6.5 Least-squares Problems

distance between vectors $a = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

$$\|a - b\| = \left((a_1 - b_1)^2 + (a_2 - b_2)^2 + (a_3 - b_3)^2\right)^{1/2}$$

Revisit the meaning of linear system

$$Ax = b$$

$$\begin{bmatrix} 3 & 4 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 11 \\ 12 \end{bmatrix} \qquad \begin{aligned} 3x_1 + 4x_2 &= 11 \\ 2x_1 - x_2 &= 12 \end{aligned}$$

$$x_1 \begin{bmatrix} 3 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 11 \\ 12 \end{bmatrix}$$

Note: we seek to write b as linear combination of the columns of A.

Thus $Ax = b$ is consistent (it has solutions)

$\iff$ b belongs $Col(A) = span\{a_1, \dots, a_n\}$

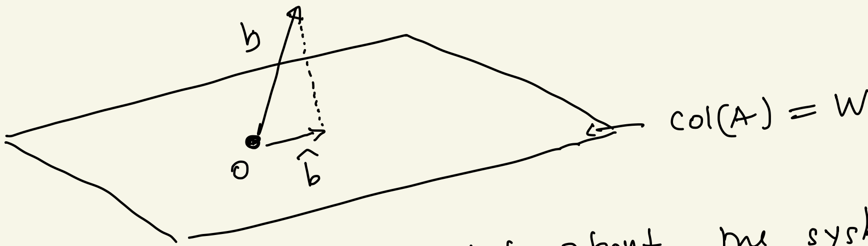
$$A = [a_1, \dots, a_n]$$

What to do when $Ax = b$ is $\boxed{\text{not consistent ?}}$

The next best thing is to search for x such that $\|b - Ax\|$ is as small as possible.

b is not in $Col(A)$

This is solved as follows



$\hat{b} = \text{proj}_W\, b$

What about the system

$$A\hat{x} = \hat{b} \quad ?$$

This is consistent (has sol's) since $\hat{b}$ belongs to $\text{Col}(A)$

If $\hat{x}$ satisfies $A\hat{x} = \hat{b}$ then

since $b - \hat{b} \perp \text{Col}(A) = W$

$b - A\hat{x} \perp \text{Col}(A)$

$\Updownarrow$

$A^T(b - A\hat{x}) = 0$

$A = \begin{pmatrix} | & \cdots & | \end{pmatrix}$

$A^T = \begin{bmatrix} \text{—} \\ \text{—} \end{bmatrix}$

$(\text{each row of } A^T)(b - A\hat{x}) = 0$

u, v in $\mathbb{R}^2$

$u = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad v = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$u \cdot v = 0$

$\underset{1\times 2}{[1, 2]} \underset{2\times 1}{\begin{bmatrix} -2 \\ 1 \end{bmatrix}} = 0 \qquad = 1\times 1$

$u^T v = 0$

Thus:

$\hat{x}$ is a solution of

$$\boxed{A^T A \hat{x} = A^T b}$$

normal system associated to $Ax = b$.

$\hat{x}$ is called a least-squares solution.

$\|b - A\hat{x}\|$

$\boxed{\text{Thm}}$ The least-squares solutions of

$$Ax = b$$

coincides with the set of solutions of the normal system $A^T A \hat{x} = A^T b$.

Any such $\hat{x}$ has the property that

$$\| b - A\hat{x} \| \leq \| b - Ax \| \quad \text{for any } x \text{ in } \mathbb{R}^n$$

Moreover the orthogonal projection

$$\operatorname{proj}_{\operatorname{Col}(A)}(b) = \hat{b} = A\hat{x}$$

where $\hat{x}$ is any least-squares sol'n.

$\boxed{\text{Ex 1}}$ Find least-squares sol's of

$$\underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \underbrace{\begin{bmatrix} -2 \\ 1 \\ 2 \\ 1 \end{bmatrix}}_{b}$$

Normal system: $A^T A \hat{x} = A^T b$

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix}}_{} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 18 \end{bmatrix} \qquad \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} \\ \frac{14}{5} \end{bmatrix}$$

solve

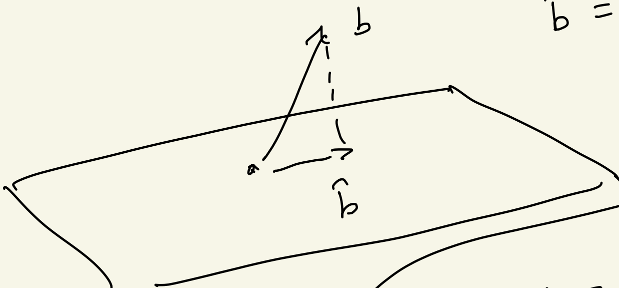
unique least-sq sol'n.

