§3.1 Introduction to determinants

Today: Introduce a "new" criterion for the invertibility of matrices using determinants.

Eg: 1×1 $\quad \det([a]) = a$

Eg: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{a\neq 0} \begin{bmatrix} a & b \\ ac & ad \end{bmatrix} \rightsquigarrow \begin{bmatrix} a & b \\ 0 & ad-bc \end{bmatrix}$

has two pivots iff $\underbrace{ad-bc}_{\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)} \neq 0$

If $a=0$, then

$\begin{bmatrix} 0 & b \\ c & d \end{bmatrix} \rightsquigarrow \begin{bmatrix} c & d \\ 0 & b \end{bmatrix}$

has two pivots iff $b\neq 0$ and $c\neq 0$

iff $bc \neq 0$

$ad-bc$ since $a=0$.

Summary: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible iff $\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$

$= ad-bc \neq 0$.

How about larger matrices? Is there some function $\det : \{n\times n \text{ matrices}\} \to \mathbb{R}$ such that A is invertible iff $\det(A)$ is nonzero.

Yes. What is it? It's really complicated.

How is it computed? Row reduction. Next time.

Eg: 3×3

$$A=\begin{bmatrix}a&b&c\\d&e&f\\g&h&i\end{bmatrix} \overset{a\neq 0}{\rightsquigarrow} \begin{bmatrix}a&b&c\\0&ae-bd&*\\g&h&i\end{bmatrix} \overset{ad-bc\neq 0}{\rightsquigarrow} \begin{bmatrix}a&b&c\\0&ae-bd&*\\0&0&?\end{bmatrix}$$

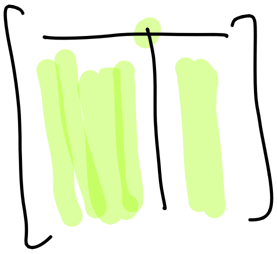
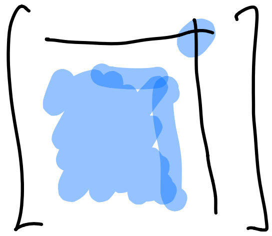
$$\begin{aligned} ? &= aei + bfg + cdh - afh - bdi - ceg \\ &= \det(A) \end{aligned}$$

$$\begin{bmatrix}a&b&c\\d&e&f\\g&h&i\end{bmatrix} - \begin{bmatrix}a&b&c\\d&e&f\\g&h&i\end{bmatrix}$$

In general, the formula is hard to remember.

$$\det(A) = a(ei-fh) - b(di-fg) + c(dh-eg)$$

$$\begin{bmatrix}a&b&c\\d&e&f\\g&h&i\end{bmatrix} \quad \begin{bmatrix}a&b&c\\d&e&f\\g&h&i\end{bmatrix} \quad \begin{bmatrix}a&b&c\\d&e&f\\g&h&i\end{bmatrix}$$

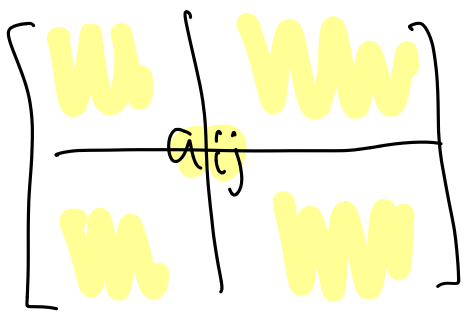
$+ \bullet \det$  $- \bullet \det$ 

Cofactor expansion

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & \cdots & \\ \vdots & \vdots & & \ddots \end{bmatrix}$$

$$C_{11} = \det \begin{bmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & \ddots & \vdots \\ a_{42} & \cdots & \end{bmatrix}$$

(1,1) cofactor



$C_{ij} = \det$ 

(i,j) cofactor

$$\det \begin{bmatrix} 0 & -2 & 0 \\ 2 & 4 & -1 \\ 1 & 5 & 0 \end{bmatrix} = -(-2) \det \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$$

$$= -(-2)\left(2 \cdot 0 - (-1)1\right)$$

$$= 2.$$

Nontrivial fact:
One can do cofactor expansion along any row or column to compute $\det(A)$.

$$\det\begin{bmatrix} 0 & -2 & 0 \\ 2 & 4 & -1 \\ 1 & 5 & 0 \end{bmatrix} = -2\det\begin{bmatrix} -2 & 0 \\ 5 & 0 \end{bmatrix} + 4\det\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$-(-1)\det\begin{bmatrix} 0 & -2 \\ 1 & 5 \end{bmatrix}$$

$$= -(-1)\left(0\cdot 5 - (-2)\cdot 1\right)$$

$$= 2$$

$$\begin{bmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \ddots \\ \vdots & & \vdots & & \ddots \end{bmatrix}$$

checkerboard sign pattern.

$$\det\begin{bmatrix} 0 & -2 & 0 \\ 2 & 4 & -1 \\ 1 & 5 & 0 \end{bmatrix} = -(-1)\det\begin{bmatrix} 0 & -2 \\ 1 & 5 \end{bmatrix}$$

$$= -(-1)\left(0\cdot 5 - (-2)\cdot 1\right)$$

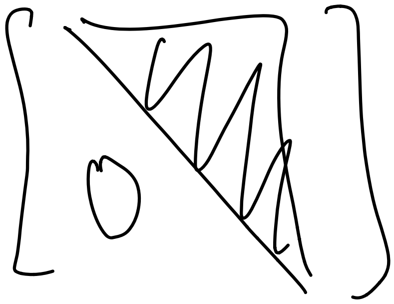
$$= 2.$$

$$\det\begin{bmatrix} 3 & -7 & 8 & 9 & -6 \\ 0 & 2 & -5 & 7 & 3 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 2 & 4 & -1 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix} = 3\det\begin{bmatrix} 2 & -5 & 7 & 3 \\ 0 & 1 & 5 & 0 \\ 0 & 2 & 4 & -1 \\ 0 & 0 & -2 & 0 \end{bmatrix}$$

$$= 3 \cdot 2 \det \begin{bmatrix} 1 & 5 & 0 \\ 2 & 4 & -1 \\ 0 & -2 & 0 \end{bmatrix} = 3 \cdot 2 \cdot 1 \det \begin{bmatrix} 2 & 4 \\ 0 & -2 \end{bmatrix}$$

$$= 3 \cdot 2 \cdot 1 \cdot 2 (-2)$$

$$= -24.$$

det  = product of diagonal entries.

eg a matrix in REF.