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Compressing Coefficients While Preserving Ideals in K-Theory for C^*-Algebras

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Abstract. An invariant based on ordered K-theory with coefficients in $\mathbb{Z} \oplus \bigoplus_{n>1} \mathbb{Z}/n$ and an infinite number of natural transformations has proved to be necessary and sufficient to classify a large class of nonsimple C^*-algebras. In this paper, we expose and explain the relations between the order structure and the ideals of the C^*-algebras in question.

As an application, we give a new complete invariant for a large class of approximately subhomogeneous C^*-algebras. The invariant is based on ordered K-theory with coefficients in $\mathbb{Z}\oplus\mathbb{Q}\oplus\mathbb{Q}/\mathbb{Z}$. This invariant is more compact (hence, easier to compute) than the invariant mentioned above, and its use requires computation of only four natural transformations.

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0. Introduction

The problem of classifying approximately homogeneous C^*-algebras of real rank zero with slow dimension growth by algebraic invariants was settled in 1994 by the first author and Guihua Gong [6]. The complete invariant $\underline{\mathbf{K}}(-)$ has three components:

– The ordered K-groups $K_0(-) \oplus K_1(-)$ (introduced in [9] and [13]).
– The ordered K-groups with coefficients $K_0(-; \mathbb{Z} \oplus \mathbb{Z}/n) \oplus K_1(-; \mathbb{Z} \oplus \mathbb{Z}/n)$ (introduced in [7]).
– The natural transformations between these groups (studied in an abstract setting in [21], integrated along with the order structure in [11], [8] and [6]).

The main result of [6] states that two real rank zero approximately subhomogeneous C^*-algebras A and B (of a certain subhomogeneity type) with *slow dimension growth* are isomorphic when $\underline{\mathbf{K}}(A)$ and $\underline{\mathbf{K}}(B)$ are isomorphic in a fashion preserving all positive cones and all the natural transformations.

The existence of a classification result based on $\underline{\mathbf{K}}(-)$ indirectly implies that $\underline{\mathbf{K}}(A)$ can be thought of as an algebraic model of the asymptotic homotopy type or shape theory type of the C^*-algebra A. It is also clear from comparing classification results based on $\underline{\mathbf{K}}(-)$ to others based on $K_0(A) \oplus K_1(A)$, that the advantage of the larger invariant is that it offers a better homological description of how the various ideals of A are glued together. To make this more precise, consider a $*$-homomorphism $\varphi: A \to B$. If φ maps some ideal I of A into some ideal J of B, the morphism $\underline{\mathbf{K}}(\varphi)$ will map the image of $\underline{\mathbf{K}}(I)$ in $\underline{\mathbf{K}}(A)$ into the image of $\underline{\mathbf{K}}(J)$ in $\underline{\mathbf{K}}(B)$. For A, B, C^*-algebras of real rank zero and stable rank one, the above property has an algebraic counterpart which can be formulated for any morphism $\Phi: \underline{\mathbf{K}}(A) \to \underline{\mathbf{K}}(B)$. A morphism satisfying that property is said to be *ideal-preserving*, a notion studied – in embryonic form – in [15].

In this paper, we give a complete description of the interrelations between positive and ideal-preserving maps. Such results give a direct, and more precise, explanation of the relevance of the order structure when studying nonsimple C^*-algebras, and have important consequences for certain types of reduction results found in [12].

As another application, we consider the question of compressing the coefficients in the invariants – without losing the information they hold concerning the ideal structure of the C^*-algebra. A proof that certain compressions of coefficients preserve the ideal-preserving maps combines with the results mentioned above to give new and more economical classification theorems.

Such reductions are of great practical importance because of the fact that while the invariant $\underline{\mathbf{K}}(-)$ is *continuous* and, hence, in theory computable for many C^*-algebras on inductive limit form, an isomorphism must intertwine an infinite family of group homomorphisms, making it somewhat difficult to work with. Furthermore, there is often quite a bit of redundancy in the invariant.

Put more precisely, our main result states that when A and B are both ASH C^*-algebras of real rank zero having slow dimension growth, and if their K-theory satisfies

$$\mathrm{Tor}(K_i(A), K_{i+1}(B)) = 0, \quad i \in \{0, 1\},$$

then there is a complete invariant based on ordered K-theory with coefficients in $\mathbb{Z} \oplus \mathbb{Q} \oplus \mathbb{Q}/\mathbb{Z}$. It was pointed out by Claude Schochet that the above condition on the vanishing of Tor is the same as the one under which the natural map $K_*(A) \otimes K_*(B) \to K_*(A \otimes B)$ is an isomorphism, see [20]. Besides eliminating redundancy in the coefficients, the main virtue of this invariant is that it is only necessary to check that a given isomorphism intertwines *four* natural transformations. Apart from the obvious practical importance of such an invariant, there are also theoretical implications of its existence.

Hoping to capture basic structural understanding which may be relevant to other applications, we proceed with our investigation as long as possible in an abstract, algebraic setting. This applies in particular to our discussion of ideal-preserving maps and compression. On the other hand, we only obtain equivalence between positivity and the ideal-preserving property in the case of approximately subhomogeneous C^*-algebras.

1. Algebraic Preliminaries

We recall a few facts about the functor Tor from, e.g., [19, Chapter 8]. For an Abelian group H, we let $\mathrm{tor}H$ denote the torsion subgroup of H and $H[n]$ the subgroup of H consisting of elements x such that $nx = 0$. First, we note that

$$\begin{aligned}
&\mathrm{Tor}(G, H) \cong \mathrm{Tor}(H, G),\\
&\mathrm{Tor}(H, \mathbb{Z}/n) \cong H[n],\\
&\mathrm{Tor}(H, \mathbb{Q}/\mathbb{Z}) \cong \mathrm{tor}H.
\end{aligned}$$

These isomorphisms are natural. Also, since $\mathrm{Tor}(-, G)$ is the right derived functor for $- \otimes G$, and Tor_2 vanishes, for any short exact sequence of groups

$$0 \longrightarrow H_1 \longrightarrow H_2 \longrightarrow H_3 \longrightarrow 0,$$

we get a long exact sequence

$$\begin{aligned}
0 \longrightarrow \mathrm{Tor}(H_1, G) \longrightarrow \mathrm{Tor}(H_2, G) \longrightarrow \mathrm{Tor}(H_3, G) \longrightarrow\\
\longrightarrow H_1 \otimes G \longrightarrow H_2 \otimes G \longrightarrow H_3 \otimes G \longrightarrow 0
\end{aligned}$$

The ensuing two lemmas rephrase known facts to fit our exposition in the following.

LEMMA 1.1. *Let G and H be Abelian groups. Then* $\mathrm{Tor}(H, G) = 0$ *if and only if whenever g and h are torsion elements of G and H, respectively, we have*

$$(\mathrm{order}(g), \mathrm{order}(h)) = 1.$$

Proof. By [14, 62.F], we have

$$\mathrm{Tor}(H, G) = \bigoplus_{p \text{ prime}} \mathrm{Tor}(H_p, G_p),$$

with H_p, G_p denoting the p-components of H, G. Hence, $\mathrm{Tor}(H, G) \neq 0$ if and only if there is a prime p with $\mathrm{Tor}(H_p, G_p) \neq 0$. The condition on the orders is equivalent to saying that one of the p-groups always vanishes, so we must see that for p-groups A, B

$$\mathrm{Tor}(A, B) = 0 \iff A = 0 \text{ or } B = 0.$$

For this, note that if $A \neq 0$, there is a short-exact sequence

$$0 \longrightarrow \mathbb{Z}/p \longrightarrow A \longrightarrow A/(x) \longrightarrow 0,$$

where x is some element of order p. Applying $\mathrm{Tor}(-, B)$, we get an embedding of $B[p]$ into $\mathrm{Tor}(A, B)$.

LEMMA 1.2. *Let $k \in \mathbb{N}$ be given. If a torsion group G has no k-torsion, G is k-divisible.*

Proof. Fix $x \in G$ and let m denote the order of x. We have $(m,k) = 1$, for if $(m,k) > 1$, $(m/(m,k))x$ is a nonzero element annihilated by k. We can find $a, b \in \mathbb{Z}$ with $am + bk = 1$ and, hence,

$$x = (am + bk)x = a(mx) + k(bx) = k(bx).$$

2. Torsion Coefficient *K*-Theory

When an integer n is given, we can define the functors $K_i(-; \mathbb{Z}/n)$ from C^*-algebras to Abelian groups in several equivalent ways, cf. [21]. For each n and m, there are also natural maps

$$\rho_n^i \colon K_i(A) \to K_i(A; \mathbb{Z}/n),$$
$$\beta_n^i \colon K_i(A; \mathbb{Z}/n) \to K_{i+1}(A),$$
$$\kappa_{m,n}^i \colon K_i(A; \mathbb{Z}/n) \to K_i(A; \mathbb{Z}/m),$$

and we have

PROPOSITION 2.1 ([21]). *The six term sequences*

(i)
$$\begin{array}{ccccc} K_0(A) & \xrightarrow{\times n} & K_0(A) & \xrightarrow{\rho_n^0} & K_0(A; \mathbb{Z}/n) \\ \beta_n^1 \uparrow & & & & \downarrow \beta_n^0 \\ K_1(A; \mathbb{Z}/n) & \xleftarrow[\rho_n^1]{} & K_1(A) & \xleftarrow[\times n]{} & K_1(A) \end{array}$$

(ii)
$$\begin{array}{ccccc} K_0(A; \mathbb{Z}/m) & \xrightarrow{\kappa_{n,m,m}^0} & K_0(A; \mathbb{Z}/nm) & \xrightarrow{\kappa_{n,nm}^0} & K_0(A; \mathbb{Z}/n) \\ \beta_{m,n}^1 \uparrow & & & & \downarrow \beta_{m,n}^0 \\ K_1(A; \mathbb{Z}/n) & \xleftarrow[\kappa_{n,nm}^1]{} & K_1(A; \mathbb{Z}/nm) & \xleftarrow[\kappa_{nm,m}^1]{} & K_1(A; \mathbb{Z}/m) \end{array}$$

are exact. Here $\beta_{m,n}^i = \rho_m^{i+1}\beta_n^i$.

LEMMA 2.2 ([21], cf. also [2]).

