§2.8 Subspaces of $\mathbb{R}^n$

Recall: $A\vec{x} = \vec{b}$

Solution sets are point, line, plane, etc.

Today: Subspaces are points, lines, planes, etc which go through the origin.

Def: A subspace of $\mathbb{R}^n$ is a subset H of vectors such that

1) If $\vec{v}, \vec{w} \in H$, then $\vec{v}+\vec{w} \in H$.
2) If $\vec{v} \in H$, $c \in \mathbb{R}$, then $c\vec{v} \in H$.
3) H is nonempty. ie $\vec{0} \in H$.

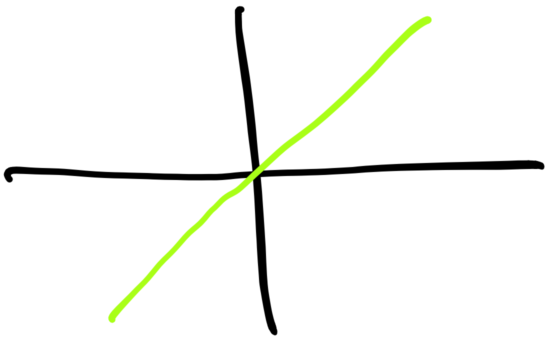
2) & 3) $\Rightarrow \vec{0} \in H$

Reason: $\vec{v} \in H \Rightarrow \underbrace{0\vec{v}}_{=\vec{0}} \in H$.

Eg: $\{\vec{0}\} \subset \mathbb{R}^n$ is a subspace.

zero subspace

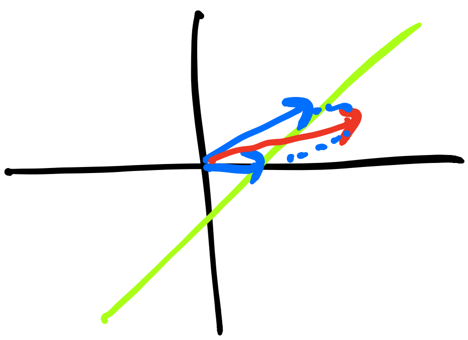
Eg: $\left\{ \begin{bmatrix} a \\ a \end{bmatrix} : a \in \mathbb{R} \right\} \subset \mathbb{R}^2$ is a subspace.



Eg: $\mathbb{R}^2 \subset \mathbb{R}^2$ is a subspace.

Non-egs: i) $\{\ \}$ is not a subspace.

ii) $L = \left\{ \begin{bmatrix} a+1 \\ a \end{bmatrix} : a \in \mathbb{R} \right\} \subset \mathbb{R}^2$ is not a subspace

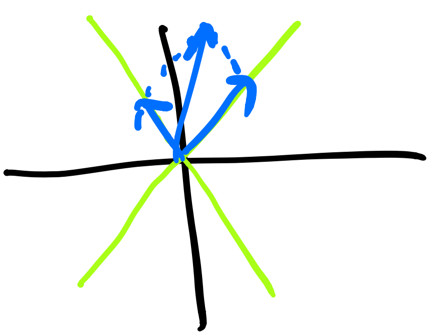


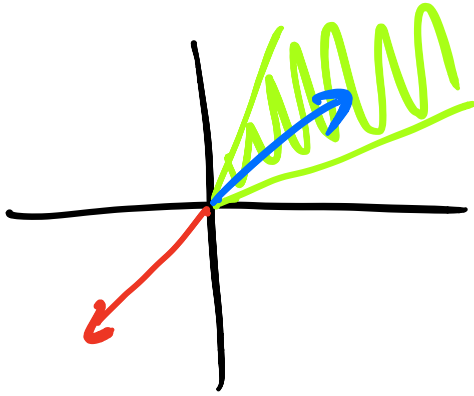
$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \in L$

$\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \notin L$

$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \in L$, but $0 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \notin L$

$2 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \notin L$

iii)  is not a subspace.

iv)  is not a subspace.

Eg: The span of a set of vectors in $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$.

