

CONNECTIVE BIEBERBACH GROUPS

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ABSTRACT. We prove that a Bieberbach group with trivial center is not connective and use this property to show that a Bieberbach group is connective if and only if it is poly- \mathbb{Z} .

1. INTRODUCTION

Connectivity [6] is a homotopy invariant property of a separable C^* -algebra A that has three interesting consequences: absence of nonzero projections, quasidiagonality, and realization of the Kasparov groups as homotopy classes of asymptotic morphisms from A to $B \otimes \mathcal{K}$ without suspensions, that is $KK(A, B) \cong [[A, B \otimes \mathcal{K}]]$, if A is nuclear. A separable amenable (nuclear) C^* -algebra is connective if it embeds in the C^* -algebra $\prod_n B_n / \bigoplus_n B_n$ where $B_n = C_0((0, 1], M_n(\mathbb{C}))$ is the C^* -algebra of continuous functions from $[0, 1]$ to $n \times n$ complex matrices vanishing at 0. A countable discrete group G is called connective if the kernel $I(G)$ of the trivial representation $\iota : C^*(G) \rightarrow \mathbb{C}$ is a connective C^* -algebra. If G is connective and amenable, then $K^0(C^*(G)) = \mathbb{Z}[\iota] \oplus [[I(G), \mathcal{K}]]$. This implies that the nontrivial part of the K -homology of $C^*(G)$ can be realized as homotopy classes of asymptotic representations $\{\pi_t : G \rightarrow U(\infty)\}_{t \in [1, \infty)}$ with $t \mapsto \pi_t(g)$ continuous, $g \in G$, and $\lim_{t \rightarrow \infty} \|\pi_t(g_1 g_2) - \pi_t(g_1) \pi_t(g_2)\| = 0$, $g_1, g_2 \in G$. Large classes of amenable connective groups were exhibited in [6], [7], [8].

Connectivity of a separable nuclear C^* -algebra A can be characterized solely in terms of its primitive spectrum $\text{Prim}(A)$. It was shown in [6] that the existence of a nonempty compact open subset of $\text{Prim}(A)$ is an obstruction to connectivity. By a remarkable result of J. Gabe [11] this is the only obstruction.

A crystallographic group of dimension $k \geq 1$ is a discrete co-compact subgroup of the isometry group $\text{Iso}(\mathbb{R}^k) = \mathbb{R}^k \rtimes O(k)$ of the Euclidean space \mathbb{R}^k . In his renowned work on Hilbert's 18th Problem, Bieberbach proved that any crystallographic group G of dimension k fits into an exact sequence

$$(1) \quad 1 \longrightarrow N \longrightarrow G \longrightarrow D \longrightarrow 1$$

where $N \cong \mathbb{Z}^k$ is a maximally abelian subgroup of G , called the lattice of G , and D is a finite group called the holonomy group. Moreover, for each fixed k , there are only finitely many isomorphism classes of crystallographic groups of dimension k and two crystallographic groups are isomorphic if and only if they are conjugate in the group $\mathbb{R}^k \rtimes GL_k(\mathbb{R})$, [5]. A torsion free crystallographic group is called a Bieberbach group. The orbit space \mathbb{R}^k/G of a Bieberbach group is a k -dimensional closed flat Riemannian manifold M with holonomy group isomorphic to D .

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Which Bieberbach groups are connective? It was shown in [6] that while any Bieberbach group with cyclic holonomy is connective [6, Thm. 3.8], the celebrated Hantzsche-Wendt group [13] (denoted here by Γ) is not connective [6, Cor. 3.2]. Γ is generated by two elements x and y subject to two relations: $x^2yx^2 = y$, $y^2xy^2 = x$, and fits into an exact sequence

$$1 \longrightarrow \mathbb{Z}^3 \longrightarrow \Gamma \longrightarrow \mathbb{Z}/2 \oplus \mathbb{Z}/2 \longrightarrow 1.$$

The non-connectivity of Γ was proved by showing that $\widehat{\Gamma} \setminus \{\iota\}$ is a compact-open subset of the unitary spectrum $\widehat{\Gamma}$. There are exactly 10 non-isomorphic 3-dimensional Bieberbach groups. The Hantzsche-Wendt group Γ is singled out among these groups by the property that it has finite first homology group, in fact, isomorphic to $\mathbb{Z}/4 \oplus \mathbb{Z}/4$ [24]. The first homology group $H_1(G, \mathbb{Z})$ of a group G can be computed as $H_1(G, \mathbb{Z}) = G/[G, G]$ where $[G, G]$ is the commutator subgroup of G . In view of the lack of connectivity of the Hantzsche-Wendt group, it is natural to ask to what extent connectivity of a Bieberbach group relates to its homology. Independently of us, Szczepański asks the same question at the end of his paper [22]. As a key step in our study of connectivity of Bieberbach groups we prove the following:

Theorem 1.1. *A Bieberbach group with finite first homology group is not connective.*

By work of Calabi [4] (see the discussion in [21], [24]) any Bieberbach group G with infinite first homology group can be written as an iterated semidirect product

$$(2) \quad G \cong ((H \rtimes \mathbb{Z}) \rtimes \cdots) \rtimes \mathbb{Z}$$

where either H is a Bieberbach group such that $H_1(H, \mathbb{Z})$ is finite, or $H = \{1\}$, in which case G is a poly- \mathbb{Z} group. We use the Calabi decomposition of G in conjunction with Theorem 1.1 to prove the following:

Theorem 1.2. *Let G be a Bieberbach group. The following assertions are equivalent.*

- (i) G is connective.
- (ii) Every nontrivial subgroup of G has a nontrivial center.
- (iii) G is a poly- \mathbb{Z} group.
- (iv) $\widehat{G} \setminus \{\iota\}$ has no nonempty compact open subsets.

