5.4 Eigenvectors of linear transformations

V vector space $T: V \to V$ linear transformation

Def'n $\vec{x}$ in V is eigenvector for T if $\vec{x} \neq \vec{0}$ and $T(\vec{x}) = \lambda \vec{x}$ for some λ in $\mathbb{R}$.

λ in $\mathbb{R}$ is eigenvalue for T if there is $\vec{x}$ in V s.t. $\vec{x} \neq \vec{0}$ and $T(\vec{x}) = \lambda \vec{x}$.

Ex 1 If A $n \times n$ square matrix and $T: \mathbb{R}^n \to \mathbb{R}^n$ $T(\vec{x}) = A\vec{x}$ then the eigenvalues and eigenvectors of A are eigenvalues and eigenvectors of T.

comment T linear?
$(x+y)' = x' + y'$
$(cx)' = cx'$

Ex 2 $V = \{$ differentiable functions on $\mathbb{R}\}$.
$T: V \to V$ $T(x) = x' = \frac{dx}{dt}$.

$x(t) = t^7$
$T(x) = \frac{d}{dt}(t^7) = 7t^6$

$x(t) = e^{2t}$ $x'(t) = 2e^{2t}$
$T(e^{2t}) = 2e^{2t}$ $x(t) = e^{2t}$ is eigenvector with eigenvalue 2

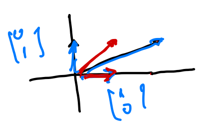
$T(e^{\lambda t}) = \lambda e^{\lambda t}$ $(e^{\lambda t})' = \lambda e^{\lambda t}$ $\lambda = 0$?

$(e^t)' = e^t$
$T(1) = 0$

Coordinates in a vector space V with basis $B = \{\vec{b}_1, \dots, \vec{b}_n\}$. Each $\vec{x}$ in V is written uniquely as $\vec{x} = r_1 \vec{b}_1 + \dots + r_n \vec{b}_n$ with r_i in $\mathbb{R}$

Thus $\vec{x}$ is given by $[\vec{x}]_B = \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix} \in \mathbb{R}^n$

$\mathbb{R}^2$ $\mathcal{E} = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$



$\vec{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\mathcal{E}$

$\vec{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ B

$[x]_B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$[x]_{\mathcal{E}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

The matrix of a linear transformation $T: V \to V$ for a finite dimensional vector space V with basis $\mathcal{B} = \{\vec{b}_1, \dots, \vec{b}_n\}$ is the $n \times n$ square matrix $M = [T]_{\mathcal{B}}$ with the property that

$$[T(\vec{x})]_{\mathcal{B}} = M \, [\vec{x}]_{\mathcal{B}}.$$

$\vec{x} \longmapsto T(\vec{x})$

$[\vec{x}]_{\mathcal{B}} \longmapsto [T(\vec{x})]_{\mathcal{B}}$

in $\mathbb{R}^n$ in $\mathbb{R}^n$

IMPORTANT OBSERVATION:

$$\underset{n \times n}{M} = \left[[T(\vec{b}_1)]_{\mathcal{B}}, \dots, [T(\vec{b}_n)]_{\mathcal{B}} \right]$$

$\boxed{\text{Ex 3}}$ Say $\dim(V) = 2$ with basis $\{\vec{b}_1, \vec{b}_2\}$ and $T: V \to V$ linear

$$T(\vec{b}_1) = 4\vec{b}_1 + 3\vec{b}_2$$
$$T(\vec{b}_2) = \vec{b}_1 - 13\vec{b}_2$$

Find $[T]_B = M = ?$

Answer: $[T(b_1)]_{\mathcal{B}} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ $[T(b_2)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -13 \end{bmatrix}$

$$M = \begin{bmatrix} 4 & 1 \\ 3 & -13 \end{bmatrix}$$

- $T(\vec{b}_1 + \vec{b}_2) = T(\vec{b}_1) + T(\vec{b}_2) = 4\vec{b}_1 + 3\vec{b}_2 + \vec{b}_1 - 13\vec{b}_2 = 5\vec{b}_1 - 10\vec{b}_2$
- $\begin{bmatrix} 4 & 1 \\ 3 & -13 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ -10 \end{bmatrix}$ $[\vec{b}_1]_B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $[\vec{b}_2] = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$\boxed{\text{Ex 4}}$ $T: \mathcal{P}_3 \to \mathcal{P}_3$ $T(p) = p'$ linear

