EXTENSIONS OF C^*-ALGEBRAS AND QUASIDIAGONALITY

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Abstract. Using extension theory and recent results of Elliott and Gong we exhibit new classes of nuclear stably finite C^*-algebras , which have real rank zero and stable rank one, and are classified by K-theoretical data. Various concepts of quasidiagonality are employed to show that these C*-algebras are not inductive limits of (sub)homogeneous C^*-algebras.

Introduction

Recently there have been far-reaching advances in the classification problems of inductive limits of subhomogeneous C^*-algebras [20-22], [47], [25], [27], [26] and Cuntz algebras [41-42]. The reader is referred to [3] for a survey on the structure of approximately homogeneous C^*-algebras (AH-algebras) and to [24] for a report on the status of the classification problems of various classes of nuclear C^*-algebras.

A recurring theme of the present article is that the concept of quasidiagonality [33], [49](a survey which includes other references) has to play a role in the project of classifying large classes of nuclear C^*-algebras. The first section of the paper is devoted to a universal coefficient theorem for the strong Ext-group and production of $*$-automorphisms with prescribed K-theory for certain extension C*-algebras. It was shown in [9] that quasidiagonality is related to torsion phenomena in K-theory. In the second section of the paper we amplify some of the ideas of [9] and use the universal coefficient theorem of [43] in order to produce non-quasidiagonal extensions

$$0 \to J \to A \to B \to 0$$

where J and B are AH-algebras and A is a stably finite C^*-algebra of real rank zero and stable rank one. The obstruction to quasidiagonality of [9] extends to arbitrary extensions and has a K_1 - analog — cf. [45], [16]. The nonvanishing of these obstructions prevents A being isomorphic to an AH-algebra or (sometimes) isomorphic to an inductive limit of subhomogeneous C^*-algebras. In particular this implies that the approximately subhomogeneous C^*-algebras do not exhaust the class of separable nuclear C^*-algebras of real rank zero and stable rank one that are embeddable into AF-algebras. This answers a question of E. G. Elliott [24]. One may conclude that the C*-algebras that are extensions of AH-algebras should be included on the list of basic building blocks that serve as local approximations of nuclear C*-algebras.

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In the third section of the paper we deal with C^*-algebras that are extensions of AH-algebras by the compact operators. Under certain technical assumptions, we classify these C^*-algebras in terms of ordered, scaled K-theory groups. The proof is based on the universal coefficient theorem for the strong Ext-group and on classification results for AH-algebras due to Elliott and Gong [26]. An interesting feature of the K-theory groups we are dealing with is the presence of a new type of perforation in real rank zero C*-algebras (see Examples 20, 23).

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1. A universal coefficient theorem for the strong Ext-group

For an account of the basic theory of extensions of C*-algebras the reader is referred to [11], [18], [10], [48], [13], [1], [38-39], [35], [43], [2].

Let H be an infinite-dimensional, separable, Hilbert space, $L(H)$ the algebra of bounded linear operators on H, $\mathcal{K} = \mathcal{K}(H)$ the ideal of compact operators and $Q(H) = L(H)/\mathcal{K}$ the Calkin algebra.

Let B be a separable C*-algebra. An essential extension of B by $\mathcal{K}$ can be described either as a $*$-monomorphism $\sigma : B \to Q(H)$ or as a short exact sequence of C*-algebras

$$0 \to \mathcal{K} \xrightarrow{j} A \xrightarrow{\pi} B \to 0$$

where $j(\mathcal{K})$ is an essential closed ideal in A. Recall that two $*$-monomorphisms $\sigma_1,\ \sigma_2 : B \to Q(H)$ are (weakly) equivalent if there is a partial isometry u in $Q(H)$ such that u^*u acts as an identity for $\sigma_1(B)$ and $\sigma_2 = u\sigma_1 u^*$. If u can be taken to be a unitary of index zero then σ_1 and σ_2 are strongly equivalent. Let $Ext(B)$ denote the weak equivalence classes of $*$-monomorphisms $\sigma : B \to Q(H)$. For non-unital C*-algebras two $*$-monomorphisms are weakly equivalent iff they are strongly equivalent, hence the two equivalence relations give rise to the same factor set. If B is unital, the strong equivalence classes divide into two disjoint sets. One set corresponds to the $*$-monomorphisms for which $\sigma(1) \neq 1$, and is parametrized by $Ext(B)$. The other set corresponds to the unital $*$-monomorphisms $\sigma : B \to Q(H)$ and is denoted by $Ext_s(B)$. It is well known that $Ext_s(B)$ and $Ext(B)$ are abelian semigroups with unit. The canonical map $Ext_s(B) \to Ext(B)$ is a unit-preserving morphism of semigroups.

Suppose that B is unital. There is a canonical action of $\mathbb{Z}$ on $Ext_s(B)$ given by $\epsilon(n)[\sigma]_s = [u\sigma u^*]_s$ where $u \in Q(H)$ is a unitary of index $-n$. Then $Ext(B)$ can be identified with the orbit space of this action. It is clear that $\epsilon(n)[\sigma_1]_s + \epsilon(m)[\sigma_2]_s = \epsilon(n+m)([\sigma_1]_s + [\sigma_2]_s)$. In particular $\epsilon(n)[\sigma]_s = [\sigma]_s + \epsilon(n)[\tau]_s$ where τ is a trivial extension. Therefore if it happens that $Ext_s(B)$ is a group (equivalently – if $Ext(B)$ is a group), then $Ext(B)$ is the quotient of $Ext_s(B)$ by the subgroup

$$\{\epsilon(n)[\tau]_s \mid n \in \mathbb{Z}\}$$

where $\tau : B \to Q(H)$ is a trivial extension. In other words the sequence

$$\mathbb{Z} \xrightarrow{\epsilon'} Ext_s(B) \longrightarrow Ext(B) \to 0$$

is exact. The map ϵ' is defined by $\epsilon'(n) = \epsilon(n)[\tau]_s$. Next we are going to identify the kernel of ϵ' under appropriate technical assumptions.

Recall that there is a natural map $\gamma_B : Ext(B) \to Hom(K_1(B), \mathbb{Z})$ with $\gamma_B[\sigma]$ given by the composition

$$K_1(B) \xrightarrow{\sigma_*} K_1(Q(H)) \xrightarrow{index} \mathbb{Z}.$$

The morphism $\gamma_B[\sigma]$ is called the index invariant of the extension σ. Let $\kappa : ker\ (\gamma_B) \to Ext(K_0(B), \mathbb{Z})$ be the natural map which takes the class of an extension

$$0 \to \mathcal{K} \to A \to B \to 0$$

with trivial index invariant to the isomorphism class of

$$0 \to \mathbb{Z} \to K_0(A) \to K_0(B) \to 0.$$

Suppose that B is a separable C*-algebra and $Ext(B)$ is a group. One says that the universal coefficient formula (UCT) is true for B if the map γ_B is surjective and the map κ is bijective. It is known that UCT is true for large classes of nuclear C*-algebras [7], [9], [43].

Proposition 1. *Let B be a unital, separable C*-algebra. Suppose that $Ext(B)$ is a group and that $\gamma_{B\otimes C(\mathbb{T})}$ is surjective. Then there is a short exact sequence of groups*

$$0 \to \mathbb{Z}/\{h[1_B] \mid h \in Hom(K_0(B), \mathbb{Z})\} \xrightarrow{\epsilon'} Ext_s(B) \to Ext(B) \to 0.$$

Proof. Set $\Gamma_B = \{h[1_B] \mid h \in Hom(K_0(B), \mathbb{Z})\}$. In the first part of the proof we show that $ker(\epsilon') \subset \Gamma_B$. If $n \in ker(\epsilon')$ then $\epsilon(n)[\tau]_s = [\tau]_s$ where τ is a trivial extension. It follows from the definition of $\epsilon(n)$ that there is a unitary u in $Q(H)$ of index equal to $-n$ such that $\tau(b)u = u\tau(b)$ for all $b \in B$. Define $\widetilde{\tau} : B \otimes C(\mathbb{T}) \to Q(H)$ by $\widetilde{\tau}(b \otimes 1) = \tau(b)$ and $\widetilde{\tau}(1 \otimes z) = u$, where z denotes the identity map of $\mathbb{T}$. Let $\beta : K_0(B) \to K_1(SB) \hookrightarrow K_1(B \otimes C(\mathbb{T}))$ be the Bott map. On classes of projections this is defined by $\beta[p] = [p \otimes z + (1-p) \otimes 1]$. Define $h \in Hom(K_0(B), \mathbb{Z})$ by $h = \gamma_{B\otimes C(\mathbb{T})}([\widetilde{\tau}]) \circ \beta$. Then $h[1_B] = \gamma_{B\otimes C(\mathbb{T})}([\widetilde{\tau}])(1 \otimes z) = index\ (u) = -n$, hence $n \in \Gamma_B$.