Auslander and Kuranishi [1] showed that any finite group is the holonomy group of some torsion free Bieberbach group. In general one cannot determine if a Bieberbach group is connective by just looking at its holonomy group. Nevertheless, by using results of [14] and [17] one can derive the following:

Corollary 1.3. *Let D be a finite group.*

- (a) *If D is not solvable, then any Bieberbach group with holonomy D is not connective.*
- (b) *If D is solvable with all Sylow subgroups cyclic (solvability is automatic in this case), then any Bieberbach group with holonomy D is connective.*
- (c) *If D is solvable and has a non-cyclic Sylow subgroup, then there are Bieberbach groups G_1 connective and G_2 not connective both with holonomy group D .*

A discrete group G is called locally indicable if every finitely generated non-trivial subgroup L of G has a quotient isomorphic to \mathbb{Z} or equivalently $H_1(L, \mathbb{Z})$ is infinite. The group G is called diffuse if every non-empty finite subset A of G has an element $a \in A$ such that for any $g \in G$, either ga or $g^{-1}a$ is not in A , [2]. Linnell and Witte Morris [18] proved that an amenable group is diffuse if and only if is locally indicable. It follows then by Theorem 1.2 that a Bieberbach group G is connective $\Leftrightarrow G$ is locally indicable $\Leftrightarrow G$ is diffuse.

2. BIEBERBACH GROUPS WITH TRIVIAL CENTER

The purpose of this section is to prove Theorem 1.1.

Hiller and Sah [14] showed that a finite group D is primitive, meaning that it can be realized as the holonomy group of a Bieberbach group with finite first homology group if and only if no cyclic Sylow subgroup of D admits a normal complement. For instance cyclic groups (including the trivial group) are not primitive but $(\mathbb{Z}/p)^m$ is primitive if $m \geq 2$ and p is prime. Recall that as a consequence of Burnside's normal p -complement theorem, if p is the smallest prime that divides the order of a finite group, then any cyclic p -Sylow subgroup admits a normal complement [20, p.138, 6.2.12]. Therefore, if all Sylow subgroups of a finite group D are cyclic, then D is not primitive.

For a Bieberbach group G the following conditions are equivalent, see [14, Prop. 1.4]:

- (i) $H_1(G, \mathbb{Z})$ is a finite group.
- (ii) G has trivial center, $Z(G) = \{0\}$.
- (iii) The action of G by conjugation on its lattice subgroup N has exactly one fixed point, $N^G = \{0\}$.

For the sake of completeness we revisit the proof of these equivalences [cf. [14]]. The rank of a discrete abelian group L , denoted $\text{rank}(L)$, is the dimension of the \mathbb{R} -vector space $L \otimes_{\mathbb{Z}} \mathbb{R}$.

Proposition 2.1. *If G is a Bieberbach group, then $\text{rank}(H_1(G, \mathbb{Z})) = \text{rank}(Z(G)) = \text{rank}(N^G)$.*

Proof. One observes that $Z(G) = N^G$ since N is maximal abelian and G acts on N by conjugation. By [3, Cor. 6.4] the exact sequence (1) induces an exact sequence

$$H_2(D, \mathbb{Z}) \longrightarrow H_1(N, \mathbb{Z})^D \longrightarrow H_1(G, \mathbb{Z}) \longrightarrow H_1(D, \mathbb{Z}) \longrightarrow 0.$$

The action of D on $H_1(N, \mathbb{Z}) = N/[N, N] = N$ is induced by the conjugation action of G on N so that $H_1(N, \mathbb{Z})^D \cong N^G$. Since D is a finite group, so are the groups $H_1(D, \mathbb{Z})$ and $H_2(D, \mathbb{Z})$. It follows that $\text{rank}(H_1(G, \mathbb{Z})) = \text{rank}(N^G)$. \square

G acts on N by conjugation: $h \mapsto ghg^{-1}$, $h \in N$. Since N is abelian, this action descends to an action of D on $N = \mathbb{Z}^k$ by automorphisms. We denote the corresponding representation by $\theta : D \rightarrow GL_k(\mathbb{Z}) = \text{Aut}(\mathbb{Z}^k)$. The map θ is injective since N is maximal abelian.

By duality G acts on $\widehat{N} \cong \mathbb{T}^k$ by automorphisms: $g \cdot \chi = \chi(g^{-1} \cdot g)$. This action descends to an action of $D \cong G/N$ which is the dual of the action θ discussed above and is denoted by $\theta^* : D \rightarrow \text{Aut}(\mathbb{T}^k)$. Let us consider the fixed points of these actions and observe that $(\mathbb{Z}^k)^D$ is a subgroup of \mathbb{Z}^k and that $(\mathbb{T}^k)^D$ is a closed subgroup of \mathbb{T}^k and hence it must be a Lie subgroup by Cartan's theorem. If Γ is the Hantzsche-Wendt group, the corresponding representation

$\theta : D = \mathbb{Z}/2 \oplus \mathbb{Z}/2 \rightarrow GL_3(\mathbb{Z})$ has the property that $(\mathbb{Z}^3)^D = \{\mathbf{0}\}$ while $(\mathbb{T}^3)^D$ consists of eight points $(\pm 1, \pm 1, \pm 1)$, see for example [6]. The finiteness of $(\mathbb{T}^3)^D$ is not coincidental. In fact we are going to see as a consequence of Proposition 2.2 that the conditions (i), (ii), (iii) from above are equivalent to the following condition which plays a crucial role in the proof of our main result.

