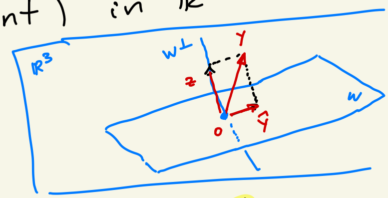
6.3 Orthogonal projections

Goal for today:

Let W be a subspace of $\mathbb{R}^n$

Let y be vector (point) in $\mathbb{R}^n$



Want to decompose y as

$$y = \hat{y} + z$$

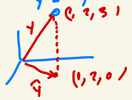
where $\hat{y}$ is in W and z in $W^\perp$

this means that $z \perp W$

$z \cdot w = 0$ for all w in W

$\mathbb{R}^3$ W = "xy-plane" $W^\perp$ = "z-axis"

$y = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ $y = \hat{y} + z$ $\hat{y} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ $z = \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}$

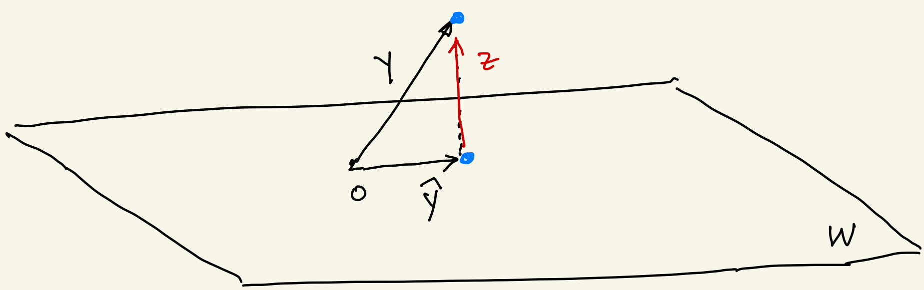


Key facts: (1) the decomposition $y = \hat{y} + z$ is unique

(2) $\hat{y}$ is the closest point in W to y

$$\text{dist}(y, W) = \|y - \hat{y}\| = \|z\|$$

Geometric insight:



Q1. How to compute $\hat{y}$ and z ?

Q2. How to compute $\operatorname{dist}(y, W)$?

answer
$= \|y - \hat{y}\| = \|z\|$

The orthogonal decomposition theorem $\longleftrightarrow$ $\boxed{y = \hat{y} + z}$

If $\{u_1, \dots, u_p\}$ is orthogonal basis of W, then

$$\hat{y} = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \cdots + \frac{y \cdot u_p}{u_p \cdot u_p} u_p$$

$$z = y - \hat{y} \qquad \left(\text{because } y = \hat{y} + z\right)$$

Notation for $\hat{y}$:

$\hat{y} = \operatorname{Proj}_W y$ = the orthogonal projection of y onto W.

Ex 1 Let W be the subspace of $\mathbb{R}^3$ spanned by $u_1 = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$ and $u_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$. Let $y = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$

Find $y = \hat{y} + z$ where $\hat{y} = \operatorname{Proj}_W y$.

<u>sol'n</u> $u_1 \cdot u_2 = 0 \qquad 2 + 0 + (2) = 0$

thus $\{u_1, u_2\}$ orthogonal basis for W

Thus $$y = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2} u_2$$

$$\frac{y \cdot u_1}{u_1 \cdot u_1} = \frac{\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}}{4 + 1 + 4} = \frac{4 - 1 - 6}{9} = -\frac{3}{9} = -\frac{1}{3}$$

$$\frac{y\cdot u_2}{u_2\cdot u_2} = \frac{\begin{bmatrix}2\\1\\3\end{bmatrix}\cdot\begin{bmatrix}1\\0\\1\end{bmatrix}}{2} = \frac{5}{2}$$

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$$\hat{y} = \left(-\frac{1}{3}\right)u_1 + \left(\frac{5}{2}\right)u_2 = -\frac{1}{3}\begin{bmatrix}2\\-1\\-2\end{bmatrix} + \frac{5}{2}\begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}\frac{11}{6}\\ \frac{1}{3}\\ \frac{19}{6}\end{bmatrix}$$

$$z = y - \hat{y} = \begin{bmatrix}2\\1\\3\end{bmatrix} - \begin{bmatrix}\frac{11}{6}\\ \frac{1}{3}\\ \frac{19}{6}\end{bmatrix} = \begin{bmatrix}1/6\\ \frac{2}{3}\\ -1/6\end{bmatrix}$$

$$y = \hat{y} + z \qquad \hat{y}\cdot z = 0$$

The best approximation theorem

$\hat{y} = \mathrm{Proj}_W\, y$ is the closest point in W to y.

