The Inverse Problem for Derivative Securities of Interest Rate

Peijun Li Guanquan Zhang

State Key Laboratory of Scientific and Engineering Computing,

Institute of Computational Mathematics and Scientific/Engineering Computing,

Chinese Academy of Sciences, Beijing, 100080

E-mail:lpj@lsec.cc.ac.cn,zgq@lsec.cc.ac.cn

May 26, 2000

Abstract

Market price for risk of interest rate reflects the close relation between risk and yield of securities dependent on interest rate. An inverse problem for derivative security of interest rate is to determine the relationship. In this paper, we reduce an inverse coefficient problem to an inverse source problem, give an approximate solution, establish the theorem of existence and uniqueness for the solution and propose an efficient iterative algorithm.

Key Words: market price for risk of interest rate, derivative security, zero-coupon bond, inverse source problem.

1 Introduction

Derivative security is a kind of security whose value depends on other more basic underlying varibles. Derivative security for interest rate is one whose payoff is determined by interest rate to some extent. Recently, the derivative securities for interest rate have been more and more fashionable in financial territory. We consider the valuation of zero-coupon bond, which is sold in somewhat higher discount and will be redeemed on its face value on the maturity date. However, our results are also applicable to certain other derivative securities.

Suppose that the value of zero-coupon bond $V(t,r)$ is a function of time t and interest r. T is the expiration date and K is a certain face value of zero-coupon bond. λ, dependent on t, is market price for risk of interest rate which reflects the relationship between risk and yield. Suppose the ITO process [1] of interest rate is that

$$dr = (-\gamma(r)r + \delta(r))dt + w(r)dz.$$

where dz is a normally distributed random varible with zero mean and variance dt. In practice the spot rate is never greater than a certain number, which is assumed to be R, and never less than or equal to zero. Therefore we assume that the interest rate $r \in [0, R]$. $\gamma(r)$ and $\delta(r)$ are both smooth bounded functions for $r \in [0, R]$ satisfying conditions $\delta(0) \geq 0$ and $-\gamma(R)R + \delta(R) \leq 0$. The above conditions are reasonable and indispensable in order to describe the mean reversion of interest rate in the ITO process. $w(r)$ is a non-negative and smooth bounded function for $r \in [0, R]$ satisfying conditions $w(0) = w(R) = 0$. These quantities can be determined by statistics and the least square method from the historic data. Following the general method for derivative security pricing [1], we get the partial differential equation for a zero-coupon bond in the form

$$\frac{\partial V}{\partial t} + \frac{w^2(r)}{2}\frac{\partial^2 V}{\partial r^2} + (-\gamma(r)r + \delta(r) + \lambda(t)w(r))\frac{\partial V}{\partial r} - rV = 0,$$
$$(t, r) \in [0, T] \times [0, R] \tag{1.1}$$

At $r = 0$ and $r = R$ the equation degenerates into a hyperbolic equation with positive and negative characteristics respectively:

$$\frac{\partial V}{\partial t} + \delta(0)\frac{\partial V}{\partial r} = 0, \quad (t, r) \in [0, T] \times \{0\} \tag{1.2}$$
$$\frac{\partial V}{\partial t} + (-\gamma(R)R + \delta(R))\frac{\partial V}{\partial r} - RV = 0, \quad (t, r) \in [0, T] \times \{R\} \tag{1.3}$$

The final condition is given by

$$V(T, r) = K \tag{1.4}$$

Usually, if we assume that R is bounded, then one has to give a boundary condition at $r = R$ in order to have a unique solution. It is difficult to give a boundary condition which has a clear and reasonable meaning in finance. The ITO process for interest rate given here avoids such a problem.