(iv) The dual action of G on \widehat{N} has finitely many fixed points i.e. \widehat{N}^G is a finite group.

The rank of a Lie group K , denoted $\text{rank}(K)$, is the dimension of any one of its Cartan subgroups. If K is abelian, then $\text{rank}(K)$ coincides with the dimension of K .

Proposition 2.2. *If $D \subset GL_k(\mathbb{Z})$ is a finite group, then $\text{rank}(\mathbb{Z}^k)^D = \text{rank}(\mathbb{T}^k)^D$. Therefore, if G is a Bieberbach group, then $\text{rank}(H_1(G, \mathbb{Z})) = \text{rank}(Z(G)) = \text{rank}(N^G) = \text{rank}(\widehat{N}^G)$.*

Proof. Write $\theta : D \rightarrow GL_k(\mathbb{Z})$ for the representation defined by the inclusion map from the statement. For $s \in D$, $\theta(s)$ is given by an $k \times k$ matrix $A(s)$ with integer coefficients. Thus D acts on \mathbb{Z}^k by $\mathbf{v} \mapsto A(s)\mathbf{v}$. The dual action maps a character $\chi : \mathbb{Z}^k \rightarrow \mathbb{T}$ to $\chi(A(s^{-1})\cdot)$.

By definition $\mathbf{H} := (\mathbb{Z}^k)^D = \{\mathbf{v} \in \mathbb{Z}^k : A(s)\mathbf{v} = \mathbf{v}, s \in D\}$. Since \mathbf{H} is a subgroup of \mathbb{Z}^k , it follows that $\mathbf{H} \cong \mathbb{Z}^n$ for some $0 \leq n \leq k$.

Similarly, $\mathbf{K} := (\mathbb{T}^k)^D = \{\chi \in \widehat{\mathbb{Z}^k} : \chi(A(s^{-1})\mathbf{v}) = \chi(\mathbf{v}), s \in D, \mathbf{v} \in \mathbb{Z}^k\}$ is a compact abelian Lie subgroup of \mathbb{T}^k . This implies that there exist a finite subgroup F of \mathbb{T}^k and a connected closed subgroup \mathbf{T} of \mathbb{T}^k isomorphic to \mathbb{T}^m for some $0 \leq m \leq k$, such that $F \cap \mathbf{T} = \{1\}$ and $\mathbf{K} = F\mathbf{T}$. Since $\text{rank}(\mathbf{K}) = \text{rank}(\mathbf{T}) = m$ and $\text{rank}(\mathbf{H}) = n$, our task is to prove that $n = m$. Let $\mathbf{W} = \{\mathbf{a} \in \mathbb{R}^k : e^{2\pi i t \mathbf{a}} \in \mathbf{T}, t \in \mathbb{R}\}$. In other words, \mathbf{W} is the Lie algebra of \mathbf{T} and its rank is m as well. If χ is the character of \mathbb{Z}^k corresponding to $e^{2\pi i t \mathbf{a}}$, with $\mathbf{a} \in \mathbf{W}$ and $t \in \mathbb{R}$, the condition $\chi(A(s^{-1})\mathbf{v}) = \chi(\mathbf{v})$ is equivalent to

$$(3) \quad e^{2\pi i \langle t \mathbf{a}, \mathbf{v} \rangle} = e^{2\pi i \langle t \mathbf{a}, A(s^{-1})\mathbf{v} \rangle} = e^{2\pi i \langle A(s^{-1})^T t \mathbf{a}, \mathbf{v} \rangle}, \quad \mathbf{v} \in \mathbb{Z}^k,$$

and therefore to $A(s^{-1})^T t \mathbf{a} - t \mathbf{a} \in \mathbb{Z}^k$, for all $s \in D$. Since \mathbb{Z}^k is a discrete space, it follows that

$$(4) \quad A(s^{-1})^T \mathbf{a} - \mathbf{a} = \mathbf{0}$$

for all $s \in D$ and $\mathbf{a} \in \mathbf{W}$. Conversely, if $\mathbf{a} \in \mathbb{R}^k$ satisfies (4) for all $s \in D$, then equation (3) shows that $\mathbf{a} \in \mathbf{W}$. This allows us to conclude that

$$\mathbf{W} = \{\mathbf{a} \in \mathbb{R}^k : A(s^{-1})^T \mathbf{a} = \mathbf{a}, s \in D\} = \{\mathbf{a} \in \mathbb{R}^k : A(s)^T \mathbf{a} = \mathbf{a}, s \in D\}.$$

On the other hand, since $A(s)$ are integral matrices acting on free abelian groups, one verifies immediately using Gaussian elimination (or the exact sequence for $\text{Tor}_{\mathbb{Z}}^{\bullet}(\cdot, \mathbb{R})$) that

$$\mathbb{R}^n \cong \mathbf{H} \otimes_{\mathbb{Z}} \mathbb{R} \cong \{\mathbf{v} \in \mathbb{R}^k : A(s)\mathbf{v} - \mathbf{v} = \mathbf{0}, s \in D\}.$$

In view of the previous discussion, we reduced the proof to showing that the vector spaces \mathbf{W} and $\mathbf{H} \otimes_{\mathbb{Z}} \mathbb{R}$ have the same dimension.

