5.7 Applications to differential equations (part 2)

Review:

$$\vec{x}' = A\vec{x}$$

$$\begin{bmatrix} x_1'(t) \\ \vdots \\ x_n'(t) \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

The solution set forms an n-dimensional vector space

Each solution is uniquely determined by an initial condition $\vec{x}(0) = \vec{x}_0$

The case when A has n distinct real eigenvalues

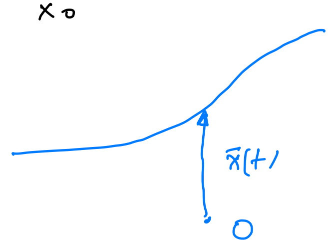
$$\lambda_1 < \lambda_2 < \cdots < \lambda_n$$

$$\vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_n \quad \leftarrow \text{eigenvectors}$$

general solution :

$$\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + \cdots + c_n e^{\lambda_n t}\vec{v}_n$$

If $\vec{x}(0) = \vec{x}_0$ then $\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = P^{-1}\vec{x}_0$



where $P = [v_1, \ldots, v_n]$

View $\vec{x}(t)$ as trajectories in $\mathbb{R}^n$ for particles with initial position x_0.

Note: $\vec{x}_0 = 0 \Rightarrow \vec{x}(t) = 0$ fixed position for all t

Qualitative study of trajectories: "PHASE PORTRAIT" (2)

Case 1 $\lambda_1 < \lambda_2 < 0$ 0 Attractor

v_1 v_2 origin

$A = \begin{bmatrix} -2 & -2 \\ 1 & -5 \end{bmatrix}$

$\lambda_1 = -4$ $\lambda_2 = -3$

$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $v_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

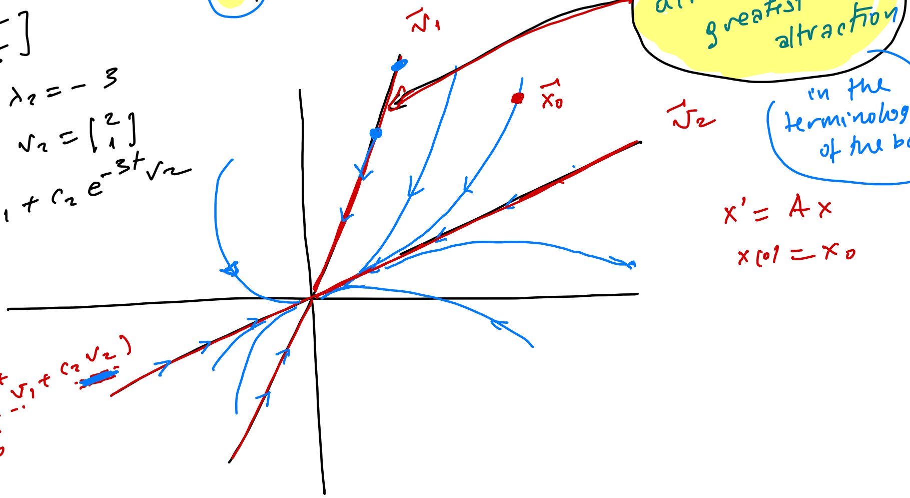
$x(t) = c_1 e^{-4t} v_1 + c_2 e^{-3t} v_2$

$t \to \infty$

$e^{-4t} \to 0$

$e^{-3t} \to 0$

$x(t) = e^{-3t}(\underbrace{c_1 e^{-t}}_{0} v_1 + c_2 v_2)$

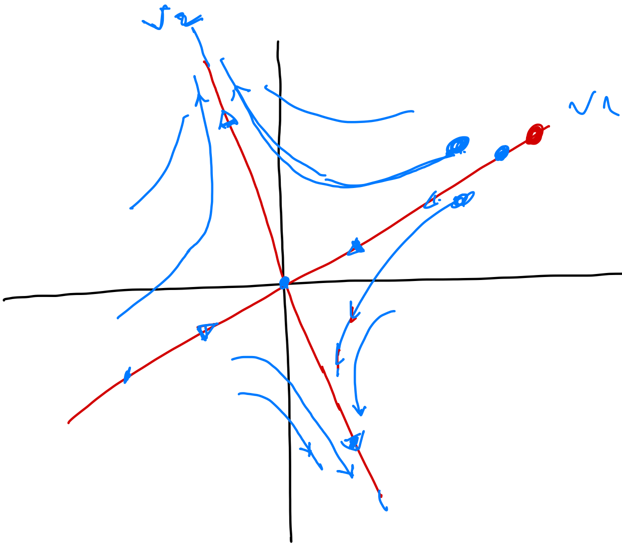


$x' = Ax$

$x(0) = x_0$

Case 2: $\lambda_1 < 0 < \lambda_2$ 0 saddle point

v_1 v_2 origin



$A = \begin{bmatrix} 1 & -1 \\ -2 & 0 \end{bmatrix}$

$\lambda_1 = -1$ $\lambda_2 = 2$

$v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$x(t) = c_1 \underbrace{e^{-t}}_{\to 0} v_1 + c_2 e^{2t} v_2$

