5.5 Complex eigenvalues

Consider A $n \times n$ matrices and $T: \mathbb{C}^n \to \mathbb{C}^n$

$T(\vec{x}) = A\vec{x}$ $\quad \vec{x}$ in $\mathbb{C}^n$.

Define eigenvalues, eigenvectors over $\mathbb{C}$.

Ex 1 Find eigenvalues and eigenvectors for

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

Sol'n $\quad \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = (\lambda - 1)^2 + 1 = 0$

no real roots
no real eigenvalues

complex eigenvalues:

$(\lambda - 1)^2 = -1 \qquad \lambda - 1 = \pm i \qquad \lambda = 1 \pm i$

$\lambda_1 = 1 + i \qquad \lambda_2 = 1 - i$

Corresponding eigenvectors:

$\text{Nul}(A - \lambda_1 I) \qquad A - \lambda_1 I = \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \xrightarrow{(-i)\cdot \text{row}} \begin{bmatrix} -1 & -i \\ -1 & -i \end{bmatrix}$

$\to \begin{bmatrix} -1 & -i \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad x_1 + i x_2 = 0$

RREF

$x_2 = s$

$x_1 = -i x_2 = -is$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -is \\ s \end{bmatrix} = s \begin{bmatrix} -i \\ 1 \end{bmatrix}$

$\lambda_1 = 1 + i \qquad \begin{bmatrix} -i \\ 1 \end{bmatrix}$

$\lambda_2 = 1 - i \qquad \text{Nul}(A - \lambda_2 I) \qquad A - \lambda_2 I = \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$\to (i x_1 + x_2 = 0)\, i \quad = -x_1 + i x_2 = 0$

$\to -x_1 + i x_2 = 0$

$\Downarrow$

$x_1 = i x_2$

$x_2 = s$

$x_1 = is$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = s \begin{bmatrix} i \\ 1 \end{bmatrix}$

$\lambda_2 = 1 - i \qquad \begin{bmatrix} i \\ 1 \end{bmatrix}$ ✓

$\begin{pmatrix} (1-i)i \\ 1-i \end{pmatrix} = \begin{pmatrix} i+1 \\ 1-i \end{pmatrix}$ ✓

Important remark

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \sqrt{2} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \sqrt{2} \underbrace{\begin{pmatrix} \cos(-\frac{\pi}{4}) & -\sin(-\frac{\pi}{4}) \\ \sin(-\frac{\pi}{4}) & \cos(-\frac{\pi}{4}) \end{pmatrix}}$$

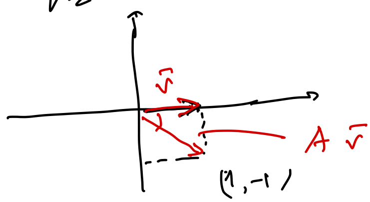
Rotation matrix by $-\pi/4$

($\pi/4$ clockwise)

How does A act on a vector $\bar{v}$ in $\mathbb{R}^2$

it rotates $\bar{v}$ clockwise by $\frac{\pi}{4}$ and then rescales it by $\sqrt{2}$

Ex. $A \begin{bmatrix} 1 \\ 0 \end{bmatrix}$



$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

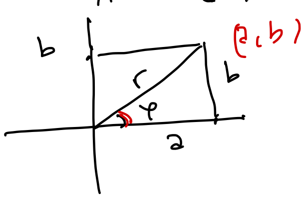
$\boxed{\text{Ex 2}}$ $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ $b \neq 0$ a, b are real numbers

has no real eigenvalues

$$\begin{vmatrix} a-\lambda & -b \\ b & a-\lambda \end{vmatrix} = 0 \qquad (\lambda - a)^2 + b^2 = 0$$

$$\lambda - a = \pm b i$$

$$\lambda = a \pm bi \qquad b \neq 0$$



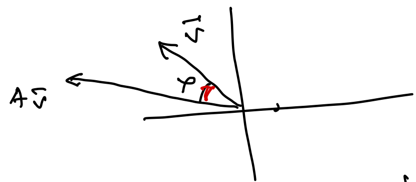
$$r = |\lambda| = \sqrt{a^2 + b^2}$$

$$A = r \begin{bmatrix} \frac{a}{r} & -\frac{b}{r} \\ \frac{b}{r} & \frac{a}{r} \end{bmatrix}$$

$$A = r \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix} \qquad -\pi < \varphi \le \pi$$

r: rescaling

$\begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$: rotation by φ counterclockwise