It is convenient to make the change of variable $\tau = T - t$. Then equations (1.1)-(1.4) can be rewritten as a parabolic equation for the first initial-boundary value with unknown coefficient function:

$$\frac{\partial V}{\partial \tau} = \frac{w^2(r)}{2}\frac{\partial^2 V}{\partial r^2} + (-\gamma(r)r + \delta(r) + \lambda(\tau)w(r))\frac{\partial V}{\partial r} - rV,$$
$$(\tau, r) \in [0, T] \times [0, R], \tag{1.5}$$
$$\frac{\partial V}{\partial \tau} = \delta(0)\frac{\partial V}{\partial r}, \quad (\tau, r) \in [0, T] \times \{0\}, \tag{1.6}$$
$$\frac{\partial V}{\partial \tau} = (-\gamma(R)R + \delta(R))\frac{\partial V}{\partial r} - RV, \quad (\tau, r) \in [0, T] \times \{R\}, \tag{1.7}$$
$$V(0, r) = K. \tag{1.8}$$

If the market price for risk of interest rate λ is assumed to be known, the initial-boundary value problem can have a unique solution for $0 \leq \tau \leq T$. However, we can not directly know the market price for risk of interest rate from financial markets because interest rate is non-traded security. Therefore we must have additional market data in order to determine the pair of functions

$V(\tau, r)$ and $\lambda(\tau)$ satisfying (1.5)-(1.8), which is called the inverse problem for derivative securities of interest rate.

This paper is organized as follows. In section 2, by reducing (1.5)-(1.8) to an inverse source problm, we obtain an approximate solution and the properly additional data which can ensure that the problem (1.5)-(1.8) has unique solution from mathematical standpoint. At the same time, we give the error estimation for the approximate solution. In section 3, we introduce the finite difference schemes and the numerical implementation for the inverse problem discussed in section 2. Furthermore, we constuct an iterative algorithm to improve the inversion accuracy. Finally we present several numerical examples and results in section 4.

2 Reduction To an Inverse Source Problem

At first, we tried to use the direct method to solve the problem (1.5)-(1.8) with the additional data $V(\tau, r_*), r_* \in (0, R)$. However, numerical experiments show that the method is no good because the inversion $\lambda(\tau)$ is unstable for the given additional data. Therefore we have to look for another method.

Consider the following problem

$$\frac{\partial \widehat{V}}{\partial \tau} = \frac{w^2(r)}{2}\frac{\partial^2 \widehat{V}}{\partial r^2} + (-\gamma(r)r + \delta(r))\frac{\partial \widehat{V}}{\partial r} - r\widehat{V}, \quad (\tau, r) \in [0, T] \times [0, R], \tag{2.1}$$

$$\widehat{V}(T, r) = K. \tag{2.2}$$

According to the assumptions that $\gamma(r)$ and $\delta(r)$ are smooth bounded functions, using one known result [2], the solution of (2.1),(2.2) is unique and continuously differential. Let $U = V - \widehat{V}$, substracting (1.5) from (2.1) and (1.8) from (2.2), then we have

$$\frac{\partial U}{\partial \tau} = \frac{w^2(r)}{2}\frac{\partial^2 U}{\partial r^2} + (-\gamma(r)r + \delta(r))\frac{\partial U}{\partial r} - rU + \lambda(\tau)w(r)\frac{\partial V}{\partial r}, \quad (\tau, r) \in [0, T] \times [0, R], \tag{2.3}$$

$$U(0, r) = 0. \tag{2.4}$$

substituting $\frac{\partial V}{\partial r}$ with $\frac{\partial \widehat{V}}{\partial r}$ in (2.3), we obtain the approximation equations

$$\frac{\partial \widehat{U}}{\partial \tau} = \frac{w^2(r)}{2}\frac{\partial^2 \widehat{U}}{\partial r^2} + (-\gamma(r)r + \delta(r))\frac{\partial \widehat{U}}{\partial r} - r\widehat{U} + \widehat{\lambda}(\tau)w(r)\frac{\partial \widehat{V}}{\partial r} \quad (\tau, r) \in [0, T] \times [0, R], \tag{2.5}$$

$$\widehat{U}(0, r) = 0 \tag{2.6}$$

The problem (2.5),(2.6) is an inverse source problem because of the unknown function $\lambda(\tau)$ in the right term. Let $\Gamma(\tau, r; s, \xi)$ be the fundamental solution of

