$\boxed{6.1}$ Inner product, Length, Orthogonality

The inner product of two vectors in $\mathbb{R}^n$ is a number.

$$\bar{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \qquad \bar{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$$

$$\bar{u} \cdot \bar{v} = u_1 v_1 + \dots + u_n v_n = \underset{1\times n}{[u_1 \ \dots \ u_n]} \underset{n\times 1}{\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}}$$

$$= u^T v$$

$\boxed{Ex\ 1}$ $\quad \bar{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} \quad \bar{v} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 1 \end{bmatrix} \qquad \bar{u} \cdot \bar{v} = -1 + 0 + 3 + 1 = 3$

$\boxed{\text{Properties}}$

$$\bar{u} \cdot \bar{v} = \bar{v} \cdot \bar{u}$$

$$(\bar{u} + \bar{v}) \cdot \bar{w} = \bar{u} \cdot \bar{w} + \bar{v} \cdot \bar{w}$$

$$(c\,\bar{u}) \cdot \bar{v} = c\ \bar{u} \cdot \bar{v} \qquad c \text{ in } \mathbb{R}$$

$$\bar{u} \cdot \bar{u} \geq 0 \quad \text{and} \quad \bar{u} \cdot \bar{u} = 0 \iff \bar{u} = 0$$

$$\bar{u} \cdot \bar{u} = u_1^2 + \dots + u_n^2 \geq 0$$

Def'n Length of $\bar{v}$ in $\mathbb{R}$ (or norm of $\bar{v}$)

$$\|v\| = \sqrt{\bar{v} \cdot \bar{v}} = \sqrt{v_1^2 + \dots + v_n^2}$$

$$\|v\|^2 = \bar{v} \cdot \bar{v}$$

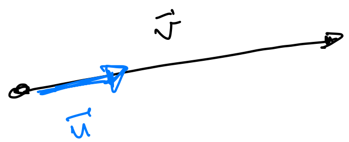
Note $\quad \|c\,\bar{v}\| = |c|\,\|\bar{v}\| \qquad c \text{ in } \mathbb{R}$

$$\|c\,\bar{v}\| = \sqrt{(c v_1)^2 + \dots + (c v_n)^2} = \sqrt{c^2 (v_1^2 + \dots + v_n^2)}$$

$$|c| \sqrt{v_1^2 + \dots + v_n^2}$$

Say that $\bar{v}$ is a <u>unit vector</u> if $\|\bar{v}\| = 1$.

$\boxed{\text{Ex 2}}$ $\bar{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ $\|\bar{v}\| = \sqrt{5}$



$\vec{u} = \frac{1}{\sqrt{5}} \vec{v} = \frac{\vec{v}}{\|\vec{v}\|}$

is a unit vector pointing in the direction of v

$\|\vec{u}\| = \|\frac{1}{\sqrt{5}} \bar{v}\|$

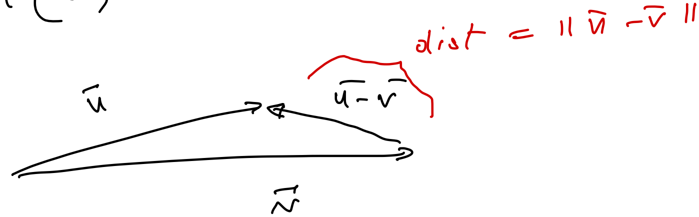
$= \frac{1}{\sqrt{5}} \|\bar{v}\| = \frac{1}{\sqrt{5}} \cdot \sqrt{5} = 1$

<u>In general</u> if $\vec{v} \neq 0$ then $\|\vec{v}\| \neq 0$

and $\bar{u} = \frac{1}{\|\bar{v}\|} \vec{v}$ is a unit vector pointing in the direction of $\bar{v}$.

<u>Distance between 2 vectors</u>

$\text{dist}(\bar{u}, \bar{v}) = \|\bar{u} - \bar{v}\|$



ORTHOGONAL VECTORS

Let $\bar{u}, \bar{v}$ be vectors in $\mathbb{R}^n$

$\vec{u}$ is **orthogonal** to $\bar{v}$ if

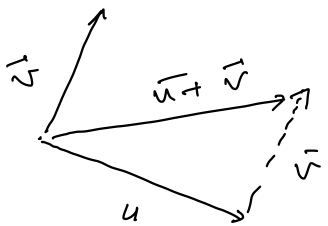
$$\vec{u} \cdot \bar{v} = 0$$

Notation $\bar{u} \perp \bar{v}$.

