|$$
$$\leq \tilde{C}(R)\frac{\kappa}{\cot(\kappa d)}\left(\epsilon\sqrt{|\ln\epsilon|}\|u_0\|_{H^2(B_R^+)}\|w\|_{H^1(B_R^+)} + \epsilon\|u_0\|_{H^3(B_R^+)}\|w\|_{H^1(B_R^+)}\right).$$

Since $u_0 = u^{\text{inc}} + u^{\text{ref}} = e^{i\kappa x \cdot d} + e^{i\kappa x \cdot d'}$, we have from straightforward calculations that

$$|b(u_0, w) - a(u_0, w)| \leq \tilde{C}(R)\frac{\kappa}{\cot(\kappa d)}\epsilon\sqrt{|\ln\epsilon|}\|w\|_{H^1(B_R^+)}. \tag{2.49}$$

Denote by $\mathscr{L}_\epsilon$ be the induced operator for the sesquilinear form (2.42) such that

$$a_\epsilon(u_\epsilon, w) = (\mathscr{L}_\epsilon u_\epsilon, w),$$

where $(\cdot, \cdot)$ is the inner product on $L^2(B_R^+)$. It follows from [28, Lemma A.3] (or [22, Lemma 4.4]) that there exists a positive constant ϵ_0 such that

$$\|\mathscr{L}_\epsilon^{-1}\|_{\mathcal{L}(H^1(B_R^+))} \leq \mathscr{C}(R) \quad \forall\, 0 < \epsilon < \epsilon_0, \tag{2.50}$$

where $\mathscr{C}(R)$ is a constant independent of ϵ. Combing (2.47)–(2.50) yields

$$\|u_\epsilon - u_0\|_{H^1(B_R^+)} \leq C(R)\frac{\kappa}{\cot(\kappa d)}\epsilon\sqrt{|\ln\epsilon|},$$

where $C(R)$ depends on $\tilde{C}(R)$ and $\mathscr{C}(R)$.

Since v_ϵ satisfies the variational problem (2.44), following the same arguments as above, we may show for sufficiently small ϵ that

$$\|v_\epsilon - u_0\|_{H^1(B_R^+)} \leq C(R)\frac{\kappa}{\cot(\kappa d)}\epsilon\sqrt{|\ln\epsilon|}. \tag{2.51}$$

Step 2: we estimate $\|v_\epsilon - u_0\|_{H^{3/2}(B_R^+)}$. Let $\xi = v_\epsilon - u_0$. It is easy to verify that ξ satisfies

$$\begin{cases} \Delta\xi + \kappa^2\xi = 0 & \text{in } B_R^+, \\ \partial_\nu\xi = 0 & \text{on } \Gamma_R^+ \setminus \Gamma_\epsilon^+, \\ \partial_{x_2}\xi = \mathscr{B}_0^{\text{PMC}}[v_\epsilon] & \text{on } \Gamma_\epsilon^+, \\ \partial_r\xi = \mathscr{T}\xi & \text{on } \partial B_R^+. \end{cases}$$

It follows from the standard regularity estimate for the Neumann problem that

$$\|\xi\|_{H^{3/2}(B_R^+)} \leq C(R)\|\mathscr{B}_0^{\text{PMC}}[v_\epsilon]\|_{L^2(\Gamma_\epsilon^+)}$$
$$\leq C(R)\frac{\kappa}{\cot(\kappa d)}\left(|u_{0,0}^+| + |v_{w,0}^+ - u_{0,0}^+|\right)\|\phi_0\|_{L^2(0,\epsilon)}$$
$$\leq C(R)\frac{\kappa}{\cot(\kappa d)}\left(\sqrt{\epsilon}\|u_0\|_{H^2(B_R^+)} + \sqrt{\epsilon|\ln\epsilon|}\|v_\epsilon - u_0\|_{H^1(B_R^+)}\right)$$
$$\leq C(R)\frac{\kappa}{\cot(\kappa d)}\sqrt{\epsilon}, \tag{2.52}$$

where the last inequality is obtained by using (2.51) and $C(R)$ may be different constant.

Step 3: This step is to estimate $\|u_\epsilon - v_\epsilon\|_{H^1(B_R^+)}$. From (2.41) and (2.44), we have

$$a(u_\epsilon - v_\epsilon, w) = a_0(v_\epsilon, w) - a(v_\epsilon, w). \tag{2.53}$$

Comparing (2.42) and (2.45) yields

$$
\begin{aligned}
|a_0(v_\epsilon, w) - a(v_\epsilon, w)| &\leq \left| \int_{\Gamma_\epsilon^+} (\mathscr{B}^{\mathrm{PMC}} - \mathscr{B}_0^{\mathrm{PMC}})[v_\epsilon] \bar{w} \, d\gamma \right| \\
&\leq \left| \int_{\Gamma_\epsilon^+} (\mathscr{B}^{\mathrm{PMC}} - \mathscr{B}_0^{\mathrm{PMC}})[u_0] \bar{w} \, d\gamma \right| \\
&\quad + \left| \int_{\Gamma_\epsilon^+} (\mathscr{B}^{\mathrm{PMC}} - \mathscr{B}_0^{\mathrm{PMC}})[v_\epsilon - u_0] \bar{w} \, d\gamma \right|.
\end{aligned}
\tag{2.54}
$$

Using the estimates (2.48)–(2.49) gives

$$
\left| \int_{\Gamma_\epsilon^+} (\mathscr{B}^{\mathrm{PMC}} - \mathscr{B}_0^{\mathrm{PMC}})[u_0] \bar{w} \, d\gamma \right| \leq C(R) \frac{\kappa}{\cot(\kappa d)} \epsilon \, \|w\|_{H^1(B_R^+)}.
$$

For the second term in (2.54), we have from the estimate of $\|v_\epsilon - u_0\|_{H^{3/2}(\Omega_R)}$ that

$$
\begin{aligned}
\left| \int_{\Gamma_\epsilon^+} (\mathscr{B}^{\mathrm{PMC}} - \mathscr{B}_0^{\mathrm{PMC}})[v_\epsilon - u_0] \bar{w} \, d\gamma \right| &\leq C(R) \frac{\kappa}{\cot(\kappa d)} \|v_\epsilon - u_0\|_{H^{1/2}(\Gamma_\epsilon^+)} \|w\|_{H^{1/2}(\Gamma_\epsilon^+)} \\
&\leq C(R) \frac{\kappa}{\cot(\kappa d)} \sqrt{\epsilon} \|v_\epsilon - u_0\|_{H^{3/2}(B_R^+)} \|w\|_{H^1(B_R^+)} \\
&\leq C(R) \frac{\kappa}{\cot(\kappa d)} \epsilon \|w\|_{H^1(B_R^+)}.
\end{aligned}
$$

For the variational problem (2.53), a combination of the above estimates and the boundedness of inverse of the linear operator $\mathscr{L}_\epsilon$ in (2.50) lead to the following estimate

$$
\|u_\epsilon - v_\epsilon\|_{H^1(B_R^+)} \leq C(R) \frac{\kappa}{\cot(\kappa d)} \epsilon.
\tag{2.55}
$$

Step 4: we estimate $\|\nabla u_\epsilon - \nabla v_\epsilon\|_{L^2(D_\epsilon)}$. We have from the expansions of u_ϵ in (2.7) and v_ϵ in (2.21) that

$$
\begin{aligned}
\|\nabla u_\epsilon - \nabla v_\epsilon\|_{L^2(D_\epsilon)}^2 &= \int_{-d}^0 \kappa^2 \left| -(\alpha_0^+ - \tilde{\alpha}_0^+) e^{-i\kappa x_2} + (\alpha_0^- - \tilde{\alpha}_0^-) e^{i\kappa(x_2+d)} \right|^2 dx_2 \\
&\quad + \int_{-d}^0 \left(\sum_{n=1}^\infty \left(\frac{n\pi}{\epsilon} \right)^2 \left| \alpha_n^+ e^{-i\beta_n x_2} + \alpha_n^- e^{i\beta_n(x_2+d)} \right|^2 \right. \\
&\quad \left. + (i\beta_n)^2 \left| -\alpha_n^+ e^{-i\beta_n x_2} + \alpha_n^- e^{i\beta_n(x_2+d)} \right|^2 \right) dx_2 \\
&\leq 2\kappa^2 d \left(|\alpha_0^+ - \tilde{\alpha}_0^+|^2 + |\alpha_0^- - \tilde{\alpha}_0^-|^2 \right) \\
&\quad + \sum_{n=1}^\infty \frac{1 + 2(n\pi/\epsilon)^2}{n\pi/2\epsilon} \left(|\alpha_n^+|^2 + |\alpha_n^-|^2 \right),
\end{aligned}
$$

where the Parseval identity is used in the last inequality.

For the first term ($n = 0$) in the above inequality, following from (2.9) and (2.22) and noting that $d \ll \lambda$, we get

$$
\begin{aligned}
|\alpha_0^+ - \tilde{\alpha}_0^+|^2 &= \frac{|u_{\epsilon,0}^+ - v_{\epsilon,0}^+|^2}{|e^{i2\kappa d} - 1|^2} = \frac{1}{4 \sin^2(\kappa d)} |u_{\epsilon,0}^+ - v_{\epsilon,0}^+|^2, \\
|\alpha_0^- - \tilde{\alpha}_0^-|^2 &= \frac{|e^{i\kappa d}(u_{\epsilon,0}^+ - v_{\epsilon,0}^+)|^2}{|e^{i2\kappa d} - 1|^2} = \frac{1}{4 \sin^2(\kappa d)} |u_{\epsilon,0}^+ - v_{\epsilon,0}^+|^2.
\end{aligned}
\tag{2.56}
$$

For the second term ($n \geq 1$), by (2.9) and $i\beta_n d = -d\sqrt{(n\pi/\epsilon)^2 - \kappa^2}$, we have

$$|\alpha_n^+|^2 = \frac{|-u_{\epsilon,n}^+|^2}{|e^{i2\beta_n d} - 1|^2} \leq 2|u_{\epsilon,n}^+|^2, \quad |\alpha_n^-|^2 = \frac{|e^{i\beta_n d}u_{\epsilon,n}^+|^2}{|e^{i2\beta_n d} - 1|^2} \leq 2|u_{\epsilon,n}^+|^2. \tag{2.57}$$

Consequently, we obtain

$$\|\nabla u_\epsilon - \nabla v_\epsilon\|_{L^2(D_\epsilon)}^2 \leq C\frac{\kappa^2 d}{\sin^2(\kappa d)}\left(|u_{\epsilon,0}^+ - v_{\epsilon,0}^+|^2 + \sum_{n=1}^{\infty}\left(1 + \left(\frac{n\pi}{\epsilon}\right)^2\right)^{1/2}|u_{\epsilon,n}^+|^2\right)$$

$$\leq C\frac{\kappa^2 d}{\sin^2(\kappa d)}\left(|u_{\epsilon,0}^+ - v_{\epsilon,0}^+|^2 + \|u_\epsilon\|_{H^{1/2}(\Gamma_\epsilon^+)}^2\right).$$

It follows from (2.52) and (2.55) that

$$\|u_\epsilon\|_{H^{1/2}(\Gamma_\epsilon^+)} \leq \|u_0\|_{H^{1/2}(\Gamma_\epsilon^+)} + \|v_\epsilon - u_0\|_{H^{1/2}(\Gamma_\epsilon^+)} + \|u_\epsilon - v_\epsilon\|_{H^{1/2}(\Gamma_\epsilon^+)}$$

$$\leq C(R)\frac{\kappa}{\cot(\kappa d)}\left(\epsilon\|u_0\|_{H^3(B_R^+)} + \sqrt{\epsilon}\|v_\epsilon - u_0\|_{H^{3/2}(B_R^+)} + \|u_\epsilon - v_\epsilon\|_{H^1(B_R^+)}\right)$$

$$\leq C(R)\frac{\kappa}{\cot(\kappa d)}\epsilon. \tag{2.58}$$

On the other hand, we have from (2.55) that

$$|u_{\epsilon,0}^+ - v_{\epsilon,0}^+| \leq C(R)\sqrt{\epsilon|\ln\epsilon|}\|u_\epsilon - v_\epsilon\|_{H^1(B_R^+)} \leq C(R)\frac{\kappa}{\cot(\kappa d)}\epsilon\sqrt{\epsilon|\ln\epsilon|}. \tag{2.59}$$

Combining the above estimates yields

$$\|\nabla u_\epsilon - \nabla v_\epsilon\|_{L^2(D_\epsilon)}^2 \leq C^2(R)\frac{\kappa^4 d}{\cos^2(\kappa d)}\epsilon^2.$$

Step 5: the last step is to estimate $\|u_\epsilon - v_\epsilon\|_{L^2(D_\epsilon)}$. Using the similar method as that in Step 4, we can obtain from the Parseval identity and (2.56)–(2.57) that

$$\|u_\epsilon - v_\epsilon\|_{L^2(D_\epsilon)}^2 \leq 2d\left(|\alpha_0^+ - \tilde{\alpha}_0^+|^2 + |\alpha_0^- - \tilde{\alpha}_0^-|^2\right) + \sum_{n=1}^{\infty}\frac{1}{n\pi/2\epsilon}\left(|\alpha_n^+|^2 + |\alpha_n^-|^2\right)$$

$$\leq C\frac{d}{\sin^2(\kappa d)}\left(|u_{\epsilon,0}^+ - v_{\epsilon,0}^+|^2 + \sum_{n=1}^{\infty}\left(\frac{\epsilon}{n\pi}\right)|u_{\epsilon,n}^+|^2\right)$$

$$\leq C\frac{d}{\sin^2(\kappa d)}\left(|u_{\epsilon,0}^+ - v_{\epsilon,0}^+|^2 + \epsilon^2\sum_{n=1}^{\infty}\left(1 + \left(\frac{n\pi}{\epsilon}\right)\right)^2|u_{\epsilon,n}^+|^2\right)$$

$$\leq C\frac{d}{\sin^2(\kappa d)}\left(|u_{\epsilon,0}^+ - v_{\epsilon,0}^+|^2 + \epsilon^2\|u_\epsilon\|_{H^{1/2}(\Gamma_\epsilon^+)}^2\right)$$

$$\leq C^2(R)\frac{\kappa^2 d}{\cos^2(\kappa d)}\left(\epsilon^3|\ln\epsilon| + \epsilon^4\right)$$

$$\leq 2C^2(R)\frac{\kappa^2 d}{\cos^2(\kappa d)}\left(\epsilon\sqrt{\epsilon|\ln\epsilon|}\right)^2,$$

where the estimates (2.58)–(2.59) are used.   □

2.5 Field enhancement without resonance

The electromagnetic field enhancement factor is defined as the ratio of the energy between the total field and the incident field in the cavity. Hence the electric and magnetic field enhancement factors are given by

$$Q_E = \frac{\|\boldsymbol{E}_\epsilon\|_{L^2(D_\epsilon)}}{\|\boldsymbol{E}^{\mathrm{inc}}\|_{L^2(D_\epsilon)}} \quad \text{and} \quad Q_H = \frac{\|\boldsymbol{H}_\epsilon\|_{L^2(D_\epsilon)}}{\|\boldsymbol{H}^{\mathrm{inc}}\|_{L^2(D_\epsilon)}}.$$

Theorem 2.8 *There exists a constant ϵ_0 depending on λ and d such that for all $0 < \epsilon < \epsilon_0$ the following estimates hold:*

$$\left(1 - C(R)\kappa\sqrt{\epsilon}\tan(\kappa d) + O(\sin(\kappa d)\kappa d\epsilon \ln \epsilon)\right)\frac{1}{\sin \kappa d} \le Q_E$$

$$\le \left(1 + C(R)\kappa\sqrt{\epsilon}\tan(\kappa d) + O(\sin(\kappa d)\kappa d\epsilon \ln \epsilon)\right)\frac{1}{\sin \kappa d}$$

and

$$\frac{4}{3} - C(R)\kappa\epsilon\sqrt{|\ln \epsilon|} + O(\epsilon \ln \epsilon) \le Q_H \le \frac{4}{3} + C(R)\kappa\epsilon\sqrt{|\ln \epsilon|} + O(\epsilon \ln \epsilon) \quad as \quad d/\lambda \to 0.$$

Remark 2.9 The electric field enhancement has an order $O\left(1/\sin(\kappa d)\right)$, which is enormous due to the length scale $d \ll \lambda$, while the magnetic field has no significant enhancement in this situation.