Let V be a finite dimensional vector space over a field whose characteristic does not divide $|D|$ and let $\theta : D \rightarrow GL(V)$ be a finite group representation. If $T : V \rightarrow V$ is a linear map, we denote by $T^* : V^* \rightarrow V^*$ its dual map. The second dual T^{**} identifies canonically with T . Let $W = \{\mathbf{v} \in V : \theta(s)\mathbf{v} = \mathbf{v}, s \in D\}$. Choose a linear map $E : V \rightarrow V$ such that $E(V) = W$ and

$E^2 = E$. Then $P : V \rightarrow V$ defined by

$$P = \frac{1}{|D|} \sum_{s \in D} \theta(s^{-1}) E \theta(s)$$

satisfies $P(V) = W$ and

$$\theta(s)P = P\theta(s) = P = P^2, \quad s \in D.$$

Passing to duals we obtain

$$\theta(s)^* P^* = P^* \theta(s)^* = P^* = (P^*)^2, \quad s \in D.$$

Moreover $P^*(V^*) = \{\mathbf{a} \in V^* : \theta(s)^* \mathbf{a} = \mathbf{a}, s \in D\}$. By elementary linear algebra, $\text{rank}(P) = \text{rank}(P^*)$. Applying all this to $\theta : D \rightarrow GL_k(\mathbb{R})$, we see that the vector spaces

$$\mathbf{H} \otimes_{\mathbb{Z}} \mathbb{R} = \{\mathbf{v} \in \mathbb{R}^k : A(s)\mathbf{v} = \mathbf{v}, s \in D\} \quad \text{and} \quad \mathbf{W} = \{\mathbf{a} \in \mathbb{R}^k : A(s)^T \mathbf{a} = \mathbf{a}, s \in D\}$$

have the same dimension. For the second part of the statement, we invoke Proposition 2.1. \square

We need some elements of representation theory and use the book of Kaniuth and Taylor [15] as a basic reference. Let G be a Bieberbach group as in (1). The unitary dual of G consists of equivalence classes of irreducible unitary representations of G and is denoted by \widehat{G} . The term character is reserved for one dimensional representations of a group. The stabilizer of a character χ of N is the subgroup of G defined by $G_\chi = \{g \in G : \chi(g^{-1} \cdot g) = \chi(\cdot)\}$. It is clear that $N \subset G_\chi$ and that there is a bijection from G/G_χ onto the orbit of χ . Mackey's theory shows that each irreducible representation $\pi \in \widehat{G}$ is supported by the orbit of some character $\chi \in \widehat{N}$, in the sense that the restriction of π to N is unitarily equivalent to some multiple of the direct sum of the characters in the orbit of χ .

$$\pi|_N \sim m_\pi \cdot \bigoplus_{g \in G/G_\chi} \chi(g^{-1} \cdot g)$$

where g runs through a set of coset representatives.

For each $\chi \in \widehat{N}$, denote by $\widehat{G}_\chi^{(\chi)}$ the subset of \widehat{G}_χ consisting of classes of irreducible representations σ of G_χ such that the restriction of σ to N is unitarily equivalent to a multiple of χ . Let $\Omega \subset \widehat{N}$ be a subset which intersects each orbit of G exactly once. We need the following basic result due to Mackey, see [15, Thm. 4.28]

Theorem 2.3. $\widehat{G} = \left\{ \text{ind}_{G_\chi}^G(\sigma) : \sigma \in \widehat{G}_\chi^{(\chi)}, \chi \in \Omega \right\}$.

The characters of G factor uniquely through $G/[G, G]$ and hence they can be identified with the characters of $H_1(G, \mathbb{Z})$. Thus, a Bieberbach group with trivial center has finitely many characters.

Theorem 2.4. *Let G be a Bieberbach group with trivial center. If ω is a character of G , then $\widehat{G} \setminus \{\omega\}$ is a compact open subset of \widehat{G} .*

Proof. The proof for the Hantzsche-Wendt group Γ from [6], with ω the trivial representation, uses an explicit calculation of the irreducible representations of Γ which is not available for an

abstract Bieberbach group. Nevertheless, we can borrow some ideas from there and adapt them to the general situation.

Points are closed in \widehat{G} so that $\widehat{G} \setminus \{\omega\}$ is open [9]. Since \widehat{G} is compact and satisfies the second axiom of countability [9], we only need to show that $\widehat{G} \setminus \{\omega\}$ is sequentially compact, [16, p. 138]. Thus it suffices to show that any sequence $(\pi_n)_n$ of points in $\widehat{G} \setminus \{\omega\}$ which converges to ω has a subsequence which is convergent to a point in $\widehat{G} \setminus \{\omega\}$. In the terminology of [6] this means that ω is a shielded point in \widehat{G} .