This means that $\|y - \hat{y}\| < \|y - w\|$ for all w in W with the exception of $w = \hat{y}$.

Thus $\mathrm{dist}(y, W) = \|y - \hat{y}\| = \|z\|$

Ex 2 For W and y as in Ex 1, find the point in W that is closest y and find $\mathrm{dist}(y, W)$.

Sol'n

this is $\hat{y} = \begin{bmatrix}\frac{11}{6}\\ \frac{1}{3}\\ \frac{19}{6}\end{bmatrix}$ (see Ex 1)

this is $\|z\| = \|y - \hat{y}\| = \left\|\begin{bmatrix}1/6\\ \frac{2}{3}\\ -1/6\end{bmatrix}\right\|$

$$= \sqrt{\left(\frac{1}{6}\right)^2 + \left(\frac{2}{3}\right)^2 + \left(-\frac{1}{6}\right)^2}.$$

Formula for $\text{proj}_W y$ simplifies if $\{u_1, \dots, u_p\}$ is orthonormal basis

$$\text{proj}_W y = (y \cdot u_1)u_1 + \dots + (y \cdot u_p)u_p$$

If we form the matrix U with columns $u_1, \dots, u_p$

$$U = [u_1, \dots, u_p]$$

then

$$\hat{y} = \text{proj}_W y = UU^T y$$

$\boxed{\text{Ex 3}}$ Revisit Ex 1 using the formula above

$$\left\{ \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

orthogonal but not orthonormal basis of W

Use $\left\{ \frac{1}{3}\begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$ $\quad U = \begin{bmatrix} \frac{2}{3} & \frac{1}{\sqrt{2}} \\ -\frac{1}{3} & 0 \\ -\frac{2}{3} & \frac{1}{\sqrt{2}} \end{bmatrix}$

$$\hat{y} = UU^T y = \underbrace{\overset{U}{\begin{bmatrix} \frac{2}{3} & \frac{1}{\sqrt{2}} \\ -\frac{1}{3} & 0 \\ -\frac{2}{3} & \frac{1}{\sqrt{2}} \end{bmatrix}}_{3\times 2} \overset{U^T}{\begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{2}{3} \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}}_{2\times 3}}_{3\times 3} \cdot \overset{y}{\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}}_{3\times 1}$$

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Ex.4 Write y in $\mathbb{R}^4$ as sum of one vector in $\text{Span}\{u_1, u_2, u_3\}$ and one in $\text{Span}\{u_4\}$.

$$u_1 = \begin{bmatrix} 0 \\ 1 \\ -5 \\ -1 \end{bmatrix} \quad u_2 = \begin{bmatrix} 4 \\ 6 \\ 1 \\ 1 \end{bmatrix} \quad u_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -5 \end{bmatrix} \quad u_4 = \begin{bmatrix} 6 \\ -4 \\ -1 \\ 1 \end{bmatrix} \quad y = \begin{bmatrix} 10 \\ -9 \\ 4 \\ 0 \end{bmatrix}$$

Sol'n

$$y = \underbrace{\frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2} u_2 + \frac{y \cdot u_3}{u_3 \cdot u_3} u_3}_{\text{in Span}\{u_1, u_2, u_3\}} + \underbrace{\frac{y \cdot u_4}{y \cdot u_4} u_4}_{\text{in Span}\{u_4\}}$$

<u>Remark:</u> easier to compute $\frac{y \cdot u_4}{u_4 \cdot u_4} u_4$

and obtain the other component as

$$y - \frac{y \cdot u_4}{u_4 \cdot u_4} u_4 .$$

<u>Remark</u>

$$y = \hat{y} + z$$

$$y = \text{proj}_W y + \text{proj}_{W^\perp} y$$

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$$z = y - \hat{y} \perp W \qquad (y - \hat{y}) \cdot \vec{u}_i \overset{?}{=} 0$$

$$y \cdot u_i = \hat{y} \cdot u_i \ ?$$

$$\hat{y} = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{y \cdot u_2}{u_2 \cdot u_2} u_2$$

$$\hat{y} \cdot u_1 = \frac{y \cdot u_1}{\cancel{u_1 \cdot u_1}} \cancel{u_1 \cdot u_1}$$

$$\hat{y} \cdot u_1 = y \cdot u_1$$