(i) $\kappa_{m,n}^i \rho_n^i = \dfrac{m}{(n,m)} \rho_m^i$,

(ii) $\beta_m^i \kappa_{m,n}^i = \dfrac{n}{(n,m)} \beta_n^i$,

(iii) $\kappa_{k,m}^i \kappa_{m,n}^i = \dfrac{m(k,n)}{(k,m)(m,n)} \kappa_{k,n}^i$.

We shall use the symbol '$\bullet$' to refer to the sum of the even and the odd part of a $\mathbb{Z}/2$-graded group, as in $K_\bullet = K_0 \oplus K_1$. Applying this convention to K-groups with coefficients in $\mathbb{Z} \oplus \mathbb{Z}/n$ we get

$$\begin{aligned} &K_\bullet(A; \mathbb{Z} \oplus \mathbb{Z}/n) \\ &\quad = K_0(A; \mathbb{Z} \oplus \mathbb{Z}/n) \oplus K_1(A; \mathbb{Z} \oplus \mathbb{Z}/n) \\ &\quad = K_0(A) \oplus K_0(A; \mathbb{Z}/n) \oplus K_1(A) \oplus K_1(A; \mathbb{Z} \oplus \mathbb{Z}/n). \end{aligned}$$

The latter decomposition actually gives a $(\mathbb{Z}/2)^2$-grading on $K_\bullet(A; \mathbb{Z} \oplus \mathbb{Z}/n)$. In [7] an order structure on these doubly graded groups was introduced by the definition

$$\begin{aligned} &K_\bullet(A; \mathbb{Z} \oplus \mathbb{Z}/n)^{++} \\ &\quad = \hom(\mathbb{I}_n^\sim, A \otimes C(S^1) \otimes \mathbb{K}) \subset KK(\mathbb{I}_n^\sim, A \otimes C(S^1)), \end{aligned}$$

where hom denotes the set of KK classes having a representative which is a $*$-homomorphism. This order structure was also used in [11]. Another order structure was considered in [8] and [6], namely

$$K_\bullet(A; \mathbb{Z} \oplus \mathbb{Z}/n)^+ = K_0(A \otimes C(W_n) \otimes C(S^1))^+,$$

where W_n is the n-Moore space. It was shown in [6] that these order structures have similar properties for the class of ASH algebras of real rank zero, and either one may be used in classifying such C^*-algebras. We shall work with the latter, but our results hold true with only minor modifications for the former.

Let Δ denote the ordered set $(\mathbb{N}, \leqslant)$, where $x \leqslant y$ iff x divides y. Note that Δ is directed, so that we may construct inductive limits over Δ. We will denote these by $\varinjlim_\Delta (G_p, f_{q,p})$, where $f_{q,p}\colon G_p \to G_q$ are the bonding maps. When a cofinal subset Δ' of Δ, is given, we may restrict attention to this, as

$$\varinjlim_\Delta G_n \cong \varinjlim_{\Delta'} G_n.$$

We define graded group homomorphisms

$$\kappa_{nm,m}\colon K_\bullet(A; \mathbb{Z} \oplus \mathbb{Z}/m) \to K_\bullet(A; \mathbb{Z} \oplus \mathbb{Z}/nm)$$

by

$$\kappa_{nm,m}^{i,\text{even}} = \chi_{mn,n}, \qquad \kappa_{nm,m}^{i,\text{odd}} = \kappa_{nm,m}^{i},$$

where $\chi_{mn,n}$ is just multiplication by m between the relevant copies of $K_i(A)$. The maps κ are positive, so we may define

DEFINITION 2.3.

$$K_\bullet(A; \mathbb{Q} \oplus \mathbb{Q}/\mathbb{Z}) = \varinjlim_\Delta \left(K_\bullet(A; \mathbb{Z} \oplus \mathbb{Z}/n), \kappa_{nm,n}\right),$$

$$K_\bullet(A; \mathbb{Q} \oplus \mathbb{Q}/\mathbb{Z})^+ = \varinjlim_\Delta \left(K_\bullet(A; \mathbb{Z} \oplus \mathbb{Z}/n)^+, \kappa_{nm,n}\right).$$

This way we obtain doubly graded ordered groups. We shall denote the odd parts by $K_i(A;\mathbb{Q}/\mathbb{Z})$. The even parts are naturally isomorphic to $K_i(A)\otimes\mathbb{Q}$ since

$$\begin{aligned}\varinjlim_{\Delta}(G,\chi_{mn,n}) &\cong \varinjlim_{\Delta}(G\otimes\mathbb{Z},\mathrm{id}\otimes\chi_{mn,n})\\ &\cong G\otimes\left(\varinjlim_{\Delta}(\mathbb{Z},\chi_{mn,n})\right)\cong G\otimes\mathbb{Q}\end{aligned}$$

naturally. We shall invoke this isomorphism tacitly in the following.

Because of Lemma 2.2(i) and (ii), we also get maps

$$\begin{aligned}&\rho^i_\infty\colon K_i(A)\otimes\mathbb{Q}\to K_i(A;\mathbb{Q}/\mathbb{Z}),\\ &\beta^i_\infty\colon K_i(A;\mathbb{Q}/\mathbb{Z})\to K_{i+1}(A),\end{aligned}$$

defined by taking inductive limits.

In what follows, we shall mainly be preoccupied with the interrelations between the K-groups with coefficients in $\mathbb{Q}/\mathbb{Z}$ and $\mathbb{Z}/n$, respectively. In charting the interplay, two maps play a role similar to the one played by the reduction and Bockstein maps in the interplay between K-theory with coefficients in $\mathbb{Z}$ and $\mathbb{Z}/n$, namely

$$\mu^i_n=\varinjlim_{\Delta}\kappa^i_{mn,n},\qquad \nu^i_n=\rho^{i+1}_n\left(\varinjlim_{\Delta}\beta^i_{mn}\right)=\rho^{i+1}_n\beta^i_\infty.$$

They are clearly natural as they are composites and limits of natural maps.

LEMMA 2.4.

(i) $\mu^i_n\rho^i_n=\rho^i_\infty\left(\mathrm{id}\otimes\dfrac{1}{n}\right)$,

(ii) $\beta^i_\infty\mu^i_n=\beta^i_n$,

(iii) $\mu^i_m\kappa^i_{m,n}=\dfrac{n}{(m,n)}\mu^i_n$.

Proof. To prove (i), we consider the diagram

$$\begin{tikzcd} K_i(A;\mathbb{Z}/n) \arrow[r, "\kappa^i_{mn,n}"] & K_i(A;\mathbb{Z}/mn) \arrow[r] & \cdots \arrow[r] & K_i(A;\mathbb{Q}/\mathbb{Z}) \\ K_i(A) \arrow[u, "\rho^i_n"'] \arrow[r, "\times m"'] & K_i(A) \arrow[u, "\rho^i_{mn}"'] \arrow[r] & \cdots \arrow[r] & K_i(A)\otimes\mathbb{Q} \arrow[u, "\rho^i_\infty"'] \end{tikzcd}$$

The composed map on the top line is μ^i_n, and one can see that under our identifications, the composed map on the lower line is exactly $\mathrm{id}\otimes 1/n$.

The remaining claims are proved by similar reasoning on the diagrams

$$\begin{tikzcd}
K_{i+1}(A) \arrow[r, equal] & K_{i+1}(A) \arrow[r, equal] & \cdots \arrow[r, equal] & K_{i+1}(A) \\
K_i(A;\mathbb{Z}/n) \arrow[u, "\beta^i_n"'] \arrow[r, "\kappa^i_{mn,n}"'] & K_i(A;\mathbb{Z}/mn) \arrow[u, "\beta^i_{mn}"'] \arrow[r] & \cdots \arrow[r] & K_i(A;\mathbb{Q}/\mathbb{Z}) \arrow[u, "\beta^i_\infty"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
K_i(A;\mathbb{Z}/m) \arrow[r, "\kappa^i_{mn,m}"] & K_i(A;\mathbb{Z}/mn) \arrow[r] & \cdots \arrow[r] & K_i(A;\mathbb{Q}/\mathbb{Z}) \\
K_i(A;\mathbb{Z}/n) \arrow[u, "\kappa^i_{m,n}"'] \arrow[r, "\kappa^i_{mn,n}"'] & K_i(A;\mathbb{Z}/mn) \arrow[u, "\times\frac{n}{(n,m)}"'] \arrow[r] & \cdots \arrow[r] & K_i(A;\mathbb{Q}/\mathbb{Z}) \arrow[u, "\times\frac{n}{(n,m)}"']
\end{tikzcd}$$

There is an alternate way, very much in the spirit of [21], of defining K-theory with coefficients in $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$. Let U be the UHF algebra with $K_0(U) = \mathbb{Q}$. Let V be the mapping cone of the unital $*$-homomorphism $\mathbb{C} \to U$. Then one has isomorphisms $K_*(A;\mathbb{Q}) \cong K_*(A \otimes U)$ and $K_*(A;\mathbb{Q}/\mathbb{Z}) \cong K_*(A \otimes V)$. Using these alternate definitions, it is not hard to derive Proposition 2.5 below from general results in [21]. However, we believe that working with Definition 2.3 makes it easier to compute the order structure in concrete cases and going back and forth between these invariants and $\underline{\mathbf{K}}(-)$.

PROPOSITION 2.5. *The six term sequences*

(i)
$$\begin{tikzcd}
K_0(A) \arrow[r, "-\otimes 1"] & K_0(A)\otimes\mathbb{Q} \arrow[r, "\rho^0_\infty"] & K_0(A;\mathbb{Q}/\mathbb{Z}) \arrow[d, "\beta^0_\infty"] \\
K_1(A;\mathbb{Q}/\mathbb{Z}) \arrow[u, "\beta^1_\infty"] & K_1(A)\otimes\mathbb{Q} \arrow[l, "\rho^1_\infty"] & K_1(A) \arrow[l, "-\otimes 1"]
\end{tikzcd}$$

(ii)
$$\begin{tikzcd}
K_0(A;\mathbb{Z}/n) \arrow[r, "\mu^0_n"] & K_0(A;\mathbb{Q}/\mathbb{Z}) \arrow[r, "\times n"] & K_0(A;\mathbb{Q}/\mathbb{Z}) \arrow[d, "\nu^0_n"] \\
K_1(A;\mathbb{Q}/\mathbb{Z}) \arrow[u, "\nu^1_n"] & K_1(A;\mathbb{Q}/\mathbb{Z}) \arrow[l, "\times n"] & K_1(A;\mathbb{Z}/n) \arrow[l, "\mu^1_n"]
\end{tikzcd}$$

are exact.