In the second part of the proof we show that $\Gamma_B \subset ker(\epsilon')$. Let $h \in Hom(K_0(B), \mathbb{Z})$ and set $n = -h[1_B] \in \Gamma_B$. Since $\gamma_{B\otimes C(\mathbb{T})}$ is surjective and $K_1(SB)$ is a direct summand in $K_1(B \otimes C(\mathbb{T}))$, there exists a unital $*$-monomorphism $\widetilde{\sigma} : B \otimes C(\mathbb{T})) \to Q(H)$ such that $\gamma_{B\otimes C(\mathbb{T})}([\widetilde{\sigma}]) = h$. Setting $u = \widetilde{\sigma}(1 \otimes z)$ one sees as above that $index\ (u) = h[1_B] = -n$. Let $\sigma : B \to Q(H)$ denote the restriction of $\widetilde{\sigma}$ to B, that is $\sigma(b) = \widetilde{\sigma}(b \otimes 1)$. Obviously $\sigma(b)$ commutes with u for all b hence $\epsilon(n)[\sigma]_s = [\sigma]_s$. Since $\epsilon(n)[\sigma]_s = [\sigma]_s + \epsilon(n)[\tau]_s$ and $Ext_s(B)$ is a group it follows that $n \in ker(\epsilon')$. □

Let H be an abelian group, $h_0 \in H$, and K an abelian group. We consider the set of all extensions of abelian groups with base point of the form

$$0 \to K \to (G, g_0) \xrightarrow{\varphi} (H, h_0) \to 0$$

where $\varphi(g_0) = h_0$. The usual group-theoretic construction of $Ext(H, K)$ makes sense also in the case of extensions with base point and gives rise to an abelian group $Ext((H,h_0),K)$. The trivial element arises from a split extension with a splitting map ψ such that $\psi(h_0)=g_0$.

It is not hard to see that the natural map $Ext((H,h_0),K)\to Ext(H,K)$ has kernel isomorphic to K/K' where $K'=\{f(h_0)\mid f\in Hom(H,K)\}$.

If B be a unital separable C*-algebra and $\widetilde{\gamma}_B : Ext_s(B)\to Hom(K_1(B),\mathbb{Z})$ is the index map, then there is a natural map $\widetilde{\kappa} : ker(\widetilde{\gamma}_B)\to Ext((K_0(B),[1_B]),\mathbb{Z})$ which takes the class of a unital extension

$$0\to\mathcal{K}\to A\to B\to 0$$

with trivial index invariant to the isomorphism class of

$$0\to\mathbb{Z}\to(K_0(A),[1_A])\to(K_0(B),[1_B])\to 0.$$

Theorem 2. *Let B be a unital separable C*-algebra. Suppose that $Ext(B)$ and $Ext(B\otimes C(\mathbb{T}))$ are groups. Suppose that UCT is true for B and $B\otimes C(\mathbb{T})$. Then there is a short exact sequence of groups*

$$0\to Ext((K_0(B),[1_B]),\mathbb{Z})\to Ext_s(B)\xrightarrow{\widetilde{\gamma}_B} Hom(K_1(B),\mathbb{Z})\to 0.$$

Proof. The proof is based on the following commutative diagram with exact columns.

$$\begin{array}{ccccccc}
0 & & 0 & & 0 & & \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathbb{Z}/\Gamma_B & = & \mathbb{Z}/\Gamma_B & = & \mathbb{Z}/\Gamma_B & & \\
\downarrow & & \downarrow & & \downarrow & & \\
Ext((K_0(B),[1_B]),\mathbb{Z}) & \xleftarrow{\widetilde{\kappa}} & ker\,\widetilde{\gamma} & \longrightarrow & Ext_s(B) & \xrightarrow{\widetilde{\gamma}} & Hom(K_1(B),\mathbb{Z}) \\
\downarrow & & \downarrow & & \downarrow & & \| \\
Ext(K_0(B),\mathbb{Z}) & \xleftarrow{\kappa} & ker\,\gamma & \longrightarrow & Ext(B) & \xrightarrow{\gamma} & Hom(K_1(B),\mathbb{Z}) \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & & 0 & & 0 & &
\end{array}$$

Since UCT is true for B it follows that γ is surjective and κ is bijective. It is then clear that $\widetilde{\gamma}$ is surjective. Moreover $\widetilde{\kappa}$ is an isomorphism by the five lemma. □

Proposition 3. *Let B be a separable C*-algebra. Let*

$$0\to\mathcal{K}\xrightarrow{j} A\xrightarrow{\pi} B\to 0$$

be an essential extension with trivial index invariant. Suppose that $Ext(B)$ and $Ext(B\otimes C(\mathbb{T}))$ are groups. Suppose that UCT is true for $B\otimes C(\mathbb{T})$. Let $\theta\in Aut(K_0(A))$ be a group-automorphism such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{j_*} & K_0(A) & \xrightarrow{\pi_*} & K_0(B) & \longrightarrow & 0 \\ & & \| & & \downarrow \theta & & \| & & \\ 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{j_*} & K_0(A) & \xrightarrow{\pi_*} & K_0(B) & \longrightarrow & 0 \end{array}$$

is commutative. Suppose that either A is non-unital or A is unital and $\theta[1_A] = [1_A]$. Then there is a $*$-automorphism $\alpha \in Aut(A)$ such that $K_0(\alpha) = \theta$ and α induces the identity map on B.

Proof. We begin by showing that $\theta = id_{K_0(A)} + j_* \circ h \circ \pi_*$ for some $h \in Hom(K_0(B), \mathbb{Z})$. Since $(\theta - id_{K_0(A)})j_* = 0$ and $image\, j_* = ker\, \pi_*$ it follows that $\theta - id_{K_0(A)}$ factors through π_*. Since $\pi_*(\theta - id_{K_0(A)}) = 0$ it follows that $image\,(\theta - id_{K_0(A)}) \subset image\, j_*$. We conclude that there is a group-homomorphism $h : K_0(B) \to \mathbb{Z}$ such that $\theta - id_{K_0(A)} = j_* \circ h \circ \pi_*$.

We deal first with the unital case. Since $\theta[1_A] = [1_A]$ one deduces that $h([1_B]) = 0$. Let $\sigma : B \to Q(H)$ be the $*$-monomorphism defined by the given extension. Next we find a unital $*$-monomorphism $\widetilde{\sigma} : B \otimes C(\mathbb{T}) \to Q(H)$ such that $\widetilde{\sigma}(b \otimes 1) = \sigma(b)$ for all $b \in B$ and $\widetilde{\gamma}_{B \otimes C(\mathbb{T})}([\sigma]_s) \circ \beta = h$. This goes as follows: Let $i : B \to B \otimes C(\mathbb{T})$ be given by $i(b) = b \otimes 1$. Let $r : B \otimes C(\mathbb{T}) \to B$ be an evaluation map at some point of $\mathbb{T}$. Since $r \circ i = id_B$ it follows that $Ext_s(B \otimes C(\mathbb{T})) \cong Ext_s(B) \oplus\ ker\ i^*$. Using UCT for $B \otimes C(\mathbb{T})$ one finds $[\sigma_0]_s \in Ker\ i^*$ with $\widetilde{\gamma}_{B \otimes C(\mathbb{T})}([\sigma_0]_s) \circ \beta = h$. Let $\widetilde{\sigma} : B \otimes C(\mathbb{T}) \to Q(H)$ be a unital $*$-monomorphism such that $[\widetilde{\sigma}]_s = [\sigma_0]_s + r^*[\sigma]_s$. Then $\widetilde{\gamma}_{B \otimes C(\mathbb{T})}([\widetilde{\sigma}]_s) \circ \beta = h$ and $[\widetilde{\sigma} \circ i]_s = [\sigma]_s$. After replacing $\widetilde{\sigma}$ by $w \widetilde{\sigma} w^*$ for a suitable unitary w we may assume that $\widetilde{\sigma} \circ i = \sigma$. Let $u = \widetilde{\sigma}(1 \otimes z)$. Then u commutes with with $\sigma(b)$ for all $b \in B$. As in the proof of Proposition 1, index $(u) = h[1_B] = 0$ hence u lifts to a unitary $v \in L(H)$. Let $\alpha(a) = vav^*$ be the $*$-automorphism of A induced by v. We now show that $\alpha_* - id_* = j_* \circ h \circ \pi_*$. Let $\widetilde{p} \in M_n(A)$ be a projection and let $p = \pi \otimes id_{M_n}(\widetilde{p}) \in M_n(B)$. We identify B with $\sigma(B)$. Then

