DEFORMATIONS OF NILPOTENT GROUPS AND HOMOTOPY SYMMETRIC C^*-ALGEBRAS

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ABSTRACT. The homotopy symmetric C^*-algebras are those separable C^*-algebras for which one can unsuspend in E-theory. We find a new simple condition that characterizes homotopy symmetric nuclear C^*-algebras and use it to show that the property of being homotopy symmetric passes to nuclear C^*-subalgebras and it has a number of other significant permanence properties.

As an application, we show that if $I(G)$ is the kernel of the trivial representation $\iota : C^*(G) \to \mathbb{C}$ for a countable discrete torsion free nilpotent group G, then $I(G)$ is homotopy symmetric and hence the Kasparov group $KK(I(G), B)$ can be realized as the homotopy classes of asymptotic morphisms $[[I(G), B \otimes \mathcal{K}]]$ for any separable C^*-algebra B.

1. INTRODUCTION

The intuitive idea of deformations of C^*-algebras was formalized by Connes and Higson who introduced the concept of asymptotic morphism [5]. These morphisms are at the heart of E-theory, the universal bifunctor from the category of separable C^*-algebras to the category abelian groups which is homotopy invariant, C^*-stable and half-exact. Asymptotic morphisms have become important tools in other areas, such as deformation quantization [22], index theory [4], [27], the Baum-Connes conjecture [14], shape theory [8], and classification theory of nuclear C*-algebras [24].

An asymptotic morphism $(\varphi_t)_{t\in[0,\infty)}$ is given by a family of maps $\varphi_t \colon A \to B$ parametrized by $t \in [0, \infty)$ such that $t \mapsto \varphi_t$ is pointwise continuous and the axioms for $*$-homomorphisms are satisfied *asymptotically* for $t \to \infty$. There is a natural notion of homotopy based on asymptotic morphisms of the form $A \to B \otimes C[0, 1]$. Homotopy classes of asymptotic morphisms from the suspension SA of A to the stabilization of SB provide a model for $E(A, B)$, i.e. $E(A, B) = [[SA, SB \otimes \mathcal{K}]]$, [5]. A similar construction using completely positive contractive asymptotic morphisms yields a realization of KK-Theory, namely $KK(A, B) \cong [[SA, SB \otimes \mathcal{K}]]^{\mathrm{cp}}$, as proven by Houghton-Larsen and Thomsen [15]. E-theory factors through KK-theory and the

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fact that the map $KK(A,B) \to E(A,B)$ is an isomorphism for nuclear C^*-algebras A can be easily seen from the Choi-Effros theorem, which implies $[[A,B]] \cong [[A,B]]^{\mathrm{cp}}$.

Note that the introduction of the suspensions and the stabilization of B are necessary to obtain a natural abelian group structure on $E(A,B)$. Equally important, but perhaps less apparent, is the fact that SA becomes quasidiagonal, as shown by Voiculescu [28], and this property implicitly assures a large supply of almost multiplicative maps $SA \to \mathcal{K}$. However, a deformation $A \to B \otimes \mathcal{K}$, without the suspensions, contains in principle more geometric information. Asymptotic morphisms of this form are a crucial tool in the classification theory of nuclear C^*-algebras. We are confronted with the dilemma of understanding $[[A, B \otimes \mathcal{K}]]$, while only $E(A,B) = [[SA, SB \otimes \mathcal{K}]]$ is computable using the tools of algebraic topology. The best case scenario in this situation is of course that the monoid homomorphism $[[A, B \otimes \mathcal{K}]] \to E(A,B)$ induced by the suspension map is an isomorphism.

Under favorable circumstances, this is in fact true: It is shown in [11] and [6] that for a connected compact metrizable space X with basepoint $x_0 \in X$, we have $[[C_0(X \setminus x_0), B \otimes \mathcal{K}]] \cong E(C_0(X \setminus x_0), B)$. In particular, $K_0(X \setminus x_0) \cong [[C_0(X \setminus x_0), \mathcal{K}]]$. On the right hand side we can replace the compact operators $\mathcal{K}$ by $\bigcup_{n=1}^{\infty} M_n(\mathbb{C})$. Thus, the reduced K-homology of X classifies deformations of $C_0(X \setminus x_0)$ into matrices. Similar deformations of commutative C^*-algebras into matrix algebras appeared in condensed matter physics, where Kitaev [18] proposed a classification of topological insulators via (real) K-homology. We refer the reader to the recent work of Loring [20] for further developments and additional references.

A full answer to the question of unsuspending in E-theory was found in [11]: The natural map $[[A, B \otimes \mathcal{K}]] \to E(A,B)$ is an isomorphism for all separable C^*-algebras B if and only if A is *homotopy symmetric*, which means that $[[\mathrm{id}_A]] \in [[A, A \otimes \mathcal{K}]]$ has an additive inverse or equivalently that $[[A, A \otimes \mathcal{K}]]$ is a group.

As nice as this condition is, it can be quite hard to check in practice. One of the two main results in this paper is Theorem 3.1, which shows that being homotopy symmetric is equivalent for separable nuclear C^*-algebras to a property which is significantly easier to verify. The proof of this theorem relies crucially on results of Thomsen [26]. The new property, which we call property (QH), see Definition 2.6 (i), makes sense for all separable C^*-algebras and has the important feature that it passes to C^*-subalgebras. This allows us to exhibit new vast classes of homotopy symmetric C^*-algebras, see Theorem 3.3 and Corollary 3.4.

