5.3 DIAGONALIZATION

Diagonal matrices:

λ_1, $\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, $\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$, ..., $\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

1×1 2×2 3×3 n×n

Our goal today:

Given A $n \times n$ matrix

find D diagonal and P invertible s.t.
$n \times n$ $n \times n$

$$\boxed{A = P D P^{-1}} \leftarrow A \text{ similar to } D.$$

If such D and P exist, we say A is diagonalizable

Facts: If this is possible then

$D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ where λ_i are eigenvalues of A

and the columns of P are linearly independent eigenvectors of A.

A is diagonalizable $\Longleftrightarrow$ A has n lin. ind. eigenvectors.

Ex 1) On 03/19/21 we saw that $A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$

has eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 3$ and
corresponding eigenvectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

A is diagonalizable and we can choose

$D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ $P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ $P^{-1} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$

$A = P D P^{-1}$ ↖ compute from P

$$\begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

One can also choose

$$D = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \quad P = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \quad P^{-1} = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix}$$

One can also choose

$$D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \quad P = \begin{bmatrix} c_1 & c_2 \\ c_1 & 2c_2 \end{bmatrix} \quad P^{-1} = [\quad]$$

$c_1 \neq 0$ $\quad$ $c_2 \neq 0$

$c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad$ $c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$\boxed{\text{Ex 2}}$ If $A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$ find A^{100}. $A^n = ?$

<u>Sol'n</u> $\quad A = PDP^{-1} \qquad A^2 = A \cdot A = PD\underbrace{P^{-1}P}_{I}DP^{-1}$

$$A^2 = PD^2P^{-1}$$

$$A^n = PD^nP^{-1} \quad \text{but } D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \quad D^n = \begin{bmatrix} 2^n & 0 \\ 0 & 3^n \end{bmatrix}$$

$$\underline{A^n} = P \begin{bmatrix} 2^n & 0 \\ 0 & 3^n \end{bmatrix} P^{-1} = \underline{\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2^n & 0 \\ 0 & 3^n \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}}$$

Algorithm for diag. process

① Find eigenvalues of A by solving $\det(A - \lambda I) = 0$

$\Rightarrow \lambda_1, \lambda_2, \dots, \lambda_p \qquad p \leq n$

say m_i = the algebraic multiplicity of λ_i

② Find a basis $\mathcal{B}_i$ of each eigenspace $\mathrm{Nul}(A - \lambda_i I)$.

$$m_i \geq \underbrace{\dim(\mathrm{Nul}(A - \lambda_i I))}_{\text{nullity}(A - \lambda_i I) \,=\, \text{number of elements of } \mathcal{B}_i} \geq 1$$

③ A is diagonalizable precisely when

$$\text{nullity}(A-\lambda_1 I)+\dots+\text{nullity}(A-\lambda_p I)=n$$

that is if the set $\mathcal{B}_1\cup\mathcal{B}_2\cup\dots\cup\mathcal{B}_p$ has n-elements.

Thm If A has n distinct real eigenvalues then A is diagonalizable

$$\left(\text{nullity}(A-\lambda_i I)=1\right)$$

$\boxed{\text{Ex 3}}$ $A=\begin{bmatrix}1&0&0\\0&0&1\\0&-1&0\end{bmatrix}$ is A d-able?

$$\det(A-\lambda I)=\begin{vmatrix}1-\lambda&0&0\\0&-\lambda&1\\0&-1&-\lambda\end{vmatrix}=(1-\lambda)\begin{vmatrix}-\lambda&1\\-1&-\lambda\end{vmatrix}=(1-\lambda)(\lambda^2+1)$$

$(1-\lambda)(\underline{\lambda^2+1})=0 \Rightarrow \lambda_1=1$ no other roots

$m_1=1$

$\dim(\text{Nul}(A-I))=1$ there is just 1 lin. indep eigenvector

Thus A is not $\underline{\text{d-able!}}$.

$\lambda^2+1=0$ no real roots.

$\boxed{\text{Ex 4}}$ $A=\begin{bmatrix}2&0&0\\0&1&1\\0&1&1\end{bmatrix}$ is A d-able?

① $\det(A-\lambda I)=\begin{vmatrix}2-\lambda&0&0\\0&1-\lambda&1\\0&1&1-\lambda\end{vmatrix}=(2-\lambda)\begin{vmatrix}1-\lambda&1\\1&1-\lambda\end{vmatrix}$

$=(2-\lambda)(\cancel{1}-2\lambda+\lambda^2-\cancel{1})=(2-\lambda)\lambda(\lambda-2)$

$=-(\lambda-2)^2\lambda$

$\lambda_1=2$ has algebraic multiplicity $=2$.

$\lambda_2=0$ has algebraic multiplicity $=1$.

(2) $\lambda_1 = 2$ $\quad$ $Nul(A-2I) = ?$ $\quad$ basis for this.

$$A - 2I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

s $\quad$ t

solve

$x_1 = s$

$x_2 - x_3 = 0$

$x_3 = t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s \\ t \\ t \end{bmatrix} = s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

basis of $Nul(A-2I)$

$$\mathcal{B}_1 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

(1) $\quad$ (2)

$\lambda_2 = 0$ $\quad$ $Nul(A - 0I) = Nul(A)$

$$A - 0I = A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

s

$\underline{x_3 = s}$ $\quad$ $x_1 = 0$ $\quad$ $x_2 = -s$

$x_2 + x_3 = 0$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -s \\ s \end{bmatrix} = s\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

basis of $Nul(A) = \left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\} = \mathcal{B}_2$

(3)

A is d-able.

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

Ex 5 $\quad$ $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\quad$ $\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix}$

$= (1-\lambda)^2 = 0$

$\lambda_1 = 1$ $\quad$ multiplicity $= 2$

$\text{Nul}(A-I) = ?$ $\qquad A-I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ RREF $\qquad$ 5

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$x_2 = 0$$

$$x_1 = s$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} s \\ 0 \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$B_1 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$$

$$\text{nullity}\,|A-I| = 1 < 2$$

A is not diagonalizable.

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