$$\begin{aligned} (\alpha_* - id_*)[\widetilde{p}]_{K_0(A)} &= [(v \otimes 1_n)\widetilde{p}(v^* \otimes 1_n)]_{K_0(A)} - [\widetilde{p}]_{K_0(A)} \\ &= j_*(index\ ((u \otimes 1_n)p + 1_n - p)) = j_* \widetilde{\gamma}_{B \otimes C(\mathbb{T})}([\widetilde{\sigma}]_s)\beta[p]_{K_0(B)} \\ &= j_* h([p]_{K_0(B)}) = j_* h \pi_*([\widetilde{p}]_{K_0(A)}) \end{aligned}$$

The second equality in the above sequence follows from the definition of the boundary map in K-theory. Finally it is clear that the restriction of α to B is the identity map since $\pi(v) = u$ commutes with $\sigma(B)$.

Let us now deal with the non-unital case. This means that the image of the $*$-monomorphism $\sigma : B \to Q(H)$ defined by the given extension does not contain the unit of $Q(H)$. Let $B^+ = B + \mathbb{C}\ 1_{Q(H)}$ and $A^+ = A + \mathbb{C}\ 1_{L(H)}$. Here we identify B with $\sigma(B)$. We extend θ to an automorphism θ^+ of $K_0(A^+) = K_0(A) \oplus \mathbb{Z}$ by setting $\theta^+ = \theta \oplus id_{\mathbb{Z}}$. Since we have already proved the Proposition in the unital case, we can use the corresponding result for the extension

$$0 \to \mathcal{K} \to A^+ \to B^+ \to 0.$$

Thus there is $\alpha^+ \in Aut(A^+)$ such that $\alpha^+_* = \theta^+$. It is clear that the restriction of α^+ to A induces θ on K_0. $\square$

2. Quasidiagonality relative to an ideal

Quasidiagonality of operators was defined by P. R. Halmos [33]. We refer the reader to a survey paper of D. Voiculescu [49] for a discussion on open problems and results on quasidiagonality.

We recall the following definition (cf. [37]).

Definition 4. *An extension of separable C^*-algebras*

$$0 \to J \to A \to B \to 0$$

is called quasidiagonal if there exists an approximate unit $(p_n)_n$ of J consisting of projections, which is quasicentral in A, i.e.

$$\lim_{n\to\infty} ||ap_n - p_n a|| = 0$$

for all $a \in A$.

A complete characterization of the quasidiagonal extensions of the form

$$0 \to \mathcal{K} \to A \to C(X) \to 0$$

was given in [9]. Such an extension is quasidiagonal if and only if all Fredholm elements in $A \otimes M_n$ have index zero and the restriction of the group extension

$$0 \to \mathbb{Z} \cong K_0(\mathcal{K}) \to K_0(A) \to K_0(C(X)) \to 0$$

to the torsion subgroup of $K_0(C(X))$ is trivial. One can show that this result remains true if $C(X)$ is replaced by an inductive limit of continuous trace C*-algebras or by an inductive limit of subhomogeneous C*-algebras. This has been further generalized by Salinas [45]. As a matter of fact we shall see that one implication in these results is valid for arbitrary quasidiagonal extensions.

Definition 5. *([29]) A subgroup K of an abelian group G is called pure if $nK = K \cap nG$ for every $n \in \mathbb{N}$.*

In other words K is a pure subgroup of G if and only if whenever $x \in K$ and $x = ng$ with $g \in G$, it follows that $x = ny$ for some $y \in K$.

Definition 6. *(cf. [29]) An extension of abelian groups*

$$0 \to K \xrightarrow{i} G \xrightarrow{\pi} H \to 0$$

is called pure if $i(K)$ is a pure subgroup of G.

It is easily seen that any extension isomorphic to a pure extension is itself pure. Thus we can talk about pure elements of $Ext(H, K)$.

The following proposition gathers some known characterizations of pure extensions (see [29]).

Proposition 7. *Let*

$$0 \to K \xrightarrow{i} G \xrightarrow{\pi} H \to 0 \tag{e}$$

be an extension of abelian groups. Then the following three conditions are equivalent.

(i) The extension e is pure.

(ii) Any torsion element of H lifts to a torsion element of G of the same order.

(iii) The restriction of e to any finitely generated subgroup of H is trivial.

If K is torsion free then the previous conditions are equivalent to

(iv) The restriction of e to the torsion subgroup of H is trivial.

A pure extension is not necessarily trivial unless H is isomorphic to a direct sum of cyclic groups. The pure extensions form a subgroup of $Ext(H,K)$. This subgroup coincides with the closure of 0 in the $\mathbb{Z}$-adic topology of $Ext(H,K)$, see [29]. Recall that for a group G, the $\mathbb{Z}$-adic topology is defined such that the subgroups nG, $n \neq 0$ form a base of neighborhoods about 0. This closure may exhaust the whole group as it happens, for instance, for $Ext(\mathbb{Q},\mathbb{Z}) \cong \mathbb{R}$ (non-canonical isomorphism). The notion of pure subgroup is intermediate between subgroup and direct summand. In view of the above properties one may think of pure extensions as being "locally trivial" extensions.

Theorem 8. *Let*

$$0 \to J \xrightarrow{j} A \xrightarrow{\pi} B \to 0$$

be a quasidiagonal extension of C^*-algebras. Then the index maps $\delta_i : K_i(B) \to K_{i+1}(J)$, $i = 0,1$ are zero and the extensions

$$0 \to K_i(J) \xrightarrow{j_*} K_i(A) \xrightarrow{\pi_*} K_i(B) \to 0$$

$i = 0,1$ are pure.

Proof. Let (e_n) be an approximate unit of J consisting of projections and which is quasicentral in A. Define $\eta_n : A \to J$ by $\eta_n(a) = e_n a e_n$. It is clear that (η_n) is a sequence of linear completely positive maps and

$$\lim_{n\to\infty} \|\eta_n(ab) - \eta_n(a)\eta_n(b)\| = \lim_{n\to\infty} \|e_n(e_n a - a e_n) b e_n\| = 0$$

for all $a, b \in A$ since each e_n is a projection and (e_n) is quasicentral in A. It follows that the sequence (η_n) defines a $*$-homomorphism $\eta : A \to \ell^\infty(J)/c_0(J)$ given by $\eta(a) = (\eta_n(a)) + c_0(J)$. On the other hand

$$\lim_{n\to\infty} \|(\eta_n \circ j)(x) - x\| = \lim_{n\to\infty} \|e_n x e_n - x\| = 0$$

for all $x \in J$ since (e_n) is an approximate unit of J. If we define $\Delta : J \to \ell^\infty(J)/c_0(J)$ by $\Delta(x) = (x, x, \ldots) + c_0(J)$ then $\eta \circ j = \Delta$. At the level of K-theory this gives a commutative diagram

$$\begin{array}{ccccc} K_*(A) & \xrightarrow{\eta_*} & K_*(\ell^\infty(J)/c_0(J)) & \xrightarrow{\mu} & \prod K_*(J)/\sum K_*(J) \\ {\scriptstyle j_*}\uparrow & & & & \\ K_*(J) & & & & \end{array}$$

where μ is the natural map. Since $\mu\Delta_*$ is injective this implies that j_* is injective. It follows from the long exact sequence in K-theory that the index maps $\delta_i : K_i(B) \to K_{i+1}(J)$, $i = 0, 1$ are zero. Next we show that $j_*(K_*(J))$ is a pure subgroup of $K_*(A)$. Assume that $j_*(x) = ng$ for some $x \in K_*(J)$, $g \in K_*(A)$ and $n \in \mathbb{N}$. Then $\Delta_*(x) = \eta_* j_*(x) = n\eta_*(g)$ hence $\mu\Delta_*(x) \in n \prod K_*(J)/\sum K_*(J)$. This implies that $x \in nK_*(J)$. $\square$