$\vec{x} = c_1\vec{v}_1 + \dots + c_p\vec{v}_p \in \text{Span}\{\vec{v}_1, \dots, \vec{v}_p\}$

$\vec{y} = d_1\vec{v}_1 + \dots + d_p\vec{v}_p \in$

$\vec{x}+\vec{y} = (c_1+d_1)\vec{v}_1 + \dots + (c_p+d_p)\vec{v}_p$

$a\vec{x} = ac_1\vec{v}_1 + \dots + ac_p\vec{v}_p \in$

$\text{Span}\{\vec{v}_1, \dots, \vec{v}_p\}$ is the subspace spanned by $\vec{v}_1, \dots, \vec{v}_p$.

explicit description of a subspace.

Eg: If A is an $m \times n$ matrix,

$\text{Col}(A)$ = span of the column vectors of A

is a subspace of $\mathbb{R}^m$.

Eg: If A is an $m \times n$ matrix,

$\text{Nul}(A) = \{\vec{x} \in \mathbb{R}^n : A\vec{x} = \vec{0}\}$ null space of A

is a subspace. implicit description of a subspace.

Why? 1) If $\vec{x}, \vec{y} \in \text{Nul}(A)$, then
$$A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y} = \vec{0} + \vec{0} = \vec{0},$$
and so $\vec{x}+\vec{y} \in \text{Nul}(A)$.

2) If $\vec{x} \in \text{Nul}(A)$, $c \in \mathbb{R}$, then
$$A(c\vec{x}) = c(A\vec{x}) = c \cdot \vec{0} = \vec{0}.$$

3) $\vec{0} \in \text{Nul}(A)$.

Def: A basis for a subspace $H \subset \mathbb{R}^n$ is a set

$\{\vec{v}_1, \dots, \vec{v}_p\} \subset H$ such that

1) $\{\vec{v}_1, \dots, \vec{v}_p\}$ is linearly independent

2) $\text{Span}\{\vec{v}_1, \dots, \vec{v}_p\} = H$.

Eg: $H = \left\{ \begin{bmatrix} a \\ a \end{bmatrix} : a \in \mathbb{R} \right\} < \mathbb{R}^2$

$H = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ is linearly independent.

So $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ is a basis for H.

Non-eg: $H = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\}$,
but $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\}$ is linearly dependent,
so $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\}$ is not a basis for H.

Example: Find a basis for the null and column spaces of $A = \begin{bmatrix} 3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & 4 & 5 & 8 & -4 \end{bmatrix}$.

$\text{Nul}(A)$ is the solution set to $A\vec{x} = \vec{0}$.

$$\begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ -3 & 6 & -1 & 1 & -7 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

$$\rightsquigarrow \begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 5 & 10 & -10 \\ 0 & 0 & 1 & 2 & -2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 5 & 10 & -10 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 2 & 3 & -1 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -2 & 0 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$x_1 = 2x_2 + x_4 - 3x_5$$

$$x_3 = -2x_4 + 2x_5$$

$$\begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}.$$

$$\text{Nul}(A) = \text{Span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

a basis

These are linearly independent.

$$\text{Col}(A) = \text{Span}\left\{ \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 6 \\ -2 \\ -4 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 8 \end{bmatrix}, \begin{bmatrix} -7 \\ -1 \\ -4 \end{bmatrix} \right\}$$

$$= \text{Span}\left\{ \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} \right\}$$

is a basis.

Take the pivot columns of A and not of RREF(A).

Why does this work?

Key: If we ignore free columns, row reduction steps are the same!

First, Col(A) is the set of $\vec{b}$ such that $A\vec{x} = \vec{b}$ is consistent.

So $A\vec{x} = \vec{b}$ is consistent iff $\begin{bmatrix} 3 & -1 \\ 1 & 2 \\ 2 & 5 \end{bmatrix}\vec{x} = \vec{b}$ is consistent.

pivot columns

Hence, $\mathrm{Col}(A) = \mathrm{Col}\left(\begin{bmatrix} 3 & -1 \\ 1 & 2 \\ 2 & 5 \end{bmatrix}\right)$

$= \mathrm{Span}\left\{\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}\right\}.$

Second, $\begin{bmatrix} 3 & -1 \\ 1 & 2 \\ 2 & 5 \end{bmatrix}$ has no free columns since we removed them, so

$$\left\{ \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} \right\}$$ is linearly independent

In conclusion, $\left\{ \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix} \right\}$ is a <u>basis</u> for $\mathrm{Col}(A)$.