Proof. The first diagram is exact by passing to inductive limits over Δ with the commutative diagram

$$\begin{tikzcd}
K_i(A) \arrow[r, "\times n"] \arrow[d, equal] & K_i(A) \arrow[r, "\rho_n^i"] \arrow[d, "\times m"] & K_i(A;\mathbb{Z}/n) \arrow[r, "\beta_n^i"] \arrow[d, "\kappa_{nm,n}"] & K_{i+1}(A) \arrow[d, equal] \\
K_i(A) \arrow[r, "\times nm"'] & K_i(A) \arrow[r, "\rho_{nm}^i"'] & K_i(A;\mathbb{Z}/nm) \arrow[r, "\beta_{nm}^i"'] & K_{i+1}(A)
\end{tikzcd}$$

and applying the natural isomorphisms mentioned above. It is straightforward to check that the induced map from $K_i(A)$ to $K_i(A)\otimes\mathbb{Q}$ becomes exactly tensoring by the unit of $\mathbb{Q}$.

For exactness of the second diagram, note that

$$\begin{tikzcd}
K_i(A;\mathbb{Z}/n) \arrow[r, "\kappa_{mn,n}^i"] \arrow[d, equal] & K_i(A;\mathbb{Z}/mn) \arrow[r, "\kappa_{m,mn}^i"] \arrow[d, "\kappa_{kmn,mn}^i"] & K_i(A;\mathbb{Z}/m) \arrow[r, "\rho_n^{i+1}\beta_m^i"] \arrow[d, "\kappa_{km,m}^i"] & K_{i+1}(A;\mathbb{Z}/n) \arrow[d, equal] \\
K_i(A;\mathbb{Z}/n) \arrow[r, "\kappa_{kmn,n}^i"'] & K_i(A;\mathbb{Z}/kmn) \arrow[r, "\kappa_{km,kmn}^i"'] & K_i(A;\mathbb{Z}/km) \arrow[r, "\rho_n^{i+1}\beta_{km}^i"'] & K_{i+1}(A;\mathbb{Z}/n)
\end{tikzcd}$$

is commutative, as is seen directly from Lemma 2.2(ii) and (iii). For commutativity of the second square, one must also verify that

$$\frac{kmn(mn,km)}{(mn,kmn)(kmn,km)} = (n,k) = \frac{m(mn,km)}{(m,mn)(m,km)}.$$

Passing to inductive limits vertically, we get a six term exact diagram as stated. To determine the endomorphisms of $K_i(A;\mathbb{Q}/\mathbb{Z})$ in this, one notes that

$$\kappa_{mn,m}\kappa_{m,mn} = \frac{m(mn,mn)}{(m,mn)^2} = n$$

for every m. The other maps are clearly the ones stated.

We end this section with a crucial technical lemma.

LEMMA 2.6. *Fix a prime p. If $K_i(A)[p]=0$, we have for any $r\in\mathbb{N}$:*

(i) $\beta_{p^r}^{i+1}=0$,

(ii) $\rho_{p^r}^{i+1}$ *is surjective,*

(iii) $\nu_{p^r}^{i+1}=0$,

(iv) $\mu_{p^r}^{i}$ *is injective.*

Proof. The exact sequences in the following diagram are derived from Proposition 2.1(i) using the Tor-sequence. The outermost maps in

$$\begin{tikzcd} 0 \arrow[r] & K_{i+1}(A)\otimes\mathbb{Q}/\mathbb{Z} \arrow[r] \arrow[d, "p^r"] & K_{i+1}(A;\mathbb{Q}/\mathbb{Z}) \arrow[r] \arrow[d, "p^r"] & \mathrm{tor}(K_i(A)) \arrow[r] \arrow[d, "p^r"] & 0 \\ 0 \arrow[r] & K_{i+1}(A)\otimes\mathbb{Q}/\mathbb{Z} \arrow[r] & K_{i+1}(A;\mathbb{Q}/\mathbb{Z}) \arrow[r] & \mathrm{tor}(K_i(A)) \arrow[r] & 0 \end{tikzcd}$$

are both onto, for the leftmost group is divisible, and the rightmost is p^r-divisible by Lemma 1.2. By the five-lemma, the center map is onto. Applying exactness of the diagram in Proposition 2.5(ii), the two last claims follow from this.

It is obvious that multiplying with p^k is an injective map in $K_i(A)$. Applying exactness of the diagram in Proposition 2.1(i), the two first claims follow directly.

Remark 2.7. There are at least two different reasons why it is desirable to work with K-theory with $\mathbb{Q}/\mathbb{Z}$-coefficients instead of the family of K-groups with all $\mathbb{Z}/n$-coefficients. One is that it is easier to compute for C^*-algebras on inductive limit form – indeed, a straightforward diagonality argument (see [10, 3.2.7]) shows that

$$K_i(\varinjlim(A_j, f_j);\mathbb{Q}/\mathbb{Z}) = \varinjlim(K_i(A_j;\mathbb{Z}/j!), \kappa_{(j+1)!,j!}\circ K_i(f_j;\mathbb{Z}/j!)).$$

Another reason is the theoretical importance of having only to argue on a finite number of groups. We shall see an example of this in Section 7 below.

3. Reduction of Torsion Coefficient *K*-Theory

We denote by $\underline{\mathbf{K}}(A)$ the direct sum of all K-groups with coefficients in $\mathbb{Z}$ or $\mathbb{Z}/n$. The symbol Λ refers to the collection of all the natural isomorphisms ρ, β, κ between those groups, and by $\mathrm{Hom}_\Lambda(\underline{\mathbf{K}}(A), \underline{\mathbf{K}}(B))$ we denote the set of families of group homomorphisms (φ^i, ψ^i_n) with

$$\varphi^i\colon K_i(A)\to K_i(B), \qquad \psi^i_n\colon K_i(A;\mathbb{Z}/n)\to K_i(B;\mathbb{Z}/n),$$

which are intertwined by the homomorphisms in Λ. The symbols $\underline{\mathbf{K}}'(-)$, Λ' refer to the subcollection for which n is a prime power.

We denote by $\mathbf{K}_\infty(A)$ the direct sum of the groups $K_i(A)$, $K_i(A;\mathbb{Q}/\mathbb{Z})$, by Λ_∞ the collection $\{\rho^i_\infty, \beta^i_\infty\}$, and by $\mathrm{Hom}_{\Lambda_\infty}(\mathbf{K}_\infty(A), \mathbf{K}_\infty(B))$ the group of 4-tuples (φ^i, ψ^i) with

$$\varphi^i\colon K_i(A)\to K_i(B), \qquad \psi^i\colon K_i(A;\mathbb{Q}/\mathbb{Z})\to K_i(B;\mathbb{Q}/\mathbb{Z})$$

intertwined by the homomorphisms in Λ_∞.

We shall consider two maps

$$R\colon \mathrm{Hom}_\Lambda(\underline{\mathbf{K}}(A), \underline{\mathbf{K}}(B)) \to \mathrm{Hom}_{\Lambda'}(\underline{\mathbf{K}}'(A), \underline{\mathbf{K}}'(B)),$$
$$N\colon \mathrm{Hom}_\Lambda(\underline{\mathbf{K}}(A), \underline{\mathbf{K}}(B)) \to \mathrm{Hom}_{\Lambda_\infty}(\mathbf{K}_\infty(A), \mathbf{K}_\infty(B)).$$

The map R, considered also in [8], is just the restriction map, and Proposition 3.2 is proved there for C^*-algebras A in the bootstrap class of Rosenberg and Schochet. Lemma 3.1 and Proposition 3.2 are proved using various properties of the Bockstein operations established in [21].

LEMMA 3.1. *Let m, n be relatively prime integers. For any C^*-algebra A, $K_i(A; \mathbb{Z}/n) \oplus K_i(A; \mathbb{Z}/m)$ is naturally isomorphic to $K_i(A; \mathbb{Z}/mn)$. An isomorphism is given by the map*

$$(x, y) \mapsto \kappa^i_{mn,n}(x) + \kappa^i_{mn,m}(y).$$

Proof. It is easily derived from Lemma 2.2(iii) that

$$\begin{aligned}
&\kappa^i_{m,mn}\kappa^i_{mn,m} = n,\\
&\kappa^i_{n,mn}\kappa^i_{mn,n} = m,\\
&\kappa^i_{m,mn}\kappa^i_{mn,n} = 0,\\
&\kappa^i_{n,mn}\kappa^i_{mn,m} = 0,\\
&\kappa^i_{mn,n}\kappa^i_{n,mn} + \kappa^i_{mn,m}\kappa^i_{m,mn} = m + n.
\end{aligned}$$

The lemma follows by noting that if m and n are relatively prime, then so are $m + n$ and mn.

PROPOSITION 3.2. *For any C^*-algebras A and B, R is a group isomorphism.*

Proof. The proof is based on the previous lemma. For if $(m, n) = 1$, then φ^i_n and φ^i_m will determine φ^i_{mn} uniquely by $\varphi^i_{mn}\kappa^i_{mn,n} = \kappa^i_{mn,n}\varphi^i_n$ and $\varphi^i_{mn}\kappa^i_{mn,m} = \kappa^i_{mn,m}\varphi^i_m$.

The map N is defined as follows. Let (φ^i, ψ^i_n) denote an element in $\mathrm{Hom}_\Lambda(\underline{\mathbf{K}}(A), \underline{\mathbf{K}}(B))$. By coherence with $\kappa^i_{m,n}$, we can define maps $\varinjlim_\Delta \psi^i_n$ and set

$$N(\varphi^i, \psi^i_n) = \left(\varphi^i, \varinjlim_\Delta \psi^i_n\right).$$

Clearly N maps into the set of complex homomorphisms by coherence with ρ^i_n and β^i_n.

Observation 3.3. We have $R(\underline{\mathbf{K}}(f)) = \underline{\mathbf{K}}'(f)$ and $N\underline{\mathbf{K}}(f) = \mathbf{K}_\infty(f)$ for any $*$-homomorphism $f\colon A \to B$.