Let $(\pi_n)_n$ be a sequence in $\widehat{G} \setminus \{\omega\}$ which converges to ω . Since $\dim(\pi_n) \leq |D|$, after passing to a subsequence, we may arrange that all the representations π_n are of the same dimension m . By Theorem 2.3 there is a sequence $(\chi_n)_n$ in Ω such that, up to unitary equivalence, $\pi_n = \text{ind}_{G_{\chi_n}}^G(\sigma_n)$ with $\sigma_n \in \widehat{G_{\chi_n}}^{(\chi_n)}$. For each character χ of N , its stabilizer G_χ is a subgroup of G that contains N . Since $D = G/N$ is a finite group, it follows that the set of stabilizers $\{G_\chi : \chi \in \Omega\}$ is finite. Thus, after passing to a subsequence of $(\pi_n)_n$, we may further assume that all stabilizers groups G_{χ_n} are equal to the same subgroup L with $N \subset L \subset G$.

We are going to show that if $L = G$, then $(\pi_n)_n$ cannot converge to ω . Indeed, suppose for a moment that $L = G$. Then each χ_n is left invariant by the action of G on \widehat{N} so that $\chi_n \in \widehat{N}^G$.

Since $N^G = Z(G) = \{\mathbf{0}\}$ by assumption, it follows by Proposition 2.2 that the group \widehat{N}^G is finite. Therefore, after passing to a subsequence of $(\pi_n)_n$, we may further assume that all χ_n are equal to the same character $\chi \in \widehat{N}^G$. Then $\pi_n = \sigma_n \in \widehat{G_\chi}^{(\chi)} = \widehat{G}^{(\chi)}$ and hence $\pi_n|_N = m \cdot \chi$ for all $n \geq 1$ (recall that $\dim(\pi_n) = m$). Since π_n converges to ω and $\pi_n|_N = m \cdot \chi$, it follows that $\omega|_N$ is weakly contained in χ . This can happen only if $\chi = \omega|_N$.

One can also argue that $\chi = \omega|_N$ as follows. Suppose $\chi \neq \omega|_N$ so that $\chi(h) - \omega(h) \neq 0$ for some $h \in N$. Then $a = \chi(h)e - h$ is an element of $C^*(G)$ with $a \in \bigcap_{n \geq 1} \text{Ker}(\pi_n)$ but $a \notin \text{Ker}(\omega)$ since $\omega(a) = \chi(h) - \omega(h) \neq 0$. This contradicts the assumption that $(\pi_n)_n$ converges to ω .

Every character of G factors through the finite group $G/[G, G]$. Thus there are only finitely many characters and their images are finite groups. Therefore there is a finite index subgroup K of G contained in N on which every character of G is trivial. Since $\pi_n|_N = m \cdot \chi = m \cdot \omega|_N$, it follows that $\pi_n|_K = m \cdot \iota|_K$ and hence each π_n factors through the homomorphism $G \rightarrow G/K$ whose image is finite. Since the unitary dual of G/K is finite, there are only finitely many distinct terms in the sequence $(\pi_n)_n$ (when viewed as elements of \widehat{G}). The points of \widehat{G} are closed and therefore $(\pi_n)_n$ can converge to ω only if the sequence is eventually constant and equal to ω . This contradicts the assumption that $\pi_n \in \widehat{G} \setminus \{\omega\}$ for all $n \geq 1$.

Thus it suffices to deal with the case when all the stabilizers G_{χ_n} are equal to a fixed subgroup $L \neq G$. Let $r = [G : L] > 1$ and choose elements e_1, \dots, e_r in G such that e_1 is the neutral element and G is the disjoint union of the cosets $e_i L$. Since all of $(\pi_n)_n$ and hence all of $(\sigma_n)_n$ are of equal dimension, each σ_n can be realized on a fixed Hilbert space $H_\sigma = \mathbb{C}^d$, independently of n . Then $\pi_n = \text{ind}_L^G(\sigma_n) : G \rightarrow L(H_{\pi_n})$ acts on

$$H_{\pi_n} = \{\xi : G \rightarrow H_\sigma : \xi(gh) = \sigma_n(h)^{-1}\xi(g), g \in G, h \in L\},$$

by $\pi_n(g)\xi = \xi(g^{-1}\cdot)$. Each element $\xi \in H_{\pi_n}$ is completely determined on the coset $e_i L$ by its value $\xi(e_i)$ since $\xi(e_i h) = \sigma_n(h)^{-1}\xi(e_i)$ for $h \in L$. Consider the Hilbert space $H_\pi = H_\sigma^{\oplus r}$. For

each n , let $V_n : H_{\pi_n} \rightarrow H_\pi$ be the unitary operator defined by

