6.2 Orthogonal sets

Recall that for $\vec{u}, \vec{v}$ in $\mathbb{R}^n$ say that $\vec{u}$ is orthogonal to $\vec{v}$ written $\bar{u} \perp \vec{v}$ if $\vec{u} \cdot \vec{v} = 0$

$$\bar{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \quad \vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

$$\vec{u} \cdot \vec{v} = u_1 v_1 + \cdots + u_n v_n$$

Remark $\vec{0}$ is orthogonal to any other vector

$S = \{ \vec{u}_1, \ldots, \vec{u}_p \}$ set of vectors in $\mathbb{R}^n$

S is an orthogonal set if $\bar{u}_i \cdot \bar{u}_j = 0$ for $i \neq j$

Fact S orthogonal and each $\bar{u}_i \neq 0$ then S linearly independent. Thus S is a basis for Span(S).

Proof: Verify S lin. indep.

Suppose that $c_1 u_1 + \cdots + c_p u_p = 0$ for some c_i in $\mathbb{R}$. Must show: all $c_i = 0$.

Compute $u_i \cdot (\overbrace{c_1 u_1 + \cdots + c_i u_i + \cdots + c_n u_n}^{0})$

$$= c_1 \underbrace{u_i \cdot u_1}_{0} + \cdots + c_i \underbrace{u_i \cdot u_i}_{\|u_i\|^2} + \cdots + c_n \underbrace{u_i \cdot u_n}_{0}$$

thus $c_i \|u_i\|^2 = 0 \Rightarrow c_i = 0$

Ex 1

$$S = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \right\}$$ ← orthogonal set

$u_1 \quad u_2 \quad u_3$

$u_1 \cdot u_2 = 0 \qquad u_2 \cdot u_3 = 0 \qquad u_1 \cdot u_3 = 0$

$\text{span}(S) = \mathbb{R}^3$

Ex 2

$$S = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$ ← orthogonal

$u_1 \quad u_2 \quad u_3$

$u_1 \cdot u_2 = 0$

Def'n An orthogonal basis for a subspace W of $\mathbb{R}^n$ is a basis of W that is orthogonal

Fact If $\{\bar{u}_1, \dots, \bar{u}_p\}$ orthogonal basis of W

for each y in W

$$y = c_1 \bar{u}_1 + \cdots + c_p \bar{u}_p$$

where $$c_i = \frac{y \cdot u_i}{u_i \cdot u_i} \qquad i = 1, \dots, p.$$

Proof: $$y = c_1 \bar{u}_1 + \cdots + c_p \bar{u}_p \Longrightarrow$$

$$y \cdot u_i = (c_1 \vec{u}_1 + \cdots + c_i \vec{u}_i + \cdots + c_p \vec{v}_p) \cdot \vec{u}_i$$

$$= c_1 \underbrace{\bar{u}_1 \cdot \bar{u}_i}_{=0} + \cdots + c_i \, \underline{\bar{u}_i \cdot \bar{u}_i} + \cdots + c_p \underbrace{\bar{u}_p \cdot \bar{u}_i}_{=0}$$

solve for c_i $$c_i = \frac{y \cdot u_i}{u_i \cdot u_i}$$

$\boxed{\text{Ex 3}}$ Write $\underset{y}{\begin{bmatrix}1\\0\\0\end{bmatrix}}$ as a linear combination of the vectors from $\boxed{\text{Ex 1}}$: $\underset{u_1}{\begin{bmatrix}1\\-1\\0\end{bmatrix}}, \underset{u_2}{\begin{bmatrix}1\\1\\1\end{bmatrix}}, \underset{u_3}{\begin{bmatrix}1\\1\\-2\end{bmatrix}}$

Sol'n:
$$\begin{bmatrix}1\\0\\0\end{bmatrix} = c_1\begin{bmatrix}1\\-1\\0\end{bmatrix} + c_2\begin{bmatrix}1\\1\\1\end{bmatrix} + c_3\begin{bmatrix}1\\1\\-2\end{bmatrix}$$

Find c_1, c_2, c_3.