Since R is invertible, we may set $N' = N \circ R^{-1}$. We shall show that N' (and hence N) is invertible provided that

$$\mathrm{Tor}(K_i(A), K_{i+1}(B)) = 0, \quad i \in \{0, 1\} \tag{1}$$

holds. This will enable us, in essence, to extract information about K-theory with coefficients in $\mathbb{Z}/n$ from K-theory with coefficients in $\mathbb{Q}/\mathbb{Z}$.

We shall prove this by defining a map

$$E\colon \mathrm{Hom}_{\Lambda_\infty}(\mathbf{K}_\infty(A), \mathbf{K}_\infty(B)) \to \mathrm{Hom}_{\Lambda'}(\underline{\mathbf{K}}'(A), \underline{\mathbf{K}}'(B)),$$

and show that it is an inverse to N' by establishing commutativity of the triangle

$$\begin{tikzcd} \mathrm{Hom}_{\Lambda}(\underline{\mathbf{K}}(A), \underline{\mathbf{K}}(B)) \arrow[r, "R"] \arrow[d, "N"'] & \mathrm{Hom}_{\Lambda'}(\underline{\mathbf{K}}'(A), \underline{\mathbf{K}}'(B)) \\ \mathrm{Hom}_{\Lambda_\infty}(\mathbf{K}_\infty(A), \mathbf{K}_\infty(B)) \arrow[ur, "E"'] & \end{tikzcd}$$

We then proceed to prove that under extra conditions on the C^*-algebras in question, each of these maps are order-preserving. For all of this, as we shall see below, we need (1) in an essential way.

PROPOSITION 3.4. *Let an element* $(\varphi^i, \psi^i) \in \mathrm{Hom}_{\Lambda_\infty}(\mathbf{K}_\infty(A), \mathbf{K}_\infty(B))$ *be given, and fix a prime* p. *If the pair of C^*-algebras* A, B *satisfies* (1), *there exist unique group homomorphisms* $\eta^i_{p^r}\colon K_i(A;\mathbb{Z}/p^r) \to K_i(B;\mathbb{Z}/p^r)$ *satisfying*

$$\eta^i_{p^r}\rho^i = \rho^i\varphi^i, \tag{2}$$

$$\mu^i\eta^i_{p^r} = \psi^i\,\mu^i. \tag{3}$$

The maps will also satisfy

$$\beta^i\eta^i_{p^r} = \varphi^{i+1}\beta^i. \tag{4}$$

Proof. Fix $i \in \{0,1\}$. As a consequence of Lemma 1.1, (1) implies

$$K_{i+1}(A)[p] = 0 \quad \text{or} \quad K_i(B)[p] = 0.$$

We shall use this to define $\eta^i_{p^r}$. In case $K_{i+1}(A)[p] = 0$, we note that ρ^i_A is onto by Lemma 2.6 and then set

$$\eta^i_{p^r}(\rho^i_A x) = \rho^i_B\varphi^i x.$$

This is defined because φ^i annihilates the kernel of ρ^i_A. Clearly (2) is satisfied by this definition, and (3) is a consequence of Lemma 2.4(i). Finally, (4) follows from the fact that $\beta^i_B = 0$ by Lemma 2.6 again.

If $K_{i+1}(A)[p] \neq 0$, we have $K_i(B)[p] = 0$. By Lemma 2.6, μ^i_B is injective, and we set $\eta^i_{p^r} = (\mu^i_B)^{-1}\psi^i\mu^i_A$. Since $\mathrm{im}\mu^i = K_i(-;\mathbb{Q}/\mathbb{Z})[p^r]$, ψ^i must send $\mathrm{im}\mu^i_A$ to $\mathrm{im}\mu^i_B$, so the map is defined, and clearly satisfies (3). We get (2) by Lemma 2.4(i) and (4) by Lemma 2.4(ii).

Uniqueness also follows from Lemma 2.6, for $\eta^i_{p^r}$ is uniquely determined by (2) if ρ^i_A is surjective, or by (3) if μ^i_B is injective. As we have seen above, at least one of these properties will always hold.

We established the following result, a consequence of Lemma 2.6, in the proof above. In our further work, we may bypass Lemma 2.6 and refer to this directly.

Observation 3.5. Suppose that the pair A, B satisfies (1). For each i and every prime power p^r, either $\rho^i_A\colon K_i(A) \to K_i(A;\mathbb{Z}/p^r)$ is surjective or $\mu^i_B\colon K_i(B;\mathbb{Z}/p^r) \to K_i(B;\mathbb{Q}/\mathbb{Z})$ is injective.

DEFINITION 3.6. When A, B is a pair of C^*-algebras satisfying (1), we define

$$E\colon \mathrm{Hom}_{\Lambda_\infty}(\mathbf{K}_\infty(A), \mathbf{K}_\infty(B)) \to \mathrm{Hom}_{\Lambda'}(\underline{\mathbf{K}}'(A), \underline{\mathbf{K}}'(B))$$

by $E\colon(\varphi^i, \psi^i) \mapsto (\varphi^i, \psi^i_{p^r})$, where $\psi^i_{p^r} = \eta^i_{p^r}$, the unique maps given in Proposition 3.4.

We claim that E maps into $\mathrm{Hom}_{\Lambda'}$. Clearly, coherence with ρ and β follows from (2) and (4). We get coherence with κ from combining (2) with Lemma 2.2(i) and (3) with Lemma 2.4(iii) to get that

$$\kappa^i_{p^s,p^r}\eta^i_{p^r}\rho^i_{p^r} = p^{s-s\wedge r}\rho^i_{p^s}\varphi^i = \eta^i_{p^s}\kappa^i_{p^s,p^r}\rho^i_{p^r},$$
$$\mu^i\kappa^i_{p^s,p^r}\eta^i_{p^r} = p^{r-s\wedge r}\psi^i\mu^i_{p^r} = \mu^i\eta^i_{p^s}\kappa^i_{p^s,p^r},$$

where $r\wedge s = \min\{r, s\}$. By Observation 3.5, either ρ^i_A is surjective or μ^i_B is injective, proving the claim.

PROPOSITION 3.7. *When A, B is a pair of C^*-algebras satisfying* (1),

(i) $R = EN$,
(ii) *E and N are group isomorphisms.*

Proof. Let (φ^i, ψ^i_n) in Hom_Λ be given. Since $(\varinjlim_\Delta \psi^i_n)\mu^i_{p^r} = \mu^i_{p^r}\psi^i_{p^r}$ for every p and every r, $\psi^i_{p^r}$ equals $\eta^i_{p^r}$ by the uniqueness established in Proposition 3.4. This shows that $R = EN$.

Arguing again via uniqueness in Proposition 3.4, it is clear that E is a group homomorphism. As a consequence of Lemma 3.2, E is surjective. And if $E(\varphi^i, \psi^i) = (0, 0, \underline{0}, \underline{0})$, then $\varphi^i = 0$ and $\psi^i\,|_{\mathrm{im}\,\mu^i} = 0$ by definition of E and (3). By definition, $K_i(A;\mathbb{Q}/\mathbb{Z})$ is generated by the images of the μ^i and so E is injective. Consequently, N is a bijection also.

4. Ideal-Preserving Maps

When I is an ideal of some C^*-algebra A, we denote by $\iota_{A,I}$ any K-theory map induced by the inclusion map $j_{A,I}\colon I \to A$. We then define

$$K_i(A\|I) \qquad K_i(A\|I;\mathbb{Z}/n) \qquad K_i(A\|I;\mathbb{Q}/\mathbb{Z})$$

as the image of $\iota_{A,I}$ in the relevant K-group. We say that an element of $\mathrm{Hom}_\Lambda(\underline{\mathbf{K}}(A), \underline{\mathbf{K}}(B))$, $\mathrm{Hom}_{\Lambda'}(\underline{\mathbf{K}}'(A), \underline{\mathbf{K}}'(B))$ or $\mathrm{Hom}_{\Lambda_\infty}(\mathbf{K}_\infty(A), \mathbf{K}_\infty(B))$ is *ideal-preserving* when $\varphi_0 K_0(A\|I) \subset K_0(B\|J)$ implies that every other element of the collection of group homomorphisms sends $K_i(A\|I)$ (resp. $K_i(A\|I;\mathbb{Z}/n)$ or $K_i(A\|I;\mathbb{Q}/\mathbb{Z})$) to $K_i(B\|J)$ (resp. $K_i(B\|J;\mathbb{Z}/n)$ or $K_i(B\|J;\mathbb{Q}/\mathbb{Z})$). The following lemma (cf. [22] and [16]) shows that the homomorphims induced on the above groups by any $*$-homomorphism between C^*-algebras of real rank zero and stable rank one are ideal-preserving.

LEMMA 4.1. *Let $f\colon A\to B$ be a $*$-homomorphism between C^*-algebras of real rank zero and stable rank one. Suppose that*

$$K_0(f)(K_0(A\|I))\subset K_0(B\|J)$$

for ideals I, J. Then $f(I\otimes\mathbb{K})\subset J\otimes\mathbb{K}$.

Proof. We may assume that A, B, I and J are all stable. Since I is generated by projections ([5]), it suffices to show that $f(p)\in J$ for any projection $p\in I$. By assumption, there is a projection $q\in J$ such that $[f(p)]=[q]\in K_0(B)$, and since B has cancellation of projections by [1] there is a partial isometry $v\in B$ such that $v^*v=f(p)$ and $vv^*=q$. Therefore, $f(p)$ must be in J.

Due to its topological flavor, the ideal-preserving property behaves very well with our reduction maps R and E. We shall explore this in the following two lemmas.