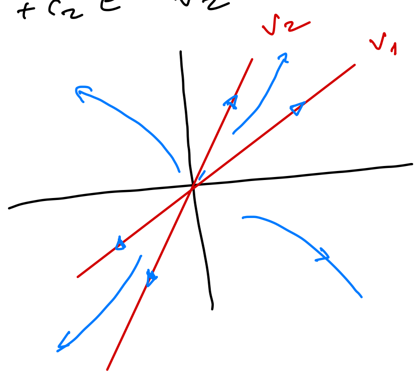
$t \to \infty$

Case 3 $\quad 0 < \lambda_1 < \lambda_2$

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix} \qquad \lambda_1 = 2 \quad \lambda_2 = 3 \qquad v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$x(t) = c_1 e^{2t} v_1 + c_2 e^{3t} v_2$$

0 repelling point



COMPLEX EIGENVALUES

$x' = Ax \qquad A \quad 2\times2 \quad$ real entries

$\lambda = a + ib \qquad \bar{\lambda} = a - ib$

$v = p + iq \qquad \bar{v} = p - iq$ > conjugated

a, b in $\mathbb{R}$

p, q in $\mathbb{R}^2$

Why?

$$Av = \lambda v$$
$$\overline{Av} = \overline{\lambda v}$$
$$\bar{A}\bar{v} = \bar{\lambda}\bar{v}$$
$$\bar{A} = A$$
$$A\bar{v} = \bar{\lambda}\bar{v}$$

General solution complex form

$$\vec{x}(t) = c_1 e^{\lambda t} v + c_2 e^{\bar{\lambda} t} \bar{v}$$

The real solution is given by

$$\boxed{c_1 \underbrace{\mathrm{Re}\left(e^{\lambda t} v\right)}_{①} + c_2 \underbrace{\mathrm{Im}\left(e^{\lambda t}\right)}_{②}}$$

More explicitly

$$e^{\lambda t} = e^{(a+ib)t} = e^{at} e^{ibt} = e^{at}\left(\cos(bt) + i\sin(bt)\right)$$

$$v = p + iq \quad \left(= \mathrm{Re}\, v + i\, \mathrm{Im}\, v\right)$$

$$e^{\lambda t} v = e^{at}\left(\cos(bt) + i\sin(bt)\right)(p + iq)$$

$$= \underbrace{e^{at}\left(\cos(bt)\,p - \sin(bt)\,q\right)}_{①} + i\, \underbrace{e^{at}\left(\sin(bt)\,p + \cos(bt)\,q\right)}_{②}$$

real solution

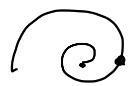
$$c_1\left(e^{at}\left(\cos(bt)\,p - \sin(bt)\,q\right)\right) + c_2\left(e^{at}\left(\sin(bt)\,p + \cos(bt)\,q\right)\right)$$

TRAJECTTORIES ? (complex eigenvalues)

$b \neq 0$

$\lambda = a + bi$

$a = 0$ trajectories are ellipses



$a > 0$ trajectories are outward spirals away from origin

$a < 0$ trajectories are "inward" spirals toward origin

Ex: $A = \begin{bmatrix} 46 & 52 & 50 \\ -34 & -40 & -50 \\ 18 & 36 & 30 \end{bmatrix}$ $\quad$ 12: $\begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$ $\quad$ $12+24i$: $\begin{bmatrix} 3-4i \\ -3+4i \\ 3 \end{bmatrix}$ $\quad$ $12-24i$: $\begin{bmatrix} 3+4i \\ -3-4i \\ 3 \end{bmatrix}$

$x' = Ax$

Complex solution:

$$x(t) = c_1 \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} e^{12t} + c_2 \underbrace{\begin{bmatrix} 3-4i \\ -3+4i \\ 3 \end{bmatrix} e^{(12+24i)t}}_{y(t)} + c_3 \begin{bmatrix} 3+4i \\ -3-4i \\ 3 \end{bmatrix} e^{(12-24i)t}$$

$\bar{x}(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2 + c_3 e^{\lambda_3} v_3$

Real solution:

$$x(t) = c_1 \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} e^{12t} + c_2 \operatorname{Re}(y(t)) + c_3 \operatorname{Im}(y(t))$$

$$y(t) = e^{12t}\left(\cos(24t) + i\sin(24t)\right)\left(\begin{bmatrix} 3 \\ -3 \\ 3 \end{bmatrix} + i\begin{bmatrix} -4 \\ 4 \\ 0 \end{bmatrix}\right)$$

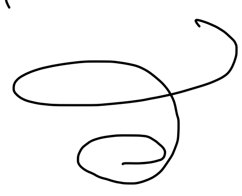
$$= e^{12t}\left(\cos(24t)\begin{bmatrix} 3 \\ -3 \\ 3 \end{bmatrix} - \sin(24t)\begin{bmatrix} -4 \\ 4 \\ 0 \end{bmatrix}\right) +$$

$$i\, e^{12t}\left(\sin(24t)\begin{bmatrix} 3 \\ -3 \\ 3 \end{bmatrix} + \cos(24t)\begin{bmatrix} -4 \\ 4 \\ 0 \end{bmatrix}\right)$$

Real sol'n

$$c_1 \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} e^{12t} + c_2 \begin{bmatrix} 3\cos(24t) + 4\sin(24t) \\ -3\cos(24t) - 4\sin(24t) \\ 3\cos(24t) \end{bmatrix} e^{12t} + c_3 \begin{bmatrix} 3\sin(24t) - 4\cos(24t) \\ -3\sin(24t) + 4\cos(24t) \\ 3\sin(24t) \end{bmatrix} e^{12t}$$

Spirals away from the origin



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