5. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation whose standard matrix is $\begin{bmatrix} t-1 & 2t-2 \\ 1 & t \end{bmatrix}$ where t is a real number. Find ALL values of t such that L is one-to-one.

A. $t \neq 1$

B. $t \neq 0, 1$

C. $t \neq 1, 2$

D. $t = 1$

E. $t = 2$

Recall $f : A \to B$ is one-to-one if $f(x) = f(y)$ is possible only when $x = y$

Thus $x \neq y$ then $f(x) \neq f(y)$.

For linear maps $T : \mathbb{R}^n \to \mathbb{R}^m$ given by A $m \times n$

$T(x) = Ax$ $\quad$ ($Ax - Ay = 0$ $\quad$ $A(x-y) = 0$)

T is one-to-one $\Longleftrightarrow$ $T(x) = 0$ happens only for $x = 0$

$Ax = 0$

$\Longleftrightarrow$ $Nul(A) = \{0\}$ $\overset{?}{\Longleftrightarrow}$ $Rank(A) = n$

For $n \times n$ matrices

$Rank(A) = n$ $\Longleftrightarrow$ A invertible

$\Longleftrightarrow$ $\det(A) \neq 0$

($Ax = 0 \Rightarrow A^{-1}Ax = A^{-1}0$)

$x = 0$

Back to question:

$$\det(A) = \begin{vmatrix} t-1 & 2t-2 \\ 1 & t \end{vmatrix} = t^2 - 3t + 2$$

$$= (t-1)(t-2)$$

$$\neq 0$$

$\Rightarrow$ $t \neq 1, t \neq 2$ $\quad$ $\underbrace{Null(A)}_{0} + Rank(A) = n$

When is $f: A \to B$ onto?

f onto if for any b in B there is a in A such that $f(a) = b$.

Ex: $f: \mathbb{R} \to \mathbb{R}$ $\quad f(x) = x^2$

not onto! there is no x such that $f(x) = -1$

$g: \mathbb{R} \to [0, \infty)$ $\quad g(x) = x^2$ is onto

For linear maps $T: \mathbb{R}^n \to \mathbb{R}^m$

$Tx = Ax$ $\quad$ A is $m \times n$

T onto if for any b in $\mathbb{R}^m$ there is a in $\mathbb{R}^n$ s.t. $Aa = b$

$\iff$ $Ax = b$ is consistent for any b in $\mathbb{R}^m$

$\iff$ $\mathrm{Col}(A) = \mathbb{R}^m \iff \dim(\mathrm{Col}(A)) = m$

$\mathrm{rank}(A) = m$

2×3

$$\begin{bmatrix} 1 & 1 & 4 \\ -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \qquad x_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$A_{2 \times 3}$ $\quad 3 \times 1$

$T: \mathbb{R}^3 \to \mathbb{R}^2$ $\quad T(x) = Ax$

Note if $\boxed{A = 2 \times 3}$

$\mathrm{rank}(A) \le 2 < 3 \implies T$ is never one-to-one

$\mathrm{nullity}(A) \geq 1 \implies$ there is $x \neq 0$ s.t. $Ax = 0$ $\quad Ax = A0$ $\quad x \neq 0$.

12. Suppose $A = PDP^{-1}$, where P is a 3×3 invertible matrix and $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$.

Let $B = 2I + 3A + A^2$, which of the following is **true**?

A. B is not diagonalizable.

$B = \underset{2PP^{-1}}{2I} + 3PDP^{-1} + PD^2P^{-1} = P(\underbrace{2I+3D+D^2}_{C})P^{-1}$

Thus B is d-able

correct

B. B is diagonalizable, and $B = PCP^{-1}$, where $C = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

$2I + 3D + D^2 = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 12 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

C. B is diagonalizable, and $B = PCP^{-1}$, where $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$.

$B = PCP^{-1}$

D. B is diagonalizable, and $B = PCP^{-1}$ for some C, but there is not enough information to determine C.

where $C = 2I + 3D + D^2$

E. There is not enough information to determine whether B is diagonalizable.

Review

A is diagonalizable (d-able) if and only if $\boxed{A \quad n \times n \text{ square}}$

