Review questions

1. The matrix below represents the augmented matrix of a system of linear equations. Assume that the variables in this system are x_1, x_2, x_3, x_4, x_5, and x_6, and let A be the coefficient matrix:

$$\begin{array}{c} \overbrace{\qquad\qquad\qquad\qquad\qquad}^{A} \quad b \\ \left(\begin{array}{cccccc|c} 1 & 0 & 1 & 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 4 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 & c \\ 0 & 0 & 0 & 0 & 0 & 1 & d \end{array}\right) \end{array}$$

$A \quad 4 \times 6$

← echelon form
not reduced echelon form.

Which of the following statements are **true**?

$\text{Rank}(A) = 4$
$\shortparallel$
$\text{Rank}(A|b) = 4$
↑
columns are in $\mathbb{R}^4$

(i) For any given c and d, the system above is consistent.

(ii) The coefficient matrix A is in reduced echelon form.

(iii) The right hand side vector is in the column space of matrix A.

(iv) The system has no solution.

(v) The system has infinitely many solutions. ← 2 free variables

$\text{nullity}(A) = 2$
$= 6 - 4$

A. (i), (ii) only

B. (i), (ii), (iii) only

C. (ii), (iii), (iv) only

D. (i), (iii), (v) only

E. all of the above

m-equations n unknown

$$\underset{m\times n}{A}\,\underset{n\times 1}{x} = \underset{m\times 1}{b} \iff \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} \iff x_1 \underbrace{\begin{bmatrix} a_{11} \\ \vdots \\ a_{m1} \end{bmatrix}}_{\mathbb{R}^m} + \cdots + x_n \underbrace{\begin{bmatrix} a_{1n} \\ \vdots \\ a_{mn} \end{bmatrix}}_{\mathbb{R}^m} = \underbrace{b}_{\mathbb{R}^m}$$

$Ax = b$ consistent (has solutions) $\iff b$ in $\text{Span}\{\vec{a}_1, \dots, \vec{a}_n\}$

$\iff b$ in $\text{Col}(A) \iff \text{Col}(A) = \text{Col}(A|b)$

$\text{Span}\{a_1, \dots, a_n\} = \text{Span}\{a_1, \dots, a_n, b\}$

$\iff \text{rank}(A) = \text{rank}(A|b)$

$\boxed{\text{rank}(A) = \dim(\text{Col}(A))}$

$\vec{a}_1, \dots, \vec{a}_n$ are linearly independent (that is $x_1\vec{a}_1 + \cdots + x_n\vec{a}_n = \vec{0} \iff x_1 = \cdots = x_n = 0$)

$\iff Ax = 0$ has a unique sol'n $\bar{x} = 0 \iff \text{Nul}(A) = \{0\} \iff \text{Rank}(A) = n$

↑

$\text{Rank}(A) + \text{Nullity}(A) = n =$ nr of columns

Key remark: Suppose $Ax = b$ has a solution x_0

Let v in $\text{Nul}(A)$ $\quad Av = 0 \quad A(x_0 + v) = Ax_0 + Av = b + 0 = b$.

If x_0 solution of $Ax = b$ that all vectors in $x_0 + \text{Nul}(A)$ are solutions

3. Which of the following statements is **false**?

False **A.** If $\{v_1, v_2, v_3\}$ is linearly dependent, then v_3 is a linear combination of v_1 and v_2.

B. Suppose that the columns of A are v_1, v_2, and v_3. Then the matrix equation $A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = b$ is equivalent to the vector equation $x_1v_1 + x_2v_2 + x_3v_3 = b$.