Proof Recalling the scale assumption $\epsilon \ll d \ll \lambda$, we have from Theorem 2.2 and Lemma 2.7 that

$$\|\nabla u_\epsilon\|_{L^2(D_\epsilon)} \le \|\nabla v_\epsilon\|_{L^2(D_\epsilon)} + \|\nabla u_\epsilon - \nabla v_\epsilon\|_{L^2(D_\epsilon)}$$

$$\le \frac{\kappa\sqrt{\epsilon d}}{\sin(\kappa d)}(1 + O(\epsilon \ln \epsilon)) + C(R)\frac{\kappa^2\sqrt{d}}{\cos(\kappa d)}\epsilon$$

$$\le \left(1 + C(R)\kappa\sqrt{\epsilon}\tan(\kappa d) + O(\sin(\kappa d)\kappa d\epsilon \ln \epsilon)\right)\frac{\kappa\sqrt{\epsilon d}}{\sin \kappa d}.$$

On the other hand,

$$\|\nabla u_\epsilon\|_{L^2(D_\epsilon)} \ge \|\nabla v_\epsilon\|_{L^2(D_\epsilon)} - \|\nabla u_\epsilon - \nabla v_\epsilon\|_{L^2(D_\epsilon)}$$

$$\ge \frac{\kappa\sqrt{\epsilon d}}{\sin(\kappa d)}(1 + O(\epsilon \ln \epsilon)) - C(R)\frac{\kappa^2\sqrt{d}}{\cos(\kappa d)}\epsilon$$

$$\ge \left(1 - C(R)\kappa\sqrt{\epsilon}\tan(\kappa d) + O(\sin(\kappa d)\kappa d\epsilon \ln \epsilon)\right)\frac{\kappa\sqrt{\epsilon d}}{\sin \kappa d}.$$

A straightforward calculation yields

$$\|\nabla u^{\mathrm{inc}}\|^2_{L^2(D_\epsilon)} = \int_{-d}^0 \int_0^\epsilon |\nabla u^{\mathrm{inc}}(\boldsymbol{x})|^2 \mathrm{d}\boldsymbol{x} = \kappa^2\epsilon d.$$

Combining the above estimates gives

$$\left(1 - C(R)\kappa\sqrt{\epsilon}\tan(\kappa d) + O(\sin(\kappa d)\kappa d\epsilon \ln \epsilon)\right)\frac{1}{\sin \kappa d} \le \frac{\|\nabla u_\epsilon\|_{L^2(D_\epsilon)}}{\|\nabla u^{\mathrm{inc}}\|_{L^2(D_\epsilon)}}$$

$$\le \left(1 + C(R)\kappa\sqrt{\epsilon}\tan(\kappa d) + O(\sin(\kappa d)\kappa d\epsilon \ln \epsilon)\right)\frac{1}{\sin \kappa d}.$$

Noting that $|\nabla \times \boldsymbol{H}_\epsilon| = |\nabla \times (0,0,u_\epsilon)| = |\nabla u_\epsilon|$ and $\nabla \times \boldsymbol{H}_\epsilon = -i\omega\varepsilon\,\boldsymbol{E}_\epsilon$, we obtain

$$\left(1 - C(R)\kappa\sqrt{\epsilon}\tan(\kappa d) + O(\sin(\kappa d)\kappa d\epsilon \ln \epsilon)\right)\frac{1}{\sin \kappa d}$$

$$\leq \frac{\|\nabla u_\epsilon\|_{L^2(D_\epsilon)}}{\|\nabla u^{\mathrm{inc}}\|_{L^2(D_\epsilon)}} = \frac{\|\boldsymbol{E}_\epsilon\|_{L^2(D_\epsilon)}}{\|\boldsymbol{E}^{\mathrm{inc}}\|_{L^2(D_\epsilon)}} = Q_E$$

$$\leq \left(1 + C(R)\kappa\sqrt{\epsilon}\tan(\kappa d) + O(\sin(\kappa d)\kappa d\epsilon \ln \epsilon)\right)\frac{1}{\sin \kappa d}.$$

For the magnetic field, it follows from Theorem 2.3 and Lemma 2.7 that

$$\|u_\epsilon\|_{L^2(D_\epsilon)} \leq \|v_\epsilon\|_{L^2(D_\epsilon)} + \|u_\epsilon - v_\epsilon\|_{L^2(D_\epsilon)}$$

$$\leq \frac{1}{\sin(\kappa d)}\sqrt{\frac{\epsilon(2\kappa d - \sin(2\kappa d))}{\kappa}}(1 + O(\epsilon \ln \epsilon)) + C(R)\frac{\kappa\sqrt{d}}{\cos(\kappa d)}\epsilon\sqrt{\epsilon|\ln \epsilon|}$$

$$\leq \sqrt{\epsilon d}\left(\frac{1}{\sin(\kappa d)}\sqrt{2 - \frac{\sin(2\kappa d)}{\kappa d}} + C(R)\frac{\kappa\epsilon\sqrt{|\ln \epsilon|}}{\cos(\kappa d)} + O(\epsilon \ln \epsilon)\right)$$

and

$$\|u_\epsilon\|_{L^2(D_\epsilon)} \geq \|v_\epsilon\|_{L^2(D_\epsilon)} + \|u_\epsilon - v_\epsilon\|_{L^2(D_\epsilon)}$$

$$\geq \sqrt{\epsilon d}\left(\frac{1}{\sin(\kappa d)}\sqrt{2 - \frac{\sin(2\kappa d)}{\kappa d}} - C(R)\frac{\kappa\epsilon\sqrt{|\ln \epsilon|}}{\cos(\kappa d)} + O(\epsilon \ln \epsilon)\right).$$

Clearly, we have

$$\|u^{\mathrm{inc}}\|_{L^2(D_\epsilon)}^2 = \int_{-d}^0 \int_0^\epsilon |u^{\mathrm{inc}}(\boldsymbol{x})|^2 \mathrm{d}\boldsymbol{x} = \epsilon d.$$

Therefore, we obtain

$$\left(\frac{1}{\sin(\kappa d)}\sqrt{2 - \frac{\sin(2\kappa d)}{\kappa d}} - C(R)\frac{\kappa\epsilon\sqrt{|\ln \epsilon|}}{\cos(\kappa d)} + O(\epsilon \ln \epsilon)\right)$$

$$\leq \frac{\|u_\epsilon\|_{L^2(D_\epsilon)}}{\|u^{\mathrm{inc}}\|_{L^2(D_\epsilon)}} = \frac{\|\boldsymbol{H}_\epsilon\|_{L^2(D_\epsilon)}}{\|\boldsymbol{H}^{\mathrm{inc}}\|_{L^2(D_\epsilon)}} = Q_H$$

$$\leq \left(\frac{1}{\sin(\kappa d)}\sqrt{2 - \frac{\sin(2\kappa d)}{\kappa d}} + C(R)\frac{\kappa\epsilon\sqrt{|\ln \epsilon|}}{\cos(\kappa d)} + O(\epsilon \ln \epsilon)\right).$$

Furthermore, the enhancement coefficients satisfy

$$\lim_{d/\lambda \to 0}\left(\frac{1}{\sin(\kappa d)}\sqrt{2 - \frac{\sin(2\kappa d)}{\kappa d}} - C(R)\frac{\kappa\epsilon\sqrt{|\ln \epsilon|}}{\cos(\kappa d)} + O(\epsilon \ln \epsilon)\right)$$

$$= \frac{4}{3} - C(R)\kappa\epsilon\sqrt{|\ln \epsilon|} + O(\epsilon \ln \epsilon),$$

$$\lim_{d/\lambda \to 0}\left(\frac{1}{\sin(\kappa d)}\sqrt{2 - \frac{\sin(2\kappa d)}{\kappa d}} + C(R)\frac{\kappa\epsilon\sqrt{|\ln \epsilon|}}{\cos(\kappa d)} + O(\epsilon \ln \epsilon)\right)$$

$$= \frac{4}{3} + C(R)\kappa\epsilon\sqrt{|\ln \epsilon|} + O(\epsilon \ln \epsilon),$$

which completes the proof.   □

3 PEC-PMC cavity with resonances

This section is to discuss the Fabry–Perot type resonance and derive the asymptotic expansions for those resonance; analyze quantitatively the field enhancement at the resonance frequencies in a narrow cavity, i.e., $\epsilon \ll \lambda$.

Recall the model problem (2.3)–(2.4):

$$
\begin{cases}
\Delta u_\epsilon + \kappa^2 u_\epsilon = 0 & \text{in } \Omega, \\
\partial_\nu u_\epsilon = 0 & \text{on } \partial\Omega \setminus \Gamma_\epsilon^-, \\
u_\epsilon = 0 & \text{on } \Gamma_\epsilon^-, \\
\lim_{r\to\infty} \sqrt{r}\left(\partial_r u_\epsilon^{\text{sc}} - i\kappa u_\epsilon^{\text{sc}}\right) = 0 & \text{in } \mathbb{R}_+^2.
\end{cases}
\tag{3.1}
$$

It is known that the scattering problem (3.1) has a unique solution for a complex wavenumber k with $\text{Im}\,k \geq 0$ (cf., [3]). By analytic continuation, the solution has a meromorphic extension to the whole complex plane, except for a countable number of points, which are poles of the resolvent associated with the scattering problem (3.1). These poles are called the resonances of the scattering problem, and the associated nontrivial solutions are called resonance modes. If the frequency of the incident wave is taken to be close to the real part of the resonance, an enhancement of the field is expected if the imaginary part of the resonance is small.

3.1 Boundary integral equation

Define by $g_\epsilon^{\text{PMC}}(x, y)$ the Green function in the PEC-PMC cavity D_ϵ, i.e., it satisfies

$$
\begin{cases}
\Delta g_\epsilon^{\text{PMC}}(x, y) + \kappa^2 g_\epsilon^{\text{PMC}}(x, y) = -\delta(x - y), & x, y \in D_\epsilon, \\
\frac{\partial g_\epsilon^{\text{PMC}}(x,y)}{\partial \nu_y} = 0 & \text{on } \partial D_\epsilon \setminus \Gamma_\epsilon^-, \\
g_\epsilon^{\text{PMC}}(x, y) = 0 & \text{on } \Gamma_\epsilon^-.
\end{cases}
$$

Since the width of the cavity ϵ tends to zero, we may assume that κ^2 is not the eigenvalue of the Laplacian with the above boundary conditions in the cavity. Thus the Green function of the Helmholtz operator in D_ϵ exists and can be expressed as [15]

$$
g_\epsilon^{\text{PMC}}(x, y) = \sum_{m,n=0}^{\infty} c_{m,n} \phi_{m,n}(x) \phi_{m,n}(y),
$$

where

$$
c_{m,n} = \frac{1}{\kappa^2 - (m\pi/\epsilon)^2 - ((n+1/2)\pi/d)^2},
$$

$$
\phi_{m,n}(x) = \sqrt{\frac{\alpha_{m,n}}{\epsilon d}} \cos\left(\frac{m\pi x_1}{\epsilon}\right) \sin\left(\frac{(n+1/2)\pi}{d}(x_2 + d)\right),
$$

$$
\alpha_{m,n} = \begin{cases} 2, & m = 0, n \geq 0, \\ 4, & m \geq 1, n \geq 0. \end{cases}
\tag{3.2}
$$

Lemma 3.1 *The scattering problem* (3.1) *is equivalent to the following boundary integral equation:*

$$\int_{\Gamma_\epsilon^+} \left(\left(-\frac{i}{2}\right) H_0^1(\kappa|x-y|) + g_\epsilon^{\mathrm{PMC}}(x,y) \right) \frac{\partial u_\epsilon(y)}{\partial \nu_y} \mathrm{d}s_y + u^{\mathrm{inc}}(x) + u^{\mathrm{ref}}(x) = 0, \quad x \in \Gamma_\epsilon^+. \tag{3.3}$$

Proof Noting that $\partial_\nu u^{\mathrm{inc}} + \partial_\nu u^{\mathrm{ref}} = 0$ on Γ_0, especially on Γ_ϵ^+, we have

$$\partial_\nu u_\epsilon^{\mathrm{sc}} = \partial_\nu u_\epsilon - \partial_\nu(u^{\mathrm{inc}} + u^{\mathrm{ref}}) = \partial_\nu u_\epsilon \quad \text{on } \Gamma_\epsilon^+.$$

It follows from Green's identity in $\mathbb{R}_2^+$ that

$$u_\epsilon(x) = u^{\mathrm{inc}}(x) + u^{\mathrm{ref}}(x) + u_\epsilon^{\mathrm{sc}}(x) = u^{\mathrm{inc}}(x) + u^{\mathrm{ref}}(x) + \int_{\Gamma_\epsilon^+} G(x,y) \frac{\partial u_\epsilon^{\mathrm{sc}}(y)}{\partial \nu_y} \mathrm{d}s_y$$

$$= u^{\mathrm{inc}}(x) + u^{\mathrm{ref}}(x) + \int_{\Gamma_\epsilon^+} G(x,y) \frac{\partial u_\epsilon(y)}{\partial \nu_y} \mathrm{d}s_y, \quad x \in \mathbb{R}_2^+, \tag{3.4}$$

where the half-space Green function G is given in (2.24). Using Green's identity inside the cavity yields

$$u_\epsilon(x) = -\int_{\Gamma_\epsilon^+} g_\epsilon^{\mathrm{PMC}}(x,y) \frac{\partial u_\epsilon(y)}{\partial \nu_y} \mathrm{d}s_y, \quad x \in D_\epsilon. \tag{3.5}$$

When x respectively tends to Γ_ϵ^+ from above or below, by the continuity of the single layer potential, we have from (3.4) and (3.5) that

$$u_\epsilon(x) = u^{\mathrm{inc}}(x) + u^{\mathrm{ref}}(x) + \int_{\Gamma_\epsilon^+} \left(-\frac{i}{2}\right) H_0^1(\kappa|x-y|) \frac{\partial u_\epsilon(y)}{\partial \nu_y} \mathrm{d}s_y, \quad x \in \Gamma_\epsilon^+$$

and

$$u_\epsilon(x) = -\int_{\Gamma_\epsilon^+} g_\epsilon^{\mathrm{PMC}}(x,y) \frac{\partial u_\epsilon(y)}{\partial \nu_y} \mathrm{d}s_y, \quad x \in \Gamma_\epsilon^+.$$

The proof is finished by imposing the continuity of the solution on the open aperture Γ_ϵ^+. $\square$

It is clear to note that

$$\partial_\nu u_\epsilon|_{\Gamma_\epsilon^+} = \partial_{x_2} u_\epsilon(x_1, 0), \quad u^{\mathrm{inc}} + u^{\mathrm{ref}} = 2e^{i\kappa \sin\theta x_1}, \quad x_1 \in (0, \epsilon).$$

Introduce new variables by rescaling with respect to ϵ: $X = x_1/\epsilon, Y = y_1/\epsilon$. Define three functions:

$$\varphi(Y) = -\partial_{y_2} u_\epsilon(\epsilon Y, 0),$$

$$G_\epsilon^e(X, Y) = \left(-\frac{i}{2}\right) H_0^{(1)}(\epsilon k|X-Y|),$$

$$G_\epsilon^i(X, Y) = g_\epsilon^{\mathrm{PMC}}(\epsilon X, 0; \epsilon Y, 0) = \sum_{m,n=0}^\infty \frac{c_{m,n}\alpha_{m,n}}{\epsilon d} \cos(m\pi X)\cos(m\pi Y), \tag{3.6}$$

and the boundary integral operators:

$$(T^e\varphi)(X) = \int_0^1 G_\epsilon^e(X, Y)\varphi(Y)\mathrm{d}Y, \quad X \in (0, 1),$$

$$(T^i \varphi)(X) = \int_0^1 G_\epsilon^i(X, Y)\varphi(Y)\mathrm{d}Y, \quad X \in (0, 1). \tag{3.7}$$

Hence the boundary integral equation (3.3) is equivalent to the following operator equation:

$$(T^e + T^i)\varphi = f/\epsilon, \tag{3.8}$$

where $f(X) := (u^{\mathrm{inc}} + u^{\mathrm{ref}})(\epsilon X, 0) = 2e^{i\kappa \sin \theta \epsilon X}$.

3.2 Asymptotics of the integral operators

In this subsection, we study the asymptotic properties of the integral operators in (3.7). For clarity, we first introduce several notation below.

$$\Gamma_1(\kappa, \epsilon) = \frac{1}{\pi}(\ln \kappa + \gamma_1) + \frac{1}{\pi}\ln \epsilon, \quad \Gamma_2(\kappa, \epsilon) = -\frac{\tan \kappa d}{\epsilon \kappa} + \frac{2\ln 2}{\pi},$$

$$R_1^\epsilon(X, Y) = -\frac{\kappa^2}{4\pi}|X - Y|^2\epsilon^2 \ln \epsilon,$$

$$R_2^\epsilon(X, Y) = \left(\gamma_2 - \frac{\ln(\kappa|X - Y|)}{4\pi}\right)\kappa^2|X - Y|^2\epsilon^2 + o((\kappa\epsilon|X - Y|)^2),$$

$$R_3^\epsilon(X, Y) = -\frac{\epsilon^2\kappa^2}{\pi}\left(\frac{1}{\pi^2}\sum_{m=1}^\infty \frac{1}{m^3} + \frac{(X + Y)^2}{4}\ln(\pi(X + Y)) + \frac{(X - Y)^2}{4}\ln(\pi(X - Y))\right.$$

$$\left. + o((X + Y)^2 + (X - Y)^2)\right),$$

where $\gamma_1 = \gamma_0 - \ln 2 - \frac{\pi i}{2}$, $\gamma_2 = -\frac{i}{4\pi} + \frac{i}{8} + \frac{1}{4\pi}(\ln 2 - \gamma_0)$, and γ_0 is the Euler constant. For the fixed depth of the cavity d and the wave number κ, if ϵ small enough, we have the following asymptotic expansions for the kernels G_ϵ^e and G_ϵ^i.

Lemma 3.2 *If $\epsilon \ll \lambda$, then the following estimates hold:*

$$G_\epsilon^e = \Gamma_1(\kappa, \epsilon) + \frac{1}{\pi}\ln|X - Y| + R_1^\epsilon(X, Y) + R_2^\epsilon(X, Y),$$

$$G_\epsilon^i = \Gamma_2(\kappa, \epsilon) + \frac{1}{\pi}\left(\ln\left|\sin\frac{\pi(X + Y)}{2}\right| + \ln\left|\sin\frac{\pi(X - Y)}{2}\right|\right) + R_3^\epsilon(X, Y),$$

where

$$|R_1^\epsilon(X, Y)| \le C_1\epsilon^2|\ln \epsilon|, \quad |R_2^\epsilon(X, Y)| \le C_2\epsilon^2, \quad |R_3^\epsilon(X, Y)| \le C_3\epsilon^2.$$

Here C_j, $j = 1, 2, 3$ are positive constants independent of ϵ.