Obs $\lambda_1 = a + bi$ $\quad \begin{bmatrix} 1 \\ -i \end{bmatrix}$ v

$\lambda_2 = a - bi$ $\quad \begin{bmatrix} 1 \\ i \end{bmatrix}$ $\bar v$

$\lambda_2 = \bar\lambda_1$

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} a+bi \\ b-ai \end{bmatrix} = (a+bi)\begin{bmatrix} 1 \\ -i \end{bmatrix}$$

Important remark

A $n \times n$ matrix with real elements

Extend conjugation to vectors and matrices by taking conjugates entrywise.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \qquad \bar{\mathbf{x}} = \begin{bmatrix} \bar x_1 \\ \vdots \\ \bar x_n \end{bmatrix}$$

If λ in $\mathbb{C}$ is eigenvalue of A with corresp. eigenvector v

then $\bar\lambda$ is also eigenvalue
eigenvectors $\bar v$

Indeed if $Av = \lambda v \Rightarrow \overline{Av} = \overline{\lambda v}$

$\bar A \bar v = \bar\lambda \bar v$

$A \bar v = \bar\lambda \bar v$

Thm A real 2×2 matrix with an eigenvalue $\lambda = a - bi$ $b \neq 0$ then $a + bi$ is also an eigenvalue and A is similar to the matrix $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.

$$A = P C P^{-1}$$

Moreover $P = [\operatorname{Re} v, \operatorname{Im} v]$

where v eigenvector for λ.

Some facts:

\#27 p 310 If $A = A^T$ A real $n\times n$ matrix then for any x in $\mathbb{C}^n$

$$\underset{1\times n}{\underline{\bar{x}^T}}\ \underset{n\times n}{\underline{A}}\ \underset{n\times 1}{\underline{x}} = q \quad \text{is a real number} \quad (1,1)$$

q is real $\iff$ $\bar{q} = q$

Say $A = [a_{ij}] = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$ $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

$$q = \bar{x}^T A x = [\bar{x}_1, \dots, \bar{x}_n] \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{n1} & & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$q = \sum_{i,j=1}^{n} a_{ij} \bar{x}_i x_j \quad \leftarrow \text{sum with } n^2 \text{ term}$$

$1 \le i \le n \quad 1 \le j \le n$

$$\bar{q} = \sum_{i,j=1}^{m} \overline{a_{ij}\, \bar{x}_i\, x_j}$$

$$= \sum_{i,j=1}^{m} a_{ij}\, x_i\, \bar{x}_j = \sum_{i,j=1}^{m} a_{ji}\, \bar{x}_j\, x_i$$

$$A = A^T \qquad a_{ij} \quad a_{ji}$$

$$= q$$

Hint for #28 : $A = A^T \Rightarrow \lambda$ eigenvalues are real numbers

Say $Ax = \lambda x$

$$q = \bar{x}^T A x = \bar{x}^T \lambda x = \lambda \bar{x}^T x$$

$$= \lambda [\bar{x}_1, \dots, \bar{x}_n] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$q = \lambda (\bar{x}_1 x_1 + \dots + \bar{x}_n x_n)$$

$$\underset{\text{real}}{q} = \lambda \underbrace{(|x_1|^2 + \dots + |x_n|^2)}_{\text{real}}$$

— END —

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ a, b, c, d real

if $\lambda = \alpha + i\beta$ eigenvalue v

then $\bar{\lambda} = \alpha - i\beta$ — " — $\bar{v}$