$Ax = b$ inconsistent

Search for x s.t. $\|b - Ax\|$ is as small as possible. There is precisely one such x since A^TA is invertible

$$\hat{x} = \begin{bmatrix} \frac{3}{5} \\ \frac{14}{5} \end{bmatrix}$$

Col(A) is 2-dimensional



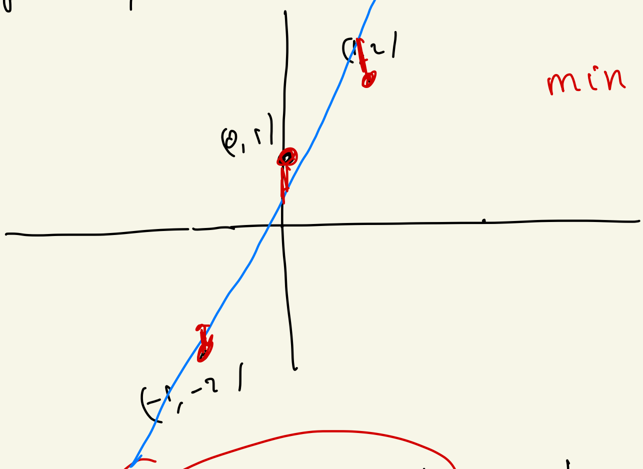
$$\hat{b} = A\hat{x} = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \frac{3}{5} \\ \frac{14}{5} \end{bmatrix} = \hat{b}$$

Error from an exact solution is

$$\|b - A\hat{x}\| = \|b - \hat{b}\| = \left\| \begin{bmatrix} \frac{1}{5} \\ -\frac{2}{5} \\ \frac{7}{5} \\ -\frac{4}{5} \end{bmatrix} \right\| = \frac{\sqrt{70}}{5}$$

(2, 7)

<u>Remark</u> Find a line which interpolates the data $(-1,-2)$ $(0,1)$ $(1,2)$ $(2,7)$ "optimally".



$\min\left(\text{sum}\left(\text{errors}^2\right)\right)$

Seek $f(t) = x_1 + x_2 t$ s.t

$f(-1) = -2$
$f(0) = 1$
$f(1) = 2$
$f(2) = 7$

$$\boxed{\begin{aligned} x_1 - x_2 &= -2 \\ x_1 &= 1 \\ x_1 + x_2 &= 2 \\ x_1 + 2x_2 &= 7 \end{aligned}}$$

Alternate method if the columns of A are orthogonal $A = [a_1, a_2, \ldots, a_n]$

$$\hat{b} = \operatorname{Proj}_{\text{Col}(A)} b = \frac{b \cdot a_1}{a_1 \cdot a_1} a_1 + \cdots + \frac{b \cdot a_n}{a_n \cdot a_n} a_n$$

How to solve $A\hat{x} = \hat{b}$?

$$\hat{x} = \begin{bmatrix} \frac{b \cdot a_1}{a_1 \cdot a_1} \\ \vdots \\ \frac{b \cdot a_n}{a_n \cdot a_n} \end{bmatrix}$$

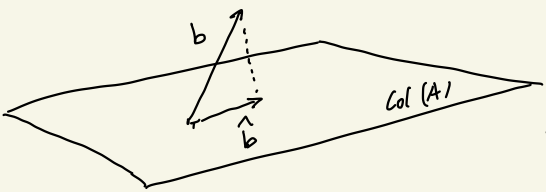
— END —

Remarks concerning Problem 13 on HW.

Recall what a least-squares solution $\hat{x}$ for $Ax = b$ does:

It satisfies the equation $A\hat{x} = \hat{b}$ where

$\hat{b} = \text{Proj}_{Col(A)}(b)$ = orthogonal projection of b onto the subspace $Col(A)$



Given A and b, the orthogonal projection $\hat{b}$ is unique.

It has the property that $\|b - \hat{b}\| < \|b - w\|$ for any w in $Col(A)$

Thus if $b_1 \neq b_2$ are in $Col(A)$ and $\|b - b_1\| = \|b - b_2\|$ then neither b_1 nor b_2 can be equal to $\hat{b}$.

Now suppose that u, v are distinct vectors such that $Au \neq Av$ and $\|b - Au\| = \|b - Av\|$

Is it possible that at least one of u or v could be a least-squares solution of $Ax = b$?