CORRECTION TO "REGULAR DIFFERENTIALS AND EQUIDIMENSIONAL SCHEME-MAPS"

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Abstract. We correct a sign error in the principal result of our paper [LS].

We noticed an error involving a sign in our paper [LS] while comparing the conventions used for Leray spectral sequences in [C] with our conventions, and comparing [C, p. 173, Cor. 3.6.6] with our principal result (viz., [LS, p. 118, Thm. (4.1)]).

All numerical references (e.g. (3.2.3)) and page numbers are to [LS] except for $(3.2.3)'$ below. Diagrams and maps in this note are labelled in a non-numerical manner (e.g. (τ)) to avoid confusion. The exception to this is again $(3.2.3)'$. Proposition (3.3.1) on p. 114 is incorrectly stated: the right side of the asserted equality needs to be multiplied by a factor of $(-1)^{nd}$. Equivalently, the equality should read

$$\nu\begin{bmatrix}\begin{bmatrix} m \\ \mathbf{t}\end{bmatrix}\\ \mathbf{s}\end{bmatrix} = \begin{bmatrix} m \\ \mathbf{t},\mathbf{s}\end{bmatrix} \qquad (m \in M).$$

Note that the order of the variables on the right side is different from the order in the original (incorrect) assertion. This modification makes the correct assertion of our principal result, Thm. (4.1) on p. 118, that the diagram displayed in its statement commutes up to a sign of $(-1)^{nd}$.

We will list all the modifications after giving the proof of (the corrected) Proposition (3.3.1).

Here are our notations and conventions. We use the convention in [EGAIII, Chap. 0, 11.4.2] for the Leray spectral sequence associated to a composite of appropriate left exact functors between appropriate abelian categories. (Cf. [C1], especially the discussion on pp. 1–2 concerning double complexes and the discussion on pp. 4–5 concerning the commutativity of (3.6.4) of [C]. Cf. also [C, p. 62, Lemma 2.6.1].) One checks that with this convention the diagram at the bottom of p. 114 (henceforth referred to as DIAG) does indeed commute.

If $Z^\bullet$ is a complex such that $\mathrm{H}^i(Z^\bullet) = 0$ for $i > r$, then, with $\tau_{\geq r} Z^\bullet$ as in lines 13–14 of p. 114, we have a natural derived category isomorphism

$$\tau_{\geq r} Z^\bullet \xrightarrow{\ \sim\ } \mathrm{H}^r(Z^\bullet)[-r] \tag{τ}$$

best described by its inverse: if $C^\bullet = \tau_{\geq r} Z^\bullet$ then the natural inclusion $\mathrm{H}^r(Z^\bullet) = \ker(C^r \to C^{r+1}) \hookrightarrow C^r$ gives a map of complexes $\mathrm{H}^r(Z^\bullet)[-r] \to C^\bullet$ which under our hypotheses is clearly a quasi-isomorphism. The displayed formula on p. 114, line -5, viz., $\tau_{\geq d}\,\mathbf{R}\,\Gamma_I(M) \xrightarrow{\ \sim\ } H^d_I(M)[-d]$ is special instance of (τ). In the arguments that follow we will be using other instances of (τ) not specifically covered by *loc.cit.* or the line following it.

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If $\mathbf{t} := (t_1, \dots, t_d)$ is a sequence in the commutative ring A and $\mathbf{K}^\bullet(\mathbf{t})$ is the associated "stable" Koszul complex (defined on p. 113, line 5) then for an A-module M we set $\mathrm{H}^i_{\mathbf{t}}(M) := \mathrm{H}^i(\mathbf{K}^\bullet(\mathbf{t}) \otimes M)$. Note that we distinguish $\mathrm{H}^i_{\mathbf{t}}(M)$ from the local cohomology module $\mathrm{H}^i_{\mathbf{t}A}(M) := \mathrm{H}^i(\Gamma_{\mathbf{t}A} I^\bullet)$ where $I^\bullet$ is a chosen injective resolution of M, though these two A-modules are canonically isomorphic (via the natural quasi-isomorphism $\mathbf{K}^\bullet(\mathbf{t}) \otimes M \to \mathbf{K}^\bullet(\mathbf{t}) \otimes I^\bullet$ and the isomorphism of Lemma (3.2.3) on p. 113) . For $m \in M$ and non-negative integers $\alpha_1, \dots, \alpha_d$

$$[m/t_1^{\alpha_1} \dots t_d^{\alpha_d}] \in H^d_{\mathbf{t}}(M)$$

will denote the cohomology class of the d–cocycle $m/t_1^{\alpha_1} \dots t_d^{\alpha_d} \in M_{t_1 \dots t_d} = \mathbf{K}^d(\mathbf{t}) \otimes M$.

