4.1 Vector spaces and subspaces (NEW RECORDING) 1

A **vector space** is a nonempty set V of objects, called *vectors*, on which are defined two operations, called *addition* and *multiplication by scalars* (real numbers), subject to the ten axioms (or rules) listed below.[1] The axioms must hold for all vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V and for all scalars c and d.

1. The sum of $\mathbf{u}$ and $\mathbf{v}$, denoted by $\mathbf{u}+\mathbf{v}$, is in V.
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.
3. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$.
4. There is a **zero vector** $\mathbf{0}$ in V such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$.
5. For each $\mathbf{u}$ in V, there is a vector $-\mathbf{u}$ in V such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.
6. The scalar multiple of $\mathbf{u}$ by c, denoted by $c\mathbf{u}$, is in V.
7. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$.
8. $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$.
9. $c(d\mathbf{u})=(cd)\mathbf{u}$.
10. $1\mathbf{u}=\mathbf{u}$.

Basic examples of vector spaces

Ex 1 $\mathbb{R}^n = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} : a_i \text{ are real numbers} \right\}$

Ex 2 $M_{m\times n}$ all $m \times n$ matrices

Ex 3 $\mathbb{P}_n$ all polynomials of degree $\leq n$

$p(t) = a_0 + a_1 t + \dots + a_n t^n$ a_i are real numbers

Ex 4 Fix a set D. Let V be all the functions $f: D \to \mathbb{R}$

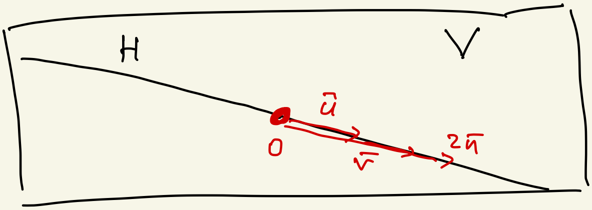
V vector space $(f+g)(x) = f(x) + g(x)$

$(cf)(x) = c f(x)$

Subspaces of vector spaces

Suppose V is a vector space with addition $+$ and scalar multiplication $\cdot$

Let H be a subset of V



$\bar{u}+\bar{v}$ in H

$c\,\bar{u}$ in H

By definition say H is a subspace of V if the following conditions are satisfied

(1) $\vec{0}_V$ is in H ($\bar{0}$ is element of V)

(2) If $\bar{u}$ and $\bar{v}$ are in H then $\bar{u}+\bar{v}$ is in H

(3) If $\bar{u}$ is in H and c in $\mathbb{R}$ then $c\,\bar{u}$ is in H.

If H is a subspace, then H is a vector space in itself.

Example of subspaces

Ex. 5 $V = \mathbb{R}^3$

$H = \left\{ \begin{bmatrix} a \\ 0 \\ b \end{bmatrix} : a, b \text{ are in } \mathbb{R} \right\}$ subset of $\mathbb{R}^3$

H is a subspace. Why? Because it satisfies (1) (2) (3)

(1) $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ belongs to H.

(2) $\begin{bmatrix} a \\ 0 \\ b \end{bmatrix} + \begin{bmatrix} a' \\ 0 \\ b' \end{bmatrix} = \begin{bmatrix} a+a' \\ 0 \\ b+b' \end{bmatrix}$ in H

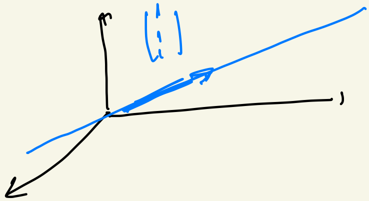
(3) c in $\mathbb{R}$ $\quad c\begin{bmatrix} a \\ 0 \\ b \end{bmatrix} = \begin{bmatrix} ca \\ 0 \\ cb \end{bmatrix}$ in H

Note that H corresponds to the xz-plane subset of $\mathbb{R}^3$

Ex 5' $H = \mathbb{R}^3$ is a subspace of $\mathbb{R}^3$

Ex 5'' $H = \{\bar{0}\}$ is a subspace of $\mathbb{R}^3$

Ex 5''' $H = \left\{ \begin{bmatrix} a \\ a \\ a \end{bmatrix} : a \text{ in } \mathbb{R} \right\}$ subspace of $\mathbb{R}^3$



Ex 6 $V = \mathbb{P}$ the vector space of all polynomials (no restriction on degree) is a vector space.

$H = \mathbb{P}_n$ = poly's of degree $\leq n$ subspace of $\mathbb{P}$.

Ex 7 $V = M_{2\times 3}$ vector space

$H = \left\{ \begin{bmatrix} a & c & e \\ b & d & 0 \end{bmatrix} : a, b, c, d, e \text{ are in } \mathbb{R} \right\}$ is a subspace

The span of a set of vectors

Let V be a vector space

Let $\vec{v}_1, v_2, \ldots, \vec{v}_p$ be elements of V

Then we define the span of these vectors

$\text{Span}\{v_1, v_2, \ldots, v_n\}$ to be the set consisting of all linear combinations

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p$$

where $c_1, c_2, \ldots, c_p$ are real numbers

$$\underbrace{\text{Span}\{\vec{v}_1, \ldots, \vec{v}_n\}}_{} = \{c_1\vec{v}_1 + \cdots + c_p\vec{v}_n : \; c_i \text{ are in } \mathbb{R}\}$$

H is a subspace of V.