$$V_n(\xi) = (\xi(e_1), \dots, \xi(e_r)).$$

Its adjoint V_n^* maps a vector $(\xi_1, \dots, \xi_r) \in H_\sigma^{\oplus r}$ to a function $\xi : G \rightarrow H_\sigma$ such that the restriction of ξ to the coset $e_i L$ is given by $\xi(e_i h) = \sigma_n(h)^{-1} \xi_i$ for $h \in L$. We will replace π_n by the unitary representation $\rho_n = V_n \pi_n(\cdot) V_n^*$. Let us observe that $\rho_n(e_r)$ maps $(\xi_1, 0, \dots, 0)$ to $(0, \dots, 0, \xi_1)$ for $\xi_1 \in H_\sigma$. Indeed, $V_n^*(\xi_1, 0, \dots, 0) = \xi$ where ξ is supported on L and $\xi(h) = \sigma_n(h)^{-1} \xi_1$. Then $\pi_n(e_r)\xi = \xi(e_r^{-1} \cdot)$ is supported entirely on the coset $e_r L$ and hence $V_n \pi_n(e_r)\xi = (0, \dots, 0, \xi_1)$. Let $E : H_\sigma^{\oplus r} \rightarrow H_\sigma^{\oplus r}$ be the orthogonal projection of H_π onto its first summand, $E(\xi_1, \dots, \xi_r) = (\xi_1, 0, \dots, 0)$. Since $\rho_n(e_r)(\xi_1, 0, \dots, 0) = (0, \dots, 0, \xi_1)$, it follows that $E\rho_n(e_r)E = 0$ for all $n \geq 1$.

Since $U(H_\pi)$ is compact and the group G is finitely presented (as an extension of finitely presented groups), it follows that $(\rho_n)_n$ has a subsequence $(\rho_{n_i})_i$ which converges in the point-norm topology to a unitary representation $\rho : G \rightarrow U(H_\pi)$. Thus $\lim_{i \rightarrow \infty} \|\rho_{n_i}(g) - \rho(g)\| = 0$ for all $g \in G$. It follows that $E\rho(e_r)E = 0$. This implies that ρ cannot be a multiple of ω since in that case ρ would commute with E and we would have $\|E\rho(e_r)E\| = \|\omega(e_r)E\| = |\omega(e_r)| = 1$. Decompose ρ into irreducible representations of G . At least one of those, denoted by γ , must be different from ω . Since $\lim_{i \rightarrow \infty} \|\rho_{n_i}(g) - \rho(g)\| = 0$ for all $g \in G$, it follows that $(\rho_{n_i})_i$ and hence $(\pi_{n_i})_i$ converges to $\gamma \in \widehat{G} \setminus \{\omega\}$. \square

Proof of Theorem 1.1. We have seen in the proof of Theorem 2.4 (applied for the trivial representation ι) that ι is a shielded point in \widehat{G} . By [6, Cor. 2.11] if G is any countable amenable group such that ι is a shielded point in \widehat{G} , then G is not connective.

One may also observe that since virtually abelian groups are type I, the primitive spectrum of $I(G)$ identifies with $\widehat{G} \setminus \{\iota\}$ and hence it is compact. Therefore $I(G)$ is not connective by [6, Prop. 2.7] as $I(G) \otimes \mathcal{O}_2$ contains a nonzero projection. Here \mathcal{O}_2 is the Cuntz algebra. \square

3. CONNECTIVE BIEBERBACH GROUPS

In this section we characterize the connective Bieberbach groups. Recall that a poly- \mathbb{Z} (or strongly polycyclic) group G is a group which admits a finite increasing series of subgroups $1 = G_0 \subset G_1 \subset \dots \subset G_k = G$ such that G_i is a normal subgroup of G_{i+1} and $G_{i+1}/G_i \cong \mathbb{Z}$ for all $0 \leq i \leq k-1$. For each i we have a split extension

$$1 \longrightarrow G_i \longrightarrow G_{i+1} \longrightarrow \mathbb{Z} \longrightarrow 1,$$

so that G_{i+1} is isomorphic to a semidirect product $G_i \rtimes_\alpha \mathbb{Z}$ for some $\alpha \in \text{Aut}(G_i)$.

Lemma 3.1 (Lemma 3.5, [6]). *Let $m > 1$ and let G and G' be countable discrete groups that fit into a short exact sequence:*

$$1 \longrightarrow G' \longrightarrow G \xrightarrow{\pi} \mathbb{Z}/m\mathbb{Z} \longrightarrow 1.$$

If G' is connective and the homomorphism π factors through \mathbb{Z} , then G is also connective.

Corollary 3.2. *Let G be a discrete countable group. Let $\alpha \in \text{Aut}(G)$ be such that $\alpha^m \in \text{Inn}(G)$ for some $m \geq 1$. If G is connective, then the semidirect product $G \rtimes_\alpha \mathbb{Z}$ is also connective.*

Proof. Since $\beta := \alpha^m$ is an inner automorphism, $G \rtimes_{\beta} \mathbb{Z} \cong G \times \mathbb{Z}$. It follows that $G \rtimes_{\beta} \mathbb{Z}$ is connective since by [8, Cor. 3.3] direct products of discrete amenable connective groups are connective. The group monomorphism $G \rtimes_{\beta} \mathbb{Z} \rightarrow G \rtimes_{\alpha} \mathbb{Z}$, $(x, k) \mapsto (x, mk)$, $x \in G$, $k \in \mathbb{Z}$, induces an exact sequence of groups

$$1 \longrightarrow G \rtimes_{\beta} \mathbb{Z} \longrightarrow G \rtimes_{\alpha} \mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/m\mathbb{Z} \longrightarrow 1 .$$

where π is the composition of the quotient map $\mathbb{Z} \rightarrow \mathbb{Z}/m\mathbb{Z}$ with $G \rtimes_{\alpha} \mathbb{Z} \rightarrow \mathbb{Z}$. We conclude that $G \rtimes_{\alpha} \mathbb{Z}$ is connective by applying Lemma 3.1. \square

