6.4 # The Gram-Schmidt Process

Goal:

Given $x_1, x_2, \dots, x_p$ linearly independent vectors in $\mathbb{R}^m$

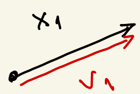
want to produce $v_1, v_2, \dots, v_p$ orthogonal (orthonormal) vectors in $\mathbb{R}^m$

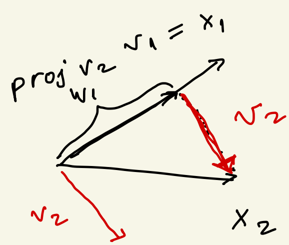
such that

$$\text{Span}\{v_1\} = \text{Span}\{x_1\}$$
$$\text{Span}\{v_1, v_2\} = \text{Span}\{x_1, x_2\}$$
$$\vdots$$
$$\text{Span}\{v_1, v_2, \dots, v_p\} = \text{Span}\{x_1, x_2, \dots, x_p\}$$

There is an algorithm for that called Gram-Schmidt

$W_1 = \text{span}\{v_1\}$





$v_1 = x_1$

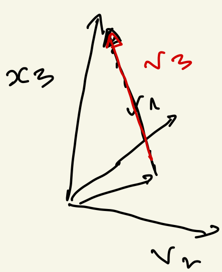
$v_1 = x_1$

$v_2 =$

v_2 must be $\perp$ v_1

$$v_2 = x_2 - \underbrace{\frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1}_{\text{proj}_{\text{span}\{v_1\}} x_2}$$

$\text{Span}\{v_1, v_2\} = \text{Span}\{x_1, x_2\}$



$v_3 = x_3 - \text{proj}_{\text{span}\{v_1, v_2\}}(x_3)$

Algorithm Gram-Schmidt:

Given $x_1, \dots, x_p$ linearly independent in $\mathbb{R}^m$

We construct $v_1, \dots, v_p$ orthogonal (orthonormal)

$$v_1 = x_1$$

$$v_2 = x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1 = x_2 - P_{W_1}(x_2)$$

$$v_2 = x_3 - \left(\frac{x_3 \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{x_3 \cdot v_2}{v_2 \cdot v_2} v_2 \right) = x_3 - P_{W_2}(x_3)$$

$$\vdots \qquad\qquad \vdots$$

$$v_p = x_p - \left(\frac{x_p \cdot v_1}{v_1 \cdot v_1} v_1 + \dots + \frac{x_p \cdot v_{p-1}}{v_{p-1} \cdot v_{p-1}} v_{p-1} \right) =$$

$$= x_p - P_{W_{p-1}}(x_p)$$

Notation:

$$W_1 = \operatorname{Span}\{v_1\}$$
$$W_2 = \operatorname{Span}\{v_1, v_2\}$$
$$\vdots$$
$$W_k = \operatorname{Span}\{v_1, \dots, v_k\}$$

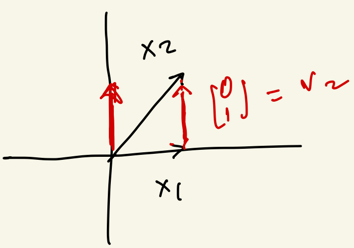
To obtain orthonormal vectors "scale"

$$\frac{v_1}{\|v_1\|}, \dots, \frac{v_p}{\|v_p\|}$$

$\boxed{\text{Ex 1}}$ $\quad x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad x_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$v_1 = x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$



$$v_2 = x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1$$

$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \frac{1}{1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = v_2$$

$\boxed{\text{Ex2}}$ Apply G-S to $\begin{bmatrix}1\\-1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\2\\0\end{bmatrix}$