Our second main result is Theorem 4.3 which states that the augmentation ideal $I(G) = \ker(\iota : C^*(G) \to \mathbb{C})$ for a torsion free countable discrete nilpotent group G satisfies property (QH) and hence it is homotopy symmetric. In particular $[[I(G), B \otimes \mathcal{K}]] \cong KK(I(G), B)$ for any separable C^*-algebra B. This confirms a conjecture of the first author [10] for the class of nilpotent groups.

2. Discrete asymptotic morphisms and Property (QH)

Definition 2.1. Let A, $(B_n)_n$ be separable C^*-algebras. A completely positive contractive (cpc) *discrete asymptotic morphism* from A to $(B_n)_n$ is a sequence of completely positive linear contractions $\{\varphi_n : A \to B_n\}_n$ such that

$$\lim_{n\to\infty} \|\varphi_n(ab) - \varphi_n(a)\varphi_n(b)\| = 0$$

for all $a, b \in A$. If, furthemore, each φ_n is unital we say (φ_n) is a *ucp discrete asymptotic morphism*. Two cpc discrete asymptotic morphisms $\{\varphi_n, \psi_n : A \to B_n\}_n$ are *equivalent*, written $\varphi_n \approx \psi_n$, if $\lim_{n\to\infty} \|\varphi_n(a) - \psi_n(a)\| = 0$ for all $a \in A$. They are *unitarily equivalent*, written $(\varphi_n) \sim (\psi_n)$, if there is a sequence of unitaries $u_n \in B_n$ such that $(\varphi_n) \approx (u_n\psi_n u_n^*)$. They are *homotopy equivalent*, written $(\varphi_n) \sim_h (\psi_n)$, if there is a cpc discrete asymptotic morphism $\{\Phi_n \colon A \to C[0,1] \otimes B_n\}_n$ such that $\Phi_n^{(0)} = \varphi_n$ and $\Phi_n^{(1)} = \psi_n$ for all $n \geq 1$. Equivalent discrete asymptotic morphisms are homotopic via the homotopy $\Phi_n^{(t)} = (1-t)\varphi_n + t\psi_n$. Homotopy of ucp asymptotic morphisms is defined similarly, by requiring that $\Phi_n(1) = 1$. The homotopy classes of cpc discrete asymptotic morphisms $\varphi_n \colon A \to B$ from A to a fixed $B = B_n$ will be denoted by $[[A, B]]_{\mathbb{N}}^{\mathrm{cp}}$. (Non-discrete) *cpc asymptotic morphisms* are defined completely analogously by replacing the indexing set $\mathbb{N}$ by $[0, \infty)$ and considering cpc maps $\varphi \colon A \to C_b([0,\infty), B)$. The homotopy classes of cpc asymptotic morphisms are denoted by $[[A, B]]^{\mathrm{cp}}$. Similarly we denote by $[[A, B]]$ (respectively $[[A, B]]_{\mathbb{N}}$ in the discrete case) the homotopy classes of general asymptotic morphisms, [5], [26].

Remark 2.2. If A and B are separable C^*-algebras, then every cpc discrete asymptotic morphism $\{\varphi_n \colon A \to B \otimes \mathcal{K}\}_n$ is equivalent to some $\{\psi_n \colon A \to M_{m(n)}(B)\}_n$ obtained by compressing φ_n by a suitable sequence of projections $1_{M(B)} \otimes p_n \in M(B) \otimes \mathcal{K}$, where $p_n \in \mathcal{K}$ converges strongly to 1.

Remark 2.3. There is a category *Asym* with objects separable C^*-algebras and with homotopy classes of (non-discrete) asymptotic morphisms as morphisms. In particular, there is a composition of asymptotic morphisms, which is well-defined up to homotopy [5]. There is a similar category $Asym^{\mathrm{cp}}$ based on cpc asymptotic morphisms. There are no analogues of these categories for homotopy classes of discrete asymptotic morphisms. Nevertheless, it was shown in [26, Thm. 7.2] that there is a well-defined pairing of the form $[[A, B]]_{\mathbb{N}} \times [[B, C]] \to [[A, C]]_{\mathbb{N}}$, $(x, y) \mapsto y \circ x$ such that $z \circ (y \circ x) = (z \circ y) \circ x$ for $z \in [[C, D]]$. The arguments of [26, Thm. 7.2] show that there is also a pairing $[[A, B]]_{\mathbb{N}}^{\mathrm{cp}} \times [[B, C]]^{\mathrm{cp}} \to [[A, C]]_{\mathbb{N}}^{\mathrm{cp}}$ with similar properties.

Any discrete asymptotic morphism $\{\varphi_n : A \to B_n\}_n$ induces a $*$-homomorphism $\mathbf{\Phi} : A \to \prod_n B_n / \bigoplus_n B_n$.

Definition 2.4. A discrete asymptotic morphism $\{\varphi_n : A \to B_n\}_n$ is called *injective* if the induced $*$-homomorphism $\mathbf{\Phi}$ is injective, or equivalently if $\limsup_n \|\varphi_n(a)\| = \|a\|$ for all $a \in A$.

The cone over a C^*-algebra B is defined as $CB = C_0(0,1] \otimes B$.