(2.5), then

$$\widehat{U}(\tau,r)=\int_0^\tau\int_0^R\Gamma(\tau,r;s,\xi)\widehat{\lambda}(s)w(\xi)\frac{\partial\widehat{V}(s,\xi)}{\partial\xi}\,d\xi ds. \tag{2.7}$$

We differeniate (2.7) in both sides with respect to τ, and using the properties of fundamental solution [2] arrive at

$$\frac{\partial\widehat{U}(\tau,r)}{\partial\tau}=\widehat{\lambda}(\tau)w(r)\frac{\partial\widehat{V}(\tau,r)}{\partial r}+\int_0^\tau\int_0^R\frac{\partial\Gamma(\tau,r;s,\xi)}{\partial\tau}\widehat{\lambda}(s)w(\xi)\frac{\partial\widehat{V}(s,\xi)}{\partial\xi}d\xi ds \tag{2.8}$$

What we need as the additional market data, $\frac{\partial\widehat{U}(\tau,r_*)}{\partial\tau}, r_*\in(0,R)$, is indispensable for the linear Volterra integral equation (2.8) to have unique solution, in this paper, which means the volatility for the difference between risk-free price and risk price of zero-coupon bond. In showing the existence and uniqueness for the solution of the integral equation, we shall make use of the following lemma.

Lemma 1 *There exists at least an $r_*\in(0,R)$ such that $\frac{\partial\widehat{V}(\tau,r_*)}{\partial\tau}$ do not vanish for all $0<\tau\le T$.*

proof: Indeed, in the contrary case there exists one time $\tau_*\in(0,T)$ such that $\frac{\partial\widehat{V}(\tau_*,r)}{\partial\tau}=0$ for all $r\in(0,R)$, that is to say, $\widehat{V}(\tau_*,r)$, not dependent on r, is a constant. Then (2.1) can be rewritten as $\frac{\partial\widehat{V}}{\partial\tau}=-r\widehat{V}$, for all $r\in(0,R)$. In practice, the formula holds only under the condition of free-risk yield when the market price for risk of interest rate $\lambda(\tau_*)$ is zero. However, $V(\tau,r)$, which satisfies equation (1.5), is the value for zero-coupon bond under the condition of risk when the market price for risk of interest rate does not vanish for all $0<\tau\le T$, which is a contradiction. □

Of course, we may assume $w(r_*)$ is not zero since $w(r)$ is a non-negative smooth one. We regard $\int_0^R\frac{\partial\Gamma(\tau,r_*;s,\xi)}{\partial\tau}w(\xi)\frac{\partial\widehat{V}(s,\xi)}{\partial\xi}d\xi$ as the integral kernel which is continous by [2], using the existence and uniqueness of the solution for linear volterra integral equation [4], thus we have proved:

Theorem 1 *Integral equation (2.8) posseses a unique continous solution for $0<\tau\le T$ if additional data $\frac{\partial\widehat{U}(\tau,r_*)}{\partial\tau}, r_*\in(0,R)$ is given, and the solution is cotinously dependent on the additional data.*

Now we shall introduce another lemma before analysing the error of the approximate solution.

