4.5 **The dimension of a vector space**

Review:

- If $\vec{v}_1, \dots, \vec{v}_p$ are vectors in vector space V, then

  $\text{span}\{\vec{v}_1, \dots, \vec{v}_p\}$ is a subspace of V

  that consists of all linear combinations

  $c_1 \vec{v}_1 + \dots + c_p \vec{v}_p$ with c_i in $\mathbb{R}$.

  One says that $\vec{v}_1, \dots, \vec{v}_p$ spans V if

  $$\text{span}\{\vec{v}_1, \dots, \vec{v}_p\} = V$$

- $\vec{v}_1, \dots, \vec{v}_p$ are linearly independent if the vector equation $c_1 \vec{v}_1 + \dots + c_p \vec{v}_p = \vec{0}$ admits only one solution $c_1 = \dots = c_p = \vec{0}$

- $\vec{v}_1, \dots, \vec{v}_p$ form a basis of V if they satisfy (i) and (ii)

  (i) $\text{span}\{\vec{v}_1, \dots, \vec{v}_p\} = V$

  (ii) $\vec{v}_1, \dots, \vec{v}_p$ are linearly independent

Remarks

$\begin{bmatrix}1\\0\\0\end{bmatrix} \begin{bmatrix}1\\1\\0\end{bmatrix} \begin{bmatrix}1\\1\\1\end{bmatrix} \begin{bmatrix}-1\\2\\100\end{bmatrix}$ span $\mathbb{R}^3$

$\begin{bmatrix}1\\0\\0\end{bmatrix} \begin{bmatrix}1\\1\\0\end{bmatrix}$ are linearly independent

$\begin{bmatrix}1\\0\\0\end{bmatrix} \begin{bmatrix}1\\1\\0\end{bmatrix} \begin{bmatrix}1\\1\\1\end{bmatrix}$ basis of $\mathbb{R}^3$

$\begin{bmatrix}1\\1\\0\end{bmatrix} \begin{bmatrix}1\\1\\1\end{bmatrix} \begin{bmatrix}-1\\2\\100\end{bmatrix}$ another basis of $\mathbb{R}^3$

- Every set of vectors that spans V contains a subset which forms a basis

- If a list of vectors $\bar{v}_1, \dots, \bar{v}_p$ in V is linearly independent and if $\bar{v}_1, \dots, \bar{v}_p, \bar{v}$ is linearly dependent for any $\bar{v}$ in V, then $\bar{v}_1, \dots, \bar{v}_p$ basis of V

· A basis is a maximal linearly independent subset

· A basis is a minimal spanning subset

$\boxed{\text{4.5 New content}}$

A vector space V is finite dimensional if it is spanned by a finite set of vectors

$\mathbb{P}_3$ is finite dimensional $\quad 1, t, t^2, t^3$ basis

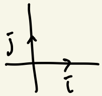
$\mathbb{P}$ is not finite dimensional

$$\text{span}\{1, t, t^2, \dots, t^n, \dots\} = \mathbb{P}$$

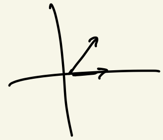
Theorem
- A vector space V has a basis
- Any two bases have the same number of elements
- The dimension of V denoted, $\dim(V)$, is defined as the number of elements in a basis.

$\dim\{0\} = 0$

$\dim(\mathbb{R}^n) = n \qquad \begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}, \begin{bmatrix}0\\1\\\vdots\\0\end{bmatrix}, \ldots, \begin{bmatrix}0\\\vdots\\0\\1\end{bmatrix} \leftarrow$ basis



$\mathbb{R}^2 \qquad \left\{ \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix} \right\} \qquad \left\{ \begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix} \right\}$



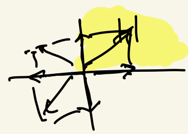
$\dim(\mathbb{P}_n) = n+1$ basis $\{1, t, \ldots, t^n\}$

$\mathbb{P}_n$ polynomials of degree $\leq n$

Properties $\dim(V) = n$ V vector space

(1) If H subspace of V then $\dim(H) \leq \dim(V)$

(2) If H subspace of V and $\dim(H) = \dim(V)$ then $H = V$



$\mathbb{R}^2$

(3) If $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_n$ are linearly independent in V then they form a basis of V.