Proposition 2.5. *For a separable C^*-algebra A the following properties are equivalent.*

(i) *There is a null-homotopic* injective *cpc discrete asymptotic morphism $\{\eta_n \colon A \to \mathcal{K}\}_n$.*
(ii) *There is a null-homotopic* injective *cpc discrete asymptotic morphism $\{\gamma_n \colon A \to L(H)\}_n$.*
(iii) *There is an injective $*$-homomorphism $\boldsymbol{\eta} : A \to \prod_n C\mathcal{K} / \bigoplus_n C\mathcal{K}$ which is liftable to a cpc map $\eta : A \to \prod_n C\mathcal{K}$.*
(iv) *There is an injective $*$-homomorphism $\boldsymbol{\gamma} : A \to \prod_n CL(H)/\bigoplus_n CL(H)$ which is liftable to a cpc map $\gamma : A \to \prod_n CL(H)$.*

Proof. (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (iv) are obvious since $\mathcal{K} \subset L(H)$. The implications (i) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (iv) are are also straightforward.

(iv) $\Rightarrow$ (i) Let γ be as in (iv) with components γ_n. Since $\boldsymbol{\gamma}$ is injective, we must have $\limsup \|\gamma_n(a)\| = \|a\|$ for all $a \in A$. Let $(a_i)_i$ be a dense sequence in A. For each i there is a strictly increasing sequence $(m(i,n))_n$ such that $\|\gamma_{m(i,n)}(a_i)\| \geq \|a_i\| - 1/n$, for all $n \geq 1$. If we define $\gamma'_n = \gamma_{m(1,n)} \oplus \gamma_{m(2,n)} \oplus \cdots \oplus \gamma_{m(n,n)}$, then $\lim \|\gamma'_n(a)\| = \|a\|$ for all $a \in A$. Therefore we may assume that (γ_n) had this property in the first place.

Since A is separable, there is a separable C^*-algebra $D \subset L(H)$ such that $\gamma_n(A) \subset D$ for all $n \geq 1$. Consider an injective $*$-homomorphism $j : C_0(0,1] \to C_0(0,1] \otimes C_0(0,1]$, for example $j(f)(s,t) = f(\min(s,t))$. Since $C_0(0,1] \otimes D$ is quasidiagonal by [28], there is a cpc asymptotic morphism $\{\theta_n : C_0(0,1] \otimes D \to \mathcal{K}\}_n$ with $\lim_n \|\theta_n(b)\| = \|b\|$ for all b. After passing to a subsequence $(\theta_n)_n$ we may arrange that the maps η_n obtained as the compositions

$$A \xrightarrow{\gamma_n} C_0(0,1] \otimes D \xrightarrow{j \otimes \mathrm{id}_D} C_0(0,1] \otimes C_0(0,1] \otimes D \xrightarrow{\mathrm{id} \otimes \theta_n} C_0(0,1] \otimes \mathcal{K}$$

define an asymptotic morphism $\{\eta_n : A \to C_0(0,1] \otimes \mathcal{K}\}_n$ such that $\lim_n \|\eta_n(a)\| = \|a\|$ for all $a \in A$.

We regard $\eta_n : A \to C\mathcal{K}$ as a continuous family of maps $\{\eta_n^{(t)} : A \to \mathcal{K}\}_{t \in [0,1]}$ with $\eta_n^{(0)} = 0$. Next we want to arrange that $\lim_n \|\eta_n^{(1)}(a)\| = \|a\|$ for all $a \in A$. Since $\lim_n \|\eta_n(a)\| = \|a\|$ for all $a \in A$, after passing to a subsequence of $(\eta_n)_n$, we may arrange for each $n \geq 1$, $\|\eta_n(a_i)\| > \|a_i\| - 1/i$ for all $1 \leq i \leq n$. Therefore for every n and i with $1 \leq i \leq n$, there is $t_{n,i} \in [0,1]$ such that $\|\eta_n^{(t_{n,i})}(a_i)\| > \|a_i\| - 1/i$. Define $E_n : A \to C\mathcal{K}$ by

$E_n^{(t)} = \bigoplus_{i=1}^n \eta_n^{(tt_{n,i})}$ for $t \in [0,1]$. It follows immediately that $\{E_n : A \to C\mathcal{K}\}_n$ is a cpc asymptotic morphism such that $\lim_n \|E_n^{(1)}(a)\| = \|a\|$ for all $a \in A$. □

Definition 2.6. (i) A separable C^*-algebra A has *property (QH)* if it satisfies one of the equivalent conditions from Proposition 2.5.

(ii) For a countable discrete group G, denote the character induced by the trivial representation by $\iota\colon C^*(G) \to \mathbb{C}$ and set $I(G) = \ker(\iota)$. We say that G has *property (QH)* if $I(G)$ has property (QH). One can reformulate this condition as follows. There is a discrete injective ucp asymptotic morphism $\{\pi_n : C^*(G) \to M_{m(n)}(\mathbb{C})\}_n$ which is homotopic to $\{\iota_n : C^*(G) \to M_{m(n)}(\mathbb{C})\}_n$, where each ι_n is the multiple $m(n) \cdot \iota$ of the trivial representation ι.

Example 2.7. If X is a compact connected metrizable space and $x_0 \in X$, then $C_0(X \setminus x_0)$ has property (QH). In particular, if G is a torsion free countable discrete abelian group, then $I(G) \cong C_0(\widehat{G} \setminus \iota)$ has property (QH).

Remark 2.8. Let us note that if A has property (QH), then A must be quasidiagonal by a result of Voiculescu [28] and A cannot have any nonzero projections. In particular, if a discrete group G has property (QH), then G must be torsion free, see Remark 4.4. It is easily verified that

(i) Property (QH) passes to C^*-subalgebras.
(ii) If a separable C^*-algebra A has property (QH) then so does $A \otimes_{min} B$ for any separable C^*-algebra B.

Other permanence properties are proved in Theorem 3.3.