Proposition 3.3 (Thm. 3.2, [21]). *Let G be a Bieberbach group and let $\alpha \in \text{Aut}(G)$. Then the semidirect product $G \rtimes_{\alpha} \mathbb{Z}$ is a Bieberbach group if and only if there exists $m \geq 1$ such that $\alpha^m \in \text{Inn}(G)$.*

Calabi (see [4], [21], [24]) introduced a reduction method in the study of flat manifolds which highlights the central role of the manifolds with the first Betti number zero. In the context of Bieberbach groups this translates as follows. If G is a k -dimensional Bieberbach group then $H_1(G, \mathbb{Z}) = G/[G, G]$ is a finitely generated abelian group. Thus if $H_1(G, \mathbb{Z})$ is not a finite group, then there is a surjective map $G \rightarrow \mathbb{Z}$. Its kernel G' is a $(k - 1)$ -dimensional Bieberbach group [21, Prop. 3.1] and hence G admits a semidirect product decomposition $G \cong G' \rtimes_{\alpha'} \mathbb{Z}$. If $H_1(G', \mathbb{Z})$ is not finite, one can repeat the procedure and “peel off” another copy of \mathbb{Z} , etc.

Proof of Theorem 1.2

(i) \Rightarrow (ii) Suppose that G is a connective Bieberbach group. Then all its subgroups are connective, since connectivity passes to subgroups [6]. It follows then by Theorem 1.1 and Proposition 2.1 that every nontrivial subgroup of G has a nontrivial center.

(ii) \Leftrightarrow (iii) A Bieberbach group G has the property that every nontrivial subgroup of G has a nontrivial center if and only if it is poly- \mathbb{Z} by [10, Thm. 23].

Alternatively, to argue that (ii) \Rightarrow (iii), one can invoke Calabi’s method as explained above and write G as an iterated semidirect product

$$G \cong ((H \rtimes \mathbb{Z}) \rtimes \cdots) \rtimes \mathbb{Z}$$

where either H is a Bieberbach group with finite first homology group (trivial center) or $H = \{1\}$. By assumption (ii) we see that H must be trivial and hence G is poly- \mathbb{Z} .

(iii) \Rightarrow (i) Assume that G is a poly- \mathbb{Z} Bieberbach group of dimension $k \geq 1$. This means that G is constructed iteratively by starting with $G_1 = \mathbb{Z}$, and then constructing $G_2, \dots, G_k = G$, where $G_{i+1} = G_i \rtimes_{\alpha_i} \mathbb{Z}$ for some $\alpha_i \in \text{Aut}(G_i)$. One must have $\alpha_i^{m_i} \in \text{Inn}(G_i)$ for some $m_i \geq 1$, as a consequence of Proposition 3.3. Therefore if G_i is connective, then G_{i+1} is also connective. This implication was previously pointed out in [22] under the stronger assumption that α_i is a finite order automorphism of G_i . Since $G_1 = \mathbb{Z}$ is connective, we use Corollary 3.2 to prove by induction that each G_i is connective and hence so is G .

(i) \Leftrightarrow (iv) Since G is a virtually abelian group, $C^*(G)$ is a type I C^* -algebra. Thus $\text{Prim}(I(G)) = \widehat{I(G)} = \widehat{G} \setminus \{t\}$, see [9]. The desired equivalence follows now from the characterization of separable nuclear connective C^* -algebras in terms of their primitive spectra as explained in the introduction, see [6] and [11]. \square

Proof of Corollary 1.3

(a) A connective Bieberbach group is poly- \mathbb{Z} by Theorem 1.2 and hence its quotient D must be solvable.

(b) Let \mathcal{G} be the class of all finite groups whose Sylow subgroups are all cyclic. If D is in \mathcal{G} , so are the normal subgroups and the quotients of D . We prove by induction on k that a k -dimensional Bieberbach group with holonomy in \mathcal{G} is poly- \mathbb{Z} and then apply Theorem 1.2 to conclude that G is connective. If $k = 1$ then $G \cong \mathbb{Z}$ and we are done. Suppose now that $k > 1$. By assumption, D belongs to the class \mathcal{G} , and hence it is not primitive, as explained in Section 2. Therefore $H_1(G, \mathbb{Z})$ is not finite and as seen earlier in the proof of Theorem 1.2, $G \cong G' \rtimes \mathbb{Z}$ where G' is a Bieberbach group of rank $k - 1$. The following diagram with exact rows and columns

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & N \cap G' & \longrightarrow & G' & \longrightarrow & D_0 \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & N & \longrightarrow & G & \longrightarrow & D \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & N/N \cap G' & \longrightarrow & \mathbb{Z} & \longrightarrow & D/D_0 \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

shows that the quotient of G' by $N \cap G' \cong \mathbb{Z}^{k-1}$, denoted by D_0 , is isomorphic to a normal subgroup of D . The finite group D_0 is not necessarily the holonomy group of G' since $N \cap G'$ is not always maximal abelian in G' . However, by [23, Thm. 3.1], the centralizer of $N \cap G'$ in G' is a maximal abelian normal subgroup $N' \cong \mathbb{Z}^{k-1}$ of G' . It follows that the holonomy group of G' is $D' \cong G'/N'$ and moreover D' is isomorphic to a quotient of D_0 because $N \cap G' \subset N'$. Since D is in \mathcal{G} , it follows that D' belongs to \mathcal{G} , as explained at the beginning of the proof. By the induction hypothesis, G' is poly- \mathbb{Z} and therefore so is G , since $G \cong G' \rtimes \mathbb{Z}$. \square