Proof It follows from the asymptotic of the Hankel function near zero (cf. [1]) that

$$G_\epsilon^e(X, Y) = \left(-\frac{i}{2}\right)H_0^1(\epsilon\kappa|X - Y|)$$

$$= \frac{1}{\pi}(\gamma_1 + \ln|\epsilon\kappa(X - Y)|) - \frac{1}{4\pi}|\epsilon k(X - Y)|^2 \ln|\epsilon\kappa(X - Y)|$$

$$+ \gamma_2(\epsilon\kappa|X - Y|)^2 + o((\kappa\epsilon|X - Y|)^2)$$

$$= \Gamma_1(\kappa, \epsilon) + \frac{1}{\pi}\ln|X - Y| + R_1^\epsilon(X, Y) + R_2^\epsilon(X, Y).$$

The integral kernel $G_\epsilon^i(X, Y)$ can be expressed as

$$G_\epsilon^i(X, Y) = \frac{1}{\epsilon d} \sum_{m,n=0}^{\infty} c_{m,n} \alpha_{m,n} \cos(m\pi X) \cos(m\pi Y). \tag{3.9}$$

Recalling the definitions of $c_{m,n}$ and $\alpha_{m,n}$ in (3.2), we set

$$C_m(\epsilon, \kappa) = \sum_{n=0}^{\infty} c_{m,n} \alpha_{m,n} = \sum_{n=0}^{\infty} \frac{\alpha_{m,n}}{\kappa^2 - \left(\frac{m\pi}{\epsilon}\right)^2 - \left(\frac{(n+\frac{1}{2})\pi}{d}\right)^2}.$$

Using the fact (cf. [20]) that

$$\tan\frac{\pi x}{2} = \frac{4x}{\pi} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2 - x^2}, \quad x \neq n + \frac{1}{2}, n \in \mathbb{Z},$$

we may check for $m = 0$ that

$$C_0(\kappa) = \sum_{n=0}^{\infty} \frac{2}{\kappa^2 - \left(\frac{(n+\frac{1}{2})\pi}{d}\right)^2} = -\frac{d\tan(\kappa d)}{\kappa}.$$

Using the identity

$$\tanh\frac{\pi x}{2} = \frac{4x}{\pi} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2 + x^2},$$

we have for $m \geq 1$ that

$$C_m(\epsilon, \kappa) = \sum_{n=0}^{\infty} \frac{4}{\kappa^2 - \left(\frac{m\pi}{\epsilon}\right)^2 - \left(\frac{(n+\frac{1}{2})\pi}{d}\right)^2}$$

$$= -\frac{2d}{\sqrt{\left(\frac{m\pi}{\epsilon}\right)^2 - \kappa^2}} \tanh\left(d\sqrt{\left(\frac{m\pi}{\epsilon}\right)^2 - \kappa^2}\right)$$

$$= -\frac{2d\epsilon}{m\pi} - \frac{\epsilon^3\kappa^2 d}{m^3\pi^3} + O\left(\frac{-2\epsilon^5\kappa^4 d}{4!m^5\pi^5}\right).$$

Using the following facts (cf. [24,26]):

$$\sum_{m=1}^{\infty} \frac{\cos m\pi x}{m} = -\left(\ln 2 + \ln\left|\sin\frac{\pi x}{2}\right|\right), \quad 0 < x < 2,$$

$$\sum_{m=1}^{\infty} \frac{\cos m\pi x}{m^3} = \sum_{m=1}^{\infty} \frac{1}{m^3} + \frac{(\pi x)^2}{2} \ln(\pi x) + O(x^2), \quad 0 < x < 2,$$

and substituting $C_0(\kappa)$, $C_m(\epsilon, \kappa)(m \geq 1)$ into (3.9), we obtain

$$G_\epsilon^i(X, Y) = \frac{1}{\epsilon d}\left(C_0(\kappa) - \sum_{m=1}^{\infty} \frac{2d\epsilon}{m\pi} \cos(m\pi X) \cos(m\pi Y)\right.$$

$$- \sum_{m=1}^{\infty} \frac{\epsilon^3 \kappa^2 d}{m^3 \pi^3} \cos(m\pi X) \cos(m\pi Y) + O\left(\frac{-2\epsilon^5 \kappa^4 d}{4! m^5 \pi^5}\right)\Bigg)$$

$$= - \frac{\tan(\kappa d)}{\epsilon \kappa} + \frac{1}{\pi}\left(2\ln 2 + \ln\left|\sin\frac{\pi(X+Y)}{2}\right| + \ln\left|\sin\frac{\pi(X-Y)}{2}\right|\right)$$

$$- \frac{\epsilon^2 \kappa^2}{\pi}\left(\frac{1}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{m^3} + \frac{(X+Y)^2}{4}\ln(\pi(X+Y)) + \frac{(X-Y)^2}{4}\ln(\pi(X-Y))\right)$$

$$+ O((X+Y)^2 + (X-Y)^2)\Bigg),$$

which complete the proof. □

Let

$$\Gamma = \Gamma_1(\kappa, \epsilon) + \Gamma_2(\kappa, \epsilon), \quad k_\infty^1(X, Y) = R_1^\epsilon(X, Y), \quad k_\infty^2 = R_2^\epsilon(X, Y) + R_3^\epsilon(X, Y),$$

$$k(X, Y) = \frac{1}{\pi}\left(\ln|X-Y| + \ln\left|\sin\frac{\pi(X+Y)}{2}\right| + \ln\left|\sin\frac{\pi(X-Y)}{2}\right|\right). \tag{3.10}$$

Denote by $K, K_\infty^1, K_\infty^2$ the integral operators corresponding to the Schwarz kernels $k(X, Y), k_\infty^1(X, Y)$ and $k_\infty^2(X, Y)$, respectively.

Let I be a bounded open interval in $\mathbb{R}$ and define

$$H^s(I) := \left\{u = U|_I, \ U \in H^s(\mathbb{R})\right\},$$

where $s \in \mathbb{R}$. Then $H^s(I)$ is a Hilbert space with the norm

$$\|u\|_{H^s(I)} = \inf\left\{\|U\|_{H^s(\mathbb{R})} \,\Big|\, U \in H^s(\mathbb{R}) \text{ and } U|_I = u\right\}.$$

Define

$$\tilde{H}^s(I) := \left\{u = U|_I \,\Big|\, U \in H^s(\mathbb{R}) \text{ and } \mathrm{supp}(U) \subset \bar{I}\right\}.$$

It can be shown that the space $\tilde{H}^s(I)$ is the dual of $H^{-s}(I)$ and the norm for $\tilde{H}^s(I)$ can be defined via the duality (cf. [2]).

We also define the operator $P : \tilde{H}^{-1/2}(0, 1) \to H^{1/2}(0, 1)$ by

$$P\phi(X) = \langle\phi, 1_{(0,1)}\rangle_{L^2(0,1)} 1_{(0,1)},$$

where the duality between $\tilde{H}^{-1/2}(0, 1)$ and $H^{1/2}(0, 1)$ is defined by $\langle\cdot, \cdot\rangle_{L^2(0,1)}$, $1_{(0,1)}$ is the function defined on the interval $(0, 1)$ and is equal to one. It is easy to note that $1_{(0,1)} \in H^{1/2}(0, 1)$.

Lemma 3.3 *The following conclusions hold:*
  (i) The operator $T^e + T^i$ admits the decomposition:

$$T^e + T^i = \Gamma P + K + K_\infty^1 + K_\infty^2.$$

Moreover, K_∞^1 and K_∞^2 are bounded from $\tilde{H}^{-1/2}(0, 1)$ to $H^{1/2}(0, 1)$ with the operator norm satisfying $\|K_\infty^1\| \le \epsilon^2 \ln|\epsilon|$, $\|K_\infty^2\| \le \epsilon^2$ uniformly for wavenumber κ.
  (ii) The operator K is invertible from $\tilde{H}^{-1/2}(0, 1)$ to $H^{1/2}(0, 1)$. Moreover, the constant $q_0 := \langle K^{-1}1_{(0,1)}, 1_{(0,1)}\rangle_{L^2(0,1)} \ne 0$.

Proof The proof of (i) follows directly from the definitions of the operators T^e, T^i in (3.7) and the asymptotic expansions of their kernels given by Lemma 3.2. The proof of (ii) follows directly from [11, Theorem 4.1, Lemma 4.2]. □

3.3 Asymptotics of the resonances

The scattering resonance of (3.1) is defined as a complex wavenumber κ with negative imaginary part such that there is a nontrivial solution to (3.1) when the incident field is zero. This is the characteristic value of the operator $\Gamma P + K + K_\infty^1 + K_\infty^2$ with respect to the variable κ. For simplicity, we write

$$\Gamma P + K + K_\infty^1 + K_\infty^2 := \mathscr{P} + \mathscr{L},$$

where $\mathscr{P} = \Gamma P$, $\mathscr{L} = K + K_\infty^1 + K_\infty^2$. By Lemma 3.3, it is easy to see that $\mathscr{L}$ is invertible for sufficiently small ϵ. Now assume that there exists φ_0 such that

$$(\mathscr{P} + \mathscr{L})\varphi_0 = 0.$$

Then

$$(\mathscr{L}^{-1}\mathscr{P} + \mathscr{I})\varphi_0 = 0,$$

where $\mathscr{I}$ is the identity operator. It is straightforward to check that the eigenvalues of operator $\mathscr{L}^{-1}\mathscr{P} + \mathscr{I}$ are

$$\lambda(\kappa, \epsilon) = 1 + \Gamma(\kappa, \epsilon)\langle \mathscr{L}^{-1}1_{(0,1)}, 1_{(0,1)}\rangle_{L^2(0,1)}.$$

Therefore, the characteristic values of the operator $\mathscr{P} + \mathscr{L}$ are the roots of the analytic functions $\lambda(\kappa, \epsilon)$, and the associated characteristic function is given by

$$\varphi_0 = \Gamma(\kappa, \epsilon)\mathscr{L}^{-1}1_{(0,1)}.$$

Lemma 3.4 *The resonance of the scattering problem* (3.1) *are the roots of the analytic functions* $\lambda(\kappa, \epsilon) = 0$. *Moreover,*

$$\mathscr{L}^{-1}1_{(0,1)} = K^{-1}1_{(0,1)} + O(\epsilon^2 \ln \epsilon) + O(\epsilon^2), \quad \langle \mathscr{L}^{-1}1_{(0,1)}, 1_{(0,1)}\rangle_{L^2(0,1)} = q_0$$
$$+O(\epsilon^2 \ln \epsilon) + O(\epsilon^2). \tag{3.11}$$

Proof For given roots of $\lambda(\kappa, \epsilon)$, it is easy to check that they are the characteristic values of the operator $\mathscr{P} + \mathscr{L}$ with corresponding characteristic function defined above.

It follows from the asymptotic expansions in Lemma 3.3 and the Neumann series that

$$\mathscr{L}^{-1} = (K + K_\infty^1 + K_\infty^2)^{-1} = K^{-1}\left(\sum_{j=0}^{\infty}(-1)^j(K^{-1}(K_\infty^1 + K_\infty^2))^j\right)$$
$$= K^{-1} + O(\epsilon^2 \ln \epsilon) + O(\epsilon^2),$$

which gives (3.11). □

Next, we present the asymptotic expansion of resonances for the scattering problem (3.1).

Theorem 3.5 *The scattering problem* (3.1) *has a set of resonances* $\{k_n\}$, *which satisfy*

$$k_n = \frac{n\pi}{d} + \frac{n\pi}{d^2}\left(\frac{1}{\pi}\epsilon \ln \epsilon + \left(\frac{1}{q_0} + \frac{1}{\pi}(2 \ln 2 + \ln \frac{n\pi}{d} + \gamma_1)\right)\epsilon\right) + O(\epsilon^2 \ln \epsilon), \quad n = 1, 2, \cdots. \tag{3.12}$$

Proof By Lemma 3.4, we consider the root of

$$\lambda(\kappa, \epsilon) = 1 + (\Gamma_1(\kappa, \epsilon) + \Gamma_2(\kappa, \epsilon))\langle \mathscr{L}^{-1} 1_{(0,1)}, 1_{(0,1)}\rangle_{L^2(0,1)} = 0.$$

Recall that $\Gamma_1(\kappa, \epsilon) = \frac{1}{\pi}(\ln \kappa + \gamma_1) + \frac{1}{\pi} \ln \epsilon$ and $\Gamma_2(\kappa, \epsilon) = -\frac{\tan \kappa d}{\epsilon \kappa} + \frac{2 \ln 2}{\pi}$. The above equation can be written as

$$1 + \left(-\frac{\tan \kappa d}{\epsilon \kappa} + \frac{1}{\pi}(2 \ln 2 + \ln \kappa + \gamma_1) + \frac{1}{\pi} \ln \epsilon\right)\langle \mathscr{L}^{-1} 1_{(0,1)}, 1_{(0,1)}\rangle_{L^2(0,1)} = 0.$$

Using Lemma 3.4, we obtain

$$p(\kappa, \epsilon) := \epsilon\lambda(\kappa, \epsilon) = \epsilon + \left(-\frac{\tan \kappa d}{\kappa} + \epsilon\rho(\kappa) + \frac{1}{\pi}\epsilon \ln \epsilon\right)\left(q_0 + O(\epsilon^2 \ln \epsilon) + O(\epsilon^2)\right), \tag{3.13}$$

where $\rho(\kappa) = \frac{1}{\pi}(2 \ln 2 + \ln \kappa + \gamma_1)$.

Note that the axis $\{z \in \mathbb{C} : \arg z = \pm\frac{\pi}{2d}\}$ is the branch cut for $p(\kappa, \epsilon)$, hence we choose a small number $\theta_0 > 0$ and consider the domain $\{z \in \mathbb{C} : -\frac{\pi}{2d} + \theta_0 < \arg z < \frac{\pi}{2d} - \theta_0 \text{ or } \frac{\pi}{2d} + \theta_0 < \arg z < \frac{3\pi}{2d} - \theta_0\}$. On the other hand, we are only interested in those resonances which are not in the high frequency regime. Therefore, we only need to find all the roots of $p(\kappa, \epsilon)$ in the domain

$$\Omega_{\theta_0, M} = \left\{z \in \mathbb{C} : |z| \leq M, \ -\frac{\pi}{2d} + \theta_0 < \arg z < \frac{\pi}{2d} - \theta_0 \text{ or } \frac{\pi}{2d} + \theta_0 < \arg z < \frac{3\pi}{2d} - \theta_0\right\}$$

for some fixed number $M > 0$. Since $p(\kappa, \epsilon)$ blows up as $\kappa \to 0$ or $\kappa \to (j+1/2)\pi/d$, $j \in \mathbb{Z}$. As a result, there exists $\delta_0 > 0$ such that all the roots of p in $\Omega_{\theta_0, M}$ actually lies in the smaller domain

$$\Omega_{\delta_0, \theta_0, M} := \{z \in \mathbb{C} : |z| \geq \delta_0 \text{ or } |z - (j+1/2)\pi/d| \geq \delta_0, \ j \in \mathbb{Z}\} \cap \Omega_{\theta_0, M}.$$

It is clear that $-\frac{\tan \kappa d}{\kappa}$ is analytic in $\Omega_{\delta_0, \theta_0, M}$ and its roots are given by $k_{n,0} = n\pi/d$, $n = 1, 2, \cdots$. Note that each root is simple. Denote by k_n the roots of $\lambda(\kappa, \epsilon)$. From Rouche's theorem, we deduce that k_n are also simple and are close to $k_{n,0}$ if ϵ is sufficiently small. We now derive the leading order asymptotic terms for these roots. Define

$$p_1(\kappa, \epsilon) = \epsilon + \left(-\frac{\tan \kappa d}{\kappa} + \epsilon\rho(\kappa) + \frac{1}{\pi}\epsilon \ln \epsilon\right)q_0. \tag{3.14}$$

Expanding $p_1(\kappa, \epsilon)$ at $k_{n,0}$ yields

$$p_1(\kappa, \epsilon) = \epsilon + \left(\left(-\frac{\tan \kappa d}{\kappa}\right)'\Big|_{\kappa = k_{n0}}(\kappa - k_{n0}) + \epsilon\rho(k_{n0}) + \epsilon\rho'(k_{n0})(\kappa - k_{n0})\right.$$

$$\left. + O(\kappa - k_{n0})^2 + \frac{1}{\pi}\epsilon \ln \epsilon\right)q_0$$

$$= \epsilon + \left(\frac{-d^2}{n\pi}(\kappa - k_{n0}) + \frac{\epsilon}{\pi}\left(2 \ln 2 + \ln \frac{n\pi}{d} + \gamma_1\right) + \frac{\epsilon d}{n\pi^2}(\kappa - k_{n0})\right)$$

$$+ O(\kappa - k_{n0})^2 + \frac{1}{\pi} \epsilon \ln \epsilon \Big) q_0.$$

We can conclude that $p_1(\kappa, \epsilon)$ has simple roots in $\Omega_{\delta_0, \theta_0, M}$ which are close to k_{n0}'s. Moreover, these roots are analytic with respect to the variables ϵ and $\epsilon \ln \epsilon$. Denoting the roots of p_1 by k_{n1} and expanding them in term of ϵ and $\epsilon \ln \epsilon$, we obtain

$$k_{n1} = k_{n0} + \frac{k_{n0}}{d} \left(\frac{1}{\pi} \epsilon \ln \epsilon + \left(\frac{1}{q_0} + \frac{1}{\pi} (2 \ln 2 + \ln \frac{n\pi}{d} + \gamma_1) \right) \epsilon \right) + O(\epsilon^2 \ln \epsilon). \quad (3.15)$$

Using Rouche's theorem, we claim that k_{n1} gives the leading order for the asymptotic expansion of the roots k_n. More precisely, we have

$$k_n = k_{n1} + O(\epsilon^2 \ln \epsilon). \quad (3.16)$$

In the following, we give a proof of the claim. Note that

$$p(\kappa, \epsilon) - p_1(\kappa, \epsilon) = \left(-\frac{\tan \kappa d}{\kappa} + \frac{1}{\pi} \epsilon \ln \epsilon + \rho(\kappa) \epsilon \right) O(\epsilon^2 \ln \epsilon)$$

and

$$p_1(\kappa, \epsilon) = -\frac{\tan \kappa d}{\kappa} q_0 + \frac{q_0}{\pi} \epsilon \ln \epsilon + (1 + \rho(\kappa) q_0) \epsilon.$$

Hence there exists a constant C_n such that

$$|p(\kappa, \epsilon) - p_1(\kappa, \epsilon)| < |p_1(\kappa, \epsilon)|, \quad \forall \kappa \in \left\{ |\kappa - k_{n1}| = C_n \epsilon^2 \ln \epsilon \right\}.$$

which implies the claim (3.16) by Rouche's theorem. $\qquad \square$

3.4 The field enhancement with resonance

First we give an asymptotic expansion of $p(\kappa, \epsilon)$ at the resonant frequencies.