Let B be a separable nuclear quasidiagonal C*-algebra. Salinas [44] proved that the isomorphism classes of quasidiagonal extensions

$$0 \to \mathcal{K} \to A \to B \to 0$$

form a subgroup $Ext_{qd}(B)$ of $Ext(B)$ that coincides with the closure of the neutral element in the natural topology of $Ext(B)$. Theorem 8 shows that the image of the natural map $Ext_{qd}(B) \to Ext(K_0(B), \mathbb{Z})$ is contained in the closure of 0 in the $\mathbb{Z}$-adic topology of $Ext(K_0(B), \mathbb{Z})$. A version of Theorem 8 and a converse to it were proved in [45] under stronger assumptions. If B is a commutative C*-algebra, the subgroup of pure extensions of $Ext(K_0(B), \mathbb{Z})$ was studied and characterized by Kaminker and Schochet in [34].

For C*-algebras A, B let $Map(A, B)$ be the space of all arbitrary maps from A to B endowed with the topology of pointwise norm convergence. Thus if (φ_k) is a sequence in $Map(A, B)$ then $\varphi_k \to \varphi$ if and only if $\|\varphi_k(a) - \varphi(a)\| \to 0$ for all $a \in A$.

A sequence (φ_k) is called an *approximate morphism* if for all $a, b \in A$ and $\lambda \in \mathbb{C}$

$$\lim_{k\to\infty} \|\varphi_k(a + \lambda b) - \varphi_k(a) - \lambda\varphi_k(b)\| = 0$$

$$\lim_{k\to\infty} \|\varphi_k(ab) - \varphi_k(a)\varphi_k(b)\| = 0$$

$$\lim_{k\to\infty} \|\varphi_k(a^*) - \varphi_k(a)^*\| = 0.$$

This is a discrete analog of the notion of asymptotic morphism of [14]. The following theorem is a kind of geometric version of Theorem 8. It shows that "locally" a quasidiagonal extension is approximately a direct sum.

Theorem 9. *Let*

$$0 \to J \xrightarrow{j} A \xrightarrow{\pi} B \to 0$$

be a quasidiagonal essential extension of separable C-algebras. Then there are approximate morphisms* $(\eta_k) : A \to J$ *and* $(\gamma_k) : B \to A$ *such that*

$$\eta_k j \to id_J, \quad \pi\gamma_k = id_B, \quad j\eta_k + \gamma_k\pi \to id_A.$$

Proof. Let (p_n) be an approximate unit of J consisting of projections, which is quasicentral in A. Arguing as in the proof of Proposition 1.22 in [50], one finds an increasing sequence of projections (q_k) such that $||q_k - p_{n_k}|| \to 0$ for a suitable sequence (n_k). Therefore we may assume that $0 = p_0 \le p_1 \le p_2 \le \dots$. Since J is an essential ideal in A it follows that (p_n) converges to 1, in the strict topology of the multiplier algebra $M(A)$. We want to show that after passing to a suitable subsequence of (p_n), the projections $e_n = p_n - p_{n-1}$ are such that for all $a \in A$, the series $\sum_{n=1}^{\infty} e_n a e_n$ is convergent in the strict topology to an element $\delta(a) \in M(A)$ and $a - \delta(a) \in J$. One mimics the proof of Theorem 2 in [1]. Let $F_1 \subset F_2 \subset \dots$ be finite subsets of A such that $F = \bigcup_{n=1}^{\infty} F_n$ is norm dense in the unit ball of A. Let $\epsilon > 0$ be given. Since $(p_n)_n$ is quasicentral, after passing to a subsequence we may assume that $||p_n a - a p_n|| \leq \frac{\epsilon}{2^{n+1}}$ for all $a \in F_n$. It follows that $||e_n a - a e_n|| = ||(p_{n+1} - p_n)a - a(p_{n+1} - p_n)|| \leq \frac{\epsilon}{2^n}$ for all $a \in F_n$, hence $\sum_{n=1}^{\infty} ||e_n a - a e_n|| < \infty$ for all $a \in F$. Since (e_n) are mutually orthogonal projections it follows that $\sum_{n=1}^{\infty} e_n a e_n$ is strictly convergent to an element $\delta(a) \in M(A)$, for all $a \in A$ and the map $a \mapsto \delta(a)$ is norm continuous. Now

$$a - \delta(a) = \sum_{n=1}^{\infty} (a e_n^2 - e_n a e_n) = \sum_{n=1}^{\infty} (a e_n - e_n a) e_n$$

in the strict topology and the latter series is norm convergent for $a \in F$. We conclude that $a - \delta(a) \in J$ for all $a \in A$ by norm continuity of δ. Moreover it is clear that $\|\delta(a) - a\| \leq \epsilon$ for $a \in F_1$. The above construction can be repeated in order to get an approximate morphism $(\delta_k) : A \to A$ such that $\delta_k \to id_A$. More precisely we can find a sequence of strictly increasing maps $\ell_k : \mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$, $\ell_k(0) = 0$, with $\ell_k(1) \to \infty$ and such that setting $e_n^k = p_{\ell_k(n+1)} - p_{\ell_k(n)}$, the sequence of maps

$$\delta_k(a) = \sum_{n=1}^{\infty} e_n^k a e_n^k$$

forms an approximate morphism convergent to id_A. Define $\eta_k : A \to J$ by $\eta_k(a) = e_1^k a e_1^k$ for $a \in A$, and $\gamma_k : B \to A$ by

$$\gamma_k(b) = \sum_{n=2}^{\infty} e_n^k s(b) e_n^k.$$

where s is any arbitrary set theoretic right inverse of π. Since $\eta_k(a) = e_1^k a e_1^k = p_{\ell_k(1)} a p_{\ell_k(1)}$ and $\ell_k(1) \to \infty$ as $k \to \infty$ we can argue as in the proof of Theorem 8 to show that (η_k) is an approximate morphism and $\eta_k j \to id_J$.

Next we prove that $j\eta_k + \gamma_k \pi \to id_A$. Actually this follows from the following estimation. For $a \in A$

$$\|j\eta_k(a) + \gamma_k \pi(a) - a\| =$$
$$\|e_1^k a e_1^k + \sum_{n=2}^{\infty} e_n^k s\pi(a) e_n^k - a\| =$$
$$\|\delta_k(a) - a + \sum_{n=2}^{\infty} e_n^k (s\pi(a) - a) e_n^k\| \leq$$
$$\|\delta_k(a) - a\| + \sup_{n \geq 2} \|(p_{\ell_k(n+1)} - p_{\ell_k(n)})(s\pi(a) - a)\|.$$

Since $\gamma_k(b) - \delta_k(s(b)) \in J$ and $\delta_k(s(b)) - s(b) \in J$, it follows that $\pi\gamma_k(b) = b$ for all $b \in B$. Finally we show that (γ_k) is an approximate morphism. Let us check that (γ_k) is approximately multiplicative. Since s is a right inverse of π, $s(bc) - s(b)s(c) \in J$ for all $b, c \in B$. Now

$$\gamma_k(bc) - \gamma_k(b)\gamma_k(c) = \sum_{n=2}^{\infty}(e_n^k s(bc)e_n^k - e_n^k s(b)e_n^k s(c)e_n^k)$$

$$= \sum_{n=2}^{\infty} e_n^k (s(bc) - s(b)s(c))e_n^k + \sum_{n=2}^{\infty} e_n^k s(b)(s(c)e_n^k - e_n^k s(c))e_n^k$$

in the strict topology, hence

$$||\gamma_k(bc) - \gamma_k(b)\gamma_k(c)|| \leq \sup_{n\geq 2}||e_n^k(s(bc) - s(b)s(c))e_n^k||$$
$$+\sup_{n\geq 2}||s(c)e_n^k - e_n^k s(c)||.$$

This implies that (γ_k) is approximately multiplicative since $e_n^k = p_{\ell_k(n+1)} - p_{\ell_k(n)}$, and p_n is an approximate unit of J which is quasicentral in A. One proves similarly that (γ_k) is approximately linear and selfadjoint. □

Recall that a homogeneous C^*-algebra A of degree n is the C^*-algebra of the continuous sections vanishing at infinity of some locally trivial M_n-bundle over a locally-compact Hausdorff space. Using the terminology of [3] an AH-algebra is a C^*-algebra isomorphic to an inductive limit of direct sums of homogeneous C^*-algebras.