C. Suppose that X_0 is a solution to the linear system $AX = b$. Then $\{X|AX = b\} = X_0 + \{X|AX = 0\}$. — Null (A)

D. The columns of A are linearly independent if and only if A has a pivot position in every column.

E. A homogeneous linear system has a non-trivial solution if and only if it has at least one free variable.

Ex: $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \end{bmatrix} \right\}$

8. Let A be an $m \times n$ matrix. Which of the following statement is necessarily **true**?

A. The nullity of A is the same as the nullity of A^T.

B. The rank of A is the same as the rank of A^T.

C. The column space of A is the same as the null space of A^T.

D. The columns of A form a basis of the column space of A.

E. The columns of A^T form a basis of the null space of A.

$$\text{Rank}(A) + \text{Nullity}(A) = n$$

$$\text{Rank}(A^T) + \text{Nullity}(A^T) = m$$

$A \quad m \times n$

$A^T \quad n \times m$

Fact: $\text{Rank}(A) = \text{Rank}(A^T) \leq \min\{m, n\}$

$\text{Col}(A) \subseteq \mathbb{R}^m$

$\dim(\text{Col}(A)) \leq m$

$\text{Rank}(A) \leq m$

have n columns $\Rightarrow \dim(\text{Col}(A)) \leq n$

18. Which of the following subsets of the vector space $\mathbb{R}^3$ are subspaces of $\mathbb{R}^3$?

(i) The set of all vectors $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ with the property $2xyz = 0$.

(ii) The set of all the solutions of the equation $x - 5y + 2z = 0$. $\leftarrow \begin{bmatrix} 1 & -5 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$

(iii) The set of all solutions for the system $\begin{bmatrix} 2 & 3 & 2 \\ 5 & 2 & 8 \\ -1 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

(iv) The set of all the solutions of the equation $x + 3y = 2z + 1$.

A. (ii) and (iii) only

B. (ii) and (iv) only

C. (iii) and (iv) only

D. (ii), (iii) and (iv) only

E. (i), (ii), (iii) and (iv)

(2) fails $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ in W

$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ not in W

$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

in W in W not in W

W subset of $\mathbb{R}^n$ when is W subspace?

if (1) 0 in W

(2) closed under addition

(3) closed under scalar multiplication

Remark The solution set of any homogeneous system $Ax = 0$ is a subspace of $\mathbb{R}^n$

A $m \times n$

called $\mathrm{Nul}(A)$.

$\boxed{\text{If } b \neq 0 \text{ sol set of } Ax = b \text{ is not a subspace}}$

$Ax = 0$ $\quad Ay = 0$ $\quad A(x+y) = 0$

$A(cx) = c\,Ax = 0$

$A\,0 = 0.$

$x - 5y + 2z = 0$

$x' - 5y' + 2z' = 0$

$(x + x') - 5(y + y') + 2(z + z') = 0$

23. Let A be an $n \times n$ matrix. Which of the following statements is/are **NOT** equivalent to that A is invertible?

(i) Columns of A are linearly independent. $\Leftrightarrow$ A invertible

(ii) A is diagonalizable. no ex $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

(iii) Columns of A is an orthonormal set. $\Longrightarrow$ A invertible

inv $\Leftrightarrow$ (iv) The dimension of the null space of A is 0. $\longrightarrow$ $\text{Rank}(A) = n - \text{nullity}(A) = n - 0 = n$

inv. $\Leftrightarrow$ (v) The linear system $AX = b$ always has solution for any $b \in \mathbb{R}^n$.

$\Downarrow$

$\text{Col}(A) = \mathbb{R}^n$

$\dim(\text{Col}(A)) = n$

$\text{Rank}(A) = n$

A. (i), (ii) and (iii) only.

B. (i) and (ii) only.

C. (ii) and (iii) only.

D. (i) and (iv) only.

E. (ii), (iv), (v) only.

some REF $= \begin{bmatrix} 1 & * & * & * \\ & 1 & * & * \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix}$

$\Uparrow$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$

Recall A $n \times n$

A invertible $\Leftrightarrow$ $\text{RREF}(A) = I_n$

$\Updownarrow$

$\dim(\text{Col}(A)) = n$ $\Leftrightarrow$ $\text{Rank}(A) = n$

— END OF CLASS —