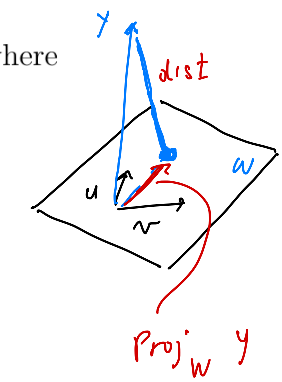
22. Find the distance from the vector $\mathbf{y}$ to the subspace $W = \text{Span}\{\mathbf{u}, \mathbf{v}\}$, where

$$\mathbf{y} = \begin{bmatrix} -1 \\ -5 \\ 10 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}.$$



A. 12.

B. $2\sqrt{2}$.

C. $3\sqrt{3}$.

D. 8.

E. $3\sqrt{5}$.

$$\text{dist}(y, W) = \|y - \text{Proj}_W y\|$$

$$\text{Proj}_W y = ? =$$

If v_1, v_2 orthogonal basis of W

$$\text{proj}_W y = \frac{y \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{y \cdot v_2}{v_2 \cdot v_2} v_2$$

In our case $u \cdot v = -2 \neq 0$ not orthogonal

Need to apply Gram-Schmidt to go from $\{u, v\} \to \{v_1, v_2\}$

$$v_1 = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix} \qquad v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$v_1 = u$$

$$v_2 = v - \frac{v \cdot u}{u \cdot u} u$$

correct

$$v_2 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} - \frac{(-2)}{4} \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$$

$$v_2 = v - \frac{v \cdot v_1}{v_1 \cdot v_1} v_1$$

$$\text{proj}_W y = \frac{y \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{y \cdot v_2}{v_2 \cdot v_2} v_2 = -\frac{1}{1} v_1 + \frac{-20}{5} v_2$$

$$= -v_1 - 4v_2$$

$$= \begin{bmatrix} -1 \\ -8 \\ 4 \end{bmatrix}$$

$$\|y - \text{proj}_W y\| = \left\| \begin{bmatrix} -1 \\ -5 \\ 10 \end{bmatrix} - \begin{bmatrix} -1 \\ -8 \\ 4 \end{bmatrix} \right\|$$

$$= \sqrt{45} = 3\sqrt{5}.$$

Say $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = I_3$

$\lambda_1 = \lambda_2 = \lambda_3 = 1$ Null $(A - I) = \mathbb{R}^3$

$A - I = 0_3$

$Q = I_3$ would work! any orthogonal matrix $A = Q I_3 Q^T = Q Q^T = I_3$.

$A = PDP^{-1}$

25. Suppose that $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = QDQ^T$ where $D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$ and Q is an orthogonal matrix. In the following select a pair of Q and D with required properties.

A. $Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}$, $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$.

B. $Q = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

C. $Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{3}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \end{bmatrix}$, $D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

D. $Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$, $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

E. $Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}$, $D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

$A = QDQ^T$

Q orthogonal

A orthogonally diagonalizable $\Updownarrow$ A symmetric

$A^T = A$

How do we find D and Q?

1. Eigenvalues $0 = \det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} = (-\lambda)\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 1 & -\lambda \end{vmatrix} + (1)\begin{vmatrix} 1 & -\lambda \\ 1 & 1 \end{vmatrix}$

$= -\lambda(\underbrace{\lambda^2 - 1}_{(\lambda-1)(\lambda+1)}) - (-\lambda - 1) + (1 + \lambda)$

$= (\lambda + 1)(-\lambda(\lambda - 1) + 2) = (\lambda+1)(\lambda+1)(2-\lambda)$

$= (\lambda+1)^2(2-\lambda) = 0$

$\lambda_1 = \lambda_2 = -1 \qquad \lambda_3 = 2$

2. Eigenvectors: $\lambda = 2 \qquad A - 2I = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Null $(A - 2I) =$ span $\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

$\lambda = -1 \qquad A - (-1)I = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ Null $(A + I) = $ 2-dim

rank = 1

$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$x_1 + x_2 + x_3 = 0$

$x_2 = s \quad x_3 = t$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -s-t \\ s \\ t \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

basis $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

Gram-Schmidt

$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

rescale to unit vectors

$$Q = \left[\begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} \right]$$

9. Let $\mathbb{P}_3$ be the space of all polynomials of degree at most 3. Which of the following sets are subspaces of $\mathbb{P}_3$?

(i) Set of all polynomials $\mathbf{p}$ in $\mathbb{P}_3$ such that $\mathbf{p}(0)\,\mathbf{p}(2) = 0$. ← not subspace

(ii) Set of all polynomials $\mathbf{p}$ in $\mathbb{P}_3$ such that $\mathbf{p}(1) = 4\,\mathbf{p}(0) + 2$. ← not subspace since 0 not in W

(iii) Set of all polynomials $\mathbf{p}$ in $\mathbb{P}_3$ such that $\mathbf{p}(1) = 0$ and $\mathbf{p}(4) = 0$. subspace

A. (i) and (ii) only

B. (i) only

C. (i) and (iii) only

D. (ii) and (iii) only

E. (iii) only

$W \subset V$ vector space

subspace if (1) contains 0

(2) closed under addition

(3) closed under scalar multiplication

$V = \mathbb{P}_3$

(i) $W = \{ p \text{ in } \mathbb{P}_3 : p(0) \cdot p(2) = 0 \}$ not closed under addition

p in $W \iff$ either 0 or 2 is a root.