LEMMA 4.2. *For any C^*-algebras A, B, $\Phi\in\mathrm{Hom}_\Lambda(\underline{\mathbf{K}}(A),\underline{\mathbf{K}}(B))$ preserves ideals if and only if $R(\Phi)\in\mathrm{Hom}_{\Lambda'}(\underline{\mathbf{K}}'(A),\underline{\mathbf{K}}'(B))$ does.*

Proof. If $\Phi\in\mathrm{Hom}_\Lambda(\underline{\mathbf{K}}(A),\underline{\mathbf{K}}(B))$ preserves ideals then it is obvious that $R(\Phi)\in\mathrm{Hom}_{\Lambda'}(\underline{\mathbf{K}}'(A),\underline{\mathbf{K}}'(B))$ preserves ideals. Let $I\subset A$, $J\subset B$ be ideals such that $\varphi(K_0(A\|I))\subset K_0(B\|J)$. For a positive integer n, we formulate the following condition

$$\varphi_n(K_i(A\|I;\mathbb{Z}/n))\subset K_i(B\|J;\mathbb{Z}/n). \tag{5}$$

To prove that if $R(\Phi)$ preserves ideals then Φ preserves ideals, it is enough to show that if m,n are relatively prime integers satisfying (5), then mn also satisfies (5). Note that the condition (5) is equivalent to the existence of a set-theoretical map $\hat\varphi_n\colon K_i(I,\mathbb{Z}/n)\to K_i(J,\mathbb{Z}/n)$ such that $\varphi_n\iota_{A,I}=\iota_{B,J}\hat\varphi_n$. Let φ_{mn} be defined as in Proposition 3.2. Every square in the diagram

$$\begin{tikzcd}
K_i(A;\mathbb{Z}/n) \arrow[rrr, "\varphi_n"] \arrow[ddd, "\kappa_{mn,n}"'] & & & K_i(B;\mathbb{Z}/n) \arrow[ddd, "\kappa_{mn,n}"] \\
& K_i(I;\mathbb{Z}/n) \arrow[ul, "\iota_{A,I}"'] \arrow[r, dashed, "\hat\varphi_n"] \arrow[d, "\kappa_{mn,n}"'] & K_i(J;\mathbb{Z}/n) \arrow[ur, "\iota_{B,J}"] \arrow[d, "\kappa_{mn,n}"] & \\
& K_i(I;\mathbb{Z}/mn) \arrow[dl, "\iota_{A,I}"'] & K_i(J;\mathbb{Z}/mn) \arrow[dr, "\iota_{B,J}"] & \\
K_i(A;\mathbb{Z}/mn) \arrow[rrr, "\varphi_{mn}"'] & & & K_i(B;\mathbb{Z}/mn)
\end{tikzcd}$$

commutes. This follows by definition of the big square, and by naturality to the left and right. We obtain $\varphi_{mn}\iota_{A,I}\kappa_{mn,n} = \iota_{B,J}\kappa_{mn,n}\hat{\varphi}_n$ and a similar equation with the roles of m and n interchanged. Adding the two equations we get $\varphi_{mn}\iota_{A,I}(\kappa_{mn,n}(x) + \kappa_{mn,m}(y)) = \iota_{B,J}(\kappa_{mn,n}\hat{\varphi}_n(x) + \kappa_{mn,m}\hat{\varphi}_m(y))$ for any $x \in K_i(I;\mathbb{Z}/n)$ and $y \in K_i(I;\mathbb{Z}/m)$. Since $(m, n) = 1$, the images of $\kappa_{mn,n}$ and $\kappa_{mn,m}$ will generate $K_i(I;\mathbb{Z}/mn)$ according to Lemma 3.1, and the above equations show that mn satisfies (5).

We say that an ideal I in a C^*-algebra A *is K-pure* if $K_i(I)$ is a pure subgroup of $K_i(A)$, $i = 0, 1$ or, equivalently, if both sequences

$$0 \to K_i(I) \to K_i(A) \to K_i(A/I) \to 0$$

are pure exact. Using Proposition 2.1(i) and the Tor-sequence we see that for any $n \geqslant 2$ the sequences

$$0 \to K_i(I;\mathbb{Z}/n) \to K_i(A;\mathbb{Z}/n) \to K_i(A/I;\mathbb{Z}/n) \to 0$$

are exact. The same conclusion remains true if $\mathbb{Z}/n$ is replaced by $\mathbb{Q}/\mathbb{Z}$. This is seen by passing to inductive limits in the above sequences. Examples of C^*-algebras whose every ideal is K-pure are given below in Proposition 4.4.

LEMMA 4.3. *Suppose that A, B is a pair of C^*-algebras satisfying* (1) *and with the property that all ideals are K-pure. Then an element Φ of* $\mathrm{Hom}_{\Lambda_\infty}(\mathbf{K}_\infty(A), \mathbf{K}_\infty(B))$ *preserves ideals if and only if $E(\Phi) \in \mathrm{Hom}_{\Lambda'}(\underline{\mathbf{K}}'(A), \underline{\mathbf{K}}'(B))$ does.*

Proof. It is clear that if $E(\Phi)$ preserves ideals, so does $\Phi = N'(E(\Phi))$. In the other direction, let (φ^i, ψ^i) be an ideal-preserving element of $\mathrm{Hom}_{\Lambda_\infty}(\mathbf{K}_\infty(A), \mathbf{K}_\infty(B))$. Let ideals I of A and J of B be given with $\varphi^0(K_0(A\|I)) \subset K_0(B\|J)$. By assumption, $\varphi^1(K_1(A\|I)) \subset K_1(B\|J)$ and $\psi^{i+1}(K_{i+1}(A\|I;\mathbb{Q}/\mathbb{Z})) \subset K_{i+1}(B\|J;\mathbb{Q}/\mathbb{Z})$, so by the observations above, (φ^i, ψ^i) induces an element $(\hat{\varphi}^i, \hat{\psi}^i)$ in $\mathrm{Hom}_{\Lambda_\infty}(\mathbf{K}_\infty(I), \mathbf{K}_\infty(J))$.

By the properties of $\mathrm{Tor}(-,-)$ combined with the asserted exactness properties, $\mathrm{Tor}(K_\bullet(I), K_{\bullet+1}(J))$ embeds into $\mathrm{Tor}(K_\bullet(I), K_{\bullet+1}(B))$, which again embeds into $\mathrm{Tor}(K_\bullet(A), K_{\bullet+1}(B))$. Consequently, the pair I, J also satisfies (1), and applying Proposition 3.4 twice, we induce maps $\eta^i_{p^r}$ and $\hat{\eta}^i_{p^r}$.

Every small square in the diagram

$$\begin{tikzcd}
K_i(I;\mathbb{Z}/p^r) \arrow[rrr, dashed, "\hat{\eta}^i"] \arrow[ddd, "\iota_{A,I}"'] & & & K_i(J;\mathbb{Z}/p^r) \arrow[ddd, "\iota_{B,J}"] \\
& K_i(I) \arrow[ul, "\hat{\rho}^i"'] \arrow[r, dashed, "\hat{\varphi}^i"] \arrow[d, "\iota_{A,I}"'] & K_i(J) \arrow[ur, "\hat{\rho}^i"] \arrow[d, "\iota_{B,J}"] & \\
& K_i(A) \arrow[dl, "\rho^i"'] \arrow[r, "\varphi^i"'] & K_i(B) \arrow[dr, "\rho^i"] & \\
K_i(A;\mathbb{Z}/p^r) \arrow[rrr, dashed, "\eta^i"'] & & & K_i(B;\mathbb{Z}/p^r)
\end{tikzcd}$$

commutes. This follows by definition in the center, by naturality to the left and right, and by (2) in the top and bottom. Replacing (2) by (3), we get commutativity of the small squares of

$$\begin{tikzcd}
K_i(I;\mathbb{Z}/p^r) \arrow[rrr, dashed, "\hat{\eta}^i"] \arrow[dr, "\hat{\mu}^i"] \arrow[ddd, "\iota_{A,I}"'] & & & K_i(J;\mathbb{Z}/p^r) \arrow[dl, "\hat{\mu}^i"'] \arrow[ddd, "\iota_{B,J}"] \\
& K_i(I;\mathbb{Q}/\mathbb{Z}) \arrow[r, dashed, "\hat{\psi}^{i+1}"] \arrow[d, "\iota_{A,I}"'] & K_i(J;\mathbb{Q}/\mathbb{Z}) \arrow[d, "\iota_{B,J}"] & \\
& K_i(A;\mathbb{Q}/\mathbb{Z}) \arrow[r, "\psi^{i+1}"'] & K_i(B;\mathbb{Q}/\mathbb{Z}) & \\
K_i(A;\mathbb{Z}/p^r) \arrow[rrr, dashed, "\eta^i"'] \arrow[ur, "\mu^i"] & & & K_i(B;\mathbb{Z}/p^r) \arrow[ul, "\mu^i"']
\end{tikzcd}$$

by similar reasoning. We cannot conclude directly from this that the large squares are commutative, but only that

$$\eta^i \iota_{A,I} \hat{\rho}^i = \iota_{B,J} \hat{\eta}^i \hat{\rho}^i,$$
$$\mu^i \eta^i \iota_{A,I} = \mu^i \iota_{B,J} \hat{\eta}^i.$$

However, we also know that the pair I, B satisfies (1), so by Observation 3.5, either $\hat{\rho}^i = \rho_I^i$ is surjective or $\mu^i = \mu_B^i$ is injective. We conclude that $\eta^i \iota_{A,I} = \iota_{B,J} \hat{\eta}^i$, and hence that $\eta_{p^r}^i(K_0(I;\mathbb{Z}/p^r)) \subset K_0(J;\mathbb{Z}/p^r)$.

It is a consequence of the following proposition that every ideal in an ASH algebra (cf. [6]) of real rank zero is K-pure. In fact, we prove that the K-theory with *any* coefficients of the extension given by an ideal I in such an A is a *pure* exact sequence. Our argument is very similar to one in [4], where such a result was proven for AH algebras.

PROPOSITION 4.4. *Suppose an extension*

$$0 \to I \to A \to A/I \to 0$$

is given. If A is ASH of real rank zero, we have

(i) *I and A/I are ASH of real rank zero.*
(ii) *The sequences*
$$0 \to K_i(I) \to K_i(A) \to K_i(A/I) \to 0$$
are pure exact.
(iii) *For any $p \geqslant 2$ the sequences*
$$0 \to K_\bullet(I;\mathbb{Z}/p) \to K_\bullet(A;\mathbb{Z}/p) \to K_\bullet(A/I;\mathbb{Z}/p) \to 0$$
are pure exact.