$\mathcal{B} = \underbrace{\{1, t, t^2, t^3\}}_{\text{basis.}}$ $[T]_{\mathcal{B}} = ?$

p in $\mathcal{P}_3$ $p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$

$$[p]_{\mathcal{B}} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

Answer: $[T]_{\mathcal{B}} = M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$$M \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} a_1 \\ 2a_2 \\ 3a_3 \\ 0 \end{bmatrix} = [\underbrace{a_1 + 2a_2 t + 3a_3 t^2}_{p'(t)}]_{\mathcal{B}}$$

$$T(a_0 + a_1 t + a_2 t^2 + a_3 t^3) = a_1 + 2a_2 t + 3a_3 t^2$$

$$[T(1)]_{\mathcal{B}} = [0]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad [T(t^2)]_{\mathcal{B}} = [2t]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

$\boxed{\text{Ex 5}}$ $T: \mathbb{R}^n \to \mathbb{R}^n$ $T(\bar{x}) = A\bar{x}$ where

A is $n \times n$ square matrix

$\mathcal{E} = \{e_1, \dots, e_n\}$ $e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ $e_n = \begin{bmatrix} 0 \\ \vdots \\ 1 \end{bmatrix}$

$$[T]_{\mathcal{E}} = A = [Te_1, Te_2, \dots, Te_n]$$

$\boxed{\text{Ex 6}}$ $T: \mathbb{R}^n \to \mathbb{R}^n$ $T(\bar{x}) = A\bar{x}$ A $n \times n$

suppose $\mathcal{B} = \{\bar{b}_1, \dots, \bar{b}_n\}$ basis of eigenvectors with eigenvalues $\lambda_1, \dots, \lambda_n$ $[T]_{\mathcal{B}} = ?$

$T(\bar{b}_1) = \lambda_1 \bar{b}_1$ $[\lambda_1 \bar{b}_1]_{\mathcal{B}} =$

$\lambda_1 \bar{b}_1 = \lambda_1 \bar{b}_1 + \dots + 0 \cdot \bar{b}_n$

$T(\bar{b}_i) = \lambda_i \bar{b}_i$

$[T]_{\mathcal{B}} = ?$

Answer: $T(\vec{b}_1) = A(b_1) = \lambda_1 b_1$

$$[T(\vec{b}_1)]_{\mathcal{B}} = [\lambda_1 \vec{b}_1]_{\mathcal{B}} = \begin{bmatrix} \lambda_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\boxed{[T]_{\mathcal{B}} = \begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \ \lambda_n \end{bmatrix} = D.}$$

Theorem Let $T: \mathbb{R}^n \to \mathbb{R}^n$

$T(\vec{x}) = A\vec{x}$ A $n \times n$ square matrix

Let $\mathcal{B} = \{b_1, \dots, b_n\}$ basis of $\mathbb{R}^n$

Note $[T]_{\mathcal{E}} = A$

Let $P = [\vec{b}_1, \dots, \vec{b}_n]$. Then

$$[T]_{\mathcal{B}} = P^{-1} A P$$

Proof: If $\vec{x}$ in $\mathbb{R}^n$ then $P[x]_{\mathcal{B}} = \vec{x}$ by definition of P and $[x]_{\mathcal{B}}$. Thus $[x]_{\mathcal{B}} = P^{-1}\vec{x}$

$$[T]_{\mathcal{B}} = \left[[T(\vec{b}_1)]_{\mathcal{B}}, \dots, [T(\vec{b}_n)]_{\mathcal{B}}\right] \quad \leftarrow \text{def'n of } [T]_{\mathcal{B}}$$

$$= \left[[A\vec{b}_1]_{\mathcal{B}}, \dots, [A\vec{b}_n]_{\mathcal{B}}\right] \quad \leftarrow \text{since } T(\vec{b}) = A\vec{b}$$

$$= [P^{-1}A\vec{b}_1, \dots, P^{-1}A\vec{b}_n] \quad \leftarrow \text{by } [x]_{\mathcal{B}} = P^{-1}\vec{x}$$

$$= P^{-1}A[\vec{b}_1, \dots, \vec{b}_n] \quad \leftarrow \text{by matrix multiplication}$$

$$= P^{-1}AP$$

$\boxed{\text{Ex } 7}$ $T : \mathbb{R}^2 \to \mathbb{R}^2$ $T(\bar{x}) = A\bar{x}$

where $A = \begin{bmatrix} -5 & -1 \\ 4 & 1 \end{bmatrix}$

Let $\mathcal{B} = \left\{ \underbrace{\begin{bmatrix} -1 \\ -2 \end{bmatrix}}_{\bar{b}_1}, \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\bar{b}_2} \right\}$.