The subspace of V generated or spanned by $\{v_1, v_2, \ldots, v_p\}$.

Ex 8. Let H be all the vectors in $\mathbb{R}^4$ of the form $\begin{bmatrix} s-t \\ s+t \\ r \\ s \end{bmatrix}$ where s, t, r are $\mathbb{R}$

$$H = \left\{ \begin{bmatrix} s-t \\ s+t \\ r \\ s \end{bmatrix} : s, t, r \in \mathbb{R} \right\}$$

Show that H is a subspace of $\mathbb{R}^4$.

Instead of verifying (1) (2), (3) we realize H as the span of some set of vectors in $\mathbb{R}^4$.

$$\begin{bmatrix} s-t \\ s+t \\ r \\ s \end{bmatrix} = \begin{bmatrix} s \\ s \\ 0 \\ s \end{bmatrix} + \begin{bmatrix} -t \\ t \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ r \\ 0 \end{bmatrix}$$

$$= s\begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + r\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Thus $H = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

hence a subspace of $\mathbb{R}^4$.

Ex 9 Find $\text{span}\left\{ \begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}2\\2\\0\end{bmatrix}, \begin{bmatrix}0\\0\\3\end{bmatrix} \right\} = H$
subspace of $\mathbb{R}^3$

Answer: $H = \mathbb{R}^3$.

Because for any vector $\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$

we can find c_1, c_2, c_3 such that

$$c_1 \begin{bmatrix}1\\0\\0\end{bmatrix} + c_2 \begin{bmatrix}2\\2\\0\end{bmatrix} + c_3 \begin{bmatrix}0\\0\\3\end{bmatrix} = \begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$$

because $\begin{vmatrix}1 & 2 & 0\\0 & 2 & 0\\0 & 0 & 3\end{vmatrix} \neq 0$
$= 6$

Ex 10 Consider the polynomials $1, t, t^2$ in $\mathbb{P}$.
Describe $\text{Span}\{1, t, t^2\}$.

Sol'n $\{ c_0 \cdot 1 + c_1 t + c_2 t^2 : c_1, c_2, c_3 \text{ are in } \mathbb{R} \}$

Thus $\text{Span}\{1, t, t^2\} = \mathbb{P}_3$ (poly's of degree ≤ 3)

Ex 11 Let H consist of all polynomials of degree $= 4$.
Is H a subspace of $\mathbb{P}$.

Answer: No! $p(t) = t^4 \quad q(t) = -t^4$

$p + q = 0 \leftarrow$ degree $0 < 4$

Moreover 0 not in H.

Ex 12 $H = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : \begin{array}{l} a, b, c \text{ are in } \mathbb{R} \\ \text{and } ab = 0 \end{array} \right\}$

is H a subspace of $\mathbb{R}^3$?

Answer: $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ in H ✓

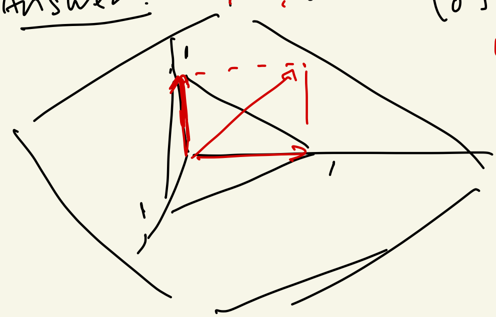
H not closed under addition $\quad 1 \cdot 1 \neq 0$

Indeed $\underset{\text{in } H}{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}} + \underset{\text{in } H}{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}} = \underset{\text{not in } H}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}$

Ex 13 $H = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} : \begin{array}{l} a, b, c \text{ are in } \mathbb{R} \\ \text{and } a+b+c=1 \end{array} \right\}$

is H a subspace of $\mathbb{R}^3$?

Answer: No! since $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ not in H.



$u = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad v = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$u + v = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$0 + 1 + 1 = 2$

Ex 14 Fix a matrix F in $M_{m \times n}$

Define $H = \{ A \text{ in } M_{n \times p} : \underset{m \times n}{F} \cdot \underset{n \times p}{A} = 0_{m \times p} \}$

Is H a subspace of $M_{n \times p}$?

Answer: Yes verify that H is closed under addition and multiplication by scalars

Say A_1, A_2 are in H

Thus $F \cdot A_1 = 0$ $\quad F \cdot A_2 = 0$

$$0 = \underbrace{F \cdot A_1}_{0} + \underbrace{F \cdot A_2}_{0} = \underbrace{F \cdot (A_1 + A_2)}$$

Thus $A_1 + A_2$ in H. ✓

If A in H and c in $\mathbb{R}$

$$F(cA) = c(\underbrace{FA}_{=0}) = c \cdot 0 = 0 \quad ✓$$

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