Lemma 2 *Assumed that*

$$f(\tau)=g(\tau)+\int_0^\tau K(\tau,s)f(s)ds, \tag{2.9}$$

$$\widehat{f}(\tau)=\widehat{g}(\tau)+\int_0^\tau\widehat{K}(\tau,s)\widehat{g}(s)ds. \tag{2.10}$$

where $g(\tau), \widehat{g}(\tau), K(\tau,s)$ *and* $\widehat{K}(\tau,s)$ *are continous functions and* $|g(\tau)-\widehat{g}(\tau)| \le \epsilon_1, |K(\tau,s)-\widehat{K}(\tau,s)| \le \epsilon_2, |\widehat{K}(\tau,s)| \le M_1$, *for all* $0 \le s \le \tau \le T$. *Then*

$$|f(\tau)-\widehat{f}(\tau)| \le (\epsilon_1 + M_2\tau\epsilon_2)\cdot e^{M_1\tau}.$$

proof: First, by our assumptions on Volterra integral equations (2.9) and (2.10) it follows that $f(\tau)$ and $\widehat{f}(\tau)$ exist and are continous for $0 \le \tau \le T$. Let $|f(\tau)| \le M_2$, subtracting (2.9) from (2.10), then we have

$$|f(\tau)-\widehat{f}(\tau)| = |g(\tau)-\widehat{g}(\tau) + \int_0^\tau (K(\tau,s)f(s) - \widehat{K}(\tau,s)\widehat{f}(\tau))ds|$$

$$\le |g(\tau)-\widehat{g}(\tau)| + \int_0^\tau |K(\tau,s)-\widehat{K}(\tau,s)|\cdot|f(s)|ds + \int_0^\tau |\widehat{K}(\tau,s)|\cdot|f(s)-\widehat{f}(s)|ds$$

$$\le (\epsilon_1 + M_2\tau\epsilon_2) + \int_0^\tau |\widehat{K}(\tau,s)|\cdot|f(s)-\widehat{f}(s)|ds$$

Using known results in [4], we obtain

$$|f(\tau)-\widehat{f}(\tau)| \le (\epsilon_1 + M_2\tau\epsilon_2)\cdot e^{M_1\tau}.$$

□

By (2.8) and lemma 1,we have

$$\widehat{\lambda}(\tau) = \frac{U_*(\tau)}{w(r_*)} / \frac{\partial \widehat{V}(\tau,r_*)}{\partial \tau} + \int_0^\tau (\int_0^R \frac{\partial \Gamma(\tau,r_*;s,\xi)}{\partial \tau} w(\xi)\frac{\partial \widehat{V}(s,\xi)}{\partial \xi}d\xi / \frac{\partial \widehat{V}(\tau,r_*)}{\partial r})\widehat{\lambda}(s)ds. \tag{2.11}$$

Similarily,

$$\lambda(\tau) = \frac{U_*(\tau)}{w(r_*)} / \frac{\partial V(\tau,r_*)}{\partial \tau} + \int_0^\tau (\int_0^R \frac{\partial \Gamma(\tau,r_*;s,\xi)}{\partial \tau} w(\xi)\frac{\partial V(s,\xi)}{\partial \xi}d\xi / \frac{\partial V(\tau,r_*)}{\partial r})\lambda(s)ds. \tag{2.12}$$

where $U_*(\tau)$ is the additional condition. Subtracting (2.12) from (2.11) and using lemma 2, we then have the following result.

Theorem 2 *The error of approximate solution satisfies*

$$|\lambda(\tau)-\widehat{\lambda}(\tau)| \le C\cdot e^{A\tau}|\frac{\partial V(\tau,r_*)}{\partial r} - \frac{\partial \widehat{V}(\tau,r_*)}{\partial r}|.$$

where C *is the bound of* $\frac{U_*(\tau)}{w(r_*)} / \frac{\partial V(\tau,r_*)}{\partial r}\frac{\partial \widehat{V}(\tau,r_*)}{\partial r}$ *and* A *is the bound of* $\int_0^R \frac{\partial \Gamma(\tau,r_*;s,\xi)}{\partial \tau} w(\xi)\frac{\partial \widehat{V}(s,\xi)}{\partial r}d\xi / \frac{\partial \widehat{V}(\tau,r_*)}{\partial r}$.

3 Numerical Implementation for the Inverse Source Problem

In this section, we describe the difference scheme for the inverse source problem, and propose an efficient iterative algorithm.