Lemma 10. *Let C be a (c_0) direct sum of homogeneous C^*-algebras. Suppose that $e \in C$ is a projection. Then the closed ideal generated by e is of the form gC for some central projection $g \in C$.*

Proof. (sketch). Let F be the compact-open subset of the spectrum of C consisting of all the points x such that $e(x) \neq 0$. Let $g = \chi_F 1$ where χ_F is the characteristic function of F. It is easily seen that g is a central projection and $\overline{CeC} = gC$. □

Proposition 11. *Let A be a separable AH-algebra. Suppose that J is a closed ideal in A and J has an approximate unit of projections. Then the extension*

$$0 \to J \to A \to A/J \to 0$$

is quasidiagonal.

Proof. Let (e_n) be an approximate unit of J consisting of projections. Write $A = \overline{\cup A_n}$ where $A_1 \subset A_2 \subset \dots$ is a sequence of finite direct sums of homogeneous C^*-algebras. By a result in [6] $J = \overline{\cup J_n}$ where $J_n = A_n \cap J$, $J_1 \subset J_2 \subset \dots$. Passing to a subsequence of (J_n) we may assume that $dist(e_n, J_n) \to 0$. Using functional calculus we find projections $f_n \in J_n$ such that $||e_n - f_n|| \to 0$. Clearly (f_n) form an approximate unit of J. Let I_n be the ideal generated by f_n in A_n. Lemma 10 shows that $I_n = g_n A_n$ for some central projection $g_n \in A_n$. Clearly $f_n \leq g_n$ and $I_n \subset J_n$ since $f_n \in J_n$. For $x \in J$

$$||x(1 - g_n)||^2 = ||x(1 - g_n)x^*|| \leq ||x(1 - f_n)x^*|| = ||x(1 - f_n)||^2,$$

hence (g_n) is an approximate unit for J. Since g_n is a central projection in A_n it follows that (g_n) is quasicentral in A. We conclude that the extension in the statement is quasidiagonal. □

Corollary 12. *Let A be any separable AH-algebra of real rank zero. Then for any closed ideal J in A, the extension*

$$0 \to J \to A \xrightarrow{\pi} A/J \to 0$$

is quasidiagonal. Consequently the index maps are zero and the corresponding extensions in K-theory

$$0 \to K_i(J) \to K_i(A) \to K_i(A/J) \to 0$$

$i = 0,\ 1$ *are pure.*

Proof. By Theorem 2.6 in [12] J has an approximate unit of projections. The statement follows from Proposition 11 combined with Theorem 8. □

Let B, C be separable AH-algebras. The universal coefficient theorem of Rosenberg and Schochet [43] gives a short exact sequence of groups

$$0 \to Ext(K_*(B), K_*(C)) \overset{i}{\hookrightarrow} KK_1(B, C) \xrightarrow{\gamma} Hom(K_*(B), K_{*+1}(C)) \to 0.$$

If

$$0 \to \mathcal{K} \otimes C \to A \to B \to 0$$

is an extension representing some element $x \in KK_1(B, C)$ then $\gamma(x) = (\delta_0, \delta_1)$ where $\delta_i : K_i(B) \to K_{i+1}(C)$ are the index maps in the six-term exact sequence in K-theory associated with the above extension of C^*-algebras. The following proposition shows how the universal coefficient theorem of [43] can be used as a source of extensions of AH-algebras that are not AH. The presence of torsion in K-theory generates situations that cannot occur in the extension theory of AF-algebras [8], [19] or for inductive limits of circle algebras [36].

Proposition 13. *Let B, C be separable AH-algebras of real rank zero and stable rank one and let x be an element of $Ext(K_*(B), K_*(C))$. Let*

$$0 \to \mathcal{K} \otimes C \to A \to B \to 0$$

be an extension with vanishing index invariants representing x. Then the following hold true.

a) A is a nuclear stably finite C^*-algebra of real rank zero and stable rank one.

b) If the extensions

$$0 \to K_i(C) \to K_i(A) \to K_i(B) \to 0$$

$i = 0, 1$, *are not both pure, then A is not isomorphic to an AH-algebra.*

Proof. a) A is nuclear being an extension of two nuclear C^*-algebras. Since the index maps δ_0, δ_1 are zero and both B and C have real rank zero and stable rank one, we conclude by Proposition 4 in [36] that so is A. Since C and B are stably finite and $\delta_1 = 0$ it follows from Lemma 1.5 in [46] that A is stably finite.

b) This follows from Corollary 12. □

In [16] it was shown that the class $A\mathcal{D}$ of inductive limit C*-algebras classified by Elliott in [20] is not closed under extensions. Example 4.5 in [16] exhibits an extension

$$0 \to A \otimes \mathcal{K} \to E \to A \to 0$$

where A is an $A\mathcal{D}$-algebra and E has real rank zero and stable rank one and such that the group extension

$$0 \to K_1(A) \to K_1(E) \to K_1(A) \to 0$$

is not pure. We conclude from Corollary 12 that E is not an AH-algebra.

An example of Ian Putnam [36] exhibits an extension E of a Bunce-Deddens algebra by an AF-algebra such that the index map δ_1 is not zero. This implies that E does not have stable rank one and therefore E is not an inductive limit of circle algebras. Moreover it follows from Corollary 12 that E is not an AH-algebra. This answers a question in [36].

3. A classification result

We shall discuss briefly the order structure on K-theory. Let A be a separable C*-algebra of real rank zero and stable rank one. Let $V(A)$ denote the semigroup of equivalence classes of projections in $M_\infty(A)$. The canonical map $V(A) \to K_0(A)$ is injective and by definition its image is equal to $K_0(A)_+$. The couple $(K_0(A), K_0(A)_+)$ is an ordered group (see [2]) with the Riesz interpolation property [51]. It is significant that one can introduce an order structure on $K_*(A) = K_0(A) \oplus K_1(A)$. This has been done independently in [15] and [20] in connection to shape and (respectively) isomorphism classifications of certain C*-algebras. In [15] one simply defined $K_*(A)_+$ to be the image of $K_0(C(\mathbb{T}, A))_+$ under the natural isomorphism $K_0(C(\mathbb{T}, A)) \to K_*(A)$. The scale $\Sigma_*(A)$ was defined in a similar way. Elliott's definition is slightly more geometric and will be used below excepting for Corollary 22. Thus $K_*(A)_+$ consists of all the pairs $([e], [u + 1 - e]) \in K_0(A) \oplus K_1(A)$ where e is a projection in $M_\infty(A)$ and u is a normal partial isometry with $uu^* = e$, $u^*u = e$. The scale $\Sigma_*(A)$ consists of those pairs $([e], [u + 1 - e])$ as above for which both e and u are in A. The definitions of [15] and [20] lead to the same positive cone $K_*(A)_+$ but to possibly different scales.