If $F \colon \mathscr{A} \to \mathscr{B}$ is a left exact functor between abelian categories and $\mathscr{A}$ has enough injectives, then for integers p, q and an object $X \in \mathscr{A}$ with injective resolution $X \to I^\bullet$, we identify $\mathrm{H}^p(\mathbf{R}\, F(X[q]))$ with $\mathrm{R}^{p+q}F(X)$ by using the natural isomorphism $\mathrm{H}^p(F(I^\bullet[q])) = \mathrm{H}^p(F(I^\bullet)[q]) \xrightarrow{\sim} \mathrm{H}^{p+q}(F(I^\bullet))$—without the intervention of signs. (See [C, p. 8, (1.3.4)]. Also compare with [C, p.35, (2.3.8)].)

We now begin the proof of the modified Prop. (3.3.1).

First, (3.2.3) is an *isomorphism of δ–functors,* in the sense that for any A-complex $X^\bullet$, if $\theta \colon \mathbf{K}^\bullet(\mathbf{t}) \otimes (X^\bullet[m]) \xrightarrow{\sim} (\mathbf{K}^\bullet \otimes X^\bullet)[m]$ is given in degree $p+q$ by

$$\theta|(\mathbf{K}^p(\mathbf{t}) \otimes X^{q+m}) = \text{multiplication by } (-1)^{pm},$$

then the following diagram commutes:

$$\begin{array}{ccc} \mathbf{R}\,\Gamma_{\mathbf{t}A}(X^\bullet[m]) & \xrightarrow[\text{(3.2.3)}]{\sim} & \mathbf{K}^\bullet(\mathbf{t}) \otimes (X^\bullet[m]) \\ \Big\| & & \simeq \Big\downarrow \theta \\ \mathbf{R}\,\Gamma_{\mathbf{t}A}(X^\bullet)[m] & \xrightarrow[\text{(3.2.3)}]{\sim} & (\mathbf{K}^\bullet(\mathbf{t}) \otimes X^\bullet)\,[m] \end{array} \qquad (3.2.3)'$$

It is enough to prove $(3.2.3)'$ for $X^\bullet$ an injective complex, so that $\mathbf{R}\,\Gamma_{\mathbf{t}A}(X^\bullet[m]) = \Gamma_{\mathbf{t}A}(X^\bullet[m]) = \Gamma_{\mathbf{t}A}(X^\bullet)[m]$. Note that θ restricted to $X^{q+m} = \mathbf{K}^0(\mathbf{t}) \otimes X^{q+m}$ is the identity. Moreover, in degree q, the map $\Gamma_{\mathbf{t}A}(X^\bullet[m]) \to \mathbf{K}^\bullet(\mathbf{t}) \otimes (X^\bullet[m])$, as well as the map $\Gamma_{\mathbf{t}A}(X^\bullet)[m] \to (\mathbf{K}^\bullet(\mathbf{t}) \otimes X^\bullet)[m]$, is induced by the inclusion $\Gamma_{\mathbf{t}A}(X^{q+m}) \hookrightarrow X^{q+m}$. The commutativity of $(3.2.3)'$ can be worked out from the just mentioned facts.