Lemma 3.6 *At the resonant frequencies* $\kappa = \mathrm{Re} k_n$ *where* k_n *is given in* (3.12), *we have*

$$p(\kappa, \epsilon) = -\frac{iq_0}{2} \epsilon + O(\epsilon^2 \ln^2 \epsilon).$$

Proof Assume that $|\kappa - \mathrm{Re} k_n| \le \epsilon \ln \epsilon$. It follows from the definition of p_1 in (3.14) that

$$\begin{aligned}
p(\kappa, \epsilon) &= p_1(\kappa, \epsilon) + O(\epsilon^2 \ln \epsilon) \\
&= p_1'(k_n)(\kappa - k_n) + O(\kappa - k_n)^2 + O(\epsilon^2 \ln \epsilon) \\
&= \left(\frac{-d}{k_{n0}} + \frac{\epsilon}{\pi k_{n0}} + O(|k_n - k_{n0}|) \right) q_0(\kappa - k_n) + O(\epsilon^2 \ln^2 \epsilon) \\
&= \left(\frac{-d}{k_{n0}} + \frac{\epsilon}{\pi k_{n0}} \right) q_0(\kappa - \mathrm{Re} k_n - i\mathrm{Im} k_n) + O(\epsilon^2 \ln^2 \epsilon).
\end{aligned}$$

Recalling $\gamma_1 = \gamma_0 - \ln 2 - \frac{\pi i}{2}$, we have from the expansion of k_{n1} in (3.15) that

$$\mathrm{Im} k_n = \mathrm{Im} k_{n1} + O(\epsilon^2 \ln \epsilon) = \frac{k_{n0}}{d\pi} (\mathrm{Im} \gamma_1) \epsilon = -\frac{k_{n0} \epsilon}{2d}.$$

When $\kappa = \mathrm{Re} k_n$, we have

$$p(\kappa, \epsilon) = iq_0 \left(-\frac{1}{2} \epsilon + \frac{1}{2d} \epsilon^2 \right) + O(\epsilon^2 \ln^2 \epsilon) = -\frac{iq_0}{2} \epsilon + O(\epsilon^2 \ln^2 \epsilon),$$

which completes the proof.   □

Lemma 3.7 *The solution φ of equation* (3.8) *has the following asymptotic expansion in* $\tilde{H}^{-1/2}(0, 1)$:

$$\varphi = K^{-1}1_{(0,1)} \cdot \frac{(2 + \sin\theta \cdot O(\kappa\epsilon))}{p} + O(\epsilon^2 \ln \epsilon).$$

Moreover,

$$\langle \varphi, 1_{(0,1)}\rangle_{L^2(0,1)} = \frac{q_0 (2 + \sin\theta \cdot O(\kappa\epsilon))}{p} + O(\epsilon^2 \ln \epsilon).$$

Proof It follows from Lemma 3.3 that equation (3.8) is equivalent to

$$(\mathscr{L}^{-1}\mathscr{P} + \mathscr{I})\varphi = \mathscr{L}^{-1}(f/\epsilon).$$

Recall that the operator $\mathscr{L}^{-1}\mathscr{P} + \mathscr{I}$ has the eigenvalue $\lambda(\kappa, \epsilon)$. Thus

$$\langle \varphi, 1_{(0,1)}\rangle_{L^2(0,1)} = \frac{1}{\lambda(\kappa, \epsilon)}\langle \mathscr{L}^{-1}(f/\epsilon), 1_{(0,1)}\rangle_{L^2(0,1)},$$

and

$$\varphi = \mathscr{L}^{-1}(f/\epsilon) - \mathscr{L}^{-1}\Gamma P\varphi = \mathscr{L}^{-1}(f/\epsilon) - \mathscr{L}^{-1}\Gamma\langle\varphi, 1_{(0,1)}\rangle 1_{(0,1)}$$

$$= \mathscr{L}^{-1}(f/\epsilon) - \mathscr{L}^{-1}1_{(0,1)}\frac{\Gamma}{\lambda(\kappa, \epsilon)}\langle \mathscr{L}^{-1}(f/\epsilon), 1_{(0,1)}\rangle_{L^2(0,1)}.$$

Recall that $\lambda(\kappa, \epsilon) = 1 + \Gamma\langle\mathscr{L}^{-1}1_{(0,1)}, 1_{(0,1)}\rangle_{L^2(0,1)}$. A simple calculation yields

$$\varphi = \mathscr{L}^{-1}(f/\epsilon) - \mathscr{L}^{-1}1_{(0,1)}\frac{(\lambda(\kappa, \epsilon)-1)/\langle\mathscr{L}^{-1}1_{(0,1)}, 1_{(0,1)}\rangle_{L^2(0,1)}}{\lambda(\kappa, \epsilon)}\langle\mathscr{L}^{-1}(f/\epsilon), 1_{(0,1)}\rangle_{L^2(0,1)}.$$

Using the definition of f and expanding it with respect to ϵ, we get

$$\frac{f}{\epsilon} = \frac{2e^{i\kappa\epsilon X \sin\theta}}{\epsilon} = \frac{2}{\epsilon} + i\kappa X \sin\theta + O(\kappa^2\epsilon),$$

which gives

$$\mathscr{L}^{-1}(f/\epsilon) = \frac{1}{\epsilon}(2 + \sin\theta \cdot O(\kappa\epsilon))\left(K^{-1}1_{(0,1)} + O(\epsilon^2 \ln \epsilon)\right).$$

Combining the above equation with (3.11), we obtain

$$\epsilon\varphi = (2 + \sin\theta \cdot O(\kappa\epsilon))\left(K^{-1}1_{(0,1)} + O(\epsilon^2 \ln \epsilon)\right)$$

$$+ \frac{(1-\lambda)/(q_0 + O(\epsilon^2 \ln \epsilon))}{\lambda} \cdot (2q_0 + q_0 \sin\theta \cdot O(\kappa\epsilon)) \cdot \left(K^{-1}1_{(0,1)} + O(\epsilon^2 \ln \epsilon)\right)$$

$$= \frac{(2 + \sin\theta \cdot O(\kappa\epsilon))}{\lambda}\left(K^{-1}1_{(0,1)} + O(\epsilon^2 \ln \epsilon)\right).$$

Thus

$$\varphi = K^{-1}1_{(0,1)} \cdot \frac{(2 + \sin\theta \cdot O(\kappa\epsilon))}{p} + O(\epsilon^2 \ln \epsilon),$$

$$\langle \varphi, 1_{(0,1)}\rangle_{L^2(0,1)} = \frac{q_0 (2 + \sin\theta \cdot O(\kappa\epsilon))}{p} + O(\epsilon^2 \ln \epsilon),$$

which completes the proof.   □

Combining Lemmas 3.6 and 3.7, we obtain at the resonant frequencies $\kappa = \mathrm{Re}k_n$ that

$$\varphi = \frac{f_1}{\epsilon} + O(\epsilon^2 \ln \epsilon), \quad \langle \varphi, 1_{(0,1)} \rangle_{L^2(0,1)} = \frac{c_2}{\epsilon} + O(\epsilon^2 \ln \epsilon), \tag{3.17}$$

where

$$f_1 = K^{-1} 1_{(0,1)} \cdot \frac{2 + \sin\theta \cdot O(\kappa\epsilon)}{-\frac{i}{2}q_0 + O(\epsilon \ln \epsilon)} \in \tilde{H}^{-1/2}(0,1), \quad c_2 = \frac{q_0\,(2 + \sin\theta \cdot O(\kappa\epsilon))}{-\frac{i}{2}q_0 + O(\epsilon \ln \epsilon)}$$

3.4.1 Enhancement in the far field

We first investigate the scattered field in the domain $\mathbb{R}_2^+ \setminus \bar{B}_R^+$. Recall that

$$u_\epsilon^{\mathrm{sc}}(x) = \int_{\Gamma_\epsilon^+} G(x, y) \frac{\partial u_\epsilon^{\mathrm{sc}}(y)}{\partial \nu_y} \mathrm{d}s_y = \int_{\Gamma_\epsilon^+} G(x, y) \frac{\partial u_\epsilon(y)}{\partial \nu_y} \mathrm{d}s_y, \quad x \in \mathbb{R}_2^+.$$

Here we use the fact $\partial_\nu u^{\mathrm{inc}} + \partial_\nu u^{\mathrm{ref}} = 0$ on Γ_0, especially on Γ_ϵ^+. Using

$$\frac{\partial u_\epsilon(y)}{\partial \nu_y}\bigg|_{\Gamma_\epsilon^+} = \frac{\partial u_\epsilon(y_1, 0)}{\partial y_2} = -\varphi\left(\frac{y_1}{\epsilon}\right),$$

we have

$$u_\epsilon^{\mathrm{sc}}(x) = -\int_{\Gamma_\epsilon^+} G(x, (y_1, 0))\varphi\left(\frac{y_1}{\epsilon}\right) \mathrm{d}y_1 = -\epsilon \int_0^1 G(x, (\epsilon Y, 0))\varphi(Y)\mathrm{d}Y.$$

Since

$$G(x, (\epsilon Y, 0)) = G(x, (0, 0))(1 + O(\epsilon)), \quad x \in \mathbb{R}_2^+ \setminus \bar{B}_R^+,$$

we get from Lemma 3.7 that

$$\int_0^1 \varphi(Y)\mathrm{d}Y = \langle \varphi, 1_{(0,1)} \rangle_{L^2(0,1)} = \frac{q_0\,(2 + \sin\theta \cdot O(\kappa\epsilon))}{p} + O(\epsilon^2 \ln \epsilon).$$

Hence

$$u_\epsilon^{\mathrm{sc}}(x) = -\epsilon G(x, (0,0))(1 + O(\epsilon)) \int_0^1 \varphi(Y)\mathrm{d}Y$$

$$= -\epsilon G(x, (0,0)) \cdot \frac{q_0\,(2 + \sin\theta \cdot O(\kappa\epsilon))}{p} + O(\epsilon^2 \ln \epsilon).$$

In the resonance case when $\kappa = \mathrm{Re}k_n$, the enhancement comes from the term $\frac{1}{p}$. It follows from Lemma 3.6 that

$$\frac{1}{p} = \frac{2\mathrm{i}}{q_0\epsilon}(1 + O(\epsilon \ln^2 \epsilon)).$$

Correspondingly, we have

$$u_\epsilon^{\mathrm{sc}}(x) = -2\mathrm{i}G(x, (0,0)) \cdot (2 + \sin\theta \cdot O(\kappa\epsilon)) + O(\epsilon \ln^2 \epsilon),$$

which shows that the enhancement of the scattered magnetic field in the far field region has an order $O(1/\epsilon)$ compared to the non-resonance case when $\kappa \neq \mathrm{Re}k_n$.

It follows from the Ampere law $\nabla \times H_\epsilon = -\mathrm{i}\omega\varepsilon E_\epsilon$ and $H_\epsilon = (0, 0, u_\epsilon)$ that the enhancement of the scattered electric field also has an order $O(1/\epsilon)$ in the far field region when $\kappa = \mathrm{Re}k_n$.

3.4.2 Enhancement in the cavity

Now we study the field enhancement in the cavity. It follows from (2.5) that the total field u_ϵ may be expanded as the sum of waveguide modes:

$$
u_\epsilon(x) = \frac{1}{\sqrt{\epsilon}} \left(\alpha_0^+ e^{-i\kappa x_2} + \alpha_0^- e^{i\kappa(x_2+d)} \right) + \sum_{m \geq 1} \sqrt{\frac{2}{\epsilon}} \alpha_m^+ \cos\left(\frac{m\pi x_1}{\epsilon} \right) e^{-i\beta_m x_2}
$$

$$
+ \sum_{m \geq 1} \sqrt{\frac{2}{\epsilon}} \alpha_m^- \cos\left(\frac{m\pi x_1}{\epsilon} \right) e^{i\beta_m(x_2+d)}, \tag{3.18}
$$

where $\beta_m = i\sqrt{(m\pi/\epsilon)^2 - \kappa^2}$.

Lemma 3.8 *The coefficients in* (3.18) *have the following expansions:*

$$
\frac{\alpha_0^+}{\sqrt{\epsilon}} = -\frac{q_0(2 + \sin\theta \cdot O(\kappa\epsilon))e^{-i\kappa d}}{2i\kappa \cos(\kappa d)p} + O(\epsilon^2 \ln \epsilon),
$$

$$
\frac{\alpha_0^-}{\sqrt{\epsilon}} = \frac{q_0(2 + \sin\theta \cdot O(\kappa\epsilon))}{2i\kappa \cos(\kappa d)p} + O(\epsilon^2 \ln \epsilon), \tag{3.19}
$$

and

$$
\sqrt{\frac{2}{\epsilon}} |\alpha_m^+| \leq \frac{C}{\sqrt{m}}, \quad \sqrt{\frac{2}{\epsilon}} |\alpha_m^-| \leq \frac{C}{\sqrt{m}},
$$

where the positive constant C independents of ϵ, κ and m.

Proof Recall that $u_\epsilon = 0$ on Γ_ϵ^- then equation (3.18) becomes

$$
u_\epsilon(x) = \frac{\alpha_0^-}{\sqrt{\epsilon}} \left(-e^{-i\kappa(x_2+d)} + e^{i\kappa(x_2+d)} \right)
$$

$$
+ \sum_{m \geq 1} \sqrt{\frac{2}{\epsilon}} \alpha_m^- \left(-e^{-i\beta_m(x_2+d)} + e^{i\beta_m(x_2+d)} \right) \cos\left(\frac{m\pi x_1}{\epsilon} \right). \tag{3.20}
$$

Taking the derivative of (3.20) with respect to x_2 yields

$$
\frac{\partial u_\epsilon(x)}{\partial x_2} = \frac{\alpha_0^-}{\sqrt{\epsilon}} i\kappa \left(e^{-i\kappa(x_2+d)} + e^{i\kappa(x_2+d)} \right)
$$

$$
+ \sum_{m \geq 1} \sqrt{\frac{2}{\epsilon}} \alpha_m^- i\beta_m \left(e^{-i\beta_m(x_2+d)} + e^{i\beta_m(x_2+d)} \right) \cos\left(\frac{m\pi x_1}{\epsilon} \right),
$$

which gives

$$
\frac{\partial u_\epsilon}{\partial x_2}(x_1, 0) = \frac{\alpha_0^-}{\sqrt{\epsilon}} 2i\kappa \cos(\kappa d) + \sum_{m \geq 1} \sqrt{\frac{2}{\epsilon}} \alpha_m^- 2i\beta_m \cos(\beta_m d) \cos\left(\frac{m\pi x_1}{\epsilon} \right). \tag{3.21}
$$

Multiplying (3.21) with $\phi_0(x_1)$ and integrating it over Γ_ϵ^+, we get

$$\frac{\alpha_0^-}{\sqrt{\epsilon}} i\kappa 2 \cos(\kappa d) = \frac{1}{\epsilon} \int_{\Gamma_\epsilon^+} \frac{\partial u_\epsilon}{\partial x_2}(x_1, 0) dx_1 = \frac{1}{\epsilon} \int_{\Gamma_\epsilon^+} -\varphi\left(\frac{x_1}{\epsilon}\right) dx_1$$

$$= \int_0^1 \varphi(X) dX = \langle \varphi, 1_{(0,1)} \rangle_{L^2(0,1)}$$

$$= \frac{q_0 (2 + \sin\theta \cdot O(\kappa\epsilon))}{p} + O(\epsilon^2 \ln\epsilon),$$

which gives the formulas for α_0^+ and α_0^- in (3.19).

For $m \geq 1$, it follows from multiplying(3.21) with $\cos\left(\frac{m\pi x_1}{\epsilon}\right)$ and integrating over Γ_ϵ^+ that

$$\sqrt{\frac{2}{\epsilon}} \alpha_m^- i\beta_m \cos(\beta_m d) = \frac{1}{\epsilon} \int_{\Gamma_\epsilon^+} \frac{\partial u_\epsilon}{\partial x_2}(x_1, 0) \cos\left(\frac{m\pi x_1}{\epsilon}\right) dx_1$$

$$= -\int_0^1 \varphi(X) \cos(m\pi X) dX.$$

Note that $i\beta_m = O\left(\frac{m}{\epsilon}\right)$ for $m \geq 1$, and by (3.17) that $\|\varphi\|_{H^{-1/2}(0,1)} \lesssim \frac{1}{\epsilon}$, $\|\cos(m\pi X)\|_{H^{1/2}(0,1)} \lesssim \sqrt{m}$. Thus, we get

$$\sqrt{\frac{2}{\epsilon}} |\alpha_m^-| \leq \frac{C}{\sqrt{m}},$$

which completes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

In the following, we give the field enhancement inside the cavity.

Theorem 3.9 *Denote by $\mathring{D}_\epsilon = \{ x \in D_\epsilon | -d + \epsilon \leq x_2 \leq -\epsilon \}$ the interior of the cavity. If $\kappa = \mathrm{Re}k_n$ where k_n is given in (3.12), then we have for $\epsilon \ll d$ that*

$$u_\epsilon(x) = \left(\frac{2}{\epsilon} + \sin\theta \cdot O(1) + O(\ln^2 \epsilon)\right) \frac{2i \sin\kappa(x_2 + d)}{\kappa \cos(\kappa d)} + O(\epsilon^2 \ln\epsilon), \quad x \in \mathring{D}_\epsilon.$$

Moreover, the electric and magnetic field enhancements are of an order $O(1/\epsilon)$.