Proposition 14. *Let*

$$0 \to \mathcal{K} \to A_i \xrightarrow{\pi_i} B_i \to 0$$

$i = 1, 2$, *be two essential extensions of separable C^*-algebras. Suppose that both A_1 and A_2 have real rank zero and stable rank one. Then every order isomorphism $\theta : (K_*(A_1), \Sigma_*(A_1)) \to (K_*(A_2), \Sigma_*(A_2))$ induces an order isomorphism $\hat{\theta} : (K_*(B_1), \Sigma_*(B_1)) \to (K_*(B_2), \Sigma_*(B_2))$ such that the following diagram is commutative*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_*(\mathcal{K}) & \longrightarrow & K_*(A_1) & \xrightarrow{\pi_{1*}} & K_*(B_1) & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\hat{\theta}} & & \\
0 & \longrightarrow & K_*(\mathcal{K}) & \longrightarrow & K_*(A_2) & \xrightarrow[\pi_{2*}]{} & K_*(B_2) & \longrightarrow & 0
\end{array}$$

Proof. Since A_i has real rank zero and stable rank one it is easily seen that B_i has real rank zero and stable rank one and the index map $\delta_1 : K_1(B_i) \to \mathbb{Z}$ is zero. We prove that θ acts identically on $\mathbb{Z} \simeq K_0(\mathcal{K})$. Let e be a minimal projection in $\mathcal{K}$. Since $\mathcal{K}$ is an hereditary subalgebra of A_1, it follows that e is a minimal projection in A_1. Thus $[e]$ is a minimal element in $K_0(A_1)_+ \cong V(A_1)$. Since θ is an order isomorphism this implies that $\theta[e]$ is minimal in $K_0(A_2)_+ \cong V(A_2)$. Thus we find a minimal projection $f \in A_2$ such that $\theta[e] = [f]$. Since $\mathcal{K}$ is an essential ideal of A_2, it follows that f is in $\mathcal{K}$. We conclude that θ acts identically on $K_0(\mathcal{K})$. As $K_1(\mathcal{K}) = 0$ this shows that θ induces a unique morphism $\hat{\theta}$ making the diagram in the statement commutative. Next we want to show that $\hat{\theta}$ is order preserving. Let $([e],[u]) \in \Sigma_*(B_1)$ be an arbitrary element of the scale of $K_*(B_1)$. Here e is a projection in B_1 and u is a partial isometry with $uu^* = e,\ u^*u = e$. By Theorem 3.14 in [12] e lifts to a projection $\overline{e} \in A_1$. All the C^*-algebras in the extension

$$0 \to \overline{e}\mathcal{K}\overline{e} \to \overline{e}A_1\overline{e} \to eB_1e \to 0$$

have real rank zero and stable rank one (see [12] and [40]). Using Proposition 4 in [36] we lift u to a unitary $\overline{u} \in \overline{e}A_2\overline{e}$. Therefore we have found $([\overline{e}],[\overline{u}]) \in \Sigma_*(A_1)$ with $\pi_{1*}([\overline{e}],[\overline{u}]) = ([e],[u])$. Since $\hat{\theta}\pi_{1*} = \pi_{2*}\theta$ and θ maps $\Sigma_*(A_1)$ onto $\Sigma_*(A_2)$ we conclude that $\hat{\theta}$ maps $\Sigma_*(B_1)$ to $\Sigma_*(B_2)$. Similarly one checks that $\hat{\theta}(K_*(B_1)_+) \subset K_*(B_2)_+$ □

Let $\mathcal{B}$ be a class of separable, nuclear C^*-algebras of real rank zero and stable rank one subject to the following two axioms.

B1) For any C*-algebra B in $\mathcal{B}$ the universal coefficient theorem for the Ext-group is true for B and $B \otimes C(\mathbb{T})$.

B2) For any two C*-algebras B_1 and B_2 in $\mathcal{B}$ and for any order preserving isomorphism $\theta : (K_*(B_1), \Sigma_*(B_1)) \to (K_*(B_2), \Sigma_*(B_2))$ there exists some $*$-isomorphism $\alpha : B_1 \to B_2$ such that $K_*(\alpha) = \theta$.

Let $\mathcal{B}_0$ be the class of all simple C^*-algebras of real rank zero and stable rank one which are isomorphic to inductive limits of C^*-algebras of the form

$$M_{n(1)}(C(X_1)) \oplus \cdots \oplus M_{n(r)}(C(X_r))$$

where X_i are polyhedra of dimension at most 3. The class $\mathcal{B}_0$ satisfies axiom B1 by [43]. A remarkable recent result of Elliott and Gong [26], asserts that $(K_*(B), \Sigma_*(B))$ is a complete invariant for the C^*-algebras $B \in \mathcal{B}_0$. More precisely they showed that the class $\mathcal{B}_0$ does satisfy the axiom B2.

Theorem 15. *Let $\mathcal{B}$ be a class of separable, nuclear C*-algebras satisfying the axioms B1 and B2. Let*

$$0 \to \mathcal{K} \to A_i \xrightarrow{\pi_i} B_i \to 0$$

be two essential extensions of C^*-algebras where $B_1, B_2 \in \mathcal{B}$. Suppose that A_1 and A_2 have stable rank one. Suppose that there exists an order preserving isomorphism $\theta : (K_*(A_1),\ \Sigma_*(A_1)) \to (K_*(A_2),\ \Sigma_*(A_2))$. Then there exists a $*$-isomorphism $\alpha : A_1 \to A_2$ such that $K_*(\alpha) = \theta$.

Proof. Since B_i has real rank zero and A_i is an extension of B_i by $\mathcal{K}$, it follows that A_i has real rank zero. By Proposition 14 there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & K_*(A_1) & \xrightarrow{\pi_{1*}} & K_*(B_1) & \longrightarrow & 0 \\ & & \| & & \downarrow \theta & & \downarrow \hat{\theta} & & \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & K_*(A_2) & \xrightarrow{\pi_{2*}} & K_*(B_2) & \longrightarrow & 0 \end{array}$$

By axiom B2 it follows that there is a $*$-isomorphism $\varphi : B_1 \to B_2$ such that $\varphi_* = \hat{\theta}$. This implies that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & K_*(A_1) & \xrightarrow{\varphi_*\pi_{1*}} & K_*(B_2) & \longrightarrow & 0 \\ & & \| & & \downarrow \theta & & \| & & \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & K_*(A_2) & \xrightarrow{\pi_{2*}} & K_*(B_2) & \longrightarrow & 0 \end{array}$$

is commutative. If A_1 is non-unital then the K_0-component of $\Sigma_*(A_1)$ has no maximal element. If A_1 is unital then $[1_{A_1}]$ is the maximum element of the scale. Since θ preserves the scale it follows that either both A_1 and A_2 are non-unital or both are unital and $\theta[1_{A_1}] = [1_{A_2}]$. By using UCT in the non-unital case and Theorem 2 and its proof in the unital case we see that the extensions

$$0 \to \mathcal{K} \to A_1 \xrightarrow{\varphi\pi_1} B_2 \to 0$$

$$0 \to \mathcal{K} \to A_2 \xrightarrow{\pi_2} B_2 \to 0$$

are strongly equivalent. In either case, we conclude that there is a unitary acting on the underlying Hilbert space which induces a $*$-isomorphism $\psi : A_1 \to A_2$ such that $\pi_2\psi = \varphi\pi_1$. By applying Proposition 3 to $\psi_*^{-1}\theta$ one shows that θ lifts to a $*$-isomorphism $\alpha : A_1 \to A_2$. $\square$

Let $\mathcal{B}_n$ consist of C*-algebras of stable rank one that are isomorphic to extensions of C*-algebras in the class $\mathcal{B}_0$ of Elliott and Gong by $\mathcal{K}^n = \mathcal{K} \oplus \ldots \oplus \mathcal{K}$. Let $\mathcal{B}_\infty$ denote the union of $\mathcal{B}_n$ with $n \geq 0$.