The following Lemma is crucial for the proof that property (QH) implies the existence of additive inverses of homotopy classes of discrete asymptotic morphisms. It is a slight generalization of [9, Lem. 5.3]. The proof is almost identical, except that one has to replace Voiculescu's theorem and Stinespring's theorem by Kasparov's generalization of these [17]. We will use the notation from [9, Sec. 5]: If A and B are C^*-algebras, E, F are right Hilbert B-modules, $\mathcal{F} \subset A$ is a finite set, $\epsilon > 0$ and $\varphi\colon A \to \mathcal{L}_B(E)$ and $\psi\colon A \to \mathcal{L}_B(F)$ are two maps, we write $\varphi \prec_{\mathcal{F},\epsilon} \psi$ if there is an isometry $v \in \mathcal{L}_B(E,F)$ such that $\|\varphi(a) - v^*\psi(a)v\| < \epsilon$ for all $a \in \mathcal{F}$. If v can be chosen to be a unitary, we write $\varphi \sim_{\mathcal{F},\epsilon} \psi$. Moreover, we write $\varphi \prec \psi$ if $\varphi \prec_{\mathcal{F},\epsilon} \psi$ for all finite sets $\mathcal{F}$ and for all $\epsilon > 0$.

Lemma 2.9. *Let A and B be separable unital C^*-algebras such that A or B is nuclear. Let $\{\varphi_n\colon A \to M_{k(n)}(B)\}_n$ and $\{\gamma_n\colon A \to M_{r(n)}(\mathbb{C})\}_n$ be ucp discrete asymptotic morphisms. Suppose that (γ_n) is injective. Then there exist a sequence $(\omega(n))$ of disjoint finite subsets of $\mathbb{N}$ with $\max\omega(n-1) < \min\omega(n)$ and a ucp discrete asymptotic morphism $\{\varphi_n'\colon A \to M_{s(n)}(B)\}_n$ such that $(\varphi_n \oplus \varphi_n') \sim (\gamma_{\omega(n)} \otimes 1_B)$, where $\gamma_{\omega(n)} = \oplus_{i\in\omega(n)}\gamma_i$.*

Proof. There exists a sequence $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \dots$ of finite selfadjoint subsets of A consisting of unitaries such that their union is dense in $U(A)$ and a sequence $\epsilon_1 \geq \epsilon_2 \geq \dots$ convergent to zero such that φ_n is $(\mathcal{F}_n, \epsilon_n)$-multiplicative. By [9, Lem. 5.1] it suffices to construct a sequence $(\omega(n))$ such that $\varphi_n \prec_{\mathcal{F}_n,\epsilon_n} \gamma_{\omega(n)}$.

We will construct $(\omega(n))$ inductively. Suppose that we already have $\omega(1), \dots, \omega(n-1)$ and choose $m > \max \omega(n-1)$ such that

$$\Gamma = \oplus_{i \geq m} \gamma_i \colon A \to \prod_{i \geq m} M_{r(i)}(\mathbb{C}) \subset \mathcal{L}(H)$$

is $(\mathcal{F}_n, \frac{\epsilon_n^2}{4})$-multiplicative. Observe that $\Gamma\colon A \to \mathcal{L}(H)$ is a unital $*$-monomorphism modulo the compact operators. Let $\pi\colon A \to \mathcal{L}(H)$ be a faithful unital representation with $\pi(A) \cap \mathcal{K}(H) = \{0\}$, then we have $\Gamma \sim_{\mathcal{F}_n,\epsilon_n} \pi$ by [3, Lem. 2.1] (or [9, Lem. 5.2]). Note that there is a natural embedding $\mathcal{L}(H) \to \mathcal{L}_B(H_B)$, which sends T to $T \otimes 1_B$. Moreover, $\Gamma \otimes 1_B \sim_{\mathcal{F}_n,\epsilon_n} \pi \otimes 1_B$. Let π_n be the Stinespring dilation of φ_n obtained via [17, Thm. 3]. We have that $\pi_n \oplus (\pi \otimes 1_B) \sim (\pi \otimes 1_B)$ by [17, Thm. 6]. Thus,

$$\varphi_n \prec \pi_n \prec \pi_n \oplus (\pi \otimes 1_B) \sim (\pi \otimes 1_B) \sim_{\mathcal{F}_n,\epsilon_n} \Gamma \otimes 1_B \ ,$$

hence $\varphi_n \prec_{\mathcal{F}_n,\epsilon_n} \Gamma \otimes 1_B$. Let $v \in \mathcal{L}_B(B^{k(n)}, H_B)$ be a partial isometry such that $\|\varphi_n(a) - v^*(\Gamma \otimes 1_B)(a)v\| < \epsilon_n$ for all $a \in \mathcal{F}_n$. The projections of $M_\infty(B)$ are dense in those of $B \otimes \mathcal{K}(H)$. Thus, we can find an isometry $w \in \mathcal{L}_B(B^{k(n)}, B^N)$ which approximates v in norm sufficiently well so that $\|\varphi_n(a) - w^*(\Gamma \otimes 1_B)(a)w\| < \epsilon_n$ for all $a \in \mathcal{F}_n$. Here we identify $B^N \subset H_B$ as the submodule of the first N coordinates. It follows that if we let $\omega(n) = \{m, m+1, \dots, m+N\}$, then $\varphi_n \prec_{\mathcal{F}_n,\epsilon_n} \gamma_{\omega(n)}$. □

Lemma 2.10. *Let $\theta : M \to G$ be a surjective morphism of monoids. If G is a group and all the elements of $\theta^{-1}(1_G)$ are invertible in M, then M is a group.*