Proof By the expansion (3.18) and Lemma 3.8, it is clear that in the region $\mathring{D}_\epsilon$

$$u_\epsilon(x) = \frac{q_0(2 + \sin\theta \cdot O(\kappa\epsilon))}{2i\kappa \cos(\kappa d) p} \left(-e^{-i\kappa(x_2+d)} + e^{i\kappa(x_2+d)}\right) + O(\epsilon^2 \ln\epsilon) + O(e^{-1/\epsilon})$$

$$= \frac{q_0(2 + \sin\theta \cdot O(\kappa\epsilon)) \sin\kappa(x_2 + d)}{\kappa \cos\kappa d} \frac{1}{p} + O(\epsilon^2 \ln\epsilon) + O(e^{-1/\epsilon}).$$

Noting when $\kappa = \mathrm{Re}k_n$, we have

$$\frac{1}{p} = \frac{2i}{q_0\epsilon}(1 + O(\epsilon \ln^2 \epsilon)),$$

which yields that

$$u_\epsilon(x) = \left(\frac{2}{\epsilon} + \sin\theta \cdot O(1) + O(\ln^2 \epsilon)\right) \frac{2i \sin\kappa(x_2 + d)}{\kappa \cos(\kappa d)} + O(\epsilon^2 \ln\epsilon).$$

Therefore, the magnetic field enhancement due to the resonance is of an order $O(1/\epsilon)$ in $\mathring{D}_\epsilon$. We conclude from Ampere's law that the electric field enhancement is also of an order $O(1/\epsilon)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

3.4.3 Field enhancement on the open aperture

Now we present the enhancement of the scattered field on the open aperture of the cavity. Recall (3.4) for the scattered field on Γ_ϵ^+:

$$u_\epsilon^{\mathrm{sc}}(x_1, 0) = \int_{\Gamma_\epsilon^+} G((x_1, 0), y) \frac{\partial u_\epsilon(y)}{\partial \nu_y} \mathrm{d}s_y.$$

Since $x_1 = \epsilon X$, $y_1 = \epsilon Y$, $\varphi = -\partial_\nu u_\epsilon |_{\Gamma_\epsilon^+}$, we have

$$u_\epsilon^{\mathrm{sc}}(x_1, 0) = -\int_0^1 G_\epsilon^e(X, Y) \epsilon \varphi(Y) \mathrm{d}Y.$$

It follows from the asymptotic expansion of G_ϵ^e in Lemma 3.2 and Lemma 3.7 that

$$
\begin{aligned}
u_\epsilon^{\mathrm{sc}}(x_1, 0) &= -\epsilon \int_0^1 \left(\Gamma_1 + \frac{1}{\pi} \ln|X - Y| + O(\epsilon^2 \ln \epsilon) \right) \varphi(Y) \mathrm{d}Y \\
&= -\epsilon \Gamma_1 \langle \varphi, 1_{(0,1)} \rangle_{L^2(0,1)} - \epsilon \frac{2 + \sin\theta \cdot O(\kappa\epsilon)}{p} \frac{1}{\pi} h_2(X) + O(\epsilon^2 \ln \epsilon) \\
&= -\epsilon \left(\frac{1}{\pi} (\ln\kappa + \gamma_1) + \frac{1}{\pi} \ln\epsilon \right) \left(\frac{q_0(2 + \sin\theta \cdot O(\kappa\epsilon))}{p} + O(\epsilon^2 \ln\epsilon) \right) \\
&\quad - \epsilon \frac{2 + \sin\theta \cdot O(\kappa\epsilon)}{p} \frac{1}{\pi} h_2(X) + O(\epsilon^2 \ln\epsilon) \\
&= -\frac{1}{\pi} \frac{2q_0}{p} \epsilon \ln\epsilon - \frac{1}{\pi} \frac{2}{p} (q_0(\ln\kappa + \gamma_1) + h_2(X)) \epsilon \\
&\quad - \frac{1}{\pi} (q_0(\ln\kappa + \gamma_1) + h_2(X)) \frac{\sin\theta \cdot O(1)}{p} \epsilon^2 \\
&\quad - \frac{1}{\pi} \frac{q_0 \sin\theta \cdot O(1)}{p} \epsilon^2 \ln\epsilon + O(\epsilon^2 \ln\epsilon).
\end{aligned}
$$

where $h_2(X) := \int_0^1 K^{-1} 1_{(0,1)}(Y) \ln|X - Y| \mathrm{d}Y$. Note that at the resonant frequencies $\kappa = \mathrm{Re} k_n$,

$$\frac{1}{p} = \frac{2\mathrm{i}}{q_0 \epsilon} (1 + O(\epsilon \ln^2 \epsilon)).$$

Thus we have

$$u_\epsilon^{\mathrm{sc}}(x_1, 0) = -\frac{4\mathrm{i}}{\pi} \ln\epsilon - \frac{4\mathrm{i}}{\pi} \left((\ln\kappa + \gamma_1) + \frac{h_2(x_1/\epsilon)}{q_0} \right) + O(\epsilon \ln^3 \epsilon).$$

It is clear to note that the leading order of the resonant mode is a constant on the open aperture with an order of $O(\ln \epsilon)$ and the magnetic field enhancement is of an order $O(1/\epsilon)$.

On the other hand, since

$$\partial_{x_1} u_\epsilon^{\mathrm{sc}}(x_1, 0) = -\frac{2 + \sin\theta \cdot O(\kappa\epsilon)}{p} \frac{1}{\pi} h_2'(x_1/\epsilon) + O(\epsilon^2 \ln\epsilon),$$

which implies that at the resonant frequencies $\kappa = \mathrm{Re} k_n$, on the top aperture Γ_ϵ^+, the scatter field satisfies

$$\partial_{x_1} u^{\mathrm{sc}}(x_1, 0) = -\frac{4\mathrm{i}(1 + O(\epsilon \ln^2 \epsilon))}{\pi q_0 \epsilon} h'(x_1/\epsilon) - \frac{2\mathrm{i} \sin\theta \cdot O(1)}{\pi q_0} h_2'(x_1/\epsilon) + O(\epsilon \ln^2 \epsilon).$$
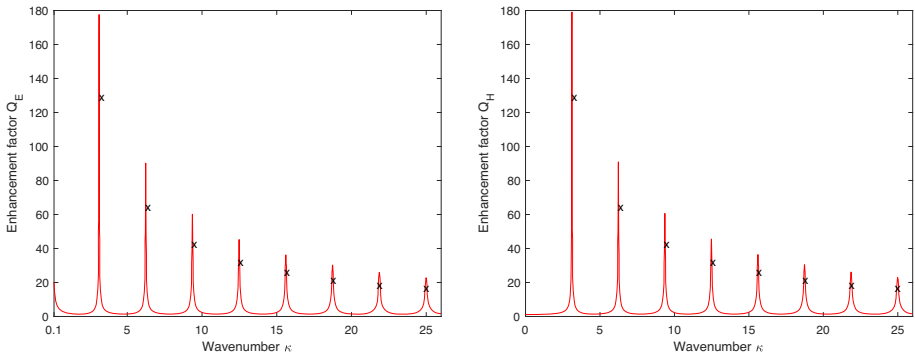
Fig. 3 The field enhancement factor is plotted against the wavenumber κ for the PEC-PMC cavity: (left) the electric field enhancement factor Q_E; (righ) the magnetic field enhancement factor Q_H

By Ampere's law, we know the electric field enhancement is also of an order $O(1/\epsilon)$ on the top aperture Γ_ϵ^+ when $\kappa = \mathrm{Re}\, k_n$.

3.5 Numerical experiments

In this section, we present some numerical experiments to verify the theoretical studies of the field enhancement for the PEC-PMC cavity. As a representative example, we take the cavity width $\epsilon = 0.005$, the cavity depth $d = 1$, and the angle of incidence $\theta = \pi/3$ in the numerics. Figure 3 (left) and (right) show the plot of the electric field enhancement Q_E the magnetic field enhancement factor Q_H against the wavenumber κ, respectively. We observe that both the electric and magnetic enhancement factors do reach peaks at the resonant frequencies (3.12), which are close to $\kappa = n\pi, n = 1, 2, \ldots$. Besides the peaks at the resonant frequencies, the electric field enhancement factor Q_E also displays a peak when the wavenumber κ is sufficiently small, i.e., when the wavelength of the incident field λ is sufficiently large (see Fig. 4). In this situation, the scales satisfy $\epsilon \ll d \ll \lambda$. The electric field should have an enormous enhancement according to our theoretical result in Theorem 2.8. In contrast, the magnetic field enhancement factor Q_H does not show a peak when the wavenumber approaches zero, which is also confirmed in Theorem 2.8 that the magnetic field has no enhancement when $\epsilon \ll d \ll \lambda$.

4 PEC-PEC cavity without resonance

In this section, assuming $\epsilon \ll d \ll \lambda$, we show that there is a weak enhancement of the electric field and there is no enhancement of the magnetic field in the PEC-PEC cavity.

4.1 Problem formulation

Consider the following model problem of the electromagnetic scattering by an open cavity:

$$
\begin{cases}
\Delta u_\epsilon + \kappa^2 u_\epsilon = 0 & \text{in } \Omega, \\
\partial_\nu u_\epsilon = 0 & \text{on } \partial\Omega, \\
\lim_{r \to \infty} \sqrt{r} \left(\partial_r u_\epsilon^{\mathrm{sc}} - \mathrm{i}\kappa u_\epsilon^{\mathrm{sc}} \right) = 0 & \text{in } \mathbb{R}_+^2,
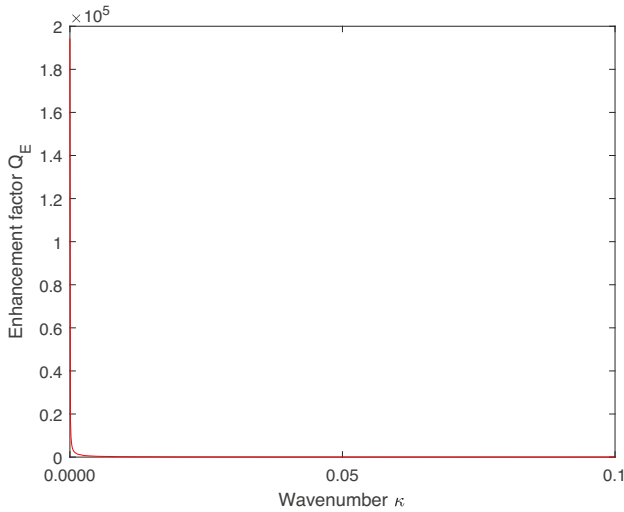\end{cases}
\tag{4.1}
$$

Fig. 4 The amplification of electric field in the PEC-PMC cavity with $\epsilon = 0.05$, $d = 1$ and $\theta = \pi/3$ when κ near zero

where $u_\epsilon^{\mathrm{sc}} = u_\epsilon - (u^{\mathrm{inc}} + u^{\mathrm{ref}})$ in $\mathbb{R}_2^+$. Here the incident field $u^{\mathrm{inc}} = e^{\mathrm{i}\kappa \boldsymbol{x} \cdot \boldsymbol{d}}$ is a plane wave propagating in the direction $\boldsymbol{d} = (\sin\theta, -\cos\theta)^\top$ and the reflected field $u^{\mathrm{ref}} = e^{\mathrm{i}\kappa \boldsymbol{x} \cdot \boldsymbol{d}'}$ with the propagating direction $d' = (\cos\theta, \sin\theta)$.

Let $\phi_0(x_1) = \frac{1}{\sqrt{\epsilon}}$ and $\phi_n(x_1) = \sqrt{\frac{2}{\epsilon}}\cos\left(\frac{n\pi x_1}{\epsilon}\right)$ $(n \geq 1)$ be an orthonormal basis on the interval $(0, \epsilon)$. Since the total field satisfies the Neumann boundary condition on the bottom and lateral sides of the cavity, we have from (4.1) that u_ϵ can be expressed as the sum of waveguide modes:

$$u_\epsilon(\boldsymbol{x}) = \sum_{n=0}^{\infty} \left(\alpha_n^+ e^{-\mathrm{i}\beta_n x_2} + \alpha_n^- e^{\mathrm{i}\beta_n(x_2+d)}\right)\phi_n(x_1), \quad \boldsymbol{x} \in D_\epsilon, \tag{4.2}$$

where the coefficients β_n are defined as

$$\beta_n = \begin{cases} \kappa, & n = 0, \\ \mathrm{i}\sqrt{(n\pi/\epsilon)^2 - \kappa^2}, & n \geq 1. \end{cases}$$

If $\epsilon < \lambda/2$, then β_n are pure imaginary numbers for all $n \geq 1$. For each n, it can be seen if $n\pi/\epsilon \leq \kappa$, the expansion consists of two propagating wave modes traveling upward and downward respectively; if $n\pi/\epsilon > \kappa$, the expansion consists of two evanescent wave modes decaying exponentially away from the bottom side and open aperture of the cavity, respectively.

Using the expansion (2.5), we may reformulate the model (4.1) into the following coupled problem:

$$
\begin{cases}
\Delta u_\epsilon + \kappa^2 u_\epsilon = 0 & \text{in } \mathbb{R}_2^+, \\
u_\epsilon = \sum\limits_{n=0}^{\infty} \left(\alpha_n^+ e^{-\mathrm{i}\beta_n x_2} + \alpha_n^- e^{\mathrm{i}\beta_n (x_2+d)} \right) \phi_n(x_1) & \text{in } D_\epsilon, \\
\partial_\nu u_\epsilon = 0 & \text{on } \partial\Omega \setminus \Gamma_\epsilon^+, \\
u_\epsilon(x_1, 0+) = u_\epsilon(x_1, 0-), \ \partial_{x_2} u_\epsilon(x_1, 0+) = \partial_{x_2} u_\epsilon(x_1, 0-) & \text{on } \Gamma_\epsilon^+, \\
\lim\limits_{r\to\infty} \sqrt{r}\left(\partial_r u_\epsilon^{\mathrm{sc}} - \mathrm{i}\kappa u_\epsilon^{\mathrm{sc}} \right) = 0 & \text{in } \mathbb{R}_2^+.
\end{cases}
\tag{4.3}
$$

By matching the expansion series (4.2) and using the continuity conditions in (4.3), we may obtain

$$
\alpha_n^+ = \frac{e^{\mathrm{i}\beta_n d} u_{\epsilon,n}^- - u_{\epsilon,n}^+}{e^{\mathrm{i}2\beta_n d} - 1}, \quad
\alpha_n^- = \frac{e^{\mathrm{i}\beta_n d} u_{\epsilon,n}^+ - u_{\epsilon,n}^-}{e^{\mathrm{i}2\beta_n d} - 1}.
$$

Therefore, the wave field

$$
u_\epsilon(\boldsymbol{x}) = \sum_{n=0}^{\infty} \left(\frac{e^{\mathrm{i}\beta_n d} u_{\epsilon,n}^- - u_{\epsilon,n}^+}{e^{\mathrm{i}2\beta_n d} - 1} e^{-\mathrm{i}\beta_n x_2} + \frac{e^{\mathrm{i}\beta_n d} u_{\epsilon,n}^+ - u_{\epsilon,n}^-}{e^{\mathrm{i}2\beta_n d} - 1} e^{\mathrm{i}\beta_n (x_2+d)} \right) \phi_n(x_1), \quad \boldsymbol{x} \in D_\epsilon.
\tag{4.4}
$$

A straightforward calculation yields

$$
\partial_{x_2} u_\epsilon(x_1, 0-) = \sum_{n=0}^{\infty} \mathrm{i}\beta_n \left(\frac{e^{\mathrm{i}\beta_n d} u_{\epsilon,n}^+ - u_{\epsilon,n}^-}{e^{\mathrm{i}2\beta_n d} - 1} e^{\mathrm{i}\beta_n d} - \frac{e^{\mathrm{i}\beta_n d} u_{\epsilon,n}^- - u_{\epsilon,n}^+}{e^{\mathrm{i}2\beta_n d} - 1} \right) \phi_n(x_1) \quad \text{on } \Gamma_\epsilon^+,
\tag{4.5}
$$

$$
\partial_{x_2} u_\epsilon(x_1, -d+) = \sum_{n=0}^{\infty} \mathrm{i}\beta_n \left(\frac{e^{\mathrm{i}\beta_n d} u_{\epsilon,n}^+ - u_{\epsilon,n}^-}{e^{\mathrm{i}2\beta_n d} - 1} - \frac{e^{\mathrm{i}\beta_n d} u_{\epsilon,n}^- - u_{\epsilon,n}^+}{e^{\mathrm{i}2\beta_n d} - 1} e^{\mathrm{i}\beta_n d} \right) \phi_n(x_1) = 0 \quad \text{on } \Gamma_\epsilon^-.
\tag{4.6}
$$

It follows from (4.6) that

$$
u_{\epsilon,n}^- = \frac{2 u_{\epsilon,n}^+}{e^{-\mathrm{i}\beta_n d} + e^{\mathrm{i}\beta_n d}}.
\tag{4.7}
$$

Substituting (4.7) into (4.5) yields

$$
\partial_{x_2} u_\epsilon(x_1, 0-) = \sum_{n=0}^{\infty} \mathrm{i}\beta_n \frac{e^{\mathrm{i}\beta_n d} - e^{-\mathrm{i}\beta_n d}}{e^{\mathrm{i}\beta_n d} + e^{-\mathrm{i}\beta_n d}} u_{\epsilon,n}^+ \phi_n(x_1) \quad \text{on } \Gamma_\epsilon^+.
$$

Given a function $u \in H^{1/2}(\Gamma_\epsilon^+)$, we let $u^+ = u(x_1, 0)$ and define a DtN map on Γ_ϵ^+:

$$
\mathscr{B}^{\mathrm{PEC}}[u] = \sum_{n=0}^{\infty} \mathrm{i}\beta_n \frac{e^{\mathrm{i}\beta_n d} - e^{-\mathrm{i}\beta_n d}}{e^{\mathrm{i}\beta_n d} + e^{-\mathrm{i}\beta_n d}} u_n^+ \phi_n(x_1),
\tag{4.8}
$$

where the Fourier coefficients $u_n^+ = \langle u^+, \phi_n \rangle_{\Gamma_\epsilon^+}$. Following a similar proof to Lemma 2.1, we may show that the DtN map $\mathscr{B}^{\mathrm{PEC}}$ is also bounded. The proof is omitted for brevity.

Lemma 4.1 *The DtN map $\mathscr{B}^{\mathrm{PEC}}$ defined by (2.15) is bounded from $H^{1/2}(\Gamma_\epsilon^+) \to H^{-1/2}(\Gamma_\epsilon^+)$, i.e.,*

$$\|\mathscr{B}^{\mathrm{PEC}}[u]\|_{H^{-1/2}(\Gamma_\epsilon^+)} \le C\|u\|_{H^{1/2}(\Gamma_\epsilon^+)} \quad \forall u \in H^{1/2}(\Gamma_\epsilon^+),$$

where the constant $C = \max\left\{\dfrac{\kappa}{\cot(\kappa d)}, \dfrac{1-e^{-2\sqrt{(n\pi/\epsilon)^2 - \kappa^2}}}{1+e^{-2\sqrt{(n\pi/\epsilon)^2 - \kappa^2}}}\right\} < 1.$

By (4.8), we derive the TBC:

$$\partial_{x_2} u_\epsilon = \mathscr{B}^{\mathrm{PEC}}[u_\epsilon] \quad \text{on } \Gamma_\epsilon^+,$$

which helps to reduce (4.1) into to a boundary value problem in $\mathbb{R}_2^+$:

$$\begin{cases} \Delta u_\epsilon + \kappa^2 u_\epsilon = 0 & \text{in } \mathbb{R}_2^+, \\ \partial_\nu u_\epsilon = 0 & \text{on } \Gamma_0 \setminus \Gamma_\epsilon^+, \\ \partial_{x_2} u_\epsilon = \mathscr{B}^{\mathrm{PEC}}[u_\epsilon] & \text{on } \Gamma_\epsilon^+, \\ \lim_{r\to\infty} \sqrt{r}\left(\partial_r u_\epsilon^{\mathrm{sc}} - i\kappa u_\epsilon^{\mathrm{sc}}\right) = 0 & \text{in } \mathbb{R}_2^+. \end{cases} \tag{4.9}$$

The solution u_ϵ in the cavity D_ϵ may be obtained from (4.4) once the Fourier coefficients $u_{\epsilon,n}^+$ are available (note that $u_{\epsilon,n}^-$ can be computed from from (4.7)).

