7.1

Diagonalization of symmetric matrices

A square $n \times n$ matrix is symmetric if

$$A^T = A$$

$\iff$ The (i,j) entry of A = the (j,i)-entry of A

Ex 1 $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ symmetric

$A = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}$ symmetric

$A = \begin{bmatrix} 1 & 3 & 10 \\ 3 & -1 & 7 \\ 10 & 7 & 2 \end{bmatrix}$ symmetric

Recall P is orthogonal (say $n \times n$)

$\iff$ the columns of P form an orthonormal basis of $\mathbb{R}^n$

$P = [u_1, u_2, \dots, u_n]$

$\|u_i\| = 1$

$u_i \cdot u_j = 0$ if $i \neq j$

$\iff$ P is invertible and $P^{-1} = P^T$

Ex 2 $P = \begin{bmatrix} 0 & -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ 1 & 0 & 0 \\ 0 & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$

$u_1 \quad u_2 \quad u_3$

$\|u_i\| = 1$

$u_1 \cdot u_2 = 0$

$u_2 \cdot u_3 = 0$

$u_1 \cdot u_3 = 0$

$P^{-1} = P^T$ $\quad P^T P = P P^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Recall A ($n \times n$) is diagonalizable if there is a basis of $\mathbb{R}^n$ consisting of eigenvectors of A

$\underbrace{v_1, \dots, v_n}_{\text{basis.}}$ in $\mathbb{R}^n$ $\quad A v_i = \lambda_i v_i$ for $i = 1, \dots, n.$

$\Longleftrightarrow$ there is P invertible and there is D diagonal matrix such that

$$A = P D P^{-1} = P \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} P^{-1}$$

<u>Key fact:</u> Can choose $P = [v_1, \dots, v_n]$

$\boxed{\text{Def'n}}$ An $n \times n$ matrix is orthogonaly diagonalizable if it is diagonalizable and moreover can choose P to be an orthogonal matrix.

$\boxed{\text{Thm}}$ An $n \times n$ matrix is orthogonaly diagonalizable $\Longleftrightarrow$ A is symmetric.

$\left((EF)^T = F^T E^T \right)$

Remark "$\Longrightarrow$" Say $A = P D P^T$

$$A^T = (P D P^T)^T = \underbrace{(P^T)^T}_{P} \underbrace{D^T}_{D} P^T$$

$$\underline{\underline{A^T}} = P \, D \, P^T = \underline{\underline{A}}$$

Thm (The spectral theorem for symmetric matrices) 3

Suppose $\underset{n\times n}{A}$ is symmetric. Then

(1) A has n real eigenvalues counting multiplicities

(2) For each eigenvalue λ

$$\dim(\underbrace{\text{Nul}(A-\lambda I)}_{\text{eigenspace}}) = \text{algebraic multiplicity of } \lambda$$

(3) Eigenspaces are mutually orthogonal

namely if $Av_1 = \lambda_1 v_1$ $\quad Av_2 = \lambda_2 v_2$

and $\lambda_1 \neq \lambda_2$ then $v_1 \cdot v_2 = 0$.

(4) A is orthogonaly diagonalizable.

$\boxed{\text{Ex 3}}$ Diagonalize $A = \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$ $\quad A = PDP^T$, P orthogonal

First find eigenvalues

$$\det(A-\lambda I) = \begin{vmatrix} -1-\lambda & 2 & 2 \\ 2 & -1-\lambda & 2 \\ 2 & 2 & -1-\lambda \end{vmatrix} =$$

$$= -(\lambda+3)^2(\lambda-3) = 0$$

$\lambda_1 = \lambda_2 = -3 \qquad \lambda_3 = 3$

Next eigenspaces

$\boxed{\text{For } \lambda = -3}$ $\quad A - \lambda I = A + 3I = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} \sim \begin{bmatrix} 2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\leftarrow$ rank $= 1$, nullity $= 2$

$$(A+3I)x=0 \qquad \begin{bmatrix}1&1&1\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$x_1+x_2+x_3=0 \qquad s \quad t$$

$$x_2=s \qquad x_1=-s-t \qquad \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}-s-t\\s\\t\end{bmatrix}$$

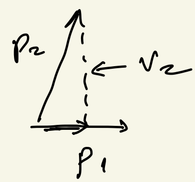
$$x_3=t$$

$$= \underbrace{s\begin{bmatrix}-1\\1\\0\end{bmatrix}+t\begin{bmatrix}-1\\0\\1\end{bmatrix}}_{\text{basis of eigenvectors for } \lambda=-3}$$

$$p_1=\begin{bmatrix}-1\\1\\0\end{bmatrix} \qquad p_2=\begin{bmatrix}-1\\0\\1\end{bmatrix}$$

basis of Null $(A+3I)$ but not orthonormal basis

Apply Gram-Schmidt



$$v_1=p_1$$

$$v_2=p_2-\frac{p_2\cdot p_1}{p_1\cdot p_1}\,p_1$$

$$v_2=\begin{bmatrix}-1\\0\\1\end{bmatrix}-\frac{1}{2}\begin{bmatrix}-1\\1\\0\end{bmatrix}=\begin{bmatrix}-\frac{1}{2}\\-\frac{1}{2}\\1\end{bmatrix} \qquad \begin{bmatrix}-1\\-1\\2\end{bmatrix}$$

v_1, v_2 orthogonal basis hence

$$\frac{v_1}{\|v_1\|}=u_1=\begin{bmatrix}-1/\sqrt{2}\\1/\sqrt{2}\\0\end{bmatrix} \qquad u_2=\begin{bmatrix}-\frac{1}{\sqrt{6}}\\-\frac{1}{\sqrt{6}}\\\frac{2}{\sqrt{6}}\end{bmatrix}=\frac{v_2}{\|v_2\|}$$

orthonormal eigenvectors for $\lambda=-3$

$\boxed{\text{For } \lambda = 3}$

multiplicity $= 1$

$$A - 3I = \begin{bmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

rank $= 2$

nullity $= 1$

$$x_1 - x_3 = 0$$
$$x_2 - x_3 = 0$$
$$x_3 = s$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s \\ s \\ s \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$u_3 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

observe that $u_3 \cdot u_1 = 0$ $\quad u_3 \cdot u_2 = 0$

$$P = [u_1, v_2, u_3] = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$$

$$A = PDP^T \qquad D = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Spectral decomposition of A symmetric

$$\boxed{A = \lambda_1 \underline{u_1 u_1^T} + \cdots + \lambda_n \underline{u_n u_n^T}}$$

Note: $\quad u_i \quad u_i^T \qquad u_i u_i^T \quad n \times n$

$\qquad n \times 1 \quad 1 \times n$

$\boxed{\text{Ex } 4}$ Find spec. decom for A from Ex 3

$$u_1 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \qquad u_1 u_1^T = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$u_2 = \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix} \qquad u_2 u_2^T = \begin{bmatrix} \frac{1}{6} & \frac{1}{6} & -\frac{1}{3} \\ \frac{1}{6} & \frac{1}{6} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

$$u_3 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \qquad u_3 u_3^T = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

$$A = (-3)\, u_1 u_1^T + (-3)\, u_2 u_2^T + 3\, u_3 u_3^T$$

Spectral decomposition of A

Remark (Intuition)

If $E_i = u_i u_i^T$

and x in $\mathbb{R}^n$ $\qquad E_i x = \text{Proj}_{\text{Span}(u_i)} x$

Ex: $A = I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} = 1 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + 2 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + 3 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\boxed{\text{- END -}}$