Deformations of spaces, groups and C*-algebras

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Almost commuting unitaries (Voiculescu)

$$u_n = \begin{pmatrix} 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \quad v_n = \begin{pmatrix} \lambda & 0 & 0 & 0 & 0 \\ 0 & \lambda^2 & 0 & 0 & 0 \\ 0 & 0 & \lambda^3 & 0 & 0 \\ 0 & 0 & 0 & \cdot & 0 \\ 0 & 0 & 0 & \cdots & \lambda^n \end{pmatrix} \quad \lambda = e^{2\pi i/n}$$

$$v_n u_n = e^{2\pi i/n} u_n v_n$$

These commuting unitaries are uniformly far from commuting unitaries.

Proof: $\quad \text{winding number of } \; t \mapsto \det((1-t)u_n v_n + t v_n u_n) = 1$

Cannot sharply localize both f and $\widehat{f}$ (Heisenberg's uncertainty principle).

A = unital Banach algebra: $\|ab\| \leq \|a\|\|b\|$.
Ex. $\ell^1(G)$, G=discrete group

A=C*-algebra: involutive Banach algebra $\|a^*a\| = \|a\|^2$
$\Rightarrow$ good spectral theory.

- $A \subset L(H)$=linear operators, H=Hilbert space
- abelian C*-algebras $C_0(X) = \{f : X \to \mathbb{C}, \text{cont. } f(\infty) = 0\}$
- group C*-algebras $C^*(G)$, $\pi : G \to U(H)$
  G abelian: $C^*(G) \cong C_0(\widehat{G})$
- Crossed products $C_0(X) \rtimes G$ induced by group actions G on X,
  = algebras of operators on orbit space X/G.

Abstract dimension theory = K-theory

Definition

If A is a ring (with unit), then $K_0(A)$ is the abelian group generated by classes $[p]$, where p is any projection in a matrix algebra over A, subject to the relations

- $[p] = [q]$ if p and q are equivalent ($p = uv$ and $q = vu$).
- $[p_1] \oplus [p_2] = \left[\begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix} \right]$

Thus $K_0(A)$ is the universal dimension group for projections in matrix algebras over A. If A is C*-algebra, $K_0(A)$ homotopy invariant.

Example

$K_0(\mathbb{C}) = \mathbb{Z}, \quad p \in M_n(\mathbb{C}), \; [p] = Tr(p).$

The hole in the circle is detected by the Moebius band.

Example (detecting the hole in the sphere)

$K_0(C(S^2)) = \mathbb{Z} \oplus \mathbb{Z}\beta, \;\; \beta = [p] - [p_0] \neq 0$

- $p_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in M_2(C(S^2))$
- $p(z) = \frac{1}{1+|z|^2} \begin{pmatrix} |z|^2 & z \\ \bar{z} & 1 \end{pmatrix} \in M_2(C(S^2))$

$K_0(C(X))$=the Atiyah-Hirzebruch K-theory group

Can detect K_0 using matricial deformations

A morphism $\pi : A \to M_n(\mathbb{C})$ induces

$\pi_* : K_0(A) \to K_0(\mathbb{C}) = \mathbb{Z}, \quad [p] \mapsto \text{Trace of } \pi(p).$

If π is only approximately multiplicative, (F, ε)-quasi-representation,

$$\|\pi(ab) - \pi(a)\pi(b)\| < \varepsilon, \; a, b \in F$$

can perturb $\pi(p)$ to a projection $\pi_\sharp(p)$.

$\pi_\sharp : K_0(A) \rightsquigarrow \mathbb{Z}, \qquad [p] \mapsto \text{Trace of } \pi_\sharp(p)$ partially defined

Can compute topological invariants via traces of matrices i.e by discrete integration!

The Exel-Loring formula

Almost commuting unitaries $u, v \in U(n)$, $\|uv - vu\| < \varepsilon$
can be viewed as correponding to a quasi-rep (matricial deformation) π of the 2-torus: $z_1 \mapsto u$, $z_2 \mapsto v$.

$K_0(C(\mathbb{T}^2)) = \mathbb{Z} \oplus \mathbb{Z}\beta \xrightarrow{\pi_\sharp} \mathbb{Z}$. ($\beta$=pure twist $\leftrightarrow$ first Chern class)

Theorem (Exel-Loring, '89)

$$\pi_\sharp(\beta) = \frac{1}{2\pi i}\mathrm{Tr}(\log[v, u]) = winding\#det((1-t)uv + tvu)$$

Deformation of algebras

Connes and Higson ('90) developed a deformation theory for C*-algebras. Homotopy classes of families quasi-reps $\pi_t : C_0(\mathbb{R}) \otimes A \to C_0(\mathbb{R}) \otimes M_{m(t)}(\mathbb{C})$ = the K-homology of A.

Deformations of suspensions $\Leftrightarrow$ K-homology!

Suspension theorem (Dad.-Loring)

The K-homology of $C_0(X)$ = homotopy classes of deformations of $C_0(X)$ into matrices if X connected.