$p(t) = t$ in W

$q(t) = t - 2$ in W

$\underbrace{p(t) + q(t)}_{r(t)} = 2t - 2$

$r(0) \cdot r(2) \neq 0$.

r not in W.

W $p(1) = 4\,p(0) + 2$ | $\underbrace{z(1)}_{0} \stackrel{?}{=} 4\,\underbrace{z(0)}_{0} + 2$

$0 = 0 + 2$ false.

0_V polynomial $= 0 = 0 \cdot 1 + 0 \cdot t + 0 \cdot t^2 + 0 \cdot t^3 = z$

$z(t) = 0$ for all t

basis $\mathbb{P}_3$: $1, t, t^2, t^3$

(ii) fails scalar multiplication

Say p is in W

Is $\underbrace{10\,p(t)}_{q(t)}$ in W?

$q(t) = 10\,p(t)$

(*) $p(1) = 4\,p(0) + 2$

$q(1) = 4\,q(0) + 2$?

(*) $10\,p(1) = 10 \cdot 4\,p(0) + 2$

$r(1) = 4\,r(0) + 2$?

Say p in W, q in W $\Longrightarrow$? $r = p + q$ in W

$p(1) + q(1) = 4\,(p(0) + q(0)) + 2$ not in W.

19. Let $C[-1,1]$ be the vector space of all continuous functions defined on [-1,1]. Define with the inner product on $C[-1,1]$ by

$$\langle f, g\rangle = \int_{-1}^{1} f(t)\, g(t)\, dt.$$

Find the orthogonal projection of $10t^3 - 5$ onto the subspace spanned by 1 and t (with respect to the above inner product on $C[-1,1]$).

A. $6t - 10$

B. $6t + 5$

C. $10t^3 - 6t$

D. $10t^3 - 5$

E. $6t - 5$

Formalize: $W = \text{span}\langle \underset{v_1}{1}, \underset{v_2}{t} \rangle$

$f(t) = 10t^3 - 5$

$\text{proj}_W(f) = ?$

Need orthogonal basis of W

are 1 and t orthogonal? Yes

$$\langle 1, t\rangle = \int_{-1}^{1} 1 \cdot t\, dt = \int_{-1}^{1} t\, dt = \frac{t^2}{2}\Big|_{-1}^{1} = 0$$

$$\text{proj}_W(f) = \frac{\langle f, 1\rangle}{\langle 1, 1\rangle} 1 + \frac{\langle f, t\rangle}{\langle t, t\rangle} t$$

$$\langle f, 1\rangle = \int_{-1}^{1} f(t)\cdot 1\, dt = \int_{-1}^{1} (10t^3 - 5)\, dt = -10$$

$$\frac{\langle f, q_1\rangle}{\langle q_1, q_1\rangle} q_1 + \frac{\langle f, q_2\rangle}{\langle q_2, q_2\rangle} q_2$$

$$\langle q_1, q_2\rangle = 0$$

$$\langle 1, 1\rangle = \int_{-1}^{1} 1\cdot 1\, dt = 2$$

$$\langle f, t\rangle = \int_{-1}^{1} (10t^3 - 5)t\, dt = 4$$

$$\langle t, t\rangle = \int_{-1}^{1} t^2\, dt = \frac{t^3}{3}\Big|_{-1}^{1} = \frac{2}{3}$$

$$\text{proj}_W(f) = -\frac{10}{2} 1 + \frac{4}{\frac{2}{3}} t = -5 + 6t$$

If instead of $1, t$ $\;P_1(t), P_2(t)$ given

need to apply G-S: $\underset{q_1}{P_1}$, $\underset{q_2}{P_2 - \frac{\langle P_2, P_1\rangle}{\langle P_1, P_1\rangle} P_1}$

1. Consider the system of linear equations

$$x + 2y + 3z = 1$$
$$3x + 5y + 4z = 2$$
$$2x + 3y + a^2 z = 0$$

For which values of a is the system inconsistent?

Remark row operation change eigenvalues

$A = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$ $\qquad$ RREF(A) $= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\lambda_1 = 2 \quad \lambda_2 = 3$

$n \times n$

A invertible $\qquad$ RREF(A) $= I_n$

rank is preserved

column space is preserved.

A singular $\iff$ not-invertible
$\qquad$ det

$\Updownarrow$

$\det(A) = 0$.

$\Updownarrow$

$\lambda = 0$ eigenvalue.

A $\quad n \times n \qquad A \neq 0$

$A = I_n \qquad Av = v$ for all v.

Review:

$$\vec{x}' = A\vec{x}$$

$$\begin{bmatrix} x_1'(t) \\ \vdots \\ x_n'(t) \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

The solution set forms an n-dimensional vector space

Each solution is uniquely determined by an initial condition $\vec{x}(0) = \vec{x}_0$

The case when A has n distinct real eigenvalues

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n$$

$$\vec{v}_1 \quad \vec{v}_2 \quad \vec{v}_n \quad \leftarrow \text{eigenvectors}$$

general solution :

$$\vec{x}(t) = c_1 e^{\lambda_1 t}\vec{v}_1 + \cdots + c_n e^{\lambda_n t}\vec{v}_n$$

If $\vec{x}(0) = \vec{x}_0$ then $\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = P^{-1}\vec{x}_0$

where $P = [v_1, \ldots, v_n]$