(c) We have seen, as a consequence of Theorem 1.2, that a Bieberbach group G is connective $\Leftrightarrow G$ is locally indicable $\Leftrightarrow G$ is diffuse. By [17, Thm. 3.5 (iii)], if a finite group D is solvable and has a non-cyclic Sylow group, then D can be realized as the holonomy of both a diffuse group G_1 and a non-diffuse group G_2 . \square

Remark 3.4. In view of the previous discussion, one can also formally derive both parts (a) and (b) of Corollary 1.3 from Theorem 1.2 and [17, Thm. 3.5 (i), (ii)]. Nevertheless, we included

a direct proof, as a mean to review Calabi's method on which the main result of our paper is based.

In [12], the authors find all Bieberbach groups up to dimension six that are non-diffuse and hence not connective. There are 38,746 six-dimensional Bieberbach groups, out of which 19,256 (almost a half) are not connective.

There are no general classification results for Bieberbach groups. To give an idea of the complexity of this question, let us mention that the number of non-isomorphic k -dimensional Bieberbach groups with holonomy $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ and finite first homology group grows as least as fast as Ck^5 for some $C > 0$, [19].

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REFERENCES

- [1] L. Auslander and M. Kuranishi. On the holonomy group of locally Euclidean spaces. *Ann. of Math. (2)*, 65:411–415, 1957. [2](#)
- [2] B. H. Bowditch. A variation on the unique product property. *J. London Math. Soc. (2)*, 62(3):813–826, 2000. [3](#)
- [3] K. S. Brown. *Cohomology of groups*, volume 87 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994. Corrected reprint of the 1982 original. [3](#)
- [4] E. Calabi. Closed, locally Euclidean 4-dimensional manifolds. *Bull. Amer. Math. Soc.*, 63(4):135–551, 1957. [2](#), [8](#)
- [5] L. S. Charlap. *Bieberbach groups and flat manifolds*. Universitext. Springer-Verlag, New York, 1986. [1](#)
- [6] M. Dadarlat and U. Pennig. Connective C^* -algebras. *J. Funct. Anal.*, 272(12):4919–4943, 2017. [1](#), [2](#), [4](#), [5](#), [6](#), [7](#), [8](#), [9](#)
- [7] M. Dadarlat and U. Pennig. Deformations of nilpotent groups and homotopy symmetric C^* -algebras. *Math. Ann.*, 367(1-2):121–134, 2017. [1](#)
- [8] M. Dadarlat, U. Pennig, and A. Schneider. Deformations of wreath products. *Bull. Lond. Math. Soc.*, 49(1):23–32, 2017. [1](#), [8](#)
- [9] J. Dixmier. *C^* -algebras*. North Holland, Amsterdam, 1982. [6](#), [9](#)
- [10] D. R. Farkas. Crystallographic groups and their mathematics. *Rocky Mountain J. Math.*, 11(4):511–551, 1981. [8](#)
- [11] J. Gabe. Traceless AF embeddings and unsuspended E -theory. arXiv:1804.08095 [math.OA]. [1](#), [9](#)
- [12] A. Gasior, R. Lutowski, and A. Szczepański. A short note about diffuse Bieberbach groups. *J. Algebra*, 494:237–245, 2018. [10](#)
- [13] W. Hantzsche and H. Wendt. Dreidimensionale euklidische Raumformen. *Math. Ann.*, 110(1):593–611, 1935. [2](#)
- [14] H. Hiller and C.-H. Sah. Holonomy of flat manifolds with $b_1 = 0$. *Quart. J. Math. Oxford Ser. (2)*, 37(146):177–187, 1986. [2](#), [3](#)

- [15] E. Kaniuth and K. F. Taylor. *Induced representations of locally compact groups*, volume 197 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2013. 5
- [16] J. L. Kelley. *General topology*. D. Van Nostrand Company, Inc., Toronto-New York-London, 1955. 6
- [17] S. Kionke and J. Raimbault. On geometric aspects of diffuse groups. *Doc. Math.*, 21:873–915, 2016. With an appendix by Nathan Dunfield. 2, 9, 10
- [18] P. Linnell and D. Witte Morris. Amenable groups with a locally invariant order are locally indicable. *Groups Geom. Dyn.*, 8(2):467–478, 2014. 3
- [19] J. P. Rossetti and P. A. Tirao. Compact Flat Manifolds with Holonomy group $Z_2 \oplus Z_2$. *Proc. Amer. Math. Soc.*, 124(8):431–474, 1996. 10
- [20] W. R. Scott. *Group theory*. Dover Publications, Inc., New York, second edition, 1987. 3
- [21] A. Szczepański. *Geometry of crystallographic groups*, volume 4 of *Algebra and Discrete Mathematics*. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2012. 2, 8
- [22] A. Szczepański. Examples of Connective C*-algebras. arXiv:1904.07134 [math.OA], 2019. 2, 8
- [23] A. T. Vasquez. Flat Riemannian manifolds. *J. Differential Geometry*, 4:367–382, 1970. 9
- [24] J. A. Wolf. *Spaces of constant curvature*. McGraw-Hill Book Co., New York-London-Sydney, 1967. 2, 8

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