Cor. All morphisms $K_0(C_0(X)) \to \mathbb{Z}$ lift to quasi-reps $\pi : C_0(X) \to M_m(\mathbb{C})$

Note: reps are not (topologically) interesting, they vanish on $K_0(C_0(X))$.

These quasi-reps (or deformations) can be obtained by compressing genuine reps to subspaces

$$\pi(f) = e \begin{pmatrix} f(x_1) & & & \\ & f(x_2) & & \\ & & \dots & \\ & & & f(x_n) \end{pmatrix} e.$$

- Can capture invariants of continuous spaces by discrete quantization
- Stability under small perturbations

Topological insulators

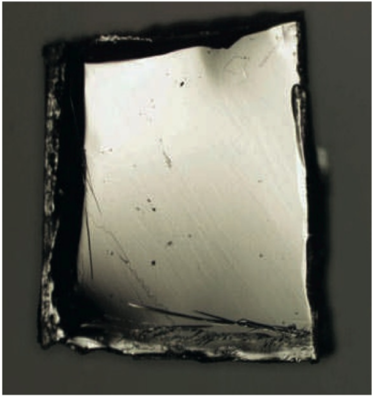
A topological insulator is a homogeneous block of material with the property that electrons move only along the surface of the block, but not through its interior.

First predicted theoretically in an effort to generalize the quantum Hall effect, TI were discovered just few years ago (Bismuth selenide Bi_2Se_3).

"The idea of a topological insulator is so strange that for a long time, physicists had no reason to believe that such a material would exist" (Nature 2010).

In a discrete lattice model one compresses position operators X_1, ..., X_k by a Fermi energy-level spectral projection e of the Hamiltonian H. Then eX_1e, ..., eX_ke only approximately commute.

Electric conductivity in the interior of a topological insulator is prevented by quantum mechanics phenomena that reflect the impossibility of perturbing quasi-representations to representations.



Electrons move along the surface of, but not through, topological insulators such as bismuth selenide.

The surface currents are topologically protected (hence the name), which means that the electrons that carry those currents don't veer off the track easily and maintain their properties over long distance even in the presence of impurities. Spintronics, Quantum Comp.

From deformations of spaces to deformations of groups

- Establish the existence of interesting matricial deformations of discrete groups G that capture $K_0(C^*(G))$ or $K_0(\ell^1(G))$.
- Seek explicit formulas for computing the pairing of these deformations with K-theory. (role of crystalline symmetries in topological insulators?)

Quasi-representations $\pi : G \to U(n)$

1) $\pi(1) = 1$
2) $\|\pi(st) - \pi(s)\pi(t)\| < \varepsilon$ for all $s, t \in F \subset G$.

Genuine reps are not topologically interesting, they kill K-theory

Results on deformations of groups

There is a class of groups with sufficiently many deformations that contains

- all amenable groups
- the free groups, $SL_2(\mathbb{Z})$
- the surface groups
- closed under free products.

Have an index theorem that computes pairings

If $\pi : G \to U(n)$ is a sufficiently multiplicative quasi-rep and D elliptic on $M = BG$:

$$\pi_\sharp(\mu[D]) = \langle ch(\sigma(D)) \cdot Td_{\mathbb{C}}(TM) \cdot ch(\ell_\pi), [M]\rangle.$$

Surface groups

M=compact Riemann surfaces of genus $g \geq 1$ (g-holed torus).
$\pi_1(M) = G = \langle s_1, t_1, ..., s_g, t_g \,;\, \prod_{i=1}^{g}[s_i, t_i] = 1\rangle$;

$$K_0(\ell^1(G)) = \mathbb{Z} \oplus \mathbb{Z}\beta$$

If $\pi : G \to U(n)$ is a sufficiently multiplicative quasi-rep, then

Theorem

$$\pi_\sharp(\beta) = \frac{1}{2\pi i}\mathrm{Tr}\log([\pi(s_1), \pi(t_1)] \cdots [\pi(s_g), \pi(t_g)])$$

The case $g = 1$ is the Exel-Loring formula.
Have generalization to tracial C*-algs, joint with my student J. Carrión.

Almost flat K-theory classes and quasi-diagonality

Connes-Moscovici-Gromov showed that the signature of a manifold with coefficients in an almost flat K-theory class is a homotopy invariant.

Theorem

Suppose G is unif. embed. Hilb. space and $C^*(G)$ is quasidiagonal. If BG is a finite simplicial complex, then all elements of $K^0(BG)$ are almost flat.

Work of Guentner-Higson-Weinberger and Borel-Serre shows that:
G arithmetic group $\Rightarrow$ G unif. embed. Hilb. space and BG finite complex.

Open problem: Let Γ_N=congruence subgroup; is $C^*(\Gamma_N)$ qd?

A is quasidiagonal if $A \subset$ (block-diagonal-algebra) $+$ (the compacts)
Thm. $\Rightarrow$ topological obstructions to quasi-diagonality