Proof. Write $A = \varinjlim(A_n, \nu_n)$ and let $\nu_{\infty,n}\colon A_n \to A$ denote the canonical map. Since A has real rank zero, it follows from [5] that I and A/I have real rank zero. Moreover, there is an increasing sequence of projections (f_n) in I, forming an approximate unit of I. By induction, we construct a sequence of projections $e_n \in A_n$ such that $\nu_{\infty,n}(e_n) = f_n$ and $\nu_n(e_n) \leqslant e_{n+1}$. Actually, in order to do this construction we need to pass to a subsequence of (A_n). Suppose that $e_1, \dots, e_n$ were constructed. Using a standard approximation argument, we find $k > n$ and a projection $g \in A_k$ such that $\nu_{\infty,k}(g) = f_{n+1} - f_n$ and g is orthogonal to $\nu_n(e_n)$. Then we set $e_{n+1} = g + \nu_n(e_n)$. This completes the construction of the sequence (e_n). Let I_n denote the closed two-sided ideal of A_n generated by e_n. Then $\nu_n(I_n) \subset I_{n+1}$ since $\nu_n(e_n) \leqslant e_{n+1}$ and $I = \varinjlim(I_n, \nu_n)$ since $\nu_\infty(e_n) = f_n$ and (f_n) is an approximate unit of I. It then clear that the given extension of C^*-algebras is the inductive limit of the sequence of extensions

$$0 \to I_n \to A_n \to A_n/I_n \to 0, \tag{6}$$

the bonding morphisms being induced by (ν_n). Since the building blocks of A_n have connected spectrum and I_n is an ideal generated by projections, it is easily seen that I_n is a direct summand of A_n, hence A_n/I_n is also a direct summand of A. It follows that I and A/I are ASH algebras, completing the proof of (i).

To prove (ii), we only need to notice that by the continuity of K-theory, the short sequence of groups from the statement is pure exact since it is the inductive limit of a sequence of short exact split sequences. To prove (iii), we take the tensor product of (6) by $B = C_0(W_p)$ or by SB and use a similar argument for the functor $K_0(-, \mathbb{Z}/p) \cong K_0(- \otimes B)$.

The last part of the proposition could also be proved using the continuity of $KK(\widetilde{\mathbb{I}}_p, -)$.

Remark 4.5. The above argument gives an alternate proof to the injectivity of $\underline{K}(\iota_{A,I})$. Let B be a C^*-algebra in the class of Rosenberg–Schochet such that $K_0(B) = G$ and $K_1(B) = 0$. Arguing as above, we can prove (iii) with G replacing $\mathbb{Q}/\mathbb{Z}$.

5. Order and Ideals

For simplicity, in most of this section we restrict our considerations to stably unital C^*-algebras with cancellation of projections. If A is such an algebra, then there is a well defined map $x_0 \mapsto I(x_0)$ from $K_0(A)^+$ to the ideals of A. Specifically, if e is a projection in $A \otimes \mathbb{K}$ with $x = [e] \in K_0(A)$, then $I(x_0) \otimes \mathbb{K}$ is the ideal of $A \otimes \mathbb{K}$ generated by e. In this case, the ideal-based order introduced in [6] can be described as follows. We will write a general element in $K_\bullet(A; \mathbb{Z} \oplus \mathbb{Z}/n)$ in the form $x = (x_0, y)$ with $x_0 \in K_0(A)$. Then by definition, $K_\bullet(A; \mathbb{Z} \oplus \mathbb{Z}/n)_+$ consists of all those elements

(x_0, y) with the property that $x_0 \in K_0(A)^+$ and $(x_0, y) \in K_\bullet(A \| I(x_0); \mathbb{Z} \oplus \mathbb{Z}/n)$. We define $K_\bullet(A, \mathbb{Q} \oplus \mathbb{Q}/\mathbb{Z})_+$ as the inductive limit of $K_\bullet(A; \mathbb{Z} \oplus \mathbb{Z}/n)_+$ (cf. Definition 2.3). Then $\mathbf{K}_\infty(A)_+$ denotes the subsemigroup of $\mathbf{K}_\infty(A)$ generated by $K_\bullet(A)_+$ and $K_\bullet(A, \mathbb{Q} \oplus \mathbb{Q}/\mathbb{Z})_+$. It is not hard to see that $x \in \mathbf{K}_\infty(A)_+$ iff x_0 i.e. its K_0 component is positive and $x \in \mathbf{K}_\infty(A \| I(x_0))$. The following notation will be used:

$$x \succeq 0 \iff x \in K_\bullet(A; \mathbb{Z} \oplus \mathbb{Z}/n)_+,$$
$$x \succeq 0 \iff x \in \mathbf{K}_\infty(A)_+,$$
$$x \geqslant 0 \iff x \in K_\bullet(A; \mathbb{Z} \oplus \mathbb{Z}/n)^+ \text{ (see Section 2)},$$
$$x \geqslant 0 \iff x \in \mathbf{K}_\infty(A)^+ \text{ (see Section 2)}.$$

If a morphism Φ preserves these orders, we will write $\Phi \succeq 0$, resp. $\Phi \geqslant 0$.

It was shown in [6] that $K_\bullet(A; \mathbb{Z} \oplus \mathbb{Z}/n)^+ \subset K_\bullet(A; \mathbb{Z} \oplus \mathbb{Z}/n)_+$. The next two chapters will be devoted to further comparative studies of these order structures, or more precisely of the conditions

(i) $\Phi \geqslant 0$
(ii) $\Phi \succeq 0$

for maps defined on the K-groups of C^*-algebras of varying generality. In this analysis, the condition

(iii) Φ is ideal-preserving

plays an interpolating role. The relations are summarized below.

Relation	Conditions on C^*-algebras	Ref.
(i) $\Longrightarrow$ (ii)	Stably unital, cancellation of projections	5.3
(ii) $\Longleftrightarrow$ (iii)	Real rank zero, stable rank one	[6, 4.12]
(i) $\Longleftarrow$ (ii)	ASH, slow dimension growth, real rank zero	6.5

These results establish a bridge between the natural (K_0-type) order structure $\mathbf{K}_\infty(-)^+$ and the less natural order structure $\mathbf{K}_\infty(-)_+$, which is important for our reduction purposes and also appears in [12].

LEMMA 5.1. *Let A be a C^*-algebra and let e, f be projections in $A \otimes \mathbb{K}$. Suppose that the ideal generated by e contains f. Then there is an integer $k \geqslant 1$ such that $[f] \leqslant k[e]$ in $K_0(A)$.*

Proof. See the proof of [1, 6.3.5].

PROPOSITION 5.2. *Let A be a stably unital C^*-algebra with cancellation of projections. Suppose that $x_0 \in K_0(A)^+$. Then $(x_0, y) \in K_\bullet(A; \mathbb{Z} \oplus \mathbb{Z}/n)_+$ iff $(kx_0, y) \in K_\bullet(A; \mathbb{Z} \oplus \mathbb{Z}/n)^+$ for some $k \geqslant 0$.*

Proof. Suppose that $(x_0, y) \succeq 0$. Let $e \in A \otimes \mathbb{K}$ be a projection with $x_0 = [e]$. Let I be the ideal of $A \otimes \mathbb{K}$ generated by e. It follows from the very definition of $\succeq$ that there is $(0, w) \in K_\bullet(I; \mathbb{Z} \oplus \mathbb{Z}/n)$ such that $\iota_{A,I}(w) = y$. We will find $k \geqslant 1$ such that $(k[e], w) \in K_\bullet(I; \mathbb{Z} \oplus \mathbb{Z}/n)^+$. I is stably unital since by Brown's Theorem ([3]) I is stably isomorphic to the unital C^*-algebra $e(A \otimes \mathbb{K})e$. By [6, 4.8], there is a projection $f \in I \otimes \mathbb{K}$ such that $([f], w) \in K_\bullet(I; \mathbb{Z} \oplus \mathbb{Z}/n)^+$. By Lemma 5.1, there is $k \geqslant 1$ such that $[f] \leqslant k[e]$ in $K_0(I)$, hence $(k[e], w) \in K_\bullet(I; \mathbb{Z} \oplus \mathbb{Z}/n)^+$. It follows that $(kx_0, y) = \iota_{A,I}(k[e], w) \geqslant 0$.

In the other direction, suppose that $x_0 \geqslant 0$ and $(kx_0, y) \geqslant 0$. Let e and I be as in the first part of the proof. We want to show that y is in the image of $\iota_{A,I}$. Let $f \in A \otimes \mathbb{K}$ be a projection such that $[f] = kx_0$ and let J be the ideal of $A \otimes \mathbb{K}$ generated by f. Since A has cancellation of projections and $(kx_0, y) \geqslant 0$, we must have $(kx_0, y) \succeq 0$ by [6, 4.7]. Therefore, there is $(0, z) \in K_\bullet(J; \mathbb{Z} \oplus \mathbb{Z}/n)$ such that $\iota_{A,J}(z) = y$. To conclude the proof, we must show that $I = J$. The projections $e \otimes 1_k$ and f have the same K-theory class. Since A has cancellation of projections, the two projections must generate the same ideal of $A \otimes \mathbb{K}$. As both e and $e \otimes 1_k$ generate the ideal I, we find that $I = J$.

COROLLARY 5.3. *Let A, B be C^*-algebras of real rank zero and stable rank one. If Φ is an element of* $\mathrm{Hom}_\Lambda(\underline{\mathbf{K}}(A), \underline{\mathbf{K}}(B))$, *then $\Phi \geqslant 0$ implies $\Phi \succeq 0$.*

Proof. Suppose $\Phi \geqslant 0$ and let $x \in K_\bullet(A; \mathbb{Z} \oplus \mathbb{Z}/n)_+$ be given. By the proposition above, there exists a k such that $(kx_0, y) \in K_\bullet(A; \mathbb{Z} \oplus \mathbb{Z}/n)^+$, so by assumption, $\Phi(kx_0, y) = (k\Phi_0(x_0), \Phi(y)) \geqslant 0$. Applying Proposition 5.2 again, we get that $\Phi(x) \succeq 0$.

It was proven in [6] that if both A and B have real rank zero and stable rank one, then $\Phi \succeq 0$ if and only if Φ is ideal-preserving and its restriction to the K_0 groups is positive. Thus, if $\Phi \geqslant 0$, then Φ is ideal-preserving. A similar reasoning combined with an inductive limit argument proves the following.