To find an approximate the model problem to (4.9), we similarly drop all the nonzero wave modes and define

$$\mathscr{B}_0^{\mathrm{PEC}}[v] = i\kappa \frac{e^{i\kappa d} - e^{-i\kappa d}}{e^{i\kappa d} + e^{-i\kappa d}} v_0^+ \phi_0(x_1). \tag{4.10}$$

Hence we may consider the following problem:

$$\begin{cases} \Delta v_\epsilon + \kappa^2 v_\epsilon = 0 & \text{in } \mathbb{R}_2^+, \\ \partial_\nu v_\epsilon = 0 & \text{on } \Gamma_0 \setminus \Gamma_\epsilon^+, \\ \partial_{x_2} v_\epsilon = \mathscr{B}_0^{\mathrm{PEC}}[v_\epsilon] & \text{on } \Gamma_\epsilon^+, \\ \lim_{r\to\infty} \sqrt{r}\left(\partial_r v_\epsilon^{\mathrm{sc}} - i\kappa v_\epsilon^{\mathrm{sc}}\right) = 0 & \text{in } \mathbb{R}_2^+, \end{cases} \tag{4.11}$$

where $v_\epsilon^s = v_\epsilon - (u^{\mathrm{inc}} + u^{\mathrm{ref}})$ in $\mathbb{R}_2^+$. Inside the cavity, we may approximate u_ϵ with one single mode:

$$v_\epsilon(\boldsymbol{x}) = \left(\tilde{\alpha}_0^+ e^{-i\kappa x_2} + \tilde{\alpha}_0^- e^{i\kappa(x_2+d)}\right)\phi_0(x_1), \tag{4.12}$$

where the coefficients $\tilde{\alpha}_0^+$ and $\tilde{\alpha}_0^-$ are given by

$$\tilde{\alpha}_0^+ = \frac{e^{i\kappa d} v_{\epsilon,0}^- - v_{\epsilon,0}^+}{e^{i2\kappa d} - 1}, \quad \tilde{\alpha}_0^- = \frac{e^{i\kappa d} v_{\epsilon,0}^+ - v_{\epsilon,0}^-}{e^{i2\kappa d} - 1}. \tag{4.13}$$

It follows from $\partial_\nu v_\epsilon = 0$ on Γ_ϵ^- that $\tilde{\alpha}_0^- = \tilde{\alpha}_0^+ e^{i\kappa d}$, which yields

$$v_{\epsilon,0}^- = \frac{2v_{\epsilon,0}^+}{e^{i\kappa d} + e^{-i\kappa d}}. \tag{4.14}$$

4.2 Enhancement of the approximated field

This section presents the estimates of the solution for the approximate model problem (4.11).

Theorem 4.2 *Let v_ϵ be the solution of* (4.11) *and be given by* (4.12) *inside the cavity, then the following holds*

$$\|\nabla v_\epsilon\|_{L^2(D_\epsilon)} = \sqrt{\kappa\epsilon(2\kappa d - \sin(2\kappa d))} + O(\kappa^3 d^{3/2}\epsilon^{3/2}).$$

Proof For the approximate model (4.11), we obtain from Green's formula that

$$v_\epsilon(\boldsymbol{x}) = u^{\text{inc}}(\boldsymbol{x}) + u^{\text{r}}(\boldsymbol{x}) + \int_{\Gamma_\epsilon^+} G(\boldsymbol{x}, \boldsymbol{y}) \frac{\partial v_\epsilon(\boldsymbol{y})}{\partial \nu_y} ds_y, \quad \boldsymbol{x} \in \mathbb{R}_2^+,$$

where G is the half-space Green function satisfying the Neumann boundary condition and is given in (2.24). It follows from the continuity of the single layer potential that

$$v_\epsilon(\boldsymbol{x}) = u^{\text{inc}}(\boldsymbol{x}) + u^{\text{ref}}(\boldsymbol{x}) - \frac{\mathrm{i}}{2} \int_{\Gamma_\epsilon^+} H_0^{(1)}(k|\boldsymbol{x} - \boldsymbol{y}|) \frac{\partial v_\epsilon(\boldsymbol{y})}{\partial \nu_y} ds_y, \quad \boldsymbol{x} \in \Gamma_\epsilon^+. \tag{4.15}$$

In light of (4.12)–(4.14), it yields that

$$\partial_{x_2} v_\epsilon = \mathrm{i}\kappa \left(-\tilde{\alpha}_0^+ + \tilde{\alpha}_0^- e^{\mathrm{i}\kappa d}\right) \phi_0(x_1) \quad \text{on } \Gamma_\epsilon^+.$$

Substituting the above equality into (4.15) and using the fact that $\phi_0(x_1) = \frac{1}{\sqrt{\epsilon}}$, we get

$$v_\epsilon(x_1, 0) = u^{\text{inc}}(x_1, 0) + u^{\text{ref}}(x_1, 0) + \frac{\kappa}{2} \left(-\tilde{\alpha}_0^+ + \tilde{\alpha}_0^- e^{\mathrm{i}\kappa d}\right) \frac{1}{\sqrt{\epsilon}} h_1(x_1), \quad x_1 \in (0, \epsilon),$$

where $h_1(x_1)$ is given in (2.26). Therefore, the Fourier coefficients $v_{\epsilon,0}^+$ may be expressed as

$$v_{\epsilon,0}^+ = \langle u^{\text{inc}}, \phi_0 \rangle_{\Gamma_\epsilon^+} + \langle u^{\text{ref}}, \phi_0 \rangle_{\Gamma_\epsilon^+} + \frac{\kappa}{2} \left(-\tilde{\alpha}_0^+ + \tilde{\alpha}_0^- e^{\mathrm{i}\kappa d}\right) \frac{1}{\sqrt{\epsilon}} \langle h_1, \phi_0 \rangle_{\Gamma_\epsilon^+}.$$

It follows from the fact $u^{\text{inc}}(x_1, 0) = u^{\text{ref}}(x_1, 0)$ and the inner product of (4.12) with $\phi_0(x_1)$ that

$$\tilde{\alpha}_0^+ + \tilde{\alpha}_0^- e^{\mathrm{i}\kappa d} = 2\langle u^{\text{inc}}, \phi_0 \rangle_{\Gamma_\epsilon^+} + \frac{\kappa}{2\sqrt{\epsilon}} \langle h_1, \phi_0 \rangle_{\Gamma_\epsilon^+} \left(-\tilde{\alpha}_0^+ + \tilde{\alpha}_0^- e^{\mathrm{i}\kappa d}\right).$$

Using $\tilde{\alpha}_0^- = \tilde{\alpha}_0^+ e^{\mathrm{i}\kappa d}$ and the above equation gives

$$\tilde{\alpha}_0^+ = \frac{2\langle u^{\text{inc}}, \phi_0 \rangle_{\Gamma_\epsilon^+}}{(1 + c_0) + (1 - c_0)e^{\mathrm{i}2\kappa d}}, \quad \tilde{\alpha}_0^- = \frac{2e^{\mathrm{i}\kappa d}\langle u^{\text{inc}}, \phi_0 \rangle_{\Gamma_\epsilon^+}}{(1 + c_0) + (1 - c_0)e^{\mathrm{i}2\kappa d}}, \tag{4.16}$$

where $c_0 = \frac{\kappa}{2\sqrt{\epsilon}} \langle h_1, \phi_0 \rangle_{\Gamma_\epsilon^+}$.

Using the assumption $\epsilon \ll d \ll \lambda$ and expression of c_0 in (2.28), it gives that

$$|(1 + c_0) + (1 - c_0)e^{\mathrm{i}2\kappa d}| = |2\cos(\kappa d) + \frac{2\kappa}{\pi} \left(\ln\frac{\kappa}{2} + \epsilon\ln\epsilon - \frac{3}{2}\epsilon\right) \sin(\kappa d)$$

$$+ \mathrm{i}\kappa\gamma_0\epsilon\sin(\kappa d) + O(\epsilon^3\ln\epsilon)|. \tag{4.17}$$

Recalling the definition of the incident wave, we get

$$|\langle u^{\text{inc}}, \phi_0 \rangle_{\Gamma_\epsilon^+}| = \left|\frac{1}{\sqrt{\epsilon}} \int_0^\epsilon e^{\mathrm{i}\kappa\sin\theta x_1} dx_1\right| = \sqrt{\epsilon} + O(\kappa^2\epsilon^{5/2}). \tag{4.18}$$

Combing (4.16)-(4.18), it yields

$$|\tilde{\alpha}_0^+| = |\tilde{\alpha}_0^-| = \sqrt{\epsilon} + O(\kappa^2 \epsilon^{5/2}).$$

It follows from (4.12) and (2.33) that

$$\|\nabla v_\epsilon\|_{L^2(D_\epsilon)}^2 \leq \int_0^\epsilon \int_{-d}^0 \left(\kappa (|\tilde{\alpha}_0^+| + |\tilde{\alpha}_0^-|)\phi_0^{x_1} \right)^2 4 \sin^2 \kappa(x_2 + d) \mathrm{d}x_1 \mathrm{d}x_2$$

$$\leq \kappa \epsilon (2\kappa d - \sin(2\kappa d)) + O(\kappa^6 d^3 \epsilon^3).$$

Substituting (4.16) into (4.12) yields

$$v_\epsilon(\boldsymbol{x}) = \left(\frac{2\langle u^{\mathrm{inc}}, \phi_0\rangle_{\Gamma_\epsilon^+}}{(1 + c_0) + (1 - c_0)e^{\mathrm{i}2\kappa d}} e^{-\mathrm{i}\kappa x_2} + \frac{2e^{\mathrm{i}\kappa d}\langle u^{\mathrm{inc}}, \phi_0\rangle_{\Gamma_\epsilon^+}}{(1 + c_0) + (1 - c_0)e^{\mathrm{i}2\kappa d}} e^{\mathrm{i}\kappa(x_2+d)} \right) \phi_0(x_1). \tag{4.19}$$

A simple calculation yields

$$\frac{\partial v_\epsilon}{\partial x_2} = \mathrm{i}\kappa e^{\mathrm{i}\kappa d} \frac{2\langle u^{\mathrm{inc}}, \phi_0\rangle_{\Gamma_\epsilon^+}}{(1 + c_0) + (1 - c_0)e^{\mathrm{i}2\kappa d}} \phi_0(x_1) \left(-e^{-\mathrm{i}\kappa(x_2+d)} + e^{\mathrm{i}\kappa(x_2+d)} \right)$$

$$= -\kappa e^{\mathrm{i}\kappa d} \frac{2\langle u^{\mathrm{inc}}, \phi_0\rangle_{\Gamma_\epsilon^+}}{(1 + c_0) + (1 - c_0)e^{\mathrm{i}2\kappa d}} 2 \sin \kappa(x_2 + d)\phi_0(x_1),$$

which gives

$$\|\nabla v_\epsilon\|_{L^2(D_\epsilon)}^2 \geq \kappa^2 \left| \frac{2\langle u^{\mathrm{inc}}, \phi_0\rangle_{\Gamma_\epsilon^+}}{(1 + c_0) + (1 - c_0)e^{\mathrm{i}2\kappa d}} \right|^2 \int_{-d}^0 4 \sin^2 \kappa(x_2 + d)\mathrm{d}x_2.$$

Noting that $d \ll \lambda$, we also have (2.33) holds. It follows from (4.17)–(4.18) that

$$\|\nabla v_\epsilon\|_{L^2(D_\epsilon)} \geq \sqrt{\kappa \epsilon (2\kappa d - \sin(2\kappa d))} + O(\kappa^3 d^{3/2} \epsilon^{3/2}),$$

which completes the proof. □

Theorem 4.3 *Let v_ϵ be the solution of (4.11) and be given by (4.12) inside the cavity, then we have the estimate*

$$\|v_\epsilon\|_{L^2(D_\epsilon)} = \sqrt{\epsilon} d \frac{\sqrt{2(1 + \sin(2\kappa d)/(2\kappa d))}}{\cos(\kappa d)} + O(\kappa \epsilon).$$

Proof By (4.19), we have

$$\|v_\epsilon\|_{L^2(D_\epsilon)}^2 = \int_0^\epsilon \int_{-d}^0 |v_\epsilon|^2 \mathrm{d}x_2 \mathrm{d}x_1$$

$$= \left| \frac{2\langle u^{\mathrm{inc}}, \phi_0\rangle_{\Gamma_\epsilon^+} e^{\mathrm{i}\kappa d}}{(1 + c_0) + (1 - c_0)e^{\mathrm{i}2\kappa d}} \right|^2 \int_{-d}^0 |e^{-\mathrm{i}\kappa(x_2+d)} + e^{\mathrm{i}\kappa(x_2+d)}|^2 \mathrm{d}x_2$$

$$= \frac{\left| 2\langle u^{\mathrm{inc}}, \phi_0\rangle_{\Gamma_\epsilon^+} \right|^2}{\left| (1 + c_0) + (1 - c_0)^{\mathrm{i}2\kappa d} \right|^2} \int_{-d}^0 4 \cos^2 \kappa(x_2 + d)\mathrm{d}x_2.$$

It follows from $d \ll \lambda$ that

$$\int_{-d}^0 4 \cos^2 \kappa(x_2 + d)\mathrm{d}x_2 = 2d(1 + \frac{\sin(2\kappa d)}{2\kappa d}).$$

In light of (4.17)–(4.18), it yields

$$\|v_\epsilon\|^2_{L^2(D_\epsilon)} = \frac{2\epsilon d}{\cos^2(\kappa d)}\left(1 + \frac{\sin(2\kappa d)}{2\kappa d}\right) + O(\kappa^2\epsilon^2),$$

which completes the proof. □

4.3 Field enhancement without resonance

Following from the same steps as those in Sect. 2.4, we may also show that

$$C(R)\frac{\kappa\sqrt{d}}{\sin(\kappa d)}\epsilon(1 - O(\sqrt{\epsilon|\ln\epsilon|})) \le \|\nabla u_\epsilon - \nabla v_\epsilon\|_{L^2(D_\epsilon)} \le C(R)\frac{\kappa\sqrt{d}}{\sin(\kappa d)}\epsilon,$$

$$\|u_\epsilon - v_\epsilon\|_{L^2(D_\epsilon)} \le C(R)\frac{\kappa\sqrt{d}}{\sin(\kappa d)}\epsilon\sqrt{\epsilon|\ln\epsilon|}, \tag{4.20}$$

where $C(R)$ is a positive constant depending on R but is independent of λ, ϵ, d. The details are omitted for brevity.

Remark 4.4 In (4.20), the constant $C(R)$ comes from two places: one is the estimate of the DtN operator $\mathscr{B}^{\mathrm{PEC}}$ in Lemma 4.1, where the constant C is independent of λ, d, and the other is the Sobolev embedding results in $\mathbb{R}^2_+$, which depends only on R.

The following estimates are the main results of the electromagnetic field enhancement for the PEC-PEC cavity when $\epsilon \ll d \ll \lambda$.

Theorem 4.5 *If ϵ is small enough such that $C(R)\sqrt{\epsilon} < \frac{2}{3}\kappa^2 d \sin(\kappa d)$, then the electric field has no enhancement. If $d \ll \lambda$ such that $\frac{d}{\lambda} \le \frac{1}{4\pi}\sqrt{\frac{3}{2}C(R)\epsilon}$, then the electric field enhancement factor satisfies*

$$\frac{C(R)}{4\pi}\frac{\lambda}{d}\sqrt{\epsilon} \le Q_E \le \frac{3C(R)}{4\pi}\frac{\lambda}{d}\sqrt{\epsilon}$$

and the magnetic field enhancement factor satisfies

$$C_3 \le Q_H \le C_4,$$

where the constants $C_3 = 2 - C(R)\epsilon\sqrt{|\ln\epsilon|}/d + O(\kappa\epsilon)$, $C_4 = 2 + C(R)\epsilon\sqrt{|\ln\epsilon|}/d + O(\kappa\epsilon)$ as $\frac{d}{\lambda} \longrightarrow 0$, and the constant $C(R)$ is given in (4.20).

Proof By Theorem 4.2 and (4.20), we can obtain

$$\|\nabla v_\epsilon\|_{L^2(D_\epsilon)} - \|\nabla u_\epsilon - \nabla v_\epsilon\|_{L^2(D_\epsilon)} \le \|\nabla u_\epsilon\|_{L^2(D_\epsilon)} \le \|\nabla v_\epsilon\|_{L^2(D_\epsilon)} + \|\nabla u_\epsilon - \nabla v_\epsilon\|_{L^2(D_\epsilon)}.$$

We consider two cases:

(i) If ϵ is small enough such that $C(R)\sqrt{\epsilon} < \frac{2}{3}\kappa^2 d \sin(\kappa d)$, then we have

$$\|\nabla u_\epsilon\|_{L^2(D_\epsilon)} \le \sqrt{\kappa\epsilon(2\kappa d - \sin(2\kappa d))} + C(R)\frac{\kappa\sqrt{d}}{\sin(\kappa d)} + O(\kappa^3 d^{3/2}\epsilon^{3/2}) \le 2\kappa^2 d^2\sqrt{\epsilon/d}$$

and

$$\|\nabla u_\epsilon\|_{L^2(D_\epsilon)} \geq C_1(\kappa d)^2 \sqrt{\frac{\epsilon}{d}} - C(R)\epsilon\sqrt{\kappa\epsilon(2\kappa d - \sin(2\kappa d))}$$

$$+ C(R)\frac{\kappa\sqrt{d}}{\sin(\kappa d)} + O(\kappa^3 d^{3/2}\epsilon^{3/2}) \geq \frac{2}{3}\kappa^2 d^2\sqrt{\epsilon/d}.$$

Since $\|\nabla u^{\mathrm{inc}}\|^2_{L^2(D_\epsilon)} = \kappa^2\epsilon d$, we obtain

$$\frac{4}{3}\pi\frac{d}{\lambda} \leq \frac{\|\nabla u_\epsilon\|_{L^2(D_\epsilon)}}{\|\nabla u^{\mathrm{inc}}\|_{L^2(D_\epsilon)}} \leq 4\pi\frac{d}{\lambda}.$$

Thus there is no enhancement in this case.