Let Ω be the domain $\Omega = \{(\tau, r) \mid 0 \le \tau \le T, 0 \le r \le R\}$, and cover Ω by the grid $\{(\tau_i, r_j) | \tau_i = i\Delta\tau, r_j = j\Delta r, \Delta\tau = \frac{T}{m}, \Delta r = \frac{R}{n}\}$, shown in Fig.1.

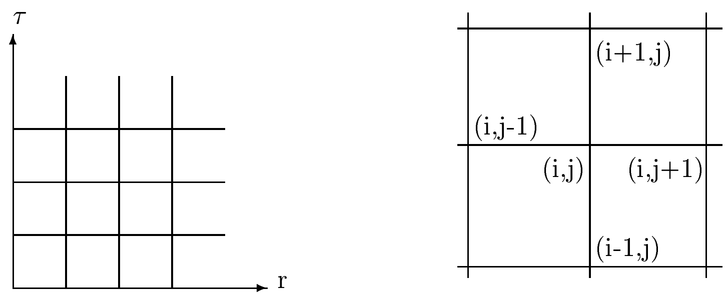


FIG.1. The grid used in numerical computation.

Denoting $f_{i,j} = f(\tau_i, r_j)$, discretizing equations (2.1),(2.2), (2.5) and (2.6), then we get

$$\frac{\widehat{V}_{i+1,j} - \widehat{V}_{i,j}}{\Delta\tau} = \frac{w^2{}_j}{2} \frac{(\widehat{V}_{i+1,j+1} - 2\widehat{V}_{i+1,j} + \widehat{V}_{i+1,j-1})}{\Delta r^2} + (-\gamma_j r_j + \delta_j)\ \frac{\widehat{V}_{i+1,j+1} - \widehat{V}_{i+1,j-1}}{2\Delta r} - r_j \widehat{V}_{i+1,j}, \tag{3.1}$$

$$\frac{\widehat{V}_{i+1,0} - \widehat{V}_{i,0}}{\Delta\tau} = \delta_0\ \frac{(\widehat{V}_{i,1} - \widehat{V}_{i,0})}{\Delta r}, \tag{3.2}$$

$$\frac{\widehat{V}_{i+1,n} - \widehat{V}_{i,n}}{\Delta\tau} = (\gamma_n R + \delta_n) \frac{\widehat{V}_{i,n} - \widehat{V}_{i,n-1}}{\Delta r} - R\widehat{V}_{i,n}, \tag{3.3}$$

$$\widehat{V}_{0,j} = K, \tag{3.4}$$

$$\frac{\widehat{U}_{i+1,j} - \widehat{U}_{i,j}}{\Delta\tau} = \frac{w^2{}_j}{2} \frac{(\widehat{U}_{i+1,j+1} - 2\widehat{U}_{i+1,j} + \widehat{U}_{i+1,j-1})}{\Delta r^2} + (-\gamma_j r_j + \delta_j) \frac{\widehat{U}_{i+1,j+1} - \widehat{U}_{i+1,j-1}}{2\Delta r} - r_j \widehat{U}_{i+1,j} + \lambda_{i+1} w_j \frac{(\widehat{V}_{i+1,j+1} - \widehat{V}_{i+1,j-1})}{2\Delta r}, \tag{3.5}$$

$$\frac{\widehat{U}_{i+1,0} - \widehat{U}_{i,0}}{\Delta\tau} = \delta_0 \frac{(\widehat{U}_{i,1} - \widehat{U}_{i,0})}{\Delta r}, \tag{3.6}$$

$$\frac{\widehat{U}_{i+1,n} - \widehat{U}_{i,n}}{\Delta\tau} = (\gamma_n R + \delta_n) \frac{\widehat{U}_{i,n} - \widehat{U}_{i,n-1}}{\Delta r} - R\widehat{U}_{i,n}, \tag{3.7}$$

$$\widehat{U}_{0,j} = 0. \tag{3.8}$$

The difference schemes,(3.1) and (3.5), are implict and have truncation errors $O(\Delta\tau + \Delta r^2)$. On boundary conditions, the difference schemes, (3.2),(3.3),(3.6) and (3.7), are explict and have truncation errors $O(\Delta\tau + \Delta r)$. The numerical computation begins at the initial time $\tau = 0$ and advances forward. At each time