Proof. For $x \in M$, choose $x' \in M$ such that $\theta(x') = \theta(x)^{-1}$. Then $\theta(xx') = \theta(x'x) = 1_G$. It follows that both xx' and $x'x$ are invertible and hence there are $y_1, y_2 \in M$ such that $x(x'y_1) = 1_M = (y_2x')x$. Thus x is invertible. □

Proposition 2.11. *Let A, B be separable C^*-algebras such that either A or B is nuclear. If A has property (QH), then $[[A, B \otimes \mathcal{K}]]_{\mathbb{N}}^{\mathrm{cp}}$ is a group.*

Proof. We will first prove the proposition for unital C^*-algebras B. Let $\{\varphi_n\colon A \to M_{r(n)}(B)\}_n$ be a discrete cpc asymptotic morphism. By Remark 2.2 it suffices to construct an additive inverse of $[[\varphi_n]]$. Since A has property (QH), there exists an injective cpc discrete asymptotic morphism $\{\eta_n\colon A \to M_{s(n)}(\mathbb{C})\}_n$ which is null-homotopic.

Let $\widetilde{A}$ denote the unitization of A and set $R(n) = r(n) + 1$. Let $\widetilde{\varphi}_n\colon \widetilde{A} \to M_{R(n)}(B)$ be the unital extension of φ_n, so that $\widetilde{\varphi}_n(1) = 1_{M_{R(n)}} \otimes 1_B$. This is a ucp asymptotic morphism. Likewise, let $\widetilde{\eta}_n\colon \widetilde{A} \to M_{S(n)}(\mathbb{C})$ for

$S(n) = s(n) + 1$ be the unitization of η_n. Note that $(\widetilde{\eta}_n)$ is still injective. From Lemma 2.9 we obtain sequences $(\omega(n))$ and $\{\widetilde{\varphi}'_n\colon A \to M_{T(n)}(B)\}_n$ such that $(\widetilde{\varphi}_n \oplus \widetilde{\varphi}'_n) \sim (\widetilde{\eta}_{\omega(n)} \otimes 1_B)$. Let $j\colon A \to \widetilde{A}$ be the inclusion map and set $\varphi'_n = \widetilde{\varphi}'_n \circ j$. Then $(\varphi_n \oplus \varphi'_n) \sim (\eta_{\omega(n)} \otimes 1_B)$ which is null-homotopic.

Now consider the case when B is nonunital. Observe that for any short exact sequence of separable C^*-algebras

$$0 \to I \to B \xrightarrow{p} D \to 0 \tag{1}$$

and an arbitrary separable C^*-algebra A there is a corresponding long exact Puppe sequence of pointed sets:

$$[[A, SB]]^{\mathrm{cp}}_{\mathbb{N}} \longrightarrow [[A, SD]]^{\mathrm{cp}}_{\mathbb{N}} \longrightarrow [[A, C_p]]^{\mathrm{cp}}_{\mathbb{N}} \longrightarrow [[A, B]]^{\mathrm{cp}}_{\mathbb{N}} \longrightarrow [[A, D]]^{\mathrm{cp}}_{\mathbb{N}} \tag{2}$$

where $C_p = \{(b, f) \in B \oplus C_0([0,1), D) \mid f(0) = p(b)\}$ is the mapping cone of the $*$-homomorphism p. The proof of exactness is entirely similar to the proof for the Puppe sequence in E-theory, see [7, Prop. 6]. Indeed as shown in the proof of [25, Thm. 3.8], the mapping cone of the map $C_p \to B$, $(b, f) \mapsto b$ is homotopic as a C^*-algebra to SD.

If the short exact sequence (1) splits, the first and the last map in (2) are surjective and we obtain the short exact sequence

$$0 \longrightarrow [[A, C_p]]^{\mathrm{cp}}_{\mathbb{N}} \longrightarrow [[A, B]]^{\mathrm{cp}}_{\mathbb{N}} \longrightarrow [[A, D]]^{\mathrm{cp}}_{\mathbb{N}} \longrightarrow 0.$$

Moreover, it was proven in [11, Prop. 3.2] that if the short exact sequence (1) splits, the canonical homomorphism $I \to C_p$ has an inverse in the category *Asym* of separable C^*-algebras and homotopy classes of (non-discrete) asymptotic morphisms. On the other hand [26, Thm. 7.2] shows that an isomorphism $B_1 \cong B_2$ in the category *Asym* induces an isomorphism $[[A, B_1]]_{\mathbb{N}} \cong [[A, B_2]]_{\mathbb{N}}$ and hence $[[A, B_1]]^{\mathrm{cp}}_{\mathbb{N}} \cong [[A, B_2]]^{\mathrm{cp}}_{\mathbb{N}}$ if A is nuclear.

If B is nuclear, then so are D, I and C_p and hence I is isomorphic to C_p in the category $Asym^{\mathrm{cp}}$. By using the version of [26, Thm. 7.2] for cpc asymptotic morphisms that was mentioned in Remark 2.3, we see that $[[A, C_p]]^{\mathrm{cp}}_{\mathbb{N}} \cong [[A, I]]^{\mathrm{cp}}_{\mathbb{N}}$ if B is nuclear. Therefore if either A or B is nuclear, the following is a short sequence of pointed sets:

$$0 \longrightarrow [[A, I]]^{\mathrm{cp}}_{\mathbb{N}} \longrightarrow [[A, B]]^{\mathrm{cp}}_{\mathbb{N}} \longrightarrow [[A, D]]^{\mathrm{cp}}_{\mathbb{N}} \longrightarrow 0. \tag{3}$$