LEMMA 5.4. *Suppose that A, B are C^*-algebras of real rank zero and stable rank one. If $\Phi \in \mathrm{Hom}_{\Lambda_\infty}(\mathbf{K}_\infty(A), \mathbf{K}_\infty(B))$, then $\Phi \succeq 0$ iff Φ is ideal-preserving and φ^0 is positive. If $\Phi \geqslant 0$, then $\Phi \succeq 0$.*

6. Order and Ideals for *ASH* Algebras

As in [6], we define the function rank: $K_0(A) \to \mathbb{Z}$ for certain A. When $A = C(Z)$, Z some compact connected Hausdorff space, rank is induced by any evaluation map $C(Z) \to \mathbb{C}$ via the canonical isomorphism $K_0(\mathbb{C}) = \mathbb{Z}$. When $A = \mathbb{I}_n^\sim \otimes C(Z)$, rank is induced by any evaluation map $\mathbb{I}_n^\sim \otimes C(Z) \to \mathbb{I}_n^\sim$ via the canonical isomorphism $K_0(\mathbb{I}_n^\sim) = \mathbb{Z}$.

LEMMA 6.1. *Let A be a C^*-algebra of the form $\mathbb{I}_n^{\sim} \otimes C(X)$ with $n \geqslant 2$ or $C(Y) \otimes C(X)$, where X and Y are finite connected CW complexes with*

$$\dim(X) \leqslant 3, \qquad \dim(Y) \leqslant 2.$$

If $x \in K_0(A)$ has rank at least two, then $x \geqslant 0$.

Proof. In the case $A = C(X) \otimes C(Y) \cong C(X \times Y)$ the implication $\operatorname{rank}(x) \geqslant 2 \Rightarrow x \geqslant 0$ is an easy consequence of the stability properties of vector bundles. For if $x = [p] - [q]$ with $\operatorname{rank}(p) - \operatorname{rank}(q) \geqslant 2$, we may conclude by [17] that $q \leqslant p$, whence x is positive. Let us deal with the case $A = \mathbb{I}_n^{\sim} \otimes B$, where $B = C(X)$. Let $\delta \colon \mathbb{I}_n^{\sim} \to \mathbb{C} \oplus \mathbb{C}$ be the restriction map at the endpoints of the spectrum on $\mathbb{I}_n^{\sim}$. After tensoring with B the exact sequence

$$0 \longrightarrow SM_n \xrightarrow{i} \mathbb{I}_n^{\sim} \xrightarrow{\delta} \mathbb{C} \oplus \mathbb{C} \longrightarrow 0$$

induces a K-theory exact sequence

$$K_0(SM_n \otimes B) \xrightarrow{i_*} K_0(\mathbb{I}_n^{\sim} \otimes B) \xrightarrow{\delta_*} K_0(B) \oplus K_0(B) \xrightarrow{\lambda} K_0(B),$$

where the connecting map λ is given by $\lambda(y_0, y_1) = n(y_0 - y_1)$ for $y_i \in K_0(B)$ since the sequence is a mapping torus sequence, cf. [1, 19.4].

Let $x \in K_0(\mathbb{I}_n^{\sim} \otimes B)$ and assume that $\operatorname{rank}(x) = r \geqslant 2$. Then $\delta_*(x) = (y_0, y_1)$ with $\operatorname{rank}(y_i) = r$. Since $B = C(X)$ with $\dim(X) \leq 3$, there are rank one projections $g_i \in B \otimes \mathbb{K}$ such that $y_i = [g_i] + (r-1)[1]$ for $i = 0, 1$ according to [18, 8.1.2]. Moreover, since $n(y_0 - y_1) = \lambda\delta_*(x) = 0$, we get $n[g_0] = n[g_1]$ by [18, 8.1.5]. Consequently, there is a homotopy of projections $G \colon [0,1] \to M_n(B \otimes \mathbb{K})$ such that $G(i) = g_i \otimes 1_n$. We can regard G as a rank one projection in $\mathbb{I}_n^{\sim} \otimes B \otimes \mathbb{K}$, and then if we let $z = [G] + (r-1)[1]$, $\delta_*(z) = (y_0, y_1)$, hence $x - z \in \ker \delta_* = \operatorname{im} i_*$. Let $w \in K_0(SM_n \otimes B)$ be such that $i_*(w) = x - z$.

Via the isomorphism $K_0(SM_n \otimes B) \cong K_0(B)$, w corresponds to an element $[p] - s[1]$ where we may think of p as a projection-valued map $p \colon SX \to M_{rn}$, of rank s, where SX denotes the unreduced topological suspension of X. We may, of course, assume that $s \geqslant 2$. By [18, 8.1.2], we can write $[p] = [p'] + (s-2)[1]$ with p' of rank 2. By [17], we may assume that in fact p' maps into the 4×4-matrices, and we may also assume that $p'(pt) = \operatorname{Diag}(1, 1, 0, 0)$. Letting

$$\tilde{p} = \operatorname{Diag}(\overbrace{1, \ldots, 1}^{n-2}, p', \overbrace{0, \ldots, 0}^{n-2}),$$

we get a projection in $M_2((SM_n \otimes B)^{\sim})$ with $\tilde{p}(pt) = \operatorname{Diag}(1_n, 0_n)$, and we may write $w = [\tilde{p}] - n[1]$. Thus, $i(\tilde{p})$ will be a rank one element and

$$\begin{aligned} x &= z + i_*(w) = [G] + (r-1)[1] + [i(\tilde{p})] - [1] \\ &= [G] + [i(\tilde{p})] + (r-2)[1] \geqslant 0. \end{aligned}$$

LEMMA 6.2. *Let A be an ASH algebra of real rank zero and slow dimension growth. Let $x \in K_\bullet(A; \mathbb{Z} \oplus \mathbb{Z}/n)_+$. Suppose that there is $z_0 \in K_0(A)^+$ such that $2z_0 \leqslant x_0 \leqslant mz_0$ for some integer $m \geqslant 0$. Then $x \geqslant 0$.*

Proof. Let such elements z_0 and x be given. By the continuity of $K_\bullet(A; \mathbb{Z}\oplus\mathbb{Z}/n)_+$ (which follows from the continuity of $K_\bullet(A; \mathbb{Z} \oplus \mathbb{Z}/n)^+$ and Proposition 5.2), we may assume that instead z_0 and x are elements of K-groups over A_n, where A_n is a finite direct sum of matrix algebras over algebras in $SH(2)$, cf. [6]. By stability of K-theory, we may assume that every summand is either $\widetilde{\mathbb{I}_k}$, $C(W_k)$ or $C(S^1)$. Let C be such a summand, and denote by y the corresponding K-element. Denoting by w_0 the corresponding component of z_0 we have $2w_0 \leqslant y_0 \leqslant mw_0$. When $w_0 = 0$, we must have $y_0 = 0$ since C is stably finite, and we get that $y = 0$ using the fact that $y \succeq 0$ and the definition of $\succeq$. When $w_0 \neq 0$, rank $y_0 \geqslant 2$, and Lemma 6.1 applies with $X = S^1 \times W_k$.

There is a version of Lemma 6.2 for the dimension-drop-based order, with 2 replaced by a suitable number depending only on n.

LEMMA 6.3. *Let A be a stably unital C^*-algebra, and let $x = (x_0, y) \in K_\bullet(A; \mathbb{Z}\oplus \mathbb{Z}/n)$. Suppose that there are integers $m \geqslant 1$ and k and there is $z_0 \in K_0(A)^+$, such that $x_0 = mz_0$ and $y = k\rho(z_0)$. Then $x \in K_\bullet(A; \mathbb{Z} \oplus \mathbb{Z}/n)^+$.*

Proof. Let e be a projection in the matrices over A, say $M_r(A)$ such that $[e] = x_0$ in $K_0(A)$. Define a *-homomorphism $\eta: \mathbb{C} \to M_r(A)$ by $\eta(1) = e$, We may assume that $0 \leqslant k < n$. Then the map $\eta_*: K_0(\mathbb{C}; \mathbb{Z}\oplus\mathbb{Z}/n) \to K_0(A; \mathbb{Z}\oplus\mathbb{Z}/n)$ takes (m, k) to x. But $(m, k) \geqslant 0$ as it corresponds to a vector bundle over W_n of rank $m \geqslant 1$. This shows that $x \geqslant 0$ since η_* is positive.

An element $x \in K_\bullet(A; \mathbb{Z} \oplus \mathbb{Z}/n)$ is called *finite dimensional* if x is a sum of elements x_i such that each x_i satisfies the hypotheses of Lemma 6.3. A finite-dimensional element is, hence, always positive. It is easily seen that if C is a finite-dimensional C^*-algebra and if $\psi: C \to A$ is a $*$-homomorphism, then all the elements of $\psi_*(K_0(C; \mathbb{Z} \oplus \mathbb{Z}/n)^+) \subset K_\bullet(A; \mathbb{Z} \oplus \mathbb{Z}/n)$ are finite dimensional. The converse is true as seen in the proof of Lemma 6.3.

LEMMA 6.4. *Let A be an ASH algebra of real rank zero and slow dimension growth. For $x \in K_\bullet(A; \mathbb{Z}\oplus\mathbb{Z}/n)^+$ and $k \geqslant 0$, there are $x', x'' \in K_\bullet(A; \mathbb{Z}\oplus\mathbb{Z}/n)^+$ such that $x = x' + x''$, x'' is finite dimensional and $kz_0 \leqslant x'_0 \leqslant mz_0$ for some $z_0 \in K_0(A)^+$ and a positive integer m.*

Proof. According to [6, 7.4], we may (re)write A as an inductive limit of a system (B_r, ξ_r) of finite direct sums of matrix algebras over building blocks such that each partial morphism of ξ_r is either strictly $(k+2)$-large or has finite-dimensional image. Given a positive x, we may assume that it is an element of $K_\bullet(B_r; \mathbb{Z}\oplus\mathbb{Z}/n)^+$. Break x up into summands x_i, and break $y_i = (\xi_r)_* x_i$ up into summands y_{ij} which we shall deal with separately. In the case where $x_i = 0$ or the corresponding partial morphism of ξ_r has finite-dimensional image, we note that y_{ij} is a finite-dimensional element.

We set $x''_{ij} = y_{ij}$ in this case. In the case where the corresponding partial morphism of ξ_r is strictly $(k+2)$-large, we have that $\operatorname{rank}(y_{ij} - k[1]) \geqslant 2$ (where 1 denotes a rank one projection), whence $y_{ij} \geqslant k[1]$ by Lemma 6.1 (with $X = \{pt\}$). We set $x'_{ij} = y_{ij}, z_{ij} = [1]$ in this case. We may also choose m such that $m[1] \geqslant (x'_{ij})_0$ for all i, j. Define x', x'' by adding up the summands found above.

