§1.9 The matrix of a linear transformation

<u>Last time</u>: Matrix transformations

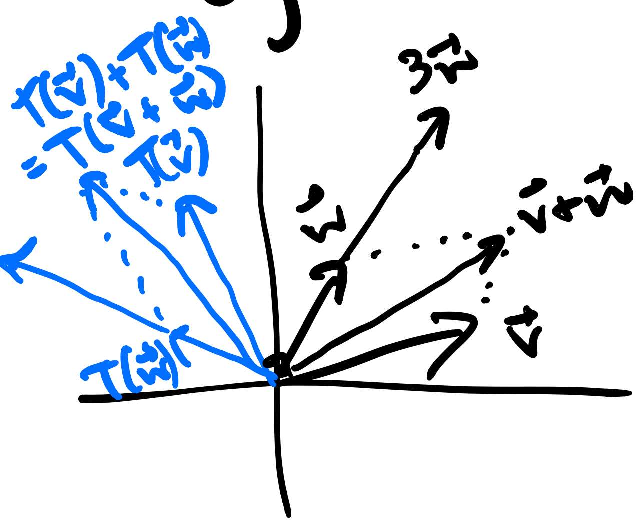
$$\vec{x} \mapsto A\vec{x}$$

These are linear transformations

<u>Today</u>: All linear transformations are matrix transformations.

$T(\vec{x}) = A\vec{x}$

Eg: $T : \mathbb{R}^2 \to \mathbb{R}^2$ is rotation by $90°$ counterclockwise.

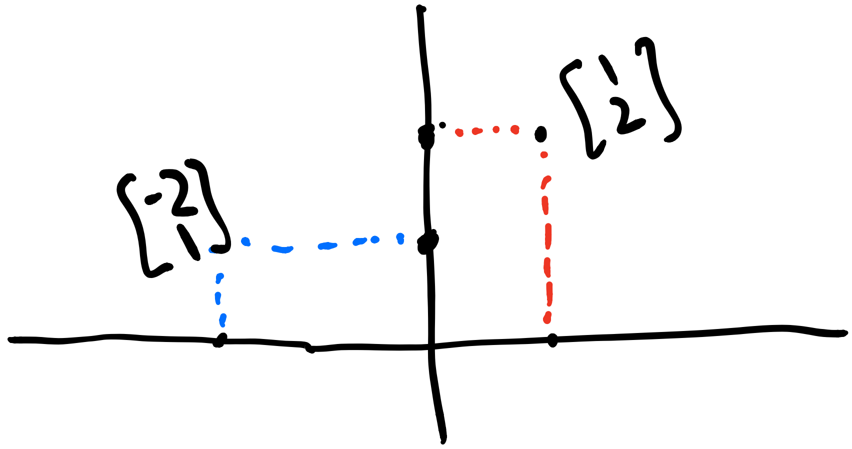


This a linear transformation

It preserves parallelograms and scaling.

What is $T\left(\begin{bmatrix}1\\2\end{bmatrix}\right)$?

$$\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} + 2\begin{bmatrix}0\\1\end{bmatrix}$$



$$
\begin{aligned}
T\left(\begin{bmatrix}1\\2\end{bmatrix}\right) &= T\left(\begin{bmatrix}1\\0\end{bmatrix} + 2\begin{bmatrix}0\\1\end{bmatrix}\right)\\
&= T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) + T\left(2\begin{bmatrix}0\\1\end{bmatrix}\right)\\
&= T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) + 2T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)\\
&= \begin{bmatrix}0\\1\end{bmatrix} + 2\begin{bmatrix}-1\\0\end{bmatrix}\\
&= \begin{bmatrix}-2\\1\end{bmatrix}.
\end{aligned}
$$

$$
\begin{aligned}
T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) &= x\begin{bmatrix}1\\0\end{bmatrix} + y\begin{bmatrix}0\\1\end{bmatrix}\\
&= T\left(x\begin{bmatrix}1\\0\end{bmatrix} + y\begin{bmatrix}0\\1\end{bmatrix}\right)
\end{aligned}
$$

$$= x\,T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) + y\,T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$$