Pythagorian theorem

If $\vec{u} \perp \bar{v}$ then

$$\boxed{\| \bar{u} + \vec{v} \|^2 = \| \bar{u} \|^2 + \| \bar{v} \|^2}$$

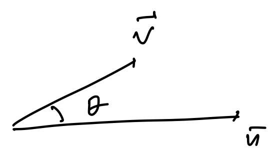


$$\|u + v\|^2 = (u+v) \cdot \overbrace{(u+v)}^{w} = u \cdot (u+v) + v \cdot (u+v)$$

$$= u \cdot u + \underbrace{u \cdot v}_{=0} + \underbrace{v \cdot u}_{=0} + v \cdot v$$

$u \perp v$

$$= \|u\|^2 + \|v\|^2$$



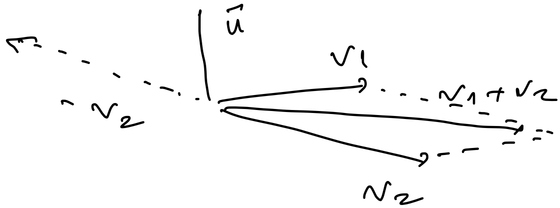
θ angle between $\bar{u}$ and $\bar{v}$

$$\cos \theta = \frac{u \cdot v}{\|u\| \, \|v\|}$$

Remark if $u \perp v_1$, $u \perp v_2$ $\Longrightarrow$ $u \perp (v_1 + v_2)$ ✓ (4)

$$u \cdot (v_1 + v_2) = \underbrace{u \cdot v_1}_{0} + \underbrace{u \cdot v_2}_{0} = 0$$

Moreover $u \perp (c_1 v_1 + c_2 v_2)$



$\bar{u} \cdot (-\bar{v}_2)$

$= \bar{u} \cdot ((-1)\bar{v}_2)$

$= (-1)\, \bar{u} \cdot \bar{v}_2 = 0$

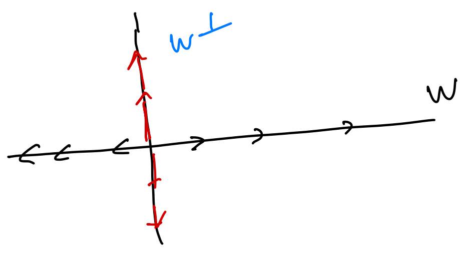
$\boxed{\text{The orthogonal complement of a subspace } W \text{ of } \mathbb{R}^n}$ denoted $W^\perp$

is $W^\perp = \{ \bar{x} \text{ in } \mathbb{R}^n : \vec{x} \cdot \vec{w} = 0 \text{ for all } \bar{w} \text{ in } W \}$

Ex 3 $\mathbb{R}^2$ $W = \left\{ \begin{bmatrix} a \\ 0 \end{bmatrix} : a \text{ in } \mathbb{R} \right\}$

$\begin{bmatrix} a \\ 0 \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \end{bmatrix} = a\, \vec{i}$

basis $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$



$W^\perp = \left\{ \begin{bmatrix} 0 \\ b \end{bmatrix} : b \text{ in } \mathbb{R} \right\} = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ ←

Reasoning search for $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ s.t

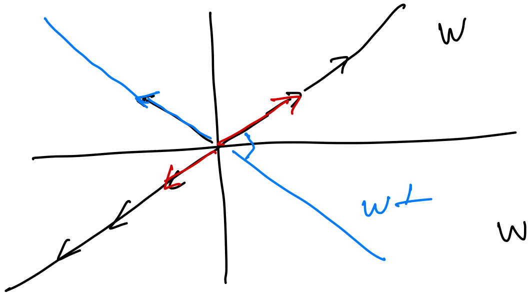
$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cdot \begin{bmatrix} a \\ 0 \end{bmatrix} = 0$ for all a.

$\bar{x} \cdot \bar{w} = 0$ for all w in W

$x_1 a = 0$ for all a. Thus $x_1 = 0$

Thus $x = \begin{bmatrix} 0 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ x_2 free.

Ex 4 $\quad \mathbb{R}^2 \quad W = \text{span}\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

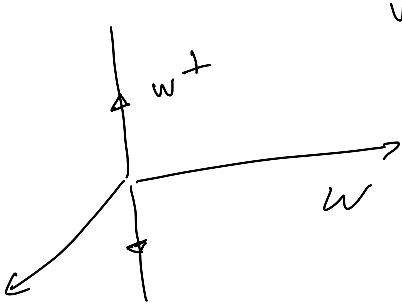


$W^\perp = \text{span}$

$\begin{bmatrix} -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 0$

Ex 5 $\quad \mathbb{R}^3 \quad W = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

$W^\perp = \text{span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$



Fact $\quad W$ subspace $\Longrightarrow$ $\boxed{W^\perp \text{ subspace of } \mathbb{R}^n}$

Ex 6 $\quad W = \{0\}$ in $\mathbb{R}^2$

$W^\perp = \mathbb{R}^2$

$W = \mathbb{R}^2 \qquad W^\perp = \{0\}$

Revisit linear systems with orthogonality in mind

$$
\begin{array}{c c}
 & \begin{array}{c}\bar{x}\end{array} \\
\begin{array}{c} r_1 \\ r_2 \\ r_3 \end{array} &
\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}
\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{array}
$$

$a_1 x + a_2 y + a_3 z = 0$

$\vdots$

$\bar{r}_1 \cdot \bar{x} = 0$ ←

$\bar{r}_2 \cdot \bar{x} = 0$ ←

$\bar{r}_3 \cdot x = 0$ ←

Solving $A\bar{x} = \bar{0}$ amounts to finding all vectors $\bar{x}$ $\perp$ all rows of A

Thus $\boxed{(\text{Row } A)^{\perp} = \text{Nul}(A)}$ ← apply this to A^T.

$(\text{Col}(A))^{\perp} = \text{Nul}(A^T)$

$\boxed{\text{END}}$