In the case of the split extension $0 \to B \otimes \mathcal{K} \to \widetilde{B} \otimes \mathcal{K} \to \mathcal{K} \to 0$, we obtain from (3) the following short exact sequence of pointed sets

$$0 \longrightarrow [[A, B \otimes \mathcal{K}]]^{\mathrm{cp}}_{\mathbb{N}} \xrightarrow{\alpha} [[A, \widetilde{B} \otimes \mathcal{K}]]^{\mathrm{cp}}_{\mathbb{N}} \xrightarrow{\beta} [[A, \mathcal{K}]]^{\mathrm{cp}}_{\mathbb{N}} \longrightarrow 0.$$

Note that $[[A, B \otimes \mathcal{K}]]^{\mathrm{cp}}_{\mathbb{N}}$ is an abelian monoid, and by the first part of the proof, $[[A, \widetilde{B} \otimes \mathcal{K}]]^{\mathrm{cp}}_{\mathbb{N}}$ and $[[A, \mathcal{K}]]^{\mathrm{cp}}_{\mathbb{N}}$ are abelian groups. All monoids are pointed by their respective neutral elements, the map α is a monoid homomorphism and β is a group homomorphism. Exactness implies that the sequence $0 \to [[A, B\otimes\mathcal{K}]]_{\mathbb{N}}^{\mathrm{cp}} \to \ker(\beta) \to 0$ is also exact. By Lemma 2.10 we conclude that $[[A, B \otimes \mathcal{K}]]_{\mathbb{N}}^{\mathrm{cp}}$ is an abelian group. □

Proposition 2.12. *Let A be a separable C^*-algebra.*
(i) A has property (QH) if and only if A is quasidiagonal and $[[A, \mathcal{K}]]_{\mathbb{N}}^{\mathrm{cp}}$ is a group.
(ii) If A is nuclear, then A has property (QH) if and only if $[[A, A\otimes\mathcal{K}]]_{\mathbb{N}}$ is a group.

Proof. (i) If A has property (QH), then A is quasidiagonal by [28]. Proposition 2.11 implies that $[[A, \mathcal{K}]]_{\mathbb{N}}^{\mathrm{cp}}$ is a group. For the other direction, since A is quasidiagonal, there is an injective cpc discrete asymptotic morphism $\{\varphi_n : A \to \mathcal{K}\}_n$. Let (φ_n') be such that $[[\varphi_n']] = -[[\varphi_n]]$ in $[[A, \mathcal{K}]]_{\mathbb{N}}^{\mathrm{cp}}$. Then $(\eta_n) = (\varphi_n \oplus \varphi_n')$ is injective and null-homotopic.

(ii) If A is nuclear, then $[[A, B]]^{\mathrm{cp}} \cong [[A, B]]$ for any separable C^*-algebra B. Suppose that A has property (QH). Proposition 2.11 implies that $[[A, A\otimes\mathcal{K}]]_{\mathbb{N}}$ is a group.

For the other direction note that since $[[A, B\otimes\mathcal{K}]]_{\mathbb{N}} \cong [[A\otimes\mathcal{K}, B\otimes\mathcal{K}]]_{\mathbb{N}}$ we may assume that $A \cong A\otimes\mathcal{K}$. Since $[[\mathrm{id}_A]]$ has an additive inverse in $[[A, A]]_{\mathbb{N}}$ it follows that there is an injective cpc discrete asymptotic morphism $\{\varphi_n : A \to A\}_n$ which is null-homotopic. By composing φ_n with a representation of A we find an injective cpc discrete asymptotic morphism $\{\theta_n : A \to L(H)\}_n$ which is null-homotopic. We conclude by applying Proposition 2.5. □

3. Nuclear homotopy symmetric C^*-algebras

Theorem 3.1. *Let A be a separable, nuclear C^*-algebra. Then A has property (QH) if and only if A is homotopy symmetric. In either case, $[[A, B\otimes\mathcal{K}]] \cong E(A, B) \cong KK(A, B)$ for any separable C^*-algebra B.*

Proof. Suppose first that A has property (QH). By Lemma 5.6 of [26] for any separable C^*-algebra B there is an exact sequence of pointed sets

$$[[A, SB\otimes\mathcal{K}]]_{\mathbb{N}} \xrightarrow{\alpha} [[A, B\otimes\mathcal{K}]] \xrightarrow{\beta} [[A, B\otimes\mathcal{K}]]_{\mathbb{N}} \xrightarrow{1-\sigma} [[A, B\otimes\mathcal{K}]]_{\mathbb{N}}.$$

Here σ is the shift map $\sigma[[\psi_n]] = [[\psi_{n+1}]]$, β is the natural restriction map and α is defined by stringing together the components of a discrete asymptotic morphism $\{\varphi_n : A \to C_0(0,1)\otimes B\otimes\mathcal{K}\}_n$ to form a continuous asymptotic morphism $\{\Phi_t : A \to B\otimes\mathcal{K}\}_{t\in[0,\infty)}$, where $\Phi_t(a) = \varphi_n(a)(t-n)$ for $t \in [n, n+1]$. Recall that if the addition operation is defined via direct sums, then $[[A, B\otimes\mathcal{K}]]$ is an abelian monoid and all the other entries are abelian groups by Proposition 2.11. It follows that $(1-\sigma)$ is a morphism of groups and both α and β are monoid homomorphisms. By Lemma 2.10, the exact sequence $[[A, SB\otimes\mathcal{K}]]_{\mathbb{N}} \to [[A, B\otimes\mathcal{K}]] \to \ker(1-\sigma) \to 0$ implies that $[[A, B\otimes\mathcal{K}]]$ is a group. In particular, taking $B = A$ we see that A is homotopy symmetric.