(ii) If $d \ll \lambda$ such that $\frac{d}{\lambda} \leq \frac{1}{4\pi}\sqrt{\frac{3}{2}C(R)\epsilon}$, then we get

$$\frac{C(R)\kappa\sqrt{d}}{2\sin(\kappa d)}\epsilon \leq \|\nabla u_\epsilon\|_{L^2(D_\epsilon)} \leq \frac{3C(R)\kappa\sqrt{d}}{2\sin(\kappa d)}\epsilon,$$

which gives

$$\frac{C(R)}{4\pi}\frac{\lambda}{d}\sqrt{\epsilon} \leq \frac{\|\nabla u_\epsilon\|_{L^2(D_\epsilon)}}{\|\nabla u^{\mathrm{inc}}\|_{L^2(D_\epsilon)}} \leq \frac{3C(R)}{4\pi}\frac{\lambda}{d}\sqrt{\epsilon}.$$

Noting that $|\nabla \times \boldsymbol{H}_\epsilon| = |\nabla u_\epsilon|$ and using Ampere's law $\nabla \times \boldsymbol{H}_\epsilon = -i\omega\varepsilon\boldsymbol{E}_\epsilon$, we obtain

$$Q_E = \frac{\|\boldsymbol{E}\|_{L^2(D_\epsilon)}}{\|\boldsymbol{E}^{\mathrm{inc}}\|_{L^2(D_\epsilon)}} = \frac{\|\nabla u_\epsilon\|_{L^2(D_\epsilon)}}{\|\nabla u^{\mathrm{inc}}\|_{L^2(D_\epsilon)}},$$

which shows that for case (i) the electric field $\boldsymbol{E}$ has no enhancement; for case (ii) the electric field enhancement factor has an order $O(\lambda\sqrt{\epsilon}/d)$.

For the magnetic field, by Theorem 4.3 and (4.20), we have

$$\sqrt{\epsilon d}\left(\frac{\sqrt{2(1 + \sin(2\kappa d)/(2\kappa d))}}{\cos(\kappa d)} - \frac{C(R)\kappa\epsilon\sqrt{|\ln\epsilon|}}{\sin(\kappa d)} + O(\kappa\epsilon)\right) \leq \|u_\epsilon\|_{L^2(D_\epsilon)}$$

$$\leq \sqrt{\epsilon d}\left(\frac{\sqrt{2(1 + \sin(2\kappa d)/(2\kappa d))}}{\cos(\kappa d)} + \frac{C(R)\kappa\epsilon\sqrt{|\ln\epsilon|}}{\sin(\kappa d)} + O(\kappa\epsilon)\right).$$

Since $\|u^{\mathrm{inc}}\|^2_{L^2(D_\epsilon)} = \epsilon d$, we obtain

$$C_3 \leq Q_H = \frac{\|\boldsymbol{H}\|_{L^2(D_\epsilon)}}{\|\boldsymbol{H}^{\mathrm{inc}}\|_{L^2(D_\epsilon)}} = \frac{\|u_\epsilon\|_{L^2(D_\epsilon)}}{\|u^{\mathrm{inc}}\|_{L^2(D_\epsilon)}} \leq C_4,$$

where

$$C_3 = \sqrt{\epsilon d}\left(\frac{\sqrt{2(1 + \sin(2\kappa d)/(2\kappa d))}}{\cos(\kappa d)} - \frac{C(R)\kappa\epsilon\sqrt{|\ln\epsilon|}}{\sin(\kappa d)} + O(\kappa\epsilon)\right)$$

$$\longrightarrow 2 - \frac{C(R)\epsilon\sqrt{|\ln\epsilon|}}{d} + O(\kappa\epsilon),$$

$$C_4 = \sqrt{\epsilon d}\left(\frac{\sqrt{2(1 + \sin(2\kappa d)/(2\kappa d))}}{\cos(\kappa d)} + \frac{C(R)\kappa\epsilon\sqrt{|\ln\epsilon|}}{\sin(\kappa d)} + O(\kappa\epsilon)\right)$$

$$\longrightarrow 2 + \frac{C(R)\epsilon\sqrt{|\ln\epsilon|}}{d} + O(\kappa\epsilon)$$

as $\frac{d}{\lambda} \longrightarrow 0$. This completes the proof. $\qquad\square$
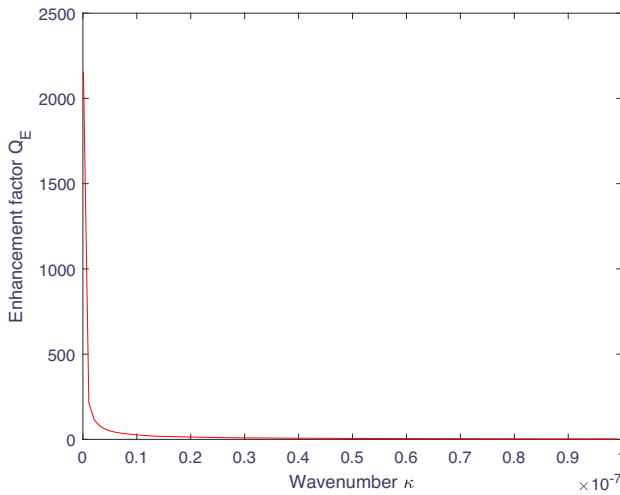
Fig. 5 The amplification of electric field in the PEC cavity with $\epsilon = 0.005$, $d = 1$ and $\theta = \pi/3$ when κ near zero

4.4 Numerical experiments

We present a numerical example to illustrate and verify the results stated in Theorem 4.5. The width of the cavity is $\epsilon = 0.005$, the depth of the cavity is $d = 1$, and the incident angle is $\theta = \pi/3$. Figure 5 plots the electric field enhancement factor Q_E against the wavenumber $\kappa \in (0, 10^{-8})$. It shows that when κ is sufficient small, i.e., when the wavelength λ is sufficiently large, the electric field does have an enhancement, which corresponds to case (ii) in Theorem 4.5; when the wavenumber κ increases (the wavelength λ decreases) but is still small, the electric field has no enhancement, which corresponds to case (i) in Theorem 4.5. The plot is not shown for the magnetic field since it has no enhancement in this region.

5 PEC-PEC cavity with resonance

In this section, we show the field enhancement at resonant frequencies for the PEC-PEC cavity. Consider the model problem:

$$\begin{cases} \Delta u_\epsilon + \kappa^2 u_\epsilon = 0 & \text{in } \Omega, \\ \partial_\nu u_\epsilon = 0 & \text{on } \partial\Omega, \\ \lim_{r \to \infty} \sqrt{r} \left(\partial_r u_\epsilon^{\text{sc}} - i\kappa u_\epsilon^{\text{sc}} \right) = 0 & \text{in } \mathbb{R}_2^+. \end{cases} \tag{5.1}$$

Here we assume that $\epsilon \ll \lambda$.

5.1 Boundary integral equations

Let $g_\epsilon^{\mathrm{PEC}}$ be the Green function for the Helmholtz equation with Neumann boundary condition in D_ϵ, i.e., it satisfies

$$
\begin{cases}
\Delta g_\epsilon^{\mathrm{PEC}}(x, y) + \kappa^2 g_\epsilon^{\mathrm{PEC}}(x, y) = -\delta(x, y), & x, y \in D_\epsilon, \\
\frac{\partial g_\epsilon^{\mathrm{PEC}}(x,y)}{\partial v_y} = 0 & \text{on } \partial D_\epsilon.
\end{cases}
$$

It can be verified that

$$
g_\epsilon^{\mathrm{PEC}}(x, y) = \sum_{m,n=0}^{\infty} c_{m,n} \phi_{m,n}(x) \phi_{m,n}(y),
$$

where

$$
\begin{aligned}
c_{m,n} &= \frac{1}{\kappa^2 - (m\pi/\epsilon)^2 - (n\pi/d)^2}, \\
\phi_{m,n}(x) &= \sqrt{\frac{\alpha_{m,n}}{\epsilon d}} \cos\left(\frac{m\pi x_1}{\epsilon}\right) \cos\left(\frac{n\pi}{d}(x_2 + d)\right), \\
\alpha_{m,n} &= \begin{cases}
1, & m = 0, n = 0, \\
2, & m = 0, n \geq 1 \text{ or } n = 0, m \geq 1 \\
4, & m \geq 1, n \geq 1.
\end{cases}
\end{aligned}
\tag{5.2}
$$

Using Green's formula, we obtain

$$
u_\epsilon(x) = \int_{\Gamma_\epsilon^+} G(x, y) \frac{\partial u_\epsilon(y)}{\partial v_y} ds_y + u^{\mathrm{inc}}(x) + u^{\mathrm{ref}}(x), \quad x \in \mathbb{R}_2^+,
$$

$$
u_\epsilon(x) = - \int_{\Gamma_\epsilon^+} g_\epsilon^{\mathrm{PEC}}(x, y) \frac{\partial u_\epsilon(y)}{\partial v_y} ds_y, \quad x \in D_\epsilon.
$$

It follows from the continuity of the single layer potential that we have

$$
u_\epsilon(x) = \int_{\Gamma_\epsilon^+} \left(-\frac{i}{2}\right) H_0^1(\kappa|x - y|) \frac{\partial u_\epsilon(y)}{\partial v_y} ds_y + u^{\mathrm{inc}}(x) + u^{\mathrm{ref}}(x), \quad x \in \Gamma_\epsilon^+
$$

and

$$
u_\epsilon(x) = - \int_{\Gamma_\epsilon^+} g_\epsilon^{\mathrm{PEC}}(x, y) \frac{\partial u_\epsilon(y)}{\partial v_y} ds_y, \quad x \in \Gamma_\epsilon^+.
$$

By imposing the continuity of the solution on Γ_ϵ^+, we have the following lemma.

Lemma 5.1 *The scattering problem* (5.1) *is equivalent to the boundary integral equation:*

$$
\int_{\Gamma_\epsilon^+} \left(\left(-\frac{i}{2}\right) H_0^1(\kappa|x - y|) + g_\epsilon^{\mathrm{PEC}}(x, y)\right) \frac{\partial u_\epsilon(y)}{\partial v_y} ds_y + u^{\mathrm{inc}}(x) + u^{\mathrm{ref}}(x) = 0, \quad x \in \Gamma_\epsilon^+.
\tag{5.3}
$$

Similarly, we introduce the rescaling variables $X = x_1/\epsilon$, $Y = y_1/\epsilon$ and define the following boundary integral operators:

$$(T^e\varphi)(X) := \int_0^2 G_\epsilon^e(X, Y)\varphi(Y)\mathrm{d}Y, \quad X \in (0, 1),$$

$$(T^i\varphi)(X) := \int_0^1 G_\epsilon^i(X, Y)\varphi(Y)\mathrm{d}Y, \quad X \in (0, 1). \tag{5.4}$$

where $G_\epsilon^e, G_\epsilon^i, \varphi$ are give in (3.6) with $g_\epsilon^{\mathrm{PMC}}$ being replaced by $g_\epsilon^{\mathrm{PEC}}$. Then the boundary integral equation (5.3) is equivalent to the operator equation:

$$(T^e + T^i)\varphi = f/\epsilon, \tag{5.5}$$

where $f = (u^{\mathrm{inc}} + u^{\mathrm{ref}})\big|_{\Gamma_\epsilon^+} = 2e^{\mathrm{i}\kappa \sin\theta\epsilon X}$.

5.2 Asymptotics of the integral operators

In this subsection, we study the asymptotic properties of the integral operators in (5.4).

Lemma 5.2 *If $\epsilon \ll \lambda$, then we have the following asymptotic formulas:*

$$G_\epsilon^e = \Gamma_1(\kappa, \epsilon) + \frac{1}{\pi} \ln|X - Y| + R_1^\epsilon(X, Y) + R_2^\epsilon(X, Y),$$

$$G_\epsilon^i = \Gamma_2(\kappa, \epsilon) + \frac{1}{\pi}\left(\ln\left|\sin\frac{\pi(X + Y)}{2}\right| + \ln\left|\sin\frac{\pi(X - Y)}{2}\right|\right) + R_3^\epsilon(X, Y),$$

where $\Gamma_2(\kappa, \epsilon) = \frac{\cot\kappa d}{\epsilon\kappa} + \frac{2\ln 2}{\pi}$, and $\Gamma_1(\kappa, \epsilon), R_1^\epsilon(X, Y), R_2^\epsilon(X, Y), R_3^\epsilon(X, Y)$ are given in Lemma 3.2.

Proof The proof is similar to that for Lemma 3.2. The integral kernels $G_\epsilon^e(X, Y)$ and $G_\epsilon^i(X, Y)$ can be expressed as

$$G_\epsilon^e(X, Y) = \Gamma_1(\kappa, \epsilon) + \frac{1}{\pi} \ln|X - Y| + R_1^\epsilon(X, Y) + R_2^\epsilon(X, Y),$$

$$G_\epsilon^i(X, Y) = \frac{1}{\epsilon d} \sum_{m,n=0}^\infty c_{m,n}\alpha_{m,n} \cos(m\pi X) \cos(m\pi Y).$$

Recalling the definitions of $c_{m,n}$ and $\alpha_{m,n}$ in (5.2), we let

$$C_m(\epsilon, \kappa) = \sum_{n=0}^\infty c_{m,n}\alpha_{m,n} = \sum_{n=0}^\infty \frac{\alpha_{m,n}}{\kappa^2 - (m\pi/\epsilon)^2 - (n\pi/d)^2}.$$

Using the fact (cf. [20]) that

$$\cot(\pi x) = \frac{1}{\pi x} + \frac{2x}{\pi} \sum_{n=1}^\infty \frac{1}{x^2 - n^2}, \quad x \neq n,$$

for $m = 0$, we get

$$C_0(\kappa) = \frac{1}{\kappa^2} + \sum_{n=1}^\infty \frac{2}{\kappa^2 - (n\pi/d)^2} = \frac{d}{\kappa}\left(\frac{1}{\pi(d\kappa/\pi)} + \frac{2d\kappa/\pi}{\pi} \sum_{n=1}^\infty \frac{1}{(d\kappa/\pi)^2 - n^2}\right)$$

$$= \frac{d\cot(d\kappa)}{\kappa}.$$

Following from the identity

$$\coth(\pi x) = \frac{1}{\pi x} + \frac{2x}{\pi} \sum_{n=1}^{\infty} \frac{1}{x^2 + n^2},$$

we have for $m \geq 1$ that

$$C_m(\epsilon, \kappa) = \frac{2}{\kappa^2 - (m\pi/\epsilon)^2} + \sum_{n \geq 1} \frac{4}{\kappa^2 - (m\pi/\epsilon)^2 - (n\pi/d)^2}$$

$$= \frac{-2d}{\sqrt{(m\pi/\epsilon)^2 - \kappa^2}} \left(\frac{2\frac{d}{\pi}\sqrt{(m\pi/\epsilon)^2 - \kappa^2}}{\pi} \sum_{n \geq 1} \frac{1}{n^2 + (\frac{d}{\pi})^2((m\pi/\epsilon)^2 - \kappa^2)} \right.$$

$$\left. + \frac{1}{d\sqrt{(m\pi/\epsilon)^2 - \kappa^2}} \right)$$

$$= \frac{-2d}{\sqrt{(m\pi/\epsilon)^2 - \kappa^2}} \coth(d\sqrt{(m\pi/\epsilon)^2 - \kappa^2})$$

$$= -\frac{2d\epsilon}{m\pi} - \frac{\epsilon^3 \kappa^2 d}{m^2 \pi^3} + O\left(\frac{\epsilon^5}{m^5}\right).$$

Here we use the asymptotic expression of $(1 - x^2)^{-1/2}$ and $\coth(d\sqrt{(m\pi/\epsilon)^2 - \kappa^2}) \to 1$ as $\epsilon \to 0$.

Substituting $C_0(\kappa)$, $C_m(\epsilon, \kappa)(m \geq 1)$ into to G_ϵ^i, we obtain

$$G_\epsilon^i(X, Y) = \frac{1}{\epsilon d} \left(\frac{d \cot(\kappa d)}{\kappa} - \sum_{m=1}^{\infty} \left(\frac{2d\epsilon}{m\pi} + \frac{\epsilon^3 \kappa^2 d}{m^3 \pi^3} \right) \cos(m\pi X) \cos(m\pi Y) + O\left(\sum_{m=1}^{\infty} \frac{\epsilon^5}{m^5} \right) \right)$$

$$= \frac{d \cot(\kappa d)}{\kappa \epsilon} + \frac{1}{\pi} \left(2 \ln 2 + \ln \left| \sin \frac{\pi(X + Y)}{2} \right| + \ln \left| \sin \frac{\pi(X - Y)}{2} \right| \right)$$

$$- \frac{\epsilon^2 \kappa^2}{\pi} \left(\frac{1}{\pi^2} \sum_{m=1}^{\infty} \frac{1}{m^3} + \frac{(X + Y)^2}{4} \ln(\pi(X + Y)) + \frac{(X - Y)^2}{4} \ln(\pi(X - Y)) \right.$$

$$\left. + O((X + Y)^2 + (X - Y)^2) \right)$$

$$= \Gamma_2(\kappa, \epsilon) + \frac{1}{\pi} \left(\ln \left| \sin \frac{\pi(X + Y)}{2} \right| + \ln \left| \sin \frac{\pi(X - Y)}{2} \right| \right) + R_3^\epsilon(X, Y),$$

which completes the proof. □

Let $\Gamma = \Gamma_1(\kappa, \epsilon) + \Gamma_2(\kappa, \epsilon)$, K, K_∞^1, K_∞^2 are defined same as in (3.10). Then for the operator $T^e + T^i$ in (5.5), we have a similar decomposition

$$T^e + T^i = \Gamma P + K + K_\infty^1 + K_\infty^2,$$

where K, K_∞^1, K_∞^2 have the same properties as those in Lemma 3.3.