Find $[T]_{\mathcal{B}}$.

<u>Sol'n</u> 1st $P = \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix}$ $P^{-1} = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$

$$[T]_{\mathcal{B}} = P^{-1} A P = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} -5 & -1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 13 & -11 \\ 20 & -17 \end{bmatrix}.$$

2nd "low tech" solution

Need to compute $[T(\bar{b}_1)]_{\mathcal{B}}$ and $[T(\bar{b}_2)]_{\mathcal{B}}$

$$T(\bar{b}_1) = \underbrace{\begin{bmatrix} -5 & -1 \\ 4 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} -1 \\ -2 \end{bmatrix}}_{\bar{b}_1} = \begin{bmatrix} 7 \\ -6 \end{bmatrix} = c\,\bar{b}_1 + d\,\bar{b}_2$$

$$T(\bar{b}_2) = \begin{bmatrix} -5 & -1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ 5 \end{bmatrix} = c'\,\bar{b}_1 + d'\,\bar{b}_2$$

$$[T]_{\mathcal{B}} = \begin{bmatrix} c & c' \\ d & d' \end{bmatrix}$$

$$\begin{bmatrix} 7 \\ -6 \end{bmatrix} = c \begin{bmatrix} -1 \\ -2 \end{bmatrix} + d \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

solve get $c = 13$, $d = 20$

$$\begin{bmatrix} -6 \\ 5 \end{bmatrix} = c' \begin{bmatrix} -1 \\ -2 \end{bmatrix} + d' \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

solve $c' = -11$, $d' = -17$

$$[T]_{\mathcal{B}} = \begin{bmatrix} 13 & -11 \\ 20 & -17 \end{bmatrix}$$

— END OF CLASS —

REMARK

$$\begin{bmatrix} 7 \\ -6 \end{bmatrix} = c\,\bar{b}_1 + d\,\bar{b}_2$$ ← can solve this using matrices

$$\begin{bmatrix} 7 \\ -6 \end{bmatrix} = [\bar{b}_1, \bar{b}_2]\begin{bmatrix} c \\ d \end{bmatrix} = P\begin{bmatrix} c \\ d \end{bmatrix}$$

$$\begin{bmatrix} c \\ d \end{bmatrix} = P^{-1}\begin{bmatrix} 7 \\ -6 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 7 \\ -6 \end{bmatrix} = \begin{bmatrix} 13 \\ 20 \end{bmatrix}$$

$$\begin{bmatrix} -6 \\ 5 \end{bmatrix} = c'\,\bar{b}_1 + d'\,\bar{b}_2 \qquad \begin{bmatrix} -6 \\ 5 \end{bmatrix} = P\begin{bmatrix} c' \\ d' \end{bmatrix}$$

$$\begin{bmatrix} c' \\ d' \end{bmatrix} = P^{-1}\begin{bmatrix} -6 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} -6 \\ 5 \end{bmatrix} = \begin{bmatrix} -11 \\ -17 \end{bmatrix}$$

$$\begin{bmatrix} c & c' \\ d & d' \end{bmatrix} = P^{-1}\underbrace{\begin{bmatrix} 7 & -6 \\ -6 & 5 \end{bmatrix}}_{AP} = P^{-1}AP$$

$$[Ab_1, Ab_2]$$

$$\left[A\begin{bmatrix} -1 \\ -2 \end{bmatrix}, A\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right]$$

Explain $[T]_B = \left[\, [T(\bar{b}_1)]_B, \dots, [T(\bar{b}_n)]_B \right]$

$$x = r_1\bar{b}_1 + \dots + r_n\bar{b}_n \iff [x]_B = \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix}$$

$$\Downarrow$$

$$T(x) = r_1 T(\bar{b}_1) + \dots + r_n T(\bar{b}_n)$$

$$\Downarrow$$

$$[T(x)]_B = r_1[T(\bar{b}_1)]_B + \dots + r_n[T(\bar{b}_n)]_B \Rightarrow \underbrace{\left[[T(\bar{b}_1)]_B \dots [T(\bar{b}_n)]\right]}_{M}\underbrace{\begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix}}_{[x]_B} = [T(x)]_B$$