The proof of the modified Proposition (3.3.1) is along the lines of the given proof on pp. 114–115, taking into account diagram $(3.2.3)'$. Our main strategy (implicit in the original proof) is to find analogues for every complex and every arrow in DIAG for the associated "stable" Koszul complexes, using (3.2.3) for comparison. This will result in a diagram DIAG$'$ involving stable Koszul complexes which is isomorphic to DIAG, from which we will read off the asserted result. In greater detail let $\nu' : \mathrm{H}^n_{\mathbf{s}}(\mathrm{H}^d_{\mathbf{t}}(M)) \to \mathrm{H}^{n+d}_{\mathbf{s},\mathbf{t}}(M)$ be defined by the commutativity of the diagram DIAG′ below

$$\begin{array}{ccccc}
\tau_{\geq n+d}\left(\mathbf{K}^\bullet(\mathbf{s},\mathbf{t})\otimes M\right) & \longrightarrow & \tau_{\geq n+d}\left(\mathbf{K}^\bullet(\mathbf{s})\otimes\mathbf{K}^\bullet(\mathbf{t})\otimes M\right) & \longrightarrow & \tau_{\geq n+d}\Big(\mathbf{K}^\bullet(\mathbf{s})\otimes\tau_{\geq d}\left(\mathbf{K}^\bullet(\mathbf{t})\otimes M\right)\Big) \\
\Big\| & & & & \Big\| \\
 & & & & \mathrm{H}^n_{\mathbf{s}}\left(\mathrm{H}^d_{\mathbf{t}}(M)\right)[-n-d] \\
 & & & & \big\downarrow {\scriptstyle (-1)^{nd}} \\
\mathrm{H}^{n+d}_{\mathbf{s},\mathbf{t}}(M)[-n-d] & & \xrightarrow{\quad\nu'^{-1}\quad} & & \mathrm{H}^n_{\mathbf{s}}\left(\mathrm{H}^d_{\mathbf{t}}(M)\right)[-n-d]
\end{array}$$

DIAG′

The two identity maps (equalities) arise from (τ). Indeed if $Z^\bullet$ is such that $Z^i = 0$ for $i > r$ then (τ) is an *equality* of complexes. We will use this fact later (in the right column of $\square(C^\bullet)$). From its definiton one checks that the map ν' is given by the rule

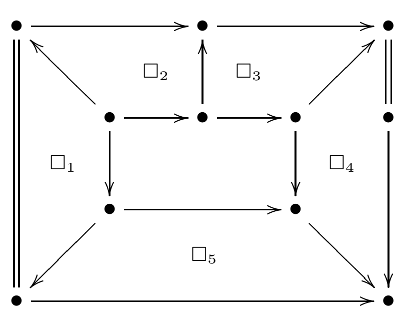
$$\left[\left[m/t_1^{\beta_1}\dots t_d^{\beta_d}\right]/s_1^{\alpha_1}\dots s_n^{\alpha_n}\right] \mapsto (-1)^{nd}\left[m/s_1^{\alpha_1}\dots s_n^{\alpha_n}t_1^{\beta_1}\dots t_d^{\beta_d}\right].$$

If we prove that

$$\begin{array}{ccc}
\mathrm{H}^{n+d}_{J+I}(M) & \xleftarrow{\ \nu\ } & \mathrm{H}^n_J(\mathrm{H}^d_I(M)) \\
{\scriptstyle (3.2.3)}\big\downarrow\simeq & & \simeq\big\downarrow{\scriptstyle (3.2.3)} \\
\mathrm{H}^{n+d}_{\mathbf{s},\mathbf{t}}(M) & \xleftarrow{\ \nu'\ } & \mathrm{H}^n_{\mathbf{s}}(\mathrm{H}^d_{\mathbf{t}}(M))
\end{array}$$

commutes, then the corrected Proposition (3.3.1) follows.

To save space what follows is the skeleton of a diagram relating DIAG and DIAG′—the outer rectangle representing DIAG′ and the inner rectangle representing DIAG and the arrows between them being given by (3.2.3).



Our interest is in showing that $\square_5$ commutes. Since the outer rectangle and the inner rectangle commute and the downward pointing arrows are isomorphisms, $\square_5$ commutes if $\square_1$, $\square_2$, $\square_3$ and $\square_4$ commute. Diagram $\square_1$ commutes by the functoriality of (τ) and $\square_3$ commutes by functoriality of truncations. Diagram $\square_2$ commutes by the second part of Proposition (3.2.3).

