6.7 Inner product spaces

Inner products are generalizations of the dot product.

Def'n An inner product on a vector space V is a function which associates to each pair of vectors u, v in V a real number

$$\langle u, v \rangle$$

Must satisfy the following axioms for all u, v, w in V and c in $\mathbb{R}$

(1) $\langle u, v \rangle = \langle v, u \rangle$

(2) $\langle u+v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$

(3) $\langle cu, v \rangle = c \langle u, v \rangle$

(4) $\langle u, u \rangle \geq 0$ and $\langle u, u \rangle = 0 \iff u = 0$.

A vector space endowed with an inner product is called an inner product space.

Ex. 1 $V = \mathbb{R}^n$ $\quad \langle u, v \rangle = u \cdot v$

Ex. 2 $V = \mathbb{R}^n$ fix an $n \times n$ invertible matrix A

$\langle u, v \rangle = (Au) \cdot (Av)$ is an inner product

Note $\langle u,u\rangle = (Au)\cdot(Au) = \|Au\|^2 \geq 0$

and $\langle u,u\rangle = 0 \iff Au = 0 \iff u = 0$

A invertible.

Ex 3 $V = \mathbb{R}^2$ $\langle u, v\rangle = 4u_1v_1 + 6u_2v_2$

corresponds to $A = \begin{bmatrix} 2 & 0 \\ 0 & \sqrt{6} \end{bmatrix}$

$$(Au)\cdot(Av) = \left(\begin{bmatrix} 2 & 0 \\ 0 & \sqrt{6} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right)\cdot\left(\begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{6} \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right)$$

$$\begin{bmatrix} 2u_1 \\ \sqrt{6}u_2 \end{bmatrix}\cdot\begin{bmatrix} 2v_1 \\ \sqrt{6}v_2 \end{bmatrix} = 4u_1v_1 + 6u_2v_2$$

Ex 4 $V = \mathbb{R}^2$ $\langle u, v\rangle = (Au)\cdot(Av)$

$A = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ $\quad A\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 2u_1 + u_2 \\ u_1 + u_2 \end{bmatrix}$

$$\langle u, v\rangle = (2u_1 + u_2)(2v_1 + v_2) + (u_1 + u_2)(v_1 + v_2)$$

$$\langle u, v\rangle = 5u_1v_1 + 3u_1v_2 + 3u_2v_1 + 2u_2v_2$$

Ex 5 $V = \mathcal{P}_n$ = polynomials of degree $\leq n$

$t_0, t_1, \dots, t_n$ distinct real numbers

$$\langle p, q\rangle = p(t_0)q(t_0) + \dots + p(t_n)q(t_n)$$

Ex 6 $V = C[a,b] = \{$ continuous functions $f: [a,b] \to \mathbb{R}\}$

$$\langle f, g\rangle = \int_a^b f(t)g(t)\,dt$$

One can define length, distance, orthogonality with respect an inner product

by def'n

$$\|u\| = \sqrt{\langle u, u \rangle}$$

$$\text{dist}(u, v) = \|u - v\|$$

definition of orthogonality

$$u \perp v \iff \langle u, v \rangle = 0$$

Ex 6 Let $\mathbb{R}^2$ have the inner product from Ex. 2:

$$\langle u, v \rangle = 5u_1v_1 + 3u_1v_2 + 3u_2v_1 + 2u_2v_2$$

If $u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\quad \|u\|^2 = \left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle = 5 \quad \|u\| = \sqrt{5}$

If $u = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ then they are NOT orthogonal

since $\langle u, v \rangle = 5 \cdot 0 \cdot 0 + \underbrace{3 \cdot 1 \cdot 1}_{3u_1v_2} + 3 \cdot 0 \cdot 1 + 2\,0 \cdot 1 = 3$

Ex 7 $\quad V = C[0, \pi] \quad \langle f, g \rangle = \int_0^\pi f(t)\,g(t)\,dt$

$$\|f\| = \left(\int_0^\pi |f(t)|^2 \, dt \right)^{1/2} = \sqrt{\langle f, f \rangle} = \sqrt{\int_0^{2\pi} f(t)\,f(t)\,dt}$$

$f(t) = \sin t \quad g(t) = \cos t \qquad f \perp g$

since $\displaystyle\int_0^\pi \sin t \cos t \, dt = \left[\frac{\sin^2 t}{2} \right]_0^\pi = 0 - 0 = 0$

One can compute orthogonal projections $\text{proj}_W x$

$$\text{proj}_W x = \frac{\langle x, v_1\rangle}{\langle v_1, v_1\rangle} v_1 + \cdots + \frac{\langle x, v_p\rangle}{\langle v_p, v_p\rangle} v_p$$

if $v_1, \ldots, v_p$ orthogonal basis of W with respect to $\langle u, v\rangle$

One can apply the Gram-Schmidt proces in any inner space. Just use $\langle u, v\rangle$ in place of usual dot product $u \cdot v$

$\boxed{\text{Ex } 8}$ Let $\mathcal{P}_2$ have the inner product

$$\langle p, q\rangle = p(-1)\,q(-1) + p(0)\,q(0) + p(1)\,q(1)$$

Find the orthogonal projection of $p(t) = t^2$ onto the subspace W of $\mathcal{P}_2$ spanned by $p_0(t) = 1$ and $p_1(t) = t$.

