§2.2, 2.3 The inverse of a matrix

Last time: Matrix operations

$$\underset{l \times m}{A} \cdot \underset{m \times n}{B} \leftarrow l \times n$$

If A, B are square matrices of the same size

$$\underset{n \times n}{A} \cdot \underset{n \times n}{B} \leftarrow n \times n$$

$$\underset{n \times n}{B} \cdot \underset{n \times n}{A} \leftarrow n \times n$$

$AB \neq BA$ always.

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Today: Matrix inverses

If A, B are square matrices $n \times n$ and $AB = I_n$ and $BA = I_n = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$, then we write $B = A^{-1}$ and say B is the inverse of A.

$a \neq 0$ $\quad ax = b \iff x = a^{-1}b \left(= \frac{b}{a}\right)$

Say A has an inverse A^{-1}. i.e. A is invertible / nonsingular

$$A\vec{x} = \vec{b} \Rightarrow \underbrace{A^{-1}A\vec{x}}_{= I\vec{x} = \vec{x}} = A^{-1}\vec{b} \Rightarrow \vec{x} = A^{-1}\vec{b}.$$

$$A\vec{x} = \vec{b} \Leftarrow A\vec{x} = AA^{-1}\vec{b} \Leftarrow \vec{x} = A^{-1}\vec{b}$$

Eg: $A = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$ $\quad A^{-1} = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$

$$AA^{-1} = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

$$A^{-1}A = I_2$$

Eg: $A\vec{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\quad A = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$

$$\vec{x} = A^{-1}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Check: $A\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Thm: An $n \times n$ matrix A is invertible iff $A\vec{x} = \vec{b}$ has a unique solution for all $\vec{b} \in \mathbb{R}^n$.

Pf: If A is invertible, $A\vec{x} = \vec{b} \Leftrightarrow \vec{x} = A^{-1}\vec{b}$.

And if $A\vec{x} = \vec{b}$ has a unique solution, define A^{-1} so that $A^{-1}\vec{b}$ is that unique solution.

$$A^{-1} = \begin{bmatrix} A^{-1}\vec{e}_1 & A^{-1}\vec{e}_2 & \cdots & A^{-1}\vec{e}_n \end{bmatrix}$$

unique sol'n to $A\vec{x} = \vec{e}_1$.

Eg: $A = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$ $\qquad A^{-1} = \begin{bmatrix} \vec{v} & \vec{w} \end{bmatrix}$

$$A^{-1}\vec{e}_1 = \vec{v} \qquad A^{-1}\vec{e}_2 = \vec{w}.$$

$$\vec{e}_1 = A\vec{v} \qquad \vec{e}_2 = A\vec{w}.$$

Solve these!

$$\left[\begin{array}{cc|c} 3 & 5 & 1 \\ 1 & 2 & 0 \end{array}\right] \qquad \left[\begin{array}{cc|c} 3 & 5 & 0 \\ 1 & 2 & 1 \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 3 & 5 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 3 & 5 & 1 & 0 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & -1 & 1 & -3 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cc|cc} 1 & 0 & 2 & -5 \\ 0 & -1 & 1 & -3 \end{array}\right]$$

$$\rightsquigarrow \begin{array}{c} \begin{array}{cccc} & & \vec{v} & \vec{w} \end{array} \\ \left[\begin{array}{cc|cc} 1 & 0 & 2 & -5 \\ 0 & 1 & -1 & 3 \end{array}\right] \\ A^{-1} \end{array}.$$

A is an $n \times n$ matrix.

The following are equivalent:

- A is invertible
- $A\vec{x} = \vec{b}$ has a unique solution for all $b \in \mathbb{R}^n$.
- A has n pivot positions (check REF(A)).
- A is row equivalent to I_n

- $A\vec{x}=\vec{0}$ has no nontrivial solutions
- The columns of A are linearly independent
- $T(\vec{x})=A\vec{x}$ is one-to-one.
- $T(\vec{x})=A\vec{x}$ is onto.
- There is a matrix C such that $CA=I$.
- There is a matrix D such that $AD=I$
- A^T is invertible.

$$(A^{-1})^T=(A^T)^{-1}$$

Check: $(A^{-1})^T A^T = (AA^{-1})^T = I^T$
$= I.$

Eg: $A=\begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}$

Is A is invertible and if so what is A^{-1}?

Row reduce

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{array}\right]$$

If $RREF(A)=I_3$, then

$$[A \mid I_3] \sim [I_3 \mid A^{-1}].$$

Otherwise A is not invertible.

$$\left[\begin{array}{ccc|ccc} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & -3 & -4 & 0 & -4 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -9/2 & 7 & -3/2 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 2 & 3 & -4 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -9/2 & 7 & -3/2 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{array}\right]$$

A^{-1}.

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible iff

$\det A = ad - bc$ is nonzero.

If so, $A^{-1} = \dfrac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$= \begin{bmatrix} \dfrac{d}{ad-bc} & -\dfrac{b}{ad-bc} \\ \dfrac{-c}{ad-bc} & \dfrac{a}{ad-bc} \end{bmatrix}.$$

Inverse matrices give another perspective on row reduction.

Row operations are linear transformations.

eg. $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \overset{R_3 \mapsto R_3 + 2R_1}{\rightsquigarrow} \begin{bmatrix} x \\ y \\ z+2x \end{bmatrix} = T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right)$.

What is the standard matrix for T?

$$\begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & T(\vec{e}_3) \end{bmatrix}$$

row operation applied to $\vec{e}_1, \vec{e}_2, \vec{e}_3$.

It is the result of applying $R_3 \mapsto R_3 + 2R_1$ to I_3 ie

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} = E.$$

elementary matrix

So applying $R_3 \mapsto R_3 + 2R_1$ is the same as multiplying by E.

Eg: $\underbrace{\begin{bmatrix} 2 & 1 & 3 \\ -1 & 4 & 6 \\ 3 & 1 & 4 \end{bmatrix}}_{B} \overset{R_3 \mapsto R_3 + 2R_1}{\rightsquigarrow} \underbrace{\begin{bmatrix} 2 & 1 & 3 \\ -1 & 4 & 6 \\ 7 & 3 & 10 \end{bmatrix}}_{EB}$.

Note that $E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$ which

corresponds to the inverse elementary row operation $R_3 \mapsto R_3 - 2R_1$.

Elementary matrices are invertible!

A <u>new perspective</u>: Row reduction is the same as repeatedly multiplying the equation $A\vec{x} = \vec{b}$ by <u>invertible</u> matrices

$$A\vec{x} = \vec{b}$$

$$E_1 A\vec{x} = E_1 \vec{b}$$

$$E_2 E_1 A\vec{x} = E_2 E_1 \vec{b}$$

$$\vdots$$

$$\text{RREF}(A) = E_k \cdots E_1 A = E_k \cdots E_1 \vec{b}$$

If A is invertible, $\text{RREF}(A) = I$

and $E_k \cdots E_1 = A^{-1}$.

$$A = E_1^{-1} \cdots E_k^{-1}.$$