It remains to prove that $\square_4$ commutes. This requires us to elaborate the map on the right column of DIAG. It is the composite

$$\begin{aligned}
\tau_{\geq n+d}\,\mathbf{R}\,\Gamma_J\left(\tau_{\geq d}\,\mathbf{R}\,\Gamma_I(M)\right) \xrightarrow[\tau]{\sim}\ & \tau_{\geq n+d}\left(\mathbf{R}\,\Gamma_J(\mathrm{H}^d_I(M)[-d])\right) \\
=\ & \tau_{\geq n+d}\left(\mathbf{R}\,\Gamma_J\left(\mathrm{H}^d_I(M)\right)[-d]\right) \\
\xrightarrow[\tau]{\sim}\ & \mathrm{H}^{n+d}\left(\mathbf{R}\,\Gamma_J\left(\mathrm{H}^d_I(M)[-d]\right)\right)[-n-d] \\
=\ & \mathrm{H}^n_J(\mathrm{H}^d_I(M))[-n-d]
\end{aligned}$$

Now suppose $C^\bullet$ is a complex of A–modules with $\mathrm{H}^i(C^\bullet) = 0$ for $i > d$. Set $T := \mathrm{H}^d(C^\bullet)$. By $(3.2.3)'$ we have a commutative diagram (horizontal arrows from (3.2.3)):

$$\begin{array}{ccc}
\tau_{\geq n+d}\,\mathbf{R}\Gamma_J(\tau_{\geq d}C^\bullet) & \xrightarrow{\quad\sim\quad} & \tau_{\geq n+d}(\mathbf{K}^\bullet(\mathbf{s})\otimes\tau_{\geq d}C^\bullet) \\
{\scriptstyle(\tau)}\big\downarrow\simeq & & \simeq\big\downarrow{\scriptstyle(\tau)} \\
\tau_{\geq n+d}\left(\mathbf{R}\Gamma_J(T[-d])\right) & \xrightarrow{\quad\sim\quad} & \tau_{\geq n+d}(\mathbf{K}^\bullet(\mathbf{s})\otimes(T[-d])) \\
\Big\| & & \simeq\big\downarrow\theta \\
\tau_{\geq n+d}\left(\mathbf{R}\Gamma_J(T)[-d]\right) & \xrightarrow{\quad\sim\quad} & \tau_{\geq n+d}\left((\mathbf{K}^\bullet(\mathbf{s})\otimes T)[-d]\right) \\
{\scriptstyle(\tau)}\big\downarrow\simeq & & {\scriptstyle(=)}\big\downarrow{\scriptstyle(\tau)} \\
\mathrm{H}^{n+d}\left((\mathbf{R}\Gamma_J T)[-d]\right)[-n-d] & \xrightarrow{\sim} & \mathrm{H}^{n+d}\left((\mathbf{K}(\mathbf{s})\otimes T)[-d]\right)[-n-d] \\
\Big\| & & \Big\| \\
\mathrm{H}^n_J(T)[-n-d] & \xrightarrow{\quad\sim\quad} & \mathrm{H}^n_{\mathbf{s}}(T)[-n-d]
\end{array} \qquad \square(C^\bullet)$$

The diagram $\square(C^\bullet)$ is functorial in $C^\bullet$ in a sense we now make precise. Denote the left column of of $\square(C^\bullet)$ by $L(C^\bullet)$ and the right column by $R(C^\bullet)$. Suppose $\alpha : C^\bullet \to D^\bullet$ is a derived category map with $\mathrm{H}^i(D^\bullet) = 0$ for $i > d$. As is standard in such situations, an arrow between two columns of the same length is to be interpreted as a commutative diagram connecting the two columns. The functoriality of $\square(C^\bullet)$ means that the natural diagram

$$\begin{array}{ccc}
L(C^\bullet) & \xrightarrow{\square(C^\bullet)} & R(C^\bullet) \\
{\scriptstyle \text{via}\quad\alpha}\big\downarrow & & \big\downarrow{\scriptstyle \text{via}\quad\alpha} \\
L(D^\bullet) & \xrightarrow[\square(D^\bullet)]{} & R(D^\bullet)
\end{array}$$

commutes. Note that the above, when unpackaged, is a three dimensional diagram. Now let $C^\bullet = \mathbf{R}\,\Gamma_I(M)$, $D^\bullet = \mathbf{K}^\bullet(\mathbf{t})\otimes M$ and $\alpha = (3.2.3)$. The arrows in $L(C^\bullet)$ compose to give the arrow in the right column of DIAG and the arrows in $R(D^\bullet)$ compose to give the arrow in the right column of DIAG$'$. The arrow $L(C^\bullet) \to R(D^\bullet)$ given by $\square(D^\bullet)\circ L(\alpha) = R(\alpha)\circ\square(C^\bullet)$ gives the commutativity of $\square_4$. This completes the proof of the corrected Proposition (3.3.1).

