5.7

Applications to differential equations

$0 \qquad x(t)$

- $\boxed{x'(t) = a\,x(t)}$ $\qquad x(t)$ numerical function

$x(t) = c\, e^{at}$ $\qquad c$ in $\mathbb{R}$ constant

$x'(t) = (c\, e^{at})' = c\, a e^{at} = a\,(c e^{at}) = a\, x(t)$

to determine c we need $x(0)$.

- $x_1(t),\ x_2(t)$

$$\begin{cases} x_1' = a_{11} x_1 + a_{12} x_2 \\ x_2' = a_{21} x_1 + a_{22} x_2 \end{cases}$$

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \qquad \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\boxed{x'(t) = A\, x(t)}$$

Important observation

If λ is an eigenvalue of A

v is a corresponding eigenvector

then $x(t) = e^{\lambda t} v$ is a solution of

$x' = Ax$

Indeed:

$$x' = \frac{d}{dt}\left(e^{\lambda t} v\right) = \lambda e^{\lambda t} v$$

$$=$$

$$Ax = A\left(e^{\lambda t} v\right) = e^{\lambda t} A v = e^{\lambda t} \lambda v$$

Suppose A 2×2 has eigenvalues $\lambda_1 \neq \lambda_2$

and eigenvectors $v_1 \quad v_2$

Then the general solution of $x' = Ax$

is $x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$

where c_1, c_2 constants in $\mathbb{R}$

Ex 1 (a) $x_1' = x_1 - x_2$
$x_2' = 2x_1 + 4x_2$ solve this

Sol'n $A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$ $\lambda_1 = 2$ $v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda_2 = 3$ $v_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$$x(t) = c_1 e^{2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{2t} + c_2 e^{3t} \\ -c_1 e^{2t} - 2c_2 e^{3t} \end{bmatrix} \quad c_1, c_2 \text{ in } \mathbb{R}$$

(b) What if we are asked to solve the same system with initial conditions

$$x(0) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ -c_1 - 2c_2 \end{bmatrix} \qquad \begin{cases} c_1 + c_2 = 3 \\ c_1 + 2c_2 = -4 \end{cases}$$

solve for c_1, c_2

$c_1 = 10$ $c_2 = -7$

More generally: $x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$ A $n \times n$ matrix

Suppose A has n distinct eigenvalues

$\lambda_1, \ldots, \lambda_n$

$v_1 \qquad v_n$ ← eigenvectors

Then general sol'n for $\boxed{x'(t) = A\,x(t)}$

is $\boxed{x(t) = c_1 e^{\lambda_1 t} v_1 + \cdots + c_n e^{\lambda_n t}}$

The constants $c_1, \ldots, c_n$ can be determined by initial conditions

Say $x(0) = x_0$ is given

$t=0$

$x(0) = x_0 = c_1 v_1 + \cdots + c_n v_n$ ← need to solve this

form $P = [v_1, \ldots, v_n]$

columns

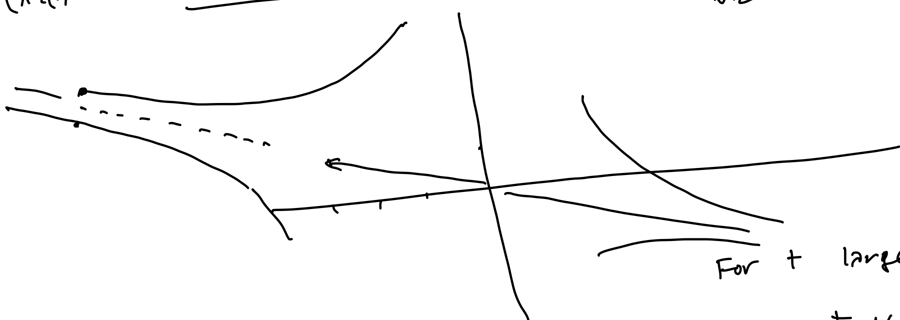
$$x_0 = P \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \Rightarrow \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = P^{-1} x_0$$

$\boxed{\text{Ex 2}}$ $A = \begin{bmatrix} 2 & 3 \\ -1 & -2 \end{bmatrix}$ $x' = Ax$

$\lambda_1 = -1$ $v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $\left(v_1 = \begin{bmatrix} -2 \\ 2 \end{bmatrix} \right)$

$\lambda_2 = 1$ $v_2 = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \boxed{x(t) = c_1 e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{t} \underbrace{\begin{bmatrix} -3 \\ 1 \end{bmatrix}}_{v_2}}$$



For t large

$\underline{x(t)} \approx \underline{c_2 e^t v_2}$

$c_2 \neq 0$

For any x_0 s.t. $c_2 \neq 0$

$\boxed{\text{saddle}}$

$\boxed{\text{Ex 3}}$ what if $x' = Ax$

leads $\bar{x}(t) = c_1 e^{-3t} v_1 + c_2 e^{-2t} v_2$

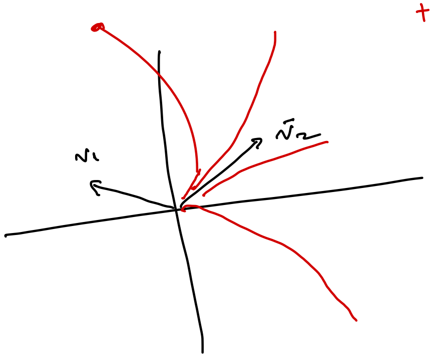
what happens when $t \to \infty$

$e^{-2t} \to 0$ $e^{-3t} \to 0$

$\Rightarrow x(t) \to 0$

$$\bar{x}(t) = e^{-2t} \Big(\underbrace{c_1 e^{-t}}_{\substack{\text{as } t\to\infty \\ \text{becomes} \\ \text{almost } 0}} v_1 + c_2 \underset{\substack{\uparrow \\ \text{constant}}}{v_2} \Big)$$

trajectories align along v_2 near the origin



origin attractor.

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