$$= x\begin{bmatrix}0\\1\end{bmatrix} + y\begin{bmatrix}-1\\0\end{bmatrix}$$

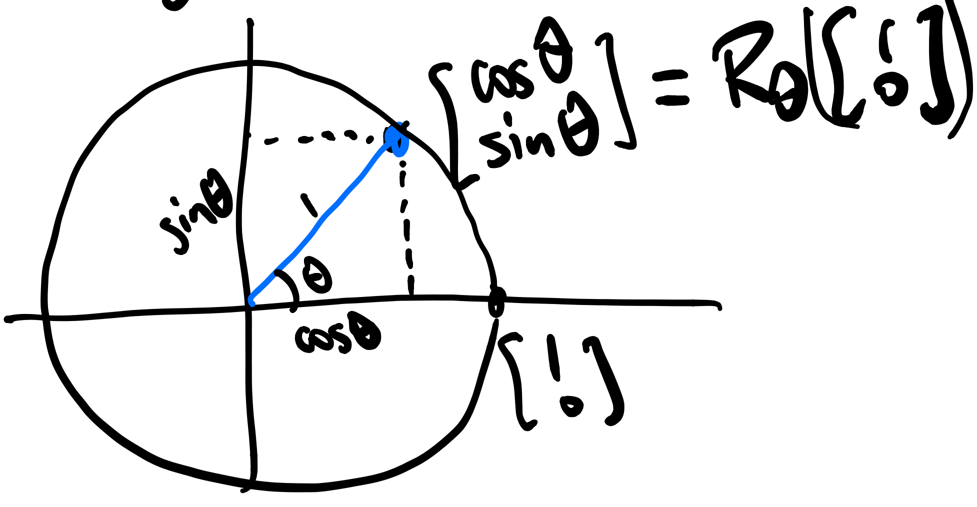
$$T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

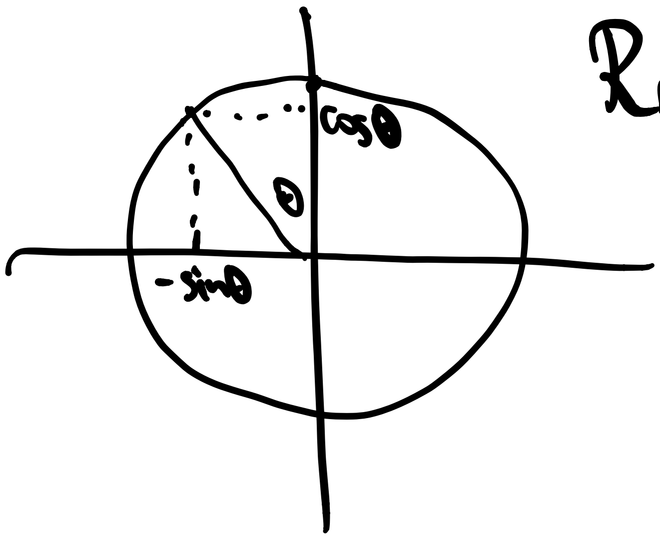
Eg: $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$

is rotation by θ counter-clockwise

What is $R_\theta\left(\begin{bmatrix}x\\y\end{bmatrix}\right)$?

$$R_\theta\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = x\,R_\theta\left(\begin{bmatrix}1\\0\end{bmatrix}\right) + y\,R_\theta\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$$





$$R_\theta\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix}$$

$$R_\theta\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{bmatrix}R_\theta\begin{bmatrix}1\\0\end{bmatrix} & R_\theta\begin{bmatrix}0\\1\end{bmatrix}\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

$$= \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

$$= \begin{bmatrix}x\cos\theta - y\sin\theta\\ x\sin\theta + y\cos\theta\end{bmatrix}$$

Eg: $T : \mathbb{R}^2 \to \mathbb{R}^n$ linear transformation

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \vec{v}, \quad T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \vec{w}.$$

$$\begin{aligned}
T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) &= T\left(x\begin{bmatrix}1\\0\end{bmatrix} + y\begin{bmatrix}0\\1\end{bmatrix}\right) \\
&= xT\left(\begin{bmatrix}1\\0\end{bmatrix}\right) + yT\left(\begin{bmatrix}0\\1\end{bmatrix}\right) \\
&= x\vec{v} + y\vec{w} \\
&= \begin{bmatrix} | & | \\ \vec{v} & \vec{w} \\ | & | \end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}.
\end{aligned}$$

<u>standard matrix</u> for the linear transformation T

<u>The general case</u>:

$T : \mathbb{R}^n \to \mathbb{R}^m$ linear transformation

$$T\left(\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}\right) = T(x_1\vec{e}_1 + \dots + x_n\vec{e}_n)$$

$$\vec{e}_i = \begin{array}{c} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \end{array} \begin{array}{l} \\ \\ \\ \leftarrow i \\ \\ \\ \\ \end{array}$$

$$= x_1T(\vec{e}_1) + x_2T(\vec{e}_2) + \dots + x_nT(\vec{e}_n)$$

$$= \underbrace{\begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) & \cdots & T(\vec{e}_n) \end{bmatrix}}_{\text{standard matrix for } T.} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

<u>Thm</u>: Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and A be its standard matrix $(m \times n)$

1) T is onto iff $Col(A) = \mathbb{R}^m$

iff there is a pivot position in every row.

range = codomain

2) T is one-to-one iff the columns are linearly independent iff there is a pivot position in every column.