- there is a basis of $\mathbb{R}^n$ consisting of eigenvectors
- there is P invertible $n \times n$ there is D diagonal $n \times n$ s.t. $A = PDP^{-1}$ (eigenvector whole the columns of A)
- charact. eqn $\det(A - \lambda I) = 0$ has n roots counting multiplicity and moreover for each eigenvalue λ_i $\dim(\mathrm{Nul}(A - \lambda_i I)) =$ algebraic multiplicity of λ_i

- If all n eigenvalues are distinct then A is d-able.
- Symmetric matrices are d-able

13. Which of the following statements are **true**?

T (i) If λ is an eigenvalue for A, then $-\lambda$ is an eigenvalue for $-A$.

T (ii) If zero is an eigenvalue of A, then A is not invertible.

F (iii) If an $n \times n$ matrix A is diagonalizable, then A has n distinct eigenvalues. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

FALSE

(iv) Let $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$, then A is both invertible and diagonalizable.

OK   not

$\begin{vmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix} = (\lambda-2)^2 = 0$

eigenvalues $\lambda_1 = \lambda_2 = 2$

$\dim(\mathrm{Nul}(A - 2I)) = 1 < 2$

$A - 2I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \leftarrow$ rank $(A-2I) = 1$

$\mathrm{Nul}(A-2I) = 1$

A. (i) and (ii) only

B. (i) and (iii) only

C. (i), (ii) and (iii) only

D. (i), (ii) and(iv) only

E. (i), (ii), (iii) and (iv)

Review: λ is eigenvalue if there is $x \neq 0$ s.t. $Ax = \lambda x$ $\iff$ $\mathrm{Nul}(A - \lambda I) \neq \{0\}$ $\iff$ $\det(A - \lambda I) = 0$

(i) Say $Av = \lambda v$ $v \neq 0$

true $(-A)v = (-1)Av = (-1)\lambda v = (-\lambda)v$

(ii) 0 eigenvalue $\iff$ $\mathrm{Nul}(A) \neq \{0\}$

$\iff$ $\det(A) = 0$

$\iff$ A not invertible.

21. Find the least squares solution to

$$\begin{bmatrix}1 & 3\\ 2 & 2\\ 1 & 5\end{bmatrix}\mathbf{x} = \begin{bmatrix}0\\ 5\\ 8\end{bmatrix}.$$

A. (0, 1)

B. (1, 1)

C. (1, 2)

D. (0, 2)

E. (2, 1)

$$\begin{bmatrix}1&2&1\\3&2&5\end{bmatrix}\begin{bmatrix}1&3\\2&2\\1&5\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}1&2&1\\3&2&5\end{bmatrix}\begin{bmatrix}0\\5\\8\end{bmatrix}$$

$$\begin{bmatrix}6&12\\12&38\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}18\\50\end{bmatrix}$$ solve

$$\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}$$

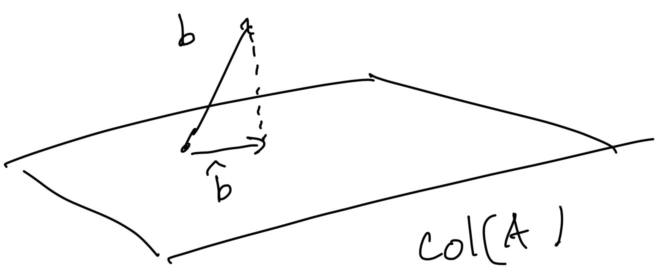
Review: least-squares sol'n of $Ax=b$ is a vector $\hat{x}$ that makes $\|A\hat{x}-b\|$ as small as possible

How to find $\hat{x}$?

Form normal system $A^TAx=A^Tb$ always consistent

its sol's are least-squares sol's.

If A^TA invertible $\hat{x}$ is unique



$\hat{b}=\text{Proj}_{\text{Col}(A)}(b)$

$A\hat{x}=\hat{b}$ is consistent.

Remark If $\text{Null}(A^TA)\neq 0$

then for any v in here

if $\hat{x}$ least-squares sol's

$\hat{x}+v$ is also a least-squares sol'n.

— END —