Modifications

We have already discussed the changes needed in the statement of Proposition (3.3.1). The most important change that ensues is in the statement of the main result, Theorem (4.1) on p. 118. This theorem should now read:

(4.1) **Theorem.** *Let $f : X \to Y$ and $g : Y \to S$ be proper maps, with $f \in \mathfrak{C}^d$ and $g \in \mathfrak{C}^n$. Then the following diagram, in which the vertical maps arise from the embedding of regular differentials into meromorphic differentials and the isomorphism on the bottom row is $\xi_1 \otimes \xi_2 \mapsto \xi_1 \wedge \xi_2$, commutes:*

$$\begin{array}{ccc}
\omega_f^d \otimes f^*\omega_g^n & \xrightarrow{\eta_f(\omega_g^n)} & \omega_{gf}^{n+d} \\
\downarrow & & \downarrow \\
\Omega^d_{k(X)/k(Y)} \otimes f^*\Omega^n_{k(Y)/k(S)} & \xrightarrow{\ \sim\ } & \Omega^{n+d}_{k(X)/k(S)}
\end{array}$$

The remaining modifications are not of statements but of definitions occurring in the proofs. They are of two types. The first group of modifications are modifications in the way we write certain modules, sheaves and maps (e.g. $B \otimes A$ rather than $A \otimes B$). These do not change the definitions of the underlying objects. The second group is less cosmetic.

Here is the first list.

(i) Throughout, the source of $\eta_f(\mathscr{F})$ (defined on pp. 105–106) should be written $\omega_f \otimes_{\mathscr{O}_X} f^*\mathscr{F}$ rather than $f^*\mathscr{F} \otimes_{\mathscr{O}_X} \omega_f$. Thus, line -6 on p. 105 should be

$$\eta_f(\mathscr{F}) : \omega_f \otimes_{\mathscr{O}_X} f^*\mathscr{F} \longrightarrow f^!\mathscr{F}.$$

(ii) Similarly the sources and targets of $F_f(\mathscr{F})$ and $G_f(\mathscr{F}, \mathscr{E})$ (defined in (2.1) pp. 105–106) need to be rewritten with the order of the tensor products switched. For example, the source of $F_f(\mathscr{F})$ should be written as $\mathrm{R}^d f_*\omega_f \otimes \mathscr{F}$ and its target as $\mathrm{R}^d(\omega_f \otimes f^*\mathscr{F})$.

(iii) Page 119, line 6. The source of $\mu(\mathscr{F}, \mathscr{E})$ should be written $\mathrm{H}^d_{\mathbf{t}R}(\mathscr{E}_x) \otimes \mathrm{H}^n_y(\mathscr{F})$ and its target as $\mathrm{H}^{n+d}_x(\mathscr{E} \otimes f^*\mathscr{F})$.

The second list is:

(i) Page 103 line 9. Write $\eta = (-1)^{nd}\varphi$ instead of $\eta = \varphi$.

(ii) Page 103 last line. The displayed relation should be

$$\begin{bmatrix} \xi_2 \otimes \xi_1 \\ \mathbf{t}, \mathbf{s} \end{bmatrix} \in \mathrm{H}^{n+d}_x(\omega_f \otimes f^*\omega_Y).$$

(iii) Page 104 line 10. Replace φ at the end of the line by $(-1)^{nd}\varphi$.

(iv) Page 119 line 8. The displayed formula should be

$$\mu\left(\begin{bmatrix} b \\ \mathbf{t} \end{bmatrix} \otimes \begin{bmatrix} a \\ \mathbf{s} \end{bmatrix}\right) = \begin{bmatrix} b \otimes a \\ \mathbf{t}, \mathbf{s}' \end{bmatrix}.$$

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