5.3 Asymptotics of the resonances

For convenience, we write

$$\Gamma P + K + K_\infty^1 + K_\infty^2 := \mathscr{P} + \mathscr{L},$$

where $\mathscr{P} = \Gamma P, \mathscr{L} = K + K_\infty^1 + K_\infty^2$. The eigenvalues of operator $\mathscr{L}^{-1}\mathscr{P} + \mathscr{I}$ are

$$\lambda(\kappa, \epsilon) = 1 + \Gamma(\kappa, \epsilon)\langle\mathscr{L}^{-1}1_{(0,1)}, 1_{(0,1)}\rangle_{L^2(0,1)}.$$

Therefore, the characteristic values of the operator $\mathscr{P} + \mathscr{L}$ are the roots of the analytic functions $\lambda(\kappa, \epsilon)$, and the associated characteristic function is given by

$$\varphi_0 = \Gamma(\kappa, \epsilon)\mathscr{L}^{-1}1_{(0,1)}.$$

Theorem 5.3 *The resonance of the scattering problem* (5.1) *are the roots of the analytic function* $\lambda(\kappa, \epsilon) = 0$. *Moreover, the resonance set* $\{k_n\}, n = 0, 1, 2, \cdots$ *satisfies the following asymptotic expansion*

$$k_n = \frac{(n + \frac{1}{2})\pi}{d} + \frac{(n + \frac{1}{2})\pi}{d^2}\left(\frac{1}{\pi}\epsilon\ln\epsilon + \left(\frac{1}{q_0} + \frac{1}{\pi}(2\ln 2 + \ln\frac{(n + \frac{1}{2})\pi}{d} + \gamma_1)\right)\epsilon\right) + O(\epsilon^2\ln\epsilon).$$

(5.6)

Proof Given roots of $\lambda(\kappa, \epsilon)$, it is easy to check that they are the characteristic values of the operator $\mathscr{P} + \mathscr{L}$ with corresponding characteristic function defined above.

Consider the roots of

$$\lambda(\kappa, \epsilon) = 1 + (\Gamma_1(\kappa, \epsilon) + \Gamma_2(\kappa, \epsilon))\langle\mathscr{L}^{-1}1_{(0,1)}, 1_{(0,1)}\rangle_{L^2(0,1)} = 0.$$

Recall that $\Gamma_1(\kappa, \epsilon) = \frac{1}{\pi}(\ln\kappa + \gamma_1) + \frac{1}{\pi}\ln\epsilon$ and $\Gamma_2(\kappa, \epsilon) = \frac{\cot\kappa d}{\epsilon\kappa} + \frac{2\ln 2}{\pi}$. The above equation can be written as

$$1 + \left(\frac{\cot\kappa d}{\epsilon\kappa} + \frac{1}{\pi}(2\ln 2 + \ln\kappa + \gamma_1) + \frac{1}{\pi}\ln\epsilon\right)\langle\mathscr{L}^{-1}1_{(0,1)}, 1_{(0,1)}\rangle_{L^2(0,1)} = 0.$$

By Lemma 3.4, similar as (3.13), we let

$$p(\kappa, \epsilon) := \epsilon\lambda(\kappa, \epsilon) = \epsilon + \left(\frac{\cot\kappa d}{\kappa} + \epsilon\rho(\kappa) + \frac{1}{\pi}\epsilon\ln\epsilon\right)(q_0 + O(\epsilon^2\ln\epsilon)),$$

where $\rho(\kappa) := \frac{1}{\pi}(2\ln 2 + \ln\kappa + \gamma_1)$. All the roots of $p(\kappa, \epsilon)$ lie in the domain $\Omega_{\delta_0, \theta_0, M} := \{z \in \mathbb{C} : |z - 2j\pi/d| \geq \delta_0, j \in \mathbb{Z}\} \cap \Omega_{\theta_0, M}$, where

$$\Omega_{\theta_0, M} = \left\{z \in \mathbb{C} : |z| \leq M, -(\frac{\pi}{d} - \theta_0) \leq \text{arg} z \leq \frac{\pi}{d} - \theta_0\right\}.$$

It is clear to note that $\frac{\cot\kappa d}{\kappa}$ is analytic in the domain $\Omega_{\delta_0, \theta_0, M}$ and the roots are given by

$$k_{n,0} = \left(n + \frac{1}{2}\right)\frac{\pi}{d}, \quad n = 0, 1, 2, \cdots.$$

Denote by k_n the roots of $\lambda(\epsilon, \kappa)$. Applying Rouche's theorem, we deduce that k_n are simple and close to $k_{n,0}$. The corresponding $p_1(\kappa, \epsilon)$ are defined by

$$p_1(\kappa, \epsilon) = \epsilon + \left(\frac{\cot\kappa d}{\kappa} + \epsilon\rho(k) + \frac{1}{\pi}\epsilon\ln\epsilon\right)q_0.$$

Expanding $p_1(\kappa, \epsilon)$ at $k_{n,0}$ yields

$$p_1(\kappa, \epsilon) = \epsilon + \left(\frac{-d^2}{(n + \frac{1}{2})\pi}(\kappa - k_{n0}) + \frac{\epsilon}{\pi} \left(2\ln 2 + \ln \frac{(n + \frac{1}{2})\pi}{d} + \gamma_1 \right) \right.$$
$$\left. + \frac{\epsilon d}{(n + \frac{1}{2})\pi^2}(\kappa - k_{n0}) + O(\kappa - k_{n0})^2 + \frac{1}{\pi}\epsilon \ln \epsilon \right) q_0.$$

We conclude that $p_1(\kappa, \epsilon)$ has simple roots in $\Omega_{\delta_0, \theta_0, M}$ which are close to k_{n0}. Moreover, these roots are analytic with respect to the variable ϵ and $\epsilon \ln \epsilon$. Denote the roots of $p_1(\kappa, \epsilon)$ by k_{n1} and expand them in term of ϵ and $\epsilon \ln \epsilon$, we obtain

$$k_{n1} = k_{n0} + \frac{k_{n0}}{d} \left(\frac{1}{\pi}\epsilon \ln \epsilon + \left(\frac{1}{q_0} + \frac{1}{\pi}(2\ln 2 + \ln \frac{(n + \frac{1}{2})\pi}{d} + \gamma_1) \right) \epsilon \right) + O(\epsilon^2 \ln \epsilon).$$

Using Rouche's theorem again, we obtain

$$k_n = k_{n1} + O(\epsilon^2 \ln \epsilon),$$

which completes the proof. $\qquad\square$

5.4 The field enhancement with resonance

In the far field region, we can follow the same steps as those in Sect. 3.4.1 with the new $p(\kappa, \epsilon)$. Then we have the following theorem with the detailed proof omitted.

Theorem 5.4 *At the resonant frequencies $\kappa = \mathrm{Re}\,k_n$ where k_n is given in (5.6), the scattered electric and magnetic fields have enhancement of an order $O(1/\epsilon)$ in the far field region.*

Next we demonstrate the field enhancement inside the cavity. It follows from (4.2) that the total field u_ϵ can be expanded as the sum of waveguide modes:

$$u_\epsilon(\boldsymbol{x}) = \frac{1}{\sqrt{\epsilon}} \left(\alpha_0^+ e^{-i\kappa x_2} + \alpha_0^- e^{i\kappa(x_2 + d)} \right)$$
$$+ \sum_{n=1}^{\infty} \sqrt{\frac{2}{\epsilon}} \left(\alpha_n^+ e^{-i\beta_n x_2} + \alpha_n^- e^{i\beta_n(x_2 + d)} \right) \cos\left(\frac{m\pi x_1}{\epsilon} \right). \tag{5.7}$$

Lemma 5.5 *The coefficients in (5.7) have the following properties:*

$$\frac{\alpha_0^+}{\sqrt{\epsilon}} = \frac{q_0(2 + \sin\theta \cdot O(\kappa\epsilon))}{i\kappa(-1 + e^{i2\kappa d})p} + O(\epsilon^2 \ln \epsilon), \quad \frac{\alpha_0^+}{\sqrt{\epsilon}} = \frac{q_0(2 + \sin\theta \cdot O(\kappa\epsilon))}{-2\kappa \sin(\kappa d)p} + O(\epsilon^2 \ln \epsilon)$$

and

$$\left| \sqrt{\frac{2}{\epsilon}}\alpha_n^+ \right| \le \frac{C}{\sqrt{n}}, \quad \left| \sqrt{\frac{2}{\epsilon}}\alpha_n^- \right| \le \frac{C}{\sqrt{n}}, \quad n \ge 1,$$

where the positive constant C is independent of λ, ϵ and n.

Proof It follows from the boundary condition $\partial_\nu u_\epsilon = 0$ on Γ_ϵ^- that the expansion of (5.7) becomes

$$u_\epsilon(\boldsymbol{x}) = \frac{\alpha_0^+ e^{i\kappa d}}{\sqrt{\epsilon}} \left(e^{-i\kappa(x_2 + d)} + e^{i\kappa(x_2 + d)} \right)$$

$$+ \sum_{n=1}^{\infty} \sqrt{\frac{2}{\epsilon}} \alpha_n^+ e^{i\beta_n d} \left(e^{-i\beta_n(x_2+d)} + e^{i\beta_n(x_2+d)} \right) \cos\left(\frac{n\pi x_1}{\epsilon}\right), \quad x \in D_\epsilon.$$

Taking the derivative of the above equality with respect to x_2 on Γ_ϵ^+ yields

$$\frac{\partial u_\epsilon(x_1,0)}{\partial x_2} = \frac{\alpha_0^+}{\sqrt{\epsilon}} i\kappa(-1 + e^{i2\kappa d}) + \sum_{n=1}^{\infty} \sqrt{\frac{2}{\epsilon}} \alpha_n^+ i\beta_n(-1 + e^{i2\beta_n d}) \cos\left(\frac{n\pi x_1}{\epsilon}\right).$$

Multiplying the above equality by $\phi_0(x_1)$ and integrating it on Γ_ϵ^+ yields

$$\frac{\alpha_0^+}{\sqrt{\epsilon}} i\kappa(-1 + e^{i2\kappa d}) = \frac{1}{\epsilon} \int_{\Gamma_\epsilon^+} \frac{\partial u_\epsilon}{\partial x_2}(x_1,0)\mathrm{d}x_1 = \frac{1}{\epsilon} \int_{\Gamma_\epsilon^+} -\varphi(x_1/\epsilon)\mathrm{d}x_1$$

$$= \int_0^1 \varphi(X)\mathrm{d}X = \frac{q_0(2 + \sin\theta \cdot O(\kappa\epsilon))}{p} + O(\epsilon^2 \ln\epsilon).$$

Here we use $\langle \varphi, 1_{(0,1)} \rangle_{L^2(0,1)} = \frac{q_0(2+\sin\theta \cdot O(\kappa\epsilon))}{p} + O(\epsilon^2 \ln\epsilon)$, which is shown in the proof of Lemma 3.7. By $\alpha_0^- = \alpha_0^+ e^{i\kappa d}$, we get

$$\frac{\alpha_0^+}{\sqrt{\epsilon}} = \frac{q_0(2 + \sin\theta \cdot O(\kappa\epsilon))}{i\kappa(-1 + e^{i2\kappa d})p} + O(\epsilon^2 \ln\epsilon),$$

$$\frac{\alpha_0^-}{\sqrt{\epsilon}} = \frac{q_0(2 + \sin\theta \cdot O(\kappa\epsilon))}{-2\kappa \sin(\kappa d)p} + O(\epsilon^2 \ln\epsilon).$$

Similarly, for $n \geq 1$, we may obtain

$$\sqrt{\frac{2}{\epsilon}} \alpha_n^+ i\beta_n(-1 + e^{i2\beta_n d}) = -\int_0^1 \varphi(X)\cos(n\pi X)\mathrm{d}X.$$

Note that $i\beta_n = O(-\frac{n}{\epsilon})$, $\|\varphi\|_{H^{-1/2}(0,1)} \lesssim \frac{1}{\epsilon}$, $\|\cos(n\pi X)\|_{H^{1/2}(0,1)} \lesssim \sqrt{n}$. Thus we have

$$\left| \sqrt{\frac{2}{\epsilon}} \alpha_n^+ \right| \leq \frac{C}{\sqrt{n}}, \quad \left| \sqrt{\frac{2}{\epsilon}} \alpha_n^- \right| \leq \frac{C}{\sqrt{n}}, \quad n \geq 1,$$

where the positive constant C is independent of λ, ϵ, n. □

The following result gives the field enhancement inside the cavity.

Theorem 5.6 *Let* $\mathring{D}_\epsilon := \{x \in D_\epsilon : -d \leq x_2 \leq -\epsilon\}$ *be the interior of the cavity* D_ϵ. *If* $\kappa = \mathrm{Re}k_n$ *where* k_n *is given in (5.6), then we have for* $\epsilon \ll \lambda$ *that*

$$u_\epsilon(x) = \left(\frac{2}{\epsilon} + \sin\theta \cdot O(1) + O(\ln^2\epsilon) \right) \frac{2i\cos\kappa(x_2+d)}{\kappa \sin(\kappa d)} + O(\epsilon^2 \ln\epsilon), \quad x \in \mathring{D}_\epsilon.$$

Moreover, the electric and magnetic field enhancements are of an order $O(1/\epsilon)$.

Proof In the region $\mathring{D}_\epsilon$, we have from the expansion of (5.7) and Lemma 5.5 that
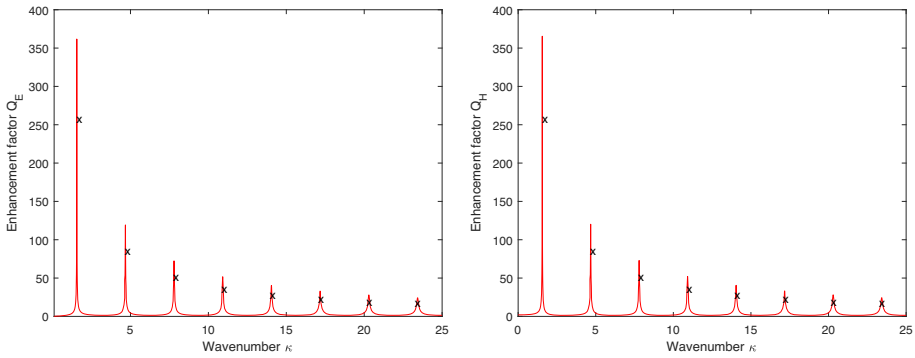
Fig. 6 The field enhancement factor is plotted against the wavenumber κ for the PEC-PEC cavity: (left) the electric field enhancement factor Q_E; (righ) the magnetic field enhancement factor Q_H

$$u_\epsilon(x) = \frac{q_0(2 + \sin\theta \cdot O(\kappa\epsilon)}{i\kappa(-e^{-i\kappa d} + e^{i\kappa d})p}\left(e^{-i\kappa(x_2+d)} + e^{i\kappa(x_2+d)}\right) + O(\epsilon^2 \ln\epsilon) + O(e^{-1/\epsilon})$$

$$= \frac{q_0(2 + \sin\theta \cdot O(\kappa\epsilon))\cos\kappa(x_2+d)}{-\kappa\sin\kappa d}\frac{1}{p} + O(\epsilon^2 \ln\epsilon) + O(e^{-1/\epsilon}).$$

Using the fact that at the resonant frequencies $\kappa = \mathrm{Re}k_n$,

$$\frac{1}{p} = \frac{2i}{q_0\epsilon}(1 + O(\epsilon \ln^2\epsilon)).$$

we obtain

$$u_\epsilon(x) = \left(\frac{2}{\epsilon} + \sin\theta \cdot O(1) + O(\ln^2\epsilon)\right)\frac{2i\cos\kappa(x_2+d)}{\kappa\sin(\kappa d)} + O(\epsilon^2 \ln\epsilon).$$

Therefore, the magnetic field enhancement has an order of $O(1/\epsilon)$. We conclude from Ampere's law that the electric field enhancement also has an order of $O(1/\epsilon)$. $\qquad\square$

Finally, on the open aperture Γ_ϵ^+, we may follow similar steps as those in Sect. 3.4.3 and show the electromagnetic field enhancement. The proof is omitted again for brevity.

Theorem 5.7 *At the resonant frequencies $\kappa = \mathrm{Re}k_n$ where k_n is given in (5.6), the enhancement of the scattered electric and magnetic fields has an order $O(1/\epsilon)$ on the open aperture Γ_ϵ^+.*

5.5 Numerical experiments

We show some numerical experiments to verify the theoretical findings of the field enhancement for the PEC-PEC cavity. We take the cavity width $\epsilon = 0.005$, the cavity depth $d = 1$, and the angle of incidence $\theta = \pi/3$. Figure 6 (left) and (right) show the plot of the electric field enhancement Q_E the the magnetic field enhancement factor Q_H against the wavenumber κ, respectively. We observe that both the electric and magnetic enhancement factors do obtain peaks at the resonant frequencies (5.6), which are close to $\kappa = (n + \frac{1}{2})\pi, n = 0, 1, 2, \ldots$.

6 Concluding remarks

We have studied the electromagnetic field enhancement for the scattering of a plane wave by a subwavelength rectangular open cavity. The TM polarized wave model problem is considered for both the PEC-PMC and the PEC-PEC cavities. The electric and magnetic field enhancements are examined in both the nonresonant and resonant regimes. These results provide an understanding on some of the mechanisms for the wave field amplification and localization in a subwavelength structure. This work focuses on the TM polarization of the electromagnetic wave in a single cavity. We will investigate the influence of multiple cavities, whether the enhancement may occur in the transverse electric polarization (TE), the more complicated three-dimensional Maxwell equations, and the elastic wave equation. The results will be reported elsewhere in the future.

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