5.2 The characteristic equation

A $n \times n$ square matrix $\qquad I_n = I = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots & 1 \end{bmatrix}$

A number λ in $\mathbb{R}$ is an eigenvalue of A if

(1) There is $\vec{x}$ in $\mathbb{R}^n$, $\vec{x} \neq \vec{0}$ s.t. $A\vec{x} = \lambda \vec{x}$
$\left((A-\lambda I)\vec{x} = 0 \right)$

$\Updownarrow$

(2) $\mathrm{Nul}(A - \lambda I) \neq \{0\}$

$\Updownarrow$

(3) $\mathrm{rank}(A - \lambda I) < n$

$\Updownarrow$

(4) $\det(A - \lambda I) = 0$

<u>Remark</u> $\lambda = 0$ is an eigenvalue $\iff \det(A) = 0$

$\det(A - \lambda I) = 0$ is an equation in λ
called the characteristic equation

$p(\lambda) = \det(A - \lambda I)$ is a polynomial in λ
of degree n in λ
called the characteristic polynomial

<u>Conclusion</u>: λ is an eigenvalue if and only if
λ is a root of the characteristic equation.

To find the eigenvalues of A, solve the eqn
$\det(A - \lambda I) = 0$.

Ex 1 (a) Find the eigenvalues of $A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$

$$p(\lambda) = |A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 \\ -2 & 4-\lambda \end{vmatrix} = (1-\lambda)(4-\lambda) + 2$$

$$= \lambda^2 - 5\lambda + 6 = 0 \quad \text{Char. eq'n.}$$

$$(\lambda - 2)(\lambda - 3) = 0 \implies \lambda_1 = 2 \quad \lambda_2 = 3$$

(b) Find the corresponding eigenvectors

Recall that if λ is an eigenvalue the corresponding eigenspace consists of all vectors $\bar{x}$ s.t. $A\bar{x} = \lambda \bar{x}$ in other words it is $\text{Nul}(A - \lambda I)$ the null space of $A - \lambda I$.

The eigenvectors for λ are the non-zero vector of $\text{Nul}(A - \lambda I)$.

Rephrase: Determine $\text{Nul}(A - \lambda_1 I) = \text{Nul}(A - 2I)$
$\text{Nul}(A - \lambda_2 I) = \text{Nul}(A - 3I)$

$\lambda_1 = 2$ $\quad A - \lambda_1 I = A - 2I = \begin{bmatrix} 1-2 & 1 \\ -2 & 4-2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix}$

$$\begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \begin{matrix} -x_1 + x_2 = 0 \\ -2x_1 + 2x_2 = 0 \end{matrix} \quad x_1 = x_2 = s$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} s \\ s \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\text{Nul}(A - 2I) = \left\{ s \begin{bmatrix} 1 \\ 1 \end{bmatrix} : s \text{ in } \mathbb{R} \right\}$$

eigenvectors $\longleftrightarrow$ $s \neq 0$.

$$\text{RREF}(A - 2I) = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad \begin{matrix} x_1 - x_2 = 0 \\ x_2 = s \quad x_1 = s \end{matrix}$$

s

$\lambda_2 = 3$ $\quad$ $\text{Nul}(A - 3I) = ?$ $\qquad$ 3

$$A - 3I = \begin{bmatrix} -2 & 1 \\ -2 & 1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}$$

$x_2 = s$

$x_1 - \frac{1}{2}x_2 = 0 \qquad x_1 = \frac{s}{2}$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{s}{2} \\ s \end{bmatrix} = s\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

$$\text{Nul}(A - 3I) = \left\{ s\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} : s \text{ in } \mathbb{R} \right\}$$

$$= \left\{ s\begin{bmatrix} 1 \\ 2 \end{bmatrix} : s \text{ in } \mathbb{R} \right\}$$

<u>Remark:</u> can replace $\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$ by any non-zero multiple of it.

$\boxed{\text{Ex 2}}$ $\quad A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ $\quad$ Find the <u>eigenvalues</u> and eigenspaces

char. equation $\quad \det(A - \lambda I) = 0 \qquad \begin{vmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$

$(1-\lambda)(1-\lambda)(2-\lambda) = 0$

$(1-\lambda)^2(2-\lambda) = 0 \implies \lambda_1 = 1 \qquad \lambda_2 = 2$

λ_1 has algebraic multiplicity equal to 2

Eigenspaces:

$\text{Nul}(A - \lambda_1 I) = \text{Nul}(A - I)$

$$(A - I)\bar{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_2 = 0 \qquad\qquad x_2 = 0$

$0 = 0 \qquad\qquad x_3 = 0$

$x_3 = 0$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{matrix} x_1 = s \\ x_2 = 0 \\ x_3 = 0 \end{matrix} \qquad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} s \\ 0 \\ 0 \end{pmatrix} = s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

s

$$\underbrace{\mathrm{Nul}\,(A - I)}_{\text{eigenspace}} = \left\{ s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} : s \text{ in } \mathbb{R} \right\}$$

1-dim

$\lambda_1 = 1$ alg. mult. $= 2$

eigenvectors corresponding to $\lambda_1 = 1$ are $\left\{ s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} s \text{ in } \mathbb{R},\ s \neq 0 \right\}$

$\lambda_2 = 2$
$$A - 2I = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{matrix} -x_1 + x_2 = 0 \\ -x_2 = 0 \\ x_3 = s \end{matrix} \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\mathrm{Nul}\,(A - 2I) = \left\{ s \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} : s \text{ in } \mathbb{R} \right\}$$

eigenvectors $s \neq 0$.

1-dim

$\lambda_2 = 2$ multiplicity $= 1$

$\boxed{\text{FACT:}}$ If λ is an eigenvalue

then its algebraic multiplicity is

$\geq$ dim (its eigenspace)

dim (Nul $(A - \lambda I)$)

$\boxed{\text{Ex 3}}$ $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$|A - \lambda I| = (1-\lambda)^2$

$\lambda_1 = 1$ alg. mult. $= 2$

$Nul(A - I) = Nul\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \mathbb{R}^2$ — dimension $= 2$

$\boxed{\text{Ex 4}}$ $A = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$ has no real eigenvalues

$\begin{vmatrix} 1-\lambda & 2 \\ -2 & 1-\lambda \end{vmatrix} = (\lambda-1)^2 + 4 = 0$

no real roots.

<u>Similar matrices</u> A and B are similar if there is P invertible such that $A = PBP^{-1}$

Note $A - \lambda I = P(B - \lambda I)P^{-1}$

$\Rightarrow \det(A - \lambda I) = \det(P)\det(B - \lambda I)\underbrace{\det(P^{-1})}_{\frac{1}{\det(P)}}$

$$\boxed{\det(A - \lambda I) = \det(B - \lambda I)}$$

<u>if A, B similar</u>

hence same eigenvalues.

Fact: A^T and A have the same eigenvalues

Recall $\det(B^T) = \det(B)$ $\quad B \quad n \times n$

$$\det(A - \lambda I) = \det\left((A - \lambda I)^T\right)$$

$$= \det(A^T - \lambda I)$$

$$\Rightarrow \quad \det(A - \lambda I) = 0 \iff \det(A^T - \lambda I) = 0$$

same roots.

—END—