$\tau, \widehat{V}(\tau,r)$ are obtained from $r=0$ to $r=R$ by solving linear equations. After solving (3.1)-(3.4), similarily, $\widehat{U}(\tau,r), \lambda(\tau)$ can also be obtained from (3.5)-(3.8) together with the additional data $\frac{\partial \widehat{U}(\tau,r_*)}{\partial \tau}, r_* \in (0,R)$. Numerical results illustrate that the market price for risk of interest rate $\lambda(\tau)$ by solving (3.1)-(3.8) do not have enough accuracy. This is mainly due to the error of substituting $\frac{\partial V}{\partial r}$ with $\frac{\partial \widehat{V}}{\partial r}$. Therefore we constuct an iterative algorithm to improve the inversion accuracy as follows:

step 1. For m=0, $\lambda_m(\tau)$=0, where m is the index of iteration;
step 2. Solve direct problem (3.1)-(3.4) to obtain $\widehat{V}$;
step 3. For the known $\widehat{V}$, solve the inverse source problem (3.5)-(3.8) to obtain $\lambda_{m+1}(\tau)$;
step 4. If

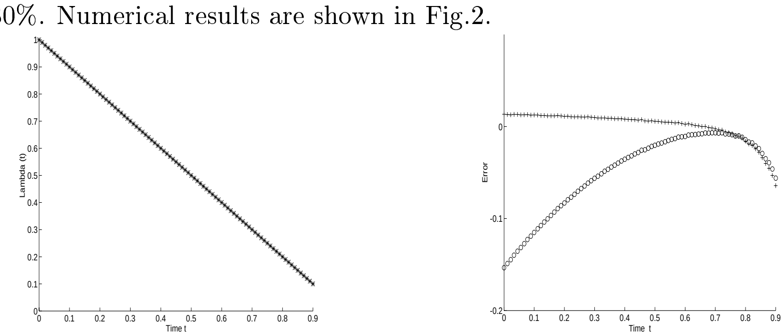
$$\epsilon = \| \lambda_{m+1}(\tau) - \lambda_m(\tau) \|$$

is small enough, stop the iteration; Otherwise, go to step 5;
step 5. Let m=m+1, go to step 2.

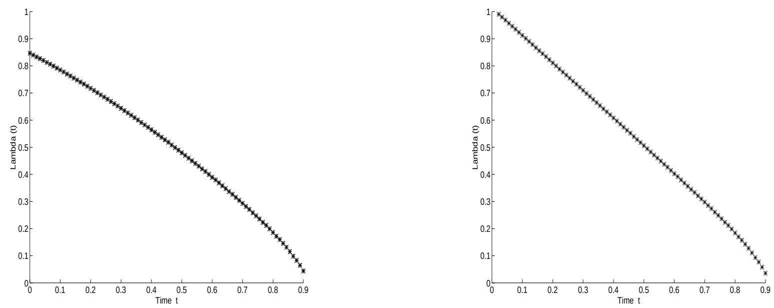
4 Numerical Tests and Results

In section 3, we construct the difference schemes and use iterative algorithm to get improved results. Here, we give two specific numerical experiments, and the computed results show that the method is effective. In computing, we can obtain ideal accuracy by only few iterative times.

Example 1: The given model of market price for risk of interest rate is $\lambda(t) = T - t$. In computing, let K=100, $\tau \in [0,1], r \in [0,1], \Delta r = \Delta\tau = 0.01$, and $-\gamma(r)r + \delta(r) = 0.5 - r, w(r) = r(1-r)$. The error of the additional data is 30%. Numerical results are shown in Fig.2.