$$\underline{\begin{bmatrix}1\\0\\0\end{bmatrix}\cdot\begin{bmatrix}1\\-1\\0\end{bmatrix}} = c_1\begin{bmatrix}1\\-1\\0\end{bmatrix}\cdot\begin{bmatrix}1\\-1\\0\end{bmatrix} + c_2\overbrace{\begin{bmatrix}1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}1\\-1\\0\end{bmatrix}}^{=0} + c_3\underbrace{\begin{bmatrix}1\\1\\-2\end{bmatrix}\cdot\begin{bmatrix}1\\-1\\0\end{bmatrix}}_{=0}$$

$$1 = c_1 \cdot 2$$

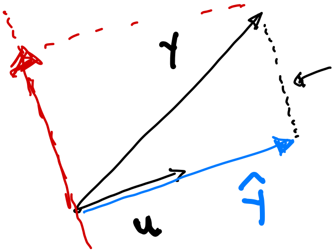
$$c_1 = \frac{1}{2}$$

$$c_1 = \frac{y\cdot u_1}{u_1\cdot u_1}$$

$$c_2 = \frac{y\cdot u_2}{u_2\cdot u_2}$$

$$c_3 = \frac{y\cdot u_3}{u_3\cdot u_3}$$

The orthogonal projection of a vector y onto another non-zero vector u in $\mathbb{R}^n$

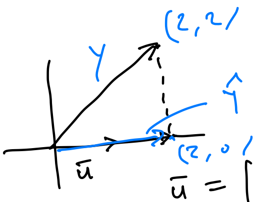


Denote this projection by $\hat{y}$

$$\boxed{\hat{y} = \frac{y \cdot u}{u \cdot u}\, u}$$

AIM: Want to write y as a sum $y = \hat{y} + z$ such that $\hat{y}$ is parallel to u, $z \perp u$

Ex



$\bar{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad y = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ $\quad \hat{y} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

$y = \hat{y} + z$ $\quad \boxed{\hat{y} = cu}$ $\quad c$ in $\mathbb{R}$

$z \perp u$ $\quad z \cdot u = 0$

Proof of formula for $\hat{y}$

Seek c in $\mathbb{R}$ such that

$y - \hat{y} \perp u$

$y - cu \perp u$

$(y - cu) \cdot u = 0$

$y \cdot u - c\, u \cdot u = 0$

$$\boxed{c = \frac{y \cdot u}{u \cdot u}}$$

Remark: If one replaces u by λu $\quad \lambda$ in $\mathbb{R}$, $\lambda \neq 0$

$\hat{y}$ does not change.

$$\frac{y \cdot u}{u \cdot u}\, u = \hat{y} = \frac{y \cdot (\cancel{\lambda} u)}{(\cancel{\lambda} u) \cdot (\cancel{\lambda} u)}\, \cancel{\lambda} u$$

Ex 4

$$\mathbf{y} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \qquad \mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\hat{y} = \frac{y \cdot u}{u \cdot u} u = \frac{\begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$= \frac{2-1}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

NOTATION if $L = \text{span}\{\bar{u}\} = \{\lambda u : \lambda \text{ in } \mathbb{R}\}$

$$\boxed{\hat{y} = \text{Proj}_L\, y}$$

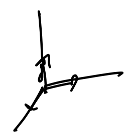
ORTHONORMAL SETS

$S = \{\mathbf{u}_1, \dots, \mathbf{u}_p\}$ is orthonormal if it is orthogonal and each $\mathbf{u}_i$ is a unit vector $\|\mathbf{u}_i\| = 1$ $i = 1, \dots, p$.

In this case S is an orthonormal basis for $W = \text{Span}(S)$

Ex 5 Find 2 different orthonormal bases of $\mathbb{R}^3$.

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$



$$\left\{ \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$\boxed{\text{Fact}}$ A $m \times n$ matrix U has orthonormal columns if and only if $U^T U = I_n$.

Proof: suppose U has orthonormal columns
The entries of (i,j) of $U^T U$

are $u_i^T u_j = u_i \cdot u_j = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$

$\boxed{\text{Fact}}$ Properties of $m \times n$ matrices with ortogonal columns

(a) $\|Ux\| = \|x\|$ for x in $\mathbb{R}^n$

(b) $(Ux) \cdot (Uy) = x \cdot y$ x, y in $\mathbb{R}^n$

(c) $(Ux) \cdot (Uy) = 0 \iff x \cdot y = 0$

If $m = n$ say U is $\boxed{\text{ORTHOGONAL}}$

ORTHOGONAL MATRICES

An $n \times n$ square matrix is called ORTHOGONAL if it has orthonormal columns.

Moreover if U is $n \times n$ then

U is orthogonal $\iff$ U is invertible and $U^{-1} = U^T$

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