PROPOSITION 6.5. *Let A, B be ASH algebras of real rank zero and slow dimension growth. If $\Phi \in \operatorname{Hom}_\Lambda(\underline{\mathbf{K}}(A), \underline{\mathbf{K}}(B))$, then $\Phi \geqslant 0$ iff $\Phi \succeq 0$.*

Proof. One direction follows from Corollary 5.3 since A and B have stable rank one by [1, 6.5.2]. For the other direction, assume that $\Phi \succeq 0$. Let $x \in K_\bullet(A; \mathbb{Z} \oplus \mathbb{Z}/n)^+$ be given and decompose $x = x' + x''$ as in Lemma 6.4 with $k = 2$. Then $\Phi(x) = \Phi(x') + \Phi(x'')$, where $\Phi(x'') \geqslant 0$ by Lemma 6.3. Since $x' \succeq 0$, we have $\Phi(x') \succeq 0$ by assumption. Moreover, $2\Phi_0(z_0) \leqslant \Phi(x')_0 = \Phi(x'_0) \leqslant m\Phi(z_0)$, hence $\Phi(x') \geqslant 0$ by Lemma 6.2.

PROPOSITION 6.6. *Let a pair of ASH algebras A, B of real rank zero and slow dimension growth, satisfying* (1), *be given. Then Φ is an element of* $\operatorname{Hom}_{\Lambda_\infty}(\mathbf{K}_\infty(A), \mathbf{K}_\infty(B))^+$ *if and only if $E(\Phi)$ is an element of* $\operatorname{Hom}_{\Lambda'}(\underline{\mathbf{K}}'(A), \underline{\mathbf{K}}'(B))^+$.

Proof. Let (φ^i, ψ^i) be a positive element of $\operatorname{Hom}_{\Lambda_\infty}(\mathbf{K}_\infty(A), \mathbf{K}_\infty(B))$. By Lemma 5.4, (φ^i, ψ^i) preserves ideals, whence so does $E(\varphi^i, \psi^i)$ by Lemma 4.3. $E(\varphi^i, \psi^i)$ is then positive by Proposition 6.5.

COROLLARY 6.7. *Let a pair of ASH algebras A, B of real rank zero and slow dimension growth, satisfying* (1), *be given. If Φ is an element of* $\operatorname{Hom}_{\Lambda_\infty}(\mathbf{K}_\infty(A), \mathbf{K}_\infty(B))$, *then $\Phi \geqslant 0$ if and only if $\Phi \succeq 0$.*

Proof. Let (φ^i, ψ^i) be an ideal-preserving in $\operatorname{Hom}_{\Lambda_\infty}(\mathbf{K}_\infty(A), \mathbf{K}_\infty(B))$. By Lemma 4.3, $E(\varphi^i, \psi^i)$ is ideal-preserving, hence positive by Proposition 6.5. Then so is $NR^{-1}E(\varphi^i, \psi^i) = (\varphi^i, \psi^i)$.

The corollary above remains true even if A and B do not satisfy (1). This can be proven along the lines of the proof of Proposition 6.5.

COROLLARY 6.8. *When A, B is a pair of ASH algebras of real rank zero and slow dimension growth, satisfying* (1), *N is an order-preserving isomorphism.*

Proof. We saw in Proposition 3.7 that N is an isomorphism of groups, and it is positive by definition. Since R is an order isomorphism by Proposition 3.2, so is $N^{-1} = R^{-1}E$ by the proposition above.

7. Classification Results

THEOREM 7.1. *The invariant $\mathbf{K}_\infty(-)$ is complete for the class of real rank zero ASH algebras with slow dimension growth and satisfying* (1). *Furthermore, the $*$-isomorphism may be chosen realizing any given order isomorphism at the level of* $\mathbf{K}_\infty(-)$.

Proof. Let A and B be C^*-algebras in this class and let an order isomorphism $\Phi\colon \mathbf{K}_\infty(A) \to \mathbf{K}_\infty(B)$ be given. By Corollary 6.8, $N^{-1}(\Phi)$ is an order isomorphism also, and by [6, 9.1], there is a $*$-isomorphism φ with $\underline{\mathbf{K}}(\varphi) = N^{-1}(\Phi)$. Consequently, $\Phi = N\underline{\mathbf{K}}(\varphi) = \mathbf{K}_\infty(\varphi)$.

In the case of real rank zero AD algebras, we always have $\mathrm{tor} K_0(-) = 0$, and it was proved in [10] that an invariant consisting of

$$\mathbf{K}^0_\infty(-)\colon\ K_0(-) \otimes \mathbb{Q} \xrightarrow{\rho^0_\infty} K_0(-;\mathbb{Q}/\mathbb{Z}) \xrightarrow{\beta^0_\infty} K_1(-)$$

equipped with order on $K_\bullet(-)$ and $K_0(-;\mathbb{Q}\oplus\mathbb{Q}/\mathbb{Z})$ was complete in this case. In fact, the reduction results for local spectra in [6] show that every ASH algebra with $\mathrm{tor} K_0(-) = 0$, slow dimension growth and real rank zero is in fact AD. This we may now bypass and prove directly:

COROLLARY 7.2. *The invariant* $\mathbf{K}^0_\infty(-)$ *is complete for the class of real rank zero* ASH *algebras with slow dimension growth and* $\mathrm{tor} K_0(-) = 0$. *Furthermore, the* $*$-*isomorphism may be chosen realizing any given order isomorphism of the invariant.*

Proof. Condition (1) is clearly satisfied. The map ρ^1_∞ vanishes and the map β^1_∞ is injective. Thus, any morphism Φ' of the above invariant corresponds to a unique morphism Φ of $\mathbf{K}_\infty(-)$. The naturality of β^1_∞ shows that if Φ' is ideal-preserving then Φ is ideal-preserving. So the part of the order structure living on $K_1(-;\mathbb{Q}/\mathbb{Z})$ is irrelevant.

Assume now that A is a simple and unital AD algebra. As in [7], combining the result above with semiprojectivity of the building blocks of the AD algebras yields a short exact sequence

$$1 \longrightarrow \overline{\mathrm{Inn}}(A) \longrightarrow \mathrm{Aut}(A) \longrightarrow \mathrm{Aut}(\mathbf{K}^0_\infty(A))^{1,+} \longrightarrow 1.$$

It is easy to compute $\mathrm{Aut}(\mathbf{K}^0_\infty(A))^{1,+}$ directly; every such automorphism consists of three automorphisms $\varphi^0, \varphi^1, \psi^0$ where φ^0 is an element of $\mathrm{Aut}(K_0(A))^{1,+}$, $\varphi^1 \in K_1(A)$ and $\psi^0 \in K_0(A;\mathbb{Q}/\mathbb{Z})$. By simplicity of A combined with Proposition 6.5, the triple is positive exactly when φ^0 is positive. Furthermore, since the unspliced sequence

$$0 \longrightarrow K_0(A) \otimes \mathbb{Q}/\mathbb{Z} \xrightarrow{\tilde{\rho}} K_0(A;\mathbb{Q}/\mathbb{Z}) \xrightarrow{\tilde{\beta}} \mathrm{tor}(K_1(A)) \longrightarrow 0$$

splits – the leftmost group being divisible – any pair of automorphisms (φ^0, φ^1) can be augmented to a coherent triple by any map ψ of the form

$$\psi = \begin{bmatrix} \varphi^0 \otimes \mathrm{id} & * \\ 0 & \mathrm{tor}(\varphi^1) \end{bmatrix}.$$

Hence, $\mathrm{Aut}(\mathbf{K}^0_\infty)^{1,+}$ is precisely

$$\mathrm{Hom}(\mathrm{tor}(K_1(A)), K_0(A) \otimes \mathbb{Q}/\mathbb{Z}) \rtimes [\mathrm{Aut}(K_0(A))^{1,+} \times \mathrm{Aut}(K_1(A))].$$

Remark 7.3. Comparing this result to [8, 3.4 ff], one sees that

$$\operatorname{ext}(K_1(A), K_0(A)) \cong \operatorname{Hom}(\operatorname{tor}(K_1(A)), K_0(A) \otimes \mathbb{Q}/\mathbb{Z})$$

for A a simple unital AD algebra of real rank zero. This is a purely algebraic phenomenon, as we are grateful to C. U. Jensen for showing us. In fact, when $\operatorname{Tor}(G_1, G_0) = 0$, there is an isomorphism

$$\operatorname{ext}(G_1, G_0) \cong \operatorname{Hom}(\operatorname{tor}(G_1), G_0 \otimes \mathbb{Q}/\mathbb{Z}).$$

Remark 7.4. Fix a prime p, and let $A = \mathbb{I}_p$, $B = C_0(W_p)$. Direct calculation shows that

$$K_0(A) = 0, \quad K_1(A) = \mathbb{Z}/p, \quad K_0(A;\mathbb{Z}/p) = K_1(A;\mathbb{Z}/p) = \mathbb{Z}/p,$$

$$K_0(A;\mathbb{Q}/\mathbb{Z}) = \mathbb{Z}/p, \quad K_1(A;\mathbb{Q}/\mathbb{Z}) = 0,$$

$$K_0(B) = \mathbb{Z}/p, \quad K_1(B) = 0, \quad K_0(B;\mathbb{Z}/p) = K_1(B;\mathbb{Z}/p) = \mathbb{Z}/p,$$

$$K_0(B;\mathbb{Q}/\mathbb{Z}) = 0, \quad K_1(B;\mathbb{Q}/\mathbb{Z}) = \mathbb{Z}/p,$$

whence $\operatorname{Hom}(\mathbf{K}_\infty(A), \mathbf{K}_\infty(B)) = 0$, while $\operatorname{Hom}_\Lambda(\underline{\mathbf{K}}(A), \underline{\mathbf{K}}(B)) \neq 0$. Consequently, N is not injective, and no map E with the property $R = EN$ can exist.

Using this example, one can produce a pair C, D of ASH algebras with real rank zero and slow dimension growth with $\mathbf{K}_\infty(C) \cong \mathbf{K}_\infty(D)$, yet $\underline{\mathbf{K}}(C) \not\cong \underline{\mathbf{K}}(D)$.

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