5.1. Eigenvectors and eigenvalues of a square matrix.

Def'n Let A be $n \times n$ square matrix

- $\vec{x}$ in $\mathbb{R}^n$ is an eigenvector of A if $\vec{x} \neq \vec{0}$ and $A\vec{x} = \lambda \vec{x}$ for some λ in $\mathbb{R}$

- λ in $\mathbb{R}$ is an eigenvalue of A if there is a non-zero vector $\bar{x}$ in $\mathbb{R}^n$ such that $A\bar{x} = \lambda \bar{x}$.

  $\bar{x}$ is an eigenvector corresponding to λ

Ex 1 (i) $\bar{x} = \begin{bmatrix}1\\1\end{bmatrix}$ is an eigenvector for $A = \begin{bmatrix}2&2\\2&2\end{bmatrix}$.

$$\underbrace{A\bar{x}}_{} = \underbrace{\begin{bmatrix}2&2\\2&2\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}1\\1\end{bmatrix}}_{\bar{x}} = \begin{bmatrix}4\\4\end{bmatrix} = 4\begin{bmatrix}1\\1\end{bmatrix} = 4\bar{x} \qquad \bar{x} \neq \bar{0}$$

it corresponds to the eigenvalue $\lambda = 4$

(ii) $\bar{x} = \begin{bmatrix}1\\-1\end{bmatrix}$ is an eigenvector for $A = \begin{bmatrix}2&2\\2&2\end{bmatrix}$

$$A\bar{x} = \begin{bmatrix}2&2\\2&2\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} = 0\begin{bmatrix}1\\-1\end{bmatrix} = 0 \cdot \bar{x}$$

$$\bar{x} = \begin{bmatrix}1\\-1\end{bmatrix} \neq \begin{bmatrix}0\\0\end{bmatrix}$$

Thus $\begin{bmatrix}1\\-1\end{bmatrix}$ eigenvector corresponding to eigenvalue $\lambda = 0$.

Note: eigenvalues can be equal to zero unlike eigenvectors

Ex 2 Is $\lambda = 3$ an eigenvalue for $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$?

Rephrase: Does the equation $A\bar{x} = 3\bar{x}$ have a non-zero solution ?

$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 3 \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$x_1 + x_2 = 3x_1$$
$$x_1 + x_2 = 3x_2$$

$$\begin{cases} -2x_1 + x_2 = 0 \\ x_1 - 2x_2 = 0 \end{cases}$$

$$2x_1 - 4x_2 = 0$$

$\Rightarrow$ $x_2 = 0$, $x_1 = 0$ $\quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Answer: 3 is not an eigenvalue of A since the equation $A\bar{x} = 3\bar{x}$ admits only the sol'n $\bar{x} = \bar{0}$.

Ex 3 Show that $\lambda = 4$ is an eigenvalue for $A = \begin{bmatrix} 4 & -3 \\ 0 & 1 \end{bmatrix}$. Find all the eigenvectors corresponding to $\lambda = 4$.

$\lambda = 4$ is an eigenvalue $\Longleftrightarrow$ $A\bar{x} = 4\bar{x}$ has non-zero solutions

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$A\bar{x} - 4\bar{x} = 0$ $\qquad$ $(A - 4I)\bar{x} = 0$

$$A - 4I = \begin{bmatrix} 4 & -3 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ 0 & -3 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -3 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-3x_2 = 0$$
$$-3x_2 = 0$$

$$x_2 = 0$$
$$x_1 = s \qquad s \in \mathbb{R}$$

$\downarrow$ RREF

$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\qquad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} s \\ 0 \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$x_1 = s$

All "non-zero" eigenvectors are $\left\{ s \begin{bmatrix} 1 \\ 0 \end{bmatrix} : s \text{ in } \mathbb{R},\ s \neq 0 \right\}$

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -101 \\ 0 \end{bmatrix}, \ldots$

Note $I = I_n = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots & 1 \end{bmatrix}$ $\qquad \bar{x}$ in $\mathbb{R}^n$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$I \bar{x} = \bar{x}$