Conversely, suppose that A is homotopy symmetric. Then $[[A\otimes\mathcal{K}, A\otimes\mathcal{K}]]$ is a group. The product $[[A, A\otimes\mathcal{K}]]_{\mathbb{N}} \times [[A\otimes\mathcal{K}, A\otimes\mathcal{K}]] \to [[A, A\otimes\mathcal{K}]]_{\mathbb{N}}$, $(x, y) \mapsto y\circ x$ has the property that $(y_1+y_2)\circ x = y_1\circ x + y_2\circ x$. By applying this property with $y_1 = [[\mathrm{id}_{A\otimes\mathcal{K}}]]$ and $y_2 = -y_1$ we obtain that $[[A, A\otimes\mathcal{K}]]_{\mathbb{N}}$ is a group. It follows that A has property (QH) by Proposition 2.12(ii).

For the last part of the statement we apply the main result of [11]. □

Remark 3.2. Let M be a homotopy associative and homotopy commutative H-space and let X be a topological space with the homotopy type of a CW-complex. By a slight extension of [29, Thm. X.2.4] the homotopy classes of continuous maps $[X, M]$ form a group if and only if $\pi_0(M) = [pt, M]$ is a group. The combination of Proposition 2.12 and Theorem 3.1 provides a counterpart for discrete asymptotic morphisms of this statement: Let A be a nuclear quasidiagonal C^*-algebra. The monoid $[[A, B\otimes\mathcal{K}]]$ is a group for any separable C^*-algebra B if and only if $[[A,\mathcal{K}]] = [[A, C(pt)\otimes\mathcal{K}]]$ is a group, if and only if $[[A,\mathcal{K}]]_{\mathbb{N}}$ is a group.

The importance of Theorem 3.1 comes from the fact that property (QH) is much easier to verify than the property of being homotopy symmetric. It allows us to vastly extend the class of known homotopy symmetric C*-algebras.

Theorem 3.3. *The class of homotopy symmetric C^*-algebras has the following permanence properties:*

(a) *A nuclear C^*-subalgebra of a separable C^*-algebra with property (QH) is homotopy symmetric.*
(b) *Let $(A_n)_n$ be a sequence of separable C^*-algebras with property (QH). Any separable nuclear C^*-subalgebra of $\prod_n A_n/\bigoplus_n A_n$ is homotopy symmetric.*
(c) *The class of separable nuclear homotopy symmetric C^*-algebras is closed under inductive limits.*
(d) *If $0\to I\to A\to B\to 0$ is a split short exact sequence of separable nuclear C^*-algebras and two of the entries are homotopy symmetric, then so is the third.*
(e) *The class of homotopy symmetric C^*-algebras is closed under tensor products by separable C^*-algebras and under (asymptotic) homotopy equivalence.*
(f) *The class of separable nuclear homotopy symmetric C^*-algebras is closed under crossed products by second countable compact groups.*

Proof. Since property (QH) passes obviously to C^*-subalgebras, statement (a) follows from Theorem 3.1.

Let A be a C^*-subalgebra of $\prod_n A_n/\bigoplus_n A_n$ as in statement (b). Using the Choi-Effros lifting theorem, we find a cpc discrete asymptotic morphism $\{\theta_n : A\to A_n\}_n$ such that $\limsup_n \|\theta_n(a)\| = \|a\|$ for all $a\in A$. Since A is separable, by replacing θ_n by finite direct sums of the form $\theta_n\oplus\theta_{n+1}\oplus\cdots\oplus\theta_N$

and A_n by $A_n \oplus A_{n+1} \oplus \cdots \oplus A_N$ we may assume that $\lim_n \|\theta_n(a)\| = \|a\|$ for all $a \in A$. Here we use the observation that the class of C^*-algebras with property (QH) is closed under finite direct sums. Let $(\Phi_n^{A_i})_n$ be the homotopy of discrete asymptotic morphisms given by Proposition 2.5(i) for A_i. Since A is separable, one can find an increasing sequence $m(n)$ of natural numbers such that $\Phi_n := \Phi_{m(n)}^{A_n} \circ \theta_n$ satisfies the condition (i) of Proposition 2.5 for A in the sense that (Φ_n) is a homotopy between an injective cpc discrete asymptotic morphism and the null map. We conclude the proof of (b) by applying Theorem 3.1.

Statement (c) follows from (b) since any inductive limit $\varinjlim A_n$ embeds as a C^*-subalgebra of $\prod_n A_n / \bigoplus_n A_n$.

For the proof of (d) first note that the cases of nuclear subalgebras and quotients follow from (a), since we assumed that the sequence splits. It remains to be proven that A is homotopy symmetric if I and B are. If B is homotopy symmetric, then $[[B \otimes \mathcal{K}, B \otimes \mathcal{K}]]$ is a group. There is an element $y \in [[B \otimes \mathcal{K}, B \otimes \mathcal{K}]]$ such that $[[\mathrm{id}_{B\otimes\mathcal{K}}]] + y = 0$. Therefore $[[A, B \otimes \mathcal{K}]]$ is a group as well, since $y \circ x$ is an additive inverse of $x \in [[A, B \otimes \mathcal{K}]]$. Likewise $[[A, I \otimes \mathcal{K}]]$ is a group if I is homotopy symmetric. By [11, Prop.3.2] we have a short exact sequence of monoids

$$0 \to [[A, I \otimes \mathcal{K}]] \to [[A, A \otimes \mathcal{K}]] \to [[A, B \otimes \mathcal{K}]] \to 0$$

and Lemma 2.10 implies that $[[A, A \otimes \mathcal{K}]]$ is a group. Thus, A is homotopy symmetric.