Sol'n observe that $\langle p, q\rangle = \begin{bmatrix} p(-1) \\ p(0) \\ p(1)\end{bmatrix} \cdot \begin{bmatrix} q(-1) \\ q(0) \\ q(1)\end{bmatrix}$. observe that $p_0 \perp p_1$

since $\langle p_0, p_1\rangle = 0$ $\quad \langle p_0, p_1\rangle = \begin{bmatrix} 1 \\ 1 \\ 1\end{bmatrix} \cdot \begin{bmatrix} -1 \\ 0 \\ 1\end{bmatrix} = 0.$

$\langle p_0, p_0\rangle = \begin{bmatrix} 1 \\ 1 \\ 1\end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1\end{bmatrix} = 3$

Thus $\text{proj}_W(p) = \dfrac{\langle p, p_0\rangle}{\langle p_0, p_0\rangle} p_0 + \dfrac{\langle p, p_1\rangle}{\langle p_1, p_1\rangle} p_1$

$\langle p, p_0\rangle = \begin{bmatrix} 1 \\ 0 \\ 1\end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 1\end{bmatrix} = 2$

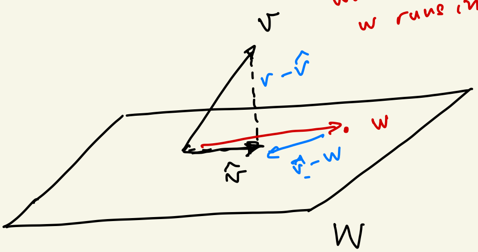
$\langle p, p_1\rangle = \begin{bmatrix} 1 \\ 0 \\ 1\end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1\end{bmatrix} = 0$

$p_0 \to \begin{bmatrix} 1 \\ 1 \\ 1\end{bmatrix} \quad p_1 \to \begin{bmatrix} -1 \\ 0 \\ 1\end{bmatrix} \quad p_2 \to \begin{bmatrix} 1 \\ 0 \\ 1\end{bmatrix}$

$\text{Proj}_W(p) = \frac{2}{3} p_0 = \frac{2}{3}$ $\qquad \text{Proj}_W(t^2) = \frac{2}{3} p_0 = \frac{2}{3}$

Want $\|v - w\|$ as small as possible where w runs in W

If v in V its best approximation by an element of W is $\hat{v} = \text{proj}_W v$ orthogonal projection



Indeed if w in W

$v - w = (v - \hat{v}) + (\hat{v} - w)$

$\|v - w\|^2 = \|v - \hat{v}\|^2 + \|\hat{v} - w\|^2$

$\Longrightarrow \|v - \hat{v}\|^2 \leq \|v - w\|^2$

Note $\langle a, b\rangle = 0 \Rightarrow \|a + b\|^2 = \|a\|^2 + \|b\|^2$ $\qquad \langle a+b, a+b\rangle = \langle a, a\rangle + \langle b, b\rangle$

Best approximation in Inner product spaces

V vector spaces consisting typically of functions endowed with an inner product

The best approximation of an element f by elements in a subspace W is $\text{proj}_W f$.

$\boxed{\text{Exg}}$ Let $V = C[-1,1]$ with $\langle f, g\rangle = \int_{-1}^{1} f(t)\, g(t)\, dt$

Find the best approximation of f by a polynomial p of degree ≤ 2.

<u>Sol'n</u> Seek $p(t)$ such that $\|f-p\|^2 = \int_{-1}^{1} |f(t) - p(t)|^2\, dt$ is as small as possible.

$p = \text{proj}_{\mathcal{P}_2}(f)$. Need orthogonal basis of $\mathcal{P}_2 = \text{span}\{1, t, t^2\}$

$\langle 1, 1\rangle = \int_{-1}^{1} 1\, dt = 2$ $\quad \langle t, t\rangle = \int_{-1}^{1} t^2\, dt = \frac{t^3}{3}\Big|_{-1}^{1} = \frac{2}{3}$

$\langle 1, t\rangle = \int_{-1}^{1} t\, dt = \frac{t^2}{2}\Big|_{-1}^{1} = 0$ $\quad 1 \perp t$ $\quad \langle t^2, 1\rangle \neq 0$

Apply Gram-Schmidt to $\{1, t, t^2\}$

$1, t, \quad t^2 - \text{Proj}_{\text{span}\{1,t\}}(t^2) = t^2 - \frac{\langle t^2, 1\rangle}{\langle 1, 1\rangle} 1 - \frac{\langle t^2, t\rangle}{\langle t, t\rangle} t$

$\langle t^2, 1\rangle = \int_{-1}^{1} t^2\, dt = \frac{t^3}{3}\Big|_{-1}^{1} = \frac{2}{3}$ $\quad \langle t^2, t\rangle = \int_{-1}^{1} t^3\, dt = \frac{t^4}{4}\Big|_{-1}^{1} = 0$

orthogonal basis of $\mathcal{P}_2$ is $\{1, t, t^2 - \frac{1}{3}\}$

$$\text{proj}_{\mathcal{P}_2}(f) = \frac{\langle f, 1\rangle}{\langle 1, 1\rangle} 1 + \frac{\langle f, t\rangle}{\langle t, t\rangle} t + \frac{\langle f, t^2 - \frac{1}{3}\rangle}{\langle t^2 - \frac{1}{3}, t^2 - \frac{1}{3}\rangle}\left(t^2 - \frac{1}{3}\right)$$

Concretely: find best approx of $f(t) = e^t$
by a degree 2 polynomial.

$$\int_{-1}^{1} (e^t - p(t))^2 \, dt \quad \text{as small as possible}$$

$p(t)$ is $= \mathrm{proj}_{P_2}(f)$

– END –

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