$\lambda I = \begin{bmatrix} \lambda & & 0 \\ & \lambda & \\ 0 & & \ddots & \lambda \end{bmatrix}$ $\qquad (\lambda I)\bar{x} = \lambda \bar{x}$

λ in $\mathbb{R}$

IMPORTANT CONCEPTUAL REMARK:

The following assertions are equivalent for A $n \times n$

1. λ is an eigenvalue of A
2. $A\bar{x} = \lambda \bar{x}$ has a non-zero solution
3. $(A - \lambda I)\bar{x} = \bar{0}$ has a non-zero solution
4. $\det(A - \lambda I) = 0$

Reminder Say B $(n \times n)$ matrix (4)

When does $B\bar{x} = \bar{0}$ has only the trivial sol'n $\bar{x} = \bar{0}$.

$\iff$ $\dim(\mathrm{Nul}(B)) = 0$

$\iff$ $\mathrm{rank}(B) = n$

$\iff$ B invertible

$\iff$ $\det(B) \neq 0$.

Apply this to $\underline{\underline{B = A - \lambda I}}$

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x in $\mathbb{R}$ $3x = 0$ $\Rightarrow$ $x = 0$

$\frac{1}{3} \cdot (3x) = \frac{1}{3} \cdot 0$ $\underbrace{\left(\frac{1}{3} \cdot 3\right)}_{1} x = \frac{1}{3} \cdot 0$

$1 \cdot x = 0$

$x = 0$ ✓

A $n \times n$ $\bar{x}$ in $\mathbb{R}^n$

$A\bar{x} = \bar{0}$

Say A invertible $A^{-1} \cdot A = I$

$\underbrace{A^{-1} A}\, \bar{x} = A^{-1} \cdot 0$

$I \cdot \bar{x} = \bar{0}$

$\bar{x} = \bar{0}$.

$\underline{\det(B) \neq 0}$ $B^{-1} = \frac{1}{\det(B)} \mathrm{adj}(B)$

Fact: The eigenvalues of a triangular matrix are the entries on the main diagonal.

Ex 4 $A = \begin{bmatrix} -1 & 10 & 5 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ has eigenvalues $-1, 3, 1$

$$A - \lambda I = \begin{bmatrix} -1-\lambda & 10 & 5 \\ 0 & 3-\lambda & 2 \\ 0 & 0 & 1-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = -(\lambda+1)(3-\lambda)(1-\lambda)$$

$$\det(A - \lambda I) = 0 \iff \text{if } \lambda = -1, 3, 1$$

$$\operatorname{rank}(A - \lambda I) \le 2$$

$$\Rightarrow \text{Nullity}(A - \lambda I) \ge 3 - 2 = 1$$

Thus there is $\bar{x} \neq \bar{0}$ s.t. $(A - \lambda I)\bar{x} = \bar{0}$

$\bar{x}$ in $\text{Null}(A - \lambda I)$

$\bar{x} \neq \bar{0}$.

Note $\bar{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$A\bar{x} = \begin{bmatrix} -1 & 10 & 5 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = (-1)\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Ex 5 Find a basis of the eigenspace of $A = \begin{bmatrix} 6 & 0 & 1 & 0 \\ 4 & 3 & 6 & 0 \\ 2 & -1 & 8 & 0 \\ 3 & -3 & 6 & 5 \end{bmatrix}$

corresponding to $\lambda = 5$

all vectors $\bar{x}$ in $\mathbb{R}^4$ such that

$A\bar{x} = 5\bar{x}$ $\quad (A - 5I)\bar{x} = \bar{0}$

Rephrase: Find a basis of $\text{Nul}(A - 5I)$.

$$A - 5I = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 4 & -2 & 6 & 0 \\ 2 & -1 & 3 & 0 \\ 3 & -3 & 6 & 0 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\qquad\qquad\qquad\qquad s \quad t$

rank $= 2$ $\quad$ nullity $= 2$

$x_3 = s \quad x_4 = t$

$x_1 + x_3 = 0$

$x_2 - x_3 = 0$

$x_3 = s$

$x_4 = t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

FIND basis. $\left\{ \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$