$\qquad\qquad\qquad\qquad\qquad\qquad x_1 \qquad\quad x_2 \qquad\quad x_3$

$$v_1 = \begin{bmatrix}1\\-1\\1\\0\end{bmatrix}$$

$$v_2 = x_2 - \frac{x_2 \cdot v_1}{v_1 \cdot v_1} v_1 = \begin{bmatrix}1\\0\\1\\0\end{bmatrix} - \frac{2}{3}\begin{bmatrix}1\\-1\\1\\0\end{bmatrix} = \begin{bmatrix}\frac{1}{3}\\ \frac{2}{3}\\ \frac{1}{3}\\ 0\end{bmatrix}$$

$$v_3 = x_3 - \frac{x_3 \cdot v_1}{v_1 \cdot v_1} v_1 - \frac{x_3 \cdot v_2}{v_2 \cdot v_2} v_2$$

$$= \begin{bmatrix}1\\1\\2\\0\end{bmatrix} - \frac{2}{3}\begin{bmatrix}1\\-1\\1\\0\end{bmatrix} - \frac{5/3}{2}\begin{bmatrix}\frac{1}{3}\\ \frac{2}{3}\\ \frac{1}{3}\\ 0\end{bmatrix} = \begin{bmatrix}-\frac{1}{2}\\ 0\\ \frac{1}{2}\\ 0\end{bmatrix}$$

$$\begin{bmatrix}1\\-1\\1\\0\end{bmatrix}, \begin{bmatrix}\frac{1}{3}\\ \frac{2}{3}\\ \frac{1}{3}\\ 0\end{bmatrix}, \begin{bmatrix}-\frac{1}{2}\\ 0\\ \frac{1}{2}\\ 0\end{bmatrix} \quad \leftarrow \text{orthogonal}$$

$\qquad\qquad v_1 \qquad\qquad v_2 \qquad\qquad v_3$

$$\begin{bmatrix}\frac{1}{\sqrt{3}}\\ -\frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{3}}\\ 0\end{bmatrix}, \begin{bmatrix}\frac{1}{\sqrt{6}}\\ \frac{2}{\sqrt{3}}\\ \frac{1}{\sqrt{6}}\\ 0\end{bmatrix}, \begin{bmatrix}-\frac{1}{\sqrt{2}}\\ 0\\ \frac{1}{\sqrt{2}}\\ 0\end{bmatrix} \quad \leftarrow \text{orthonormal}$$

$\qquad\qquad u_1 \qquad\qquad u_2 \qquad\qquad u_3$

The QR Factorization of a matrix A $m \times n$

with linearly independent columns

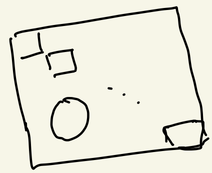
$$A = QR \qquad \left(\Rightarrow Q^{-1}A = R \Rightarrow Q^T A = R \right)$$

Q is orthogonal $m \times n$

$\underbrace{\phantom{Q \text{ is orthogonal}}}$

its columns form an orthonormal set

and R is invertible upper triangular $n \times n$ with positive diagonal entries



$r_{kk} > 0$

One can use Gram-Schmidt for obtaining QR-factorization

More precisely apply G-S to the columns of A to obtain n orthonormal vectors $u_1, \dots, u_n$

form $Q = [u_1, \dots, u_n]$ $\quad$ $m \times n$ matrix with columns $u_1, \dots, u_n$

since Q orthonormal $\quad Q^T Q = I_n$

$$R = Q^T A$$

(if $r_{kk} < 0$ replace u_k by $-u_k$ in Q, r_{kk} by $-r_{kk}$)

$\boxed{\text{Ex 3}}$ $\quad A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad$ QR-factorization?

by $\boxed{\text{Ex 2}}$ $\quad Q = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}} & 0 \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & 0 \end{bmatrix}$

$$R = Q^T A = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & 0 \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{3}} & \frac{4}{\sqrt{3}} \\ 0 & \frac{\sqrt{6}}{3} & \frac{5}{\sqrt{6}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}}_{R}$$

HW # 23

A $m \times n$

Q orthonormal $m \times n$

R $n \times n$

<u>Hint:</u> $A = QR$

A lin. indep. columns.

Argue / prove that R is invertible.

R invertible $\iff$ $Rx = 0$ has only trivial sol'n.

Say $\underline{Ry = 0}$ for $y \neq 0$

$\Rightarrow$ $\underline{QRy = 0}$ $\quad Ay = 0 \Rightarrow y = 0.$

(A lin. ind columns)

or $\quad \underline{Rx = 0} \Rightarrow QRx = 0 \Rightarrow Ax = 0$

$\Rightarrow \underline{x = 0}$ since

— END —