Corollary 16. *Let A_1 and A_2 be C*-algebras in $\mathcal{B}_\infty$. Suppose that there exists an order preserving isomorphism $\theta : (K_*(A_1),\ \Sigma_*(A_1)) \to (K_*(A_2),\ \Sigma_*(A_2))$. Then there is a $*$-isomorphism $\alpha : A_1 \to A_2$ such that $K_*(\alpha) = \theta$.*

Proof. The C*-algebras in $\mathcal{B}_\infty$ are stably finite and have real rank zero (see Proposition 4 in [36] and Lemma 1.5 in [46]). If A is a stably finite C*-algebra of real rank zero then by Theorem 10.9 in [31] the lattice of closed ideals of A is isomorphic to the lattice of order ideals of $K_0(A)$. Since the C*-algebras in $\mathcal{B}_0$ are simple it follows that if A_1 and A_2 are as in the statement of the Corollary then A_1 and A_2 are in the same class $\mathcal{B}_n$ for some n. Notice that for A in $\mathcal{B}_n$ there is a sequence of extensions

$$0 \to \mathcal{K} \to B_{i+1} \to B_i \to 0$$

$0 \leq i \leq n-1$, with $B_0 \in \mathcal{B}_0$ and $B_n \cong A$. The proof is completed by an inductive argument based on Theorem 15. $\square$

The class of unital C^*-algebras classified by Theorem 15 can be written as a disjoint union of classes $\{\mathcal{E}_B\}_{B\in\mathcal{B}}$ where for a fixed $B \in \mathcal{B}$, $\mathcal{E}_B$ consists of C^*-algebras A which are unital, have stable rank one and are essential extensions of B by $\mathcal{K}$

$$0 \to \mathcal{K} \to A \to B \to 0.$$

It is then natural to ask how many non-isomorphic C^*-algebras A as above do exist for a given B. This is clarified by Proposition 17.

Let $Aut(K_*(B), \Sigma_*(B))$ consist of all the automorphisms $\hat{\theta}$ of $K_*(B)$ that preserve the order and the scale. If B is unital then $\hat{\theta}[1_B] = [1_B]$. Via the pullback operation we thus obtain a natural action of $Aut(K_*(B), \Sigma_*(B))$ on $Ext((K_0(B), [1_B]), \mathbb{Z})$.

Proposition 17. *Let*

$$0 \to \mathcal{K} \to A_i \xrightarrow{\pi_i} B \to 0$$

$i = 1, 2$ be two essential extensions where $B \in \mathcal{B}$ and both A_1 and A_2 are unital. Suppose that A_1 and A_2 have stable rank one. Let $x_i \in Ext((K_0(B), [1_B]), \mathbb{Z})$ be the isomorphism class of the group extension

$$0 \to \mathbb{Z} \to (K_0(A_i), [1_{A_i}]) \to (K_0(B), [1_B]) \to 0.$$

Then A_1 is isomorphic to A_2 if and only if x_1 and x_2 lie on same orbit of the action of $Aut(K_*(B), \Sigma_*(B))$ on $Ext((K_0(B), [1_B]), \mathbb{Z})$. One has an analogous result in the non-unital case.

Proof. Let $\chi \in Aut(K_*(B), \Sigma_*(B))$ such that $\chi(x_2) = x_1$. By axiom B2 there is $\varphi \in Aut(B)$ inducing χ on K-theory. It follows that the group extensions with base points

$$0 \to \mathbb{Z} \to K_0(A_1) \xrightarrow{(\varphi\pi_1)_*} K_0(B) \to 0$$

$$0 \to \mathbb{Z} \to K_0(A_2) \xrightarrow{\pi_{2*}} K_0(B) \to 0$$

are isomorphic. Reasoning as in the proof of Theorem 15 we conclude that A_1 is isomorphic to A_2. The argument showing that if $A_1 \cong A_2$ then x_1 and x_2 lie on the same orbit is implicit in the proof of Theorem 15. Indeed any isomorphism $\psi : A_1 \to A_2$ induces an isomorphism
$\theta \overset{def}{=} \psi_* : (K_*(A_1),\ \Sigma_*(A_1)) \to (K_*(A_2),\ \Sigma_*(A_2))$ and it is apparent that $\hat{\theta}(x_2) = x_1$. $\square$

We will see a little bit later (Examples 20,23) that there are simple C^*-algebras $B \in \mathcal{B}_0$ such that the action of $Aut(K_*(B), \Sigma_*(B))$ on $Ext((K_0(B), [1_B]), \mathbb{Z})$ has infinitely many orbits.

The following result improves on Proposition 13 in the case of extensions by $\mathcal{K}$.

Proposition 18. *Let B be a separable AH-algebra of real rank zero and stable rank one. Let*

$$0 \to \mathcal{K} \to A \to B \to 0$$

be an extension representing some element $x \in Ext(K_0(B), \mathbb{Z})$. Let T be the torsion subgroup of $K_0(B)$. Suppose that the image of x into $Ext(T, \mathbb{Z})$ is nonzero. Then A is a nuclear, stably-finite C^*-algebra of real rank zero and stable rank one which is not isomorphic to any inductive limit of subhomogeneous C^*-algebras.

Proof. Recall that a separable C^*-algebra D is called strongly quasidiagonal if all of its separable representations $\pi : D \to L(H)$ are quasidiagonal (see [32]). That means that for all π the extension

$$0 \to \mathcal{K} \to \pi(D) + \mathcal{K} \to (\pi(D) + \mathcal{K})/\mathcal{K} \to 0$$

is quasidiagonal in the sense of Definition 1. By hypothesis x is not pure. By Theorem 8 this implies that the given extension is not quasidiagonal and therefore A is not strongly quasidiagonal. On the other hand, any inductive limit of subhomogeneous C^*-algebras is strongly quasidiagonal. For instance this follows from Propositions 5 and 8 in [32]. The same argument generalizes to show that A is not isomorphic to an inductive limit of CCR C*-algebras. □

Remark 19.

a) Let A, B, x be as in Proposition 18. If in addition $B \in \mathcal{B}_0$ then A can be embedded into an AF-algebra (see [46]). Actually even more is true for if U is the UHF-algebra with $K_0(U) \cong \mathbb{Q}$ then $U \otimes A$ is an $A\mathbb{T}$-algebra i.e. an inductive limit of circle algebras. Indeed, $U \otimes B$ is an $A\mathbb{T}$-algebra by the classification theorem of [26]. Hence $U \otimes A$ is an extension of two $A\mathbb{T}$-algebras. We conclude that $U \otimes A$ is an $A\mathbb{T}$-algebra by a theorem of Lin and Rørdam [36]. Using results in [20] is easily seen that any real rank zero $A\mathbb{T}$-algebra is embeddable into an AF-algebra.

b) Elliott [24] proposed a list of inductive limits of subhomogeneous C^*-algebras that could conceivably exhaust the separable nuclear stably finite C^*-algebras of real rank zero. However it follows from Propositions 10, 18 that there are large classes of nuclear C^*-subalgebras of AF-algebras having real rank zero and stable rank one and which are not approximately subhomogeneous.

Example 20. For $m \geq 2$, let X be a two-dimensional space obtained by attaching, with degree m, the boundary of the unit disk to the unit circle. Thus $K_0(C(X)) = \mathbb{Z} \oplus \mathbb{Z}/m$ and $K_1(C(X)) = 0$. Let q be a prime number. Using diagonal embeddings one constructs as in [30] a unital AH-algebra $B = \lim\limits_{\to} C(X, M_n)$ such that B is simple of real rank zero and stable rank one, with

$$\begin{aligned} K_0(B) &= \mathbb{Z}[1/q] \oplus \mathbb{Z}/m, \quad K_1(B) = 0 \\ K_0(B)_+ &= \{(r,x) \in \mathbb{Z}[1/q] \oplus \mathbb{Z}/m \mid r > 0\} \cup \{(0,0)\} \\ \Sigma_*(B) &= \{(r,x) \mid 0 < r < 1\} \cup \{(0,0)\} \cup \{(1,0)\} \\ [1_B] &= (1,0). \end{aligned}$$

Here $\mathbb{Z}[1/q]$ denotes the group of all rational numbers whose denominators are a power of q. Let $\overline{\mathbb{Z}}_q$ denote the ring of $q-$adic integers. Then

$$Ext((K_0(B),[1_B]),\mathbb{Z}) = Ext((\mathbb{Z}[1/q],1),\mathbb{Z}) \oplus Ext(\mathbb{Z}/m,\mathbb{Z}) \cong \overline{\mathbb{Z}}_q \oplus \mathbb{Z}/m.$$

A little algebra shows that

$$Aut(K_*(B),\Sigma_*(B)) = \{id_{\mathbb{Z}[1/q]} \times \beta \mid \beta \in Aut(\mathbb{Z}/m)\}$$

It follows that

$$Ext((K_0(B),[1_B]),\mathbb{Z})/Aut(K_*(B),\Sigma_*(B)) \cong (\overline{\mathbb{Z}}_q \oplus \mathbb{Z}/m) \ / \ Aut(\mathbb{Z}/m).$$