The statement (e) is an immediate consequence of the definition as noted in [11].

For the proof of (f) let A be a separable C^*-algebra and let G be a second countable compact group that acts on A by automorphisms. Then

$$A \rtimes G \subset (A \otimes C(G)) \rtimes G \cong A \otimes \mathcal{K}(L^2(G))$$

by [13, Cor. 2.9]. We conclude the proof by applying (a). □

The following corollary exhibits a new large class of homotopy symmetric C^*-algebras.

Corollary 3.4. *Let A be a separable continuous field of nuclear C^*-algebras over a compact connected metrizable space X. If one of the fibers of A is homotopy symmetric, then A is homotopy symmetric.*

Proof. A has nuclear fibers and hence it is a nuclear C^*-algebra. A embeds in $C(X) \otimes \mathcal{O}_2$ by [2]. Fix $x_0 \in X$ such that $A(x_0)$ is homotopy symmetric. Furthermore, since the embedding is $C(X)$-linear, it follows that A embeds in $E = \{f \in C(X) \otimes \mathcal{O}_2 : f(x_0) \in D\}$, where $D \subset \mathcal{O}_2$ is a C^*-subalgebra isomorphic to $A(x_0)$. Thus, E fits into a short exact sequence

$$0 \longrightarrow C_0(X \setminus \{x_0\}) \otimes \mathcal{O}_2 \longrightarrow E \xrightarrow{ev_{x_0}} D \longrightarrow 0$$

This sequence splits via the $*$-homomorphism $D \to E$ that maps d to the constant function $f(x) = d$ on X. Since both $C_0(X \setminus \{x_0\}) \otimes \mathcal{O}_2$ and D are homotopy symmetric, the statement now follows from Theorem 3.3 (d). □

Remark 3.5. Statement (c) in Theorem 3.3 strengthens the main result of [6] in the case of nuclear C^*-algebras.

4. Group C^*-algebras

In this section we prove that any countable torsion free nilpotent group has property (QH).

Theorem 4.1. *Let $1 \to N \to G \to H \to 1$ be a central extension of discrete countable amenable groups where N is torsion free. If H has property (QH) then so does G.*

Proof. By [21, Thm. 1.2], (see also [12, Lemma 6.3]), $C^*(G)$ is a nuclear continuous field of C^*-algebras over the spectrum $\widehat{N}$ of $C^*(N)$. Moreover, the fiber over the trivial character ι of N is isomorphic to $C^*(H)$. It follows that $I(G)$ is a nuclear continuous field of C^*-algebras over the spectrum $\widehat{N}$ whose fiber at ι is isomorphic to $I(H)$. Since N is torsion free, its Pontriagin dual is connected. We conclude the proof by applying Cor. 3.4. □

Lemma 4.2. *Suppose that a countable discrete amenable group G is the union of an increasing sequence of subgroups $(G_i)_i$ each of which has property (QH). Then G has property (QH).*

Proof. Since G is amenable, so is each G_i and the associated group C^*-algebras are nuclear. We may regard $C^*(G_i)$ as a C^*-subalgebra of $C^*(G)$. Then the union of $C^*(G_i)$ is dense in $C^*(G)$ and hence the union of $I(G_i)$ is dense in $I(G)$. The conclusion follows from Thm. 3.3 (c). □

Theorem 4.3. *If G is a countable torsion free nilpotent group, then $I(G)$ is a homotopy symmetric C^*-algebra.*

Proof. Subgroups of nilpotent groups are nilpotent. Consequently, we may assume by Lemma 4.2 that G is finitely generated. Since G is nilpotent it has a finite upper central series $(Z_i)_{i=0}^n$ consisting of subgroups

$$\{1\} = Z_0 \subset Z_1 \subset \cdots \subset Z_{n-1} \subset Z_n = G, \tag{4}$$

where Z_1 is the center of G and for $i \geq 1$, Z_{i+1} is the unique subgroup of G such that Z_{i+1}/Z_i is the center of G/Z_i. We argue by induction on the length n of the central series of G. Suppose that property (QH) holds for all finitely generated nilpotent groups with upper central series of length $n-1$. If G is a finitely generated, torsion free and satisfies (4), then G/Z_1 is finitely generated, torsion free and nilpotent by [16, Cor. 1.3] and $(Z_{i+1}/Z_1)_{i=0}^{n-1}$ is a central series of length $n-1$ for G/Z_1. Since the upper central series is the shortest central series [23, 5.1.9], we conclude the proof by applying Theorem 4.1 to the central extension $1 \to Z_1 \to G \to G/Z_1 \to 1$. □

Remark 4.4. (i) The assumption that G is torsion free is essential. Indeed if $s \in G$ is an element of order $n > 1$, then $1 - \frac{1}{n}(1 + s + \cdots + s^{n-1})$ is a nonzero projection contained in $I(G)$.

(ii) If $\pi : \mathbb{H}_3 \to U(n)$ is a representation of the Heisenberg group whose restriction to the center is non-trivial, then there is no continuous path of representations connecting π to a multiple of the trivial representation, [1].

(iii) The K-homology of nilpotent groups such as $\mathbb{H}_{2n+3}$, $n \geq 1$, has non-trivial torsion, [19]. In view of Theorem 4.3, there are matricial deformations of $C^*(\mathbb{H}_{2n+3})$ which detect this torsion.

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