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A GENERALIZATION OF CEVA'S THEOREM

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Ceva's theorem in elementary geometry deals with a triangle $A_1A_2A_3$ and a point P in general position in its plane. The lines A_1P, A_2P, A_3P, intersect the sides A_2A_3, A_3A_1, A_1A_2, respectively, in points I_1, I_2, I_3. The theorem states that

$$\frac{A_2I_1}{A_3I_1}\cdot\frac{A_3I_2}{A_1I_2}\cdot\frac{A_1I_3}{A_2I_3} = (-1)^3.$$

We propose to generalize this result by considering a plane polygon $A_1(x_1, y_1)$, $\cdots$, $A_n(x_n, y_n)$, and $\frac{1}{2}d(d+3)-1$ points (P) in general position in its plane.

A preliminary example, with $n=5$, $d=2$, will help to clarify what follows. Each vertex A_r of a pentagon $A_1A_2A_3A_4A_5$ determines with four fixed points $P_1P_2P_3P_4$ a unique conic Q_r. Name the six points in which A_1A_2 intersects Q_3, Q_4, and Q_5, P_{12}^i $(i=1, \cdots, 6)$; name the six points in which A_2A_3 intersects Q_4, Q_5, and Q_1, P_{23}^i; define similarly P_{34}^i, P_{45}^i, P_{51}^i. What is to be proved is that the product

$$(A_1P_{12}^1 \cdots A_1P_{12}^6)(A_2P_{23}^1 \cdots A_2P_{23}^6) \cdots (A_5P_{51}^1 \cdots A_5P_{51}^6)$$

is equal to $(-1)^5$ times the similar product taken in the opposite direction, *i.e.*, to

$$(-1)^5(A_1P_{51}^1 \cdots A_1P_{51}^6)(A_5P_{45}^1 \cdots A_5P_{45}^6) \cdots (A_2P_{12}^1 \cdots A_2P_{12}^6).$$

The theorem can now be stated in general terms.

THEOREM. *Let $Q_r(x, y)=0$ be the unique curve of degree d determined by the points (P) and the vertex* A_r of a plane polygon $A_1, \cdots, A_n$. Let $A_{r(t,t+1)}^s$ denote the product of all the signed lengths of the segments joining A_s to the d points of intersection of $Q_r=0$ with the line A_tA_{t+1}. Then*

$$\prod_{r=1}^{n}\prod_{s=r+1}^{r+n-2}\frac{A_{r(s,s+1)}^{s}}{A_{r(s,s+1)}^{s+1}} = (-1)^n.$$

Proof. The proof that follows uses a result of Newton's which we recall.† Let the two points $B(b_1, \cdots, b_m)$, $C(c_1, \cdots, c_m)$ determine a line in euclidean

* We shall identify A_h and A_k provided $h\equiv k \pmod n$.

† See G. Salmon, Higher Plane Curves, 3rd ed., Dublin 1879, p. 108.

m-space which intersects $H(x_1, \cdots, x_m)=0$, a hypersurface of order g. Then

$$(1) \qquad P_b/P_c = H(b_1, \cdots, b_m)/H(c_1, \cdots, c_m),$$

where P_b is the product of the directed distances from B to the g points of intersection and P_c, the corresponding product for C. When this last formula is applied to A_s, A_{s+1}, and the intersection of the line passing through them with the curve $Q_r=0$, we obtain

$$\frac{A^{s}_{r(s,s+1)}}{A^{s+1}_{r(s,s+1)}} = \frac{Q_r(x_s, y_s)}{Q_r(x_{s+1}, y_{s+1})}$$

and hence

$$\prod_{s=r+1}^{r+n-2} \frac{A^{s}_{r(s,s+1)}}{A^{s+1}_{r(s,s+1)}} = \frac{Q_r(x_{r+1}, y_{r+1})}{Q_r(x_{r+n-1}, y_{r+n-1})} = \frac{Q_r(x_{r+1}, y_{r+1})}{Q_r(x_{r-1}, y_{r-1})} = J_r.$$

To show that $\prod_1^n J_r=(-1)^n$, write Q_r as $\sum_{i+j\le d} {}_ra_{ij}x^iy^j$. Since all the Q_r together form a linear pencil, the points $\bar{x}_r$ in $\frac{1}{2}d(d+3)$-space, whose homogeneous coordinates are ${}_ra_{ij}$, lie on a straight line. Furthermore, $\bar{x}_r$ is the unique point of intersection of this line (L) with the hyperplane

$$W_r = \sum_{i+j\le d} (x_r)^i(y_r)^jX_{ij} = 0.$$

It is also clear that $W_p(\bar{x}_r)=Q_r(x_p, y_p)$ for any r, p. Applying (1) to the points $\bar{x}_{r+1}$, $\bar{x}_{r-1}$ and the intersection $\bar{x}_r$ of the line (L) through them with the hyperplane $W_r=0$, we obtain

$$\prod_1^n \frac{\bar{x}_{r-1}\bar{x}_r}{\bar{x}_{r+1}\bar{x}_r} = \prod_1^n \frac{W_r(\bar{x}_{r-1})}{W_r(\bar{x}_{r+1})} = \prod_1^n \frac{Q_{r-1}(x_r, y_r)}{Q_{r+1}(x_r, y_r)} = \prod_1^n J_r.$$

Hence $\prod_1^n J_r$ turns out to be the product of the n distances $\bar{x}_r\bar{x}_{r+1}$ taken in one direction divided by the same product taken in the opposite direction, and so, must equal $(-1)^n$.

The same method may readily be extended to prove the theorem for $A_1, \cdots, A_n$ in three or more dimensions so long as the number of points (P) (which would now be in general position in the space determined by the A's) is such that the points (P) will determine with each A_r a unique hypersurface.

We can also deduce two geometrical interpretations of the cross ratio (c) of any linear pencil of four d-ic hypersurfaces. First it is clear that $c=(\bar{x}_1\bar{x}_2\bar{x}_3\bar{x}_4)$, where the $\bar{x}_i$ are related to the four hypersurfaces Q_i as above. Secondly, choose A_1 arbitrarily on Q_1 and A_2 on Q_2. Then

$$\frac{A^1_{3(1,2)}}{A^2_{3(1,2)}} \div \frac{A^1_{4(1,2)}}{A^2_{4(1,2)}} = \frac{Q_3(A_1)}{Q_3(A_2)} \div \frac{Q_4(A_1)}{Q_4(A_2)} = \frac{\bar{x}_1\bar{x}_3}{\bar{x}_1\bar{x}_4} \div \frac{\bar{x}_2\bar{x}_3}{\bar{x}_2\bar{x}_4} = (\bar{x}_1\bar{x}_2\bar{x}_3\bar{x}_4) = c.$$