For $m = p^n$, p a prime number, this orbit space is canonically identified with $\overline{\mathbb{Z}}_q \times \{0, 1, p, p^2, \ldots, p^{n-1}\}$. For any $(t, z) \in \overline{\mathbb{Z}}_q \times \{0, 1, p, p^2, \ldots, p^{n-1}\}$ there is a unital essential extension

$$0 \to \mathcal{K} \to A_{(t,z)} \to B \to 0.$$

such that the C^*-algebras $A_{(t,z)}$ are mutually non-isomorphic (see Proposition 17). Each $A_{(t,z)}$ is nuclear, has real rank zero and stable one. For $z \neq 0$, $A_{(t,z)}$ is not isomorphic to an inductive limit of subhomogeneous C^*-algebras (see Proposition 18). Let U be a UHF-algebra with dimension group $\mathbb{Q}$. Then $U \otimes A_{(t,z)}$ is an AF-algebra (cf. [28]). Let us specialize further and take $m = 2$, $t = 0$ with z corresponding to the group extension

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2 \to 0.$$

Setting $A = A_{(0,z)}$, it is not hard to see that

$$\begin{aligned} K_0(A) &= \mathbb{Z}[1/q] \oplus \mathbb{Z} \\ K_0(A)_+ &= \{(0, 2n) : n \in \mathbb{N}\} \cup \{(r, k) \mid r \in \mathbb{Z}[1/q],\ m > 0 \text{ and } k \in \mathbb{Z}\}. \end{aligned}$$

Therefore if $x = (0, 1)$, then $nx > 0$ if and only if n is even. We conclude that $K_0(A)$ is a torsion free perforated dimension group. This kind of perforation can not occur for AH-algebras with torsion free dimension groups. This is a consequence of Corollary 22.

Lemma 21. *Let A be a homogeneous C^*-algebra with spectrum isomorphic to a connected polyhedron. Let $x \in K_0(A)$ and suppose that $mx > 0$ for some m. Then there is k such that $nx > 0$ for all $n \geq k$.*

Proof. Write $x = [p] - [q]$ for some projections p, q in $M_r(A)$. Since $mx > 0$ and the spectrum $\hat{A}$ of A is connected it follows that $trace(\pi(p)) - trace(\pi(q)) > 0$ for any nonzero irreducible representation π. Therefore there is k such that if $n \geq k$ then $trace(\pi(p) \otimes 1_n) - trace(\pi(q) \otimes 1_n) > \frac{1}{2} dim(\hat{A})$ for all π. By induction over the number of cells of $\hat{A}$ one shows that $q \otimes 1_n$ is a subprojection of $p \otimes 1_n$ hence $nx > 0$. This argument uses the fact that the Stiefel manifold $V_s(\mathbb{C}^{s+r})$ is $2r$ connected. □

Corollary 22. *Let A be an inductive limit of continuous trace C^*-algebras and let $x \in K_*(A)$. Suppose that $mx > 0$ for some $m \in \mathbb{N}$. Then there are $y, z \in K_*(A)$ and $k \in \mathbb{N}$ such that $x = y + z$, z is a torsion element and $ny > 0$ for all $n \geq k$.*

Proof. (sketch) We may assume that A is a continuous trace C*-algebra. First we deal with the case $x \in K_0(A)$. Since $mx > 0$ it follows that $mx = [p]$ for some nonzero projection $p \in M_r(A)$. There is a direct sum decomposition $A = A_1 \oplus A_0$ where $\hat{A}_1$ is the compact-open subset of $\hat{A}$ given by the support of p. Accordingly $x = x_1 + x_0$ with $x_i \in K_0(A_i)$ and $mx_0 = 0$. Since A_1 is stably isomorphic to $pM_r(A_1)p$ we may assume that A_1 is a unital continuous trace C*-algebras. Hence A_1 is isomorphic to a finite direct sum of unital homogeneous C*-algebras. Furthermore A_1 can be written as an inductive limit of homogeneous C*-algebras with spectra homeomorphic to finite polyhedra. These spectra can have at most finitely many connected components. Using a standard approximation argument and Lemma 21 one derives the desired conclusion.

The general case is reduced to the previous situation by using the isomorphism $K_*(A)_+ \cong K_0(A \otimes C(\mathbb{T}))_+$. $\square$

Next we compute an example where the C*-algebra B is simple, the torsion subgroup of $K_0(B)$ is of type p^∞ and the group $Aut(K_*(B), \Sigma_*(B))$ is infinite.

Example 23. Let p and q be distinct prime numbers. Let X and Y be two-dimensional connected polyhedra. By a result in [15], the homotopy classes of unital $*$-homomorphisms from $C(X)$ to $M_m(C(Y))$ is given by

$$[C(X), M_m(C(Y))] \cong KK(C_0(X \setminus \{pt\}), C_0(Y \setminus \{pt\}))$$

for any $m \geq 6$. This fact together with techniques from [4], [17] or [5] can be used to construct a unital AH-algebra B of real rank zero and stable rank one with trivial K_1-group and

$$\begin{aligned} K_0(B) &= \mathbb{Z}[1/q] \oplus \mathbb{Z}(p^\infty) \\ K_0(B)_+ &= \{(r,x) \in \mathbb{Z}[1/q] \oplus \mathbb{Z}(p^\infty) \mid \quad r > 0\} \cup \{(0,0)\} \\ [1_B] &= (1,0). \end{aligned}$$

Here $\mathbb{Z}(p^\infty)$ denotes the group of all $p^n th$ complex roots of unity, with n running over all non-negative integers. In view of Proposition 17 we need to determine $Aut(K_*(B), \Sigma_*(B))$ and the orbit space of its action on $Ext((K_0(B), [1_B]), \mathbb{Z})$. Since p and q are distinct primes, it is easily seen that $Aut(K_*(B), \Sigma_*(B)) \cong id_{\mathbb{Z}[1/q]} \times Aut(\mathbb{Z}(p^\infty))$. The ring structure of $End(\mathbb{Z}(p^\infty))$ is relevant for our discussion. By Ex 3 at page 106 in [29] $End(\mathbb{Z}(p^\infty))$ is isomorphic to the ring of p-adic integers denoted here by $\overline{\mathbb{Z}}_p$. Under this isomorphism $Aut(\mathbb{Z}(p^\infty))$ corresponds to $\overline{\mathbb{Z}}_p^*$, the group of units of $\overline{\mathbb{Z}}_p$.

There are isomorphisms of groups

$$\begin{aligned} Ext((K_*(B), [1_B]), \mathbb{Z}) &\cong Ext((\mathbb{Z}[1/q], 1), \mathbb{Z}) \oplus Ext(\mathbb{Z}(p^\infty), \mathbb{Z}) \\ &\cong \overline{\mathbb{Z}}_q \oplus Hom(\mathbb{Z}(p^\infty), \mathbb{T}) \\ &\cong \overline{\mathbb{Z}}_q \oplus End(\mathbb{Z}(p^\infty)) \\ &\cong \overline{\mathbb{Z}}_q \oplus \overline{\mathbb{Z}}_p \end{aligned}$$

In view of the above discussion the action of $Aut(K_*(B), \Sigma_*(B))$ on $Ext((K_0(B), [1_B]), \mathbb{Z})$ can be identified with the action by multiplication of $1 \times \overline{\mathbb{Z}}_p^*$ on $\overline{\mathbb{Z}}_q \times \overline{\mathbb{Z}}_p$. Let $|\ |_p : \overline{\mathbb{Z}}_p \to \{0, 1, 1/p, 1/p^2, \ldots\}$ denote the p-adic valuation. The group of units $\overline{\mathbb{Z}}_p^*$ consists exactly of those p-adic integers $u \in \overline{\mathbb{Z}}_p$ for which $|u|_p = 1$. Since $|xy|_p = |x|_p |y|_p$ it is clear that $ux = y$ for some $u \in \overline{\mathbb{Z}}_p^*$ if and only if $|x|_p = |y|_p$. We conclude that the orbit space of this action is

$$\overline{\mathbb{Z}}_q \times \{0, 1, 1/p, 1/p^2, \ldots\}.$$

This space parametrizes the non-isomorphic C*-algebras in $\mathcal{E}_B$.

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