(4) If $b_1, \ldots, b_n$ are in V and $\text{Span}\{b_1, \ldots, b_n\} = V$ then $b_1, \ldots, b_n$ are linearly indep.

Ex1 Find a basis for the subspace of $\mathbb{R}^4$:

$$\left\{ \begin{bmatrix} a+b+3c \\ -b-c-d \\ a+2b+4c+d \\ b+c+d \end{bmatrix} : a, b, c, d \text{ in } \mathbb{R} \right\}$$

4×1

$$a\begin{bmatrix}1\\0\\1\\0\end{bmatrix} + b\begin{bmatrix}1\\-1\\2\\1\end{bmatrix} + c\begin{bmatrix}3\\-1\\4\\1\end{bmatrix} + d\begin{bmatrix}0\\-1\\1\\1\end{bmatrix}$$

basis

Form matrix $A = \begin{bmatrix} 1 & 1 & 3 & 0 \\ 0 & -1 & -1 & -1 \\ 1 & 2 & 4 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$

subspace $= \text{Col}(A)$ $\quad$ $\text{RREF}(A) = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\dim(\text{Col}(A)) = \text{rank}(A) = 3$

basis

Answer basis is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix} \right\}$

Ex 2 Do the polynomials

$1-t, \; 1+t, \; t^2-1, \; t^3+t^2$

span $\mathbb{P}_3$? Do they form a basis?

Sol'n $\quad \dim(\mathbb{P}_3) = 4 \quad$ basis $1, t, t^2, t^3$

If we show that the given poly's are linearly independent then they form a basis.

$c_1(1-t) + c_2(1+t) + c_3(t^2-1) + c_4(t^3+t^2) = 0$

$(\underbrace{c_1 + c_2 - c_3}_{=0}) + (\underbrace{-c_1 + c_2}_{=0})t + (\underbrace{c_3 + c_4}_{=0})t^2 + \underbrace{c_4}_{=0} t^3 = 0$

$\rightarrow \begin{bmatrix} 1 & 1 & -1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$\text{rank}(A) = 4 \qquad \det(A) \neq 0$

unique soln $\quad c_1 = c_2 = c_3 = c_4 = 0$

thus linear indep. thus basis.

In general know this $\quad A \quad m \times n$

$\text{Col}(A) \quad - \quad \dim(\text{Col}(A)) = \text{rank}(A)$

$\qquad\qquad\qquad\qquad \shortparallel$

$\text{Row}(A) \quad \leftarrow \quad \dim(\text{Row}(A))$

$\text{Nul}(A) \quad \leftarrow \quad \dim(\text{Nul}(A)) = \text{nullity}(A)$

$\boxed{\text{Rank theorem:}}$

$$\boxed{\text{rank}(A) + \text{nullity}(A) = \text{number of columns}}$$

$\text{rank}(A) \leq \min\{m, n\}$

$\text{rank}(A) \leq m \quad$ since columns of A are vectors in $\mathbb{R}^m$

$\text{rank}(A) \leq n \quad$ since there are n-columns

$\text{rank}(A) = \text{rank}(A^T)$

$\text{Col}(A^T) = \text{Row}(A)$

— END —

$\text{Nul}(A) = \{\bar{x} \text{ in } \mathbb{R}^n : A\bar{x} = \bar{0}\}$ subspace of $\mathbb{R}^n$

$\text{nullity}(A) = \dim(\text{Nul}(A)) = n - \text{rank}(A)$

$= n - \{\text{number of pivots in RREF}(A)\}$