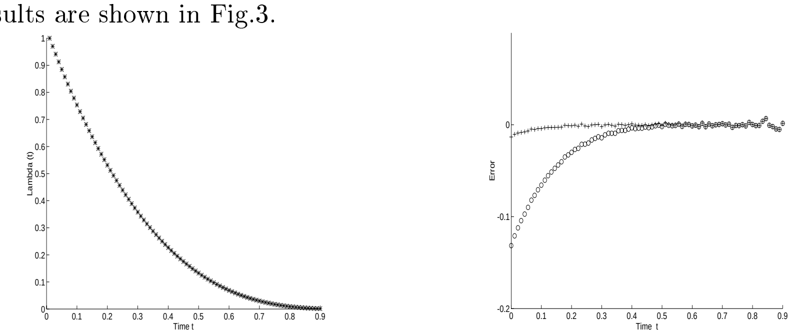
a:Risk price model $\lambda(t) = 1 - t$ b:The error of iteration



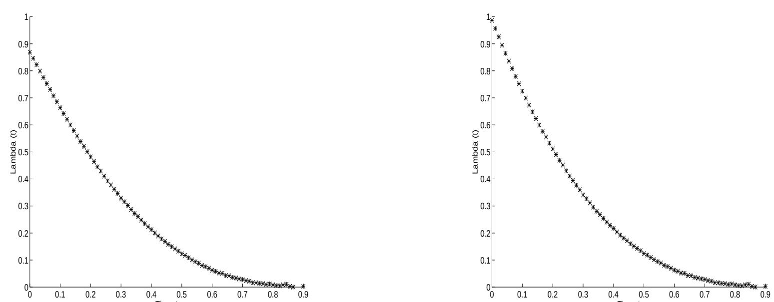
c:Result of 1st iteration d:Result of 2nd iteration

FIG.2. Numerical results for example 1.

Example 2: The given model of market price for risk of interest rate is $\lambda(t) = (T-t)^3$. The other variables are the same as the example 1. Numerical results are shown in Fig.3.



a:Risk price model $\lambda(t) = (1-t)^3$ b:The error of iteration



c:Result of 1st iteration d:Result of 2nd iteration

FIG.3. Numerical results for example 2.

FIG.2 FIG.3. (a) the market price for risk of interest rate in example; (b) the error of the iteration, o: the error of the first iteration, +: the error of the second iteration; (c) the inversion result of the first iteration; (d) the inversion result of the second itration.

5 Conclusions

By reducing the original inverse problem with unknown coefficient $\lambda(t)$ to an inverse source problem, we obtain an approximate solution, establish the theorem for its existence and uniqueness and give its error estimation. The iterative method proposed in section 3 remarkly improves the inversion accuracy. From the numerical experiment, we see this method is efficient. However, we must point out that we give a effective method to solve market price for risk of interest rate which only depends on time t. If market price for risk of interest rate is the function of interest rate r, we can obtain a Fredholm integral equation of the first kind which, as anybody can see, generally, is very ill-posed, and it is worse that the kernel is smooth. One will make little progress in solving inverse problem of $\lambda(r)$ if not seeking a regularization method.

References

[1] Hull J, *options,futures, and other derivative securities*, 3rd edn(upper saddle river, NJ:Prentice-Hall, 1997).
[2] Friedman A, *partial differential equations of parabolic type* (Englewood cliffs, NJ:Prentice-Hall, 1964).
[3] Bouchouev I and Isakov V, *The inverse problem of option pricing*, Inverse Problems 13, 11(1997).
[4] Linz P, *Analytical and numerical methods for Volterra equations,* (Philadeiphia: S.I.A.M, 1985).
[5] Isakov V, *Inverse problems for partial differential equations*, (New York, springer-verlag, 1998).