MA265 Linear Algebra — Practice Exam 1

Date: Spring 2021 *Duration:* 60 min

Name: ______________________

PUID: ______________________

- All answers must be justified and you must show all your work in order to get credit.
- The exam is open book. Each students should work independently, Academic integrity is strictly observed.

Problem	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total:	100	

1. Consider the matrices $A = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$. Let S be the subspace of $\mathbb{R}^3$ consisting of those vectors $\mathbf{x}$ such that $A\mathbf{x} = B\mathbf{x}$. Find a basis of S. [10pt]

$Ax = Bx \Rightarrow (A-B)x = 0 \qquad A - B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

S ~~consists~~ is the null space of $C = A - B$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & 0 \\ 0 & -1 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

introduce parameter corresponding to this column $x_3 = s$

rank $(C) = 2$ (2 pivots)

$\dim(\mathrm{Nul}(C)) = 3 - 2 = 1$

$x_3 = s$

$$x_1 + \frac{1}{2}s = 0$$
$$x_2 - \frac{1}{2}s = 0$$
$$x_3 = s$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

Answer: $\left\{ \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix} \right\}$ basis of S

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any non-zero multiple of this vector will form a basis as well

$\left\{ \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \right\}$ ← also correct.

$$\left(\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right)$$

double check ✓

2. Consider the vectors [10pt]

$$\mathbf{u} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad \mathbf{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Let A be a 3×3 matrix such that $A\mathbf{u} = \mathbf{0}$, $A\mathbf{v} = \mathbf{0}$ and $A\mathbf{w} = \mathbf{w}$. What is the rank of A?

Use the rank theorem (p. 165)

$\text{rank}(A) + \dim(\text{Nul}(A)) = 3$

Nul(A) is a subspace of $\mathbb{R}^3$ so that $\dim(\text{Nul}(A)) \leq 3$

$\vec{u}$ and $\vec{v}$ are linearly independent and

$A\vec{u} = \vec{0}$ $A\vec{v} = \vec{0}$ $\Rightarrow$ $\vec{u}, \vec{v}$ belong to Nul(A)

Thus $\dim(\text{Nul}(A)) \geq 2$. Since $A\vec{w} = \vec{w} \neq 0$

$\vec{w}$ does not belong to Nul(A). Thus

$\text{Nul}(A) \neq \mathbb{R}^3$ so that $\dim(\text{Nul}(A)) = 2$

Then $\text{rank}(A) = 3 - \dim(\text{Nul}(A)) = 3 - 2 = 1$

Answer: $\boxed{\text{rank}(A) = 1}$

3. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear map such that $T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ (1) and [10pt]

$T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ (2). Compute $T\left(\begin{bmatrix} 3 \\ 4 \end{bmatrix}\right)$.

$$\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$ Since T is linear

$$T\left(\begin{bmatrix} 3 \\ 4 \end{bmatrix}\right) = T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + 2T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 2\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+2 \\ 2-2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}$$

<u>Answer:</u> $T\left(\begin{bmatrix} 3 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}$

Another method is to find the matrix A with the property that $T(\bar{x}) = A\bar{x}$ for all $\bar{x}$ in $\mathbb{R}^2$

$T : \mathbb{R}^2 \to \mathbb{R}^3 \Rightarrow A$ is 3×2 matrix

$$A = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \end{bmatrix}$$

Find A using (1) and (2)

$$T\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$T\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} a_1 + 2a_2 \\ b_1 + 2b_2 \\ c_1 + 2c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$a_2 = 0$

$b_2 = -\frac{3}{2}$

$c_2 = -\frac{3}{2}$

$2 + 2b_2 = -1$

$3 + 2c_2 = 0$

$$\boxed{A = \begin{bmatrix} 1 & 0 \\ 2 & -3/2 \\ 3 & -3/2 \end{bmatrix}} \Rightarrow \begin{bmatrix} 1 & 0 \\ 2 & -3/2 \\ 3 & -3/2 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}$$

using this matrix can compute T(any vector).

4. Let A be a 3×5 matrix. Which of the following statements are true? Indicate clearly <u>**all**</u> correct answers. [10pt]

F A. The rank of A is 3.

T B. The null space of A has dimension at least 2.

F C. $A\mathbf{x} = \mathbf{0}$ has only one solution, the trivial solution.

T D. There exists two linearly independent vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^5$ such that $A\mathbf{u} = A\mathbf{v} = \mathbf{0}$.

T E. The columns of A are linearly dependent.

A. FALSE take for example $A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ rank(A) = 1

B. $\operatorname{rank}(A) \leq 3$ (since it has 3-rows)

$\Rightarrow$ $\operatorname{rank}(A) + \dim(\operatorname{Nul}(A)) = 5$

$\dim(\operatorname{Nul}(A)) = 5 - \operatorname{rank}(A) \geq 5 - 3 = 2$

Thus B is correct

C. False, Indeed if $A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Then $A\bar{x} = 0$ has infinitely many solutions

any vector in $\mathbb{R}^3$ is a solution.

D. Correct: Indeed, we have seen that $\dim(\operatorname{Nul}(A)) \geq 2$ and so any basis of $\operatorname{Nul}(A)$ has at least 2 linearly independent vectors $\bar{u}$ and $\bar{v}$.

E. Correct Any 5 vectors in $\mathbb{R}^3$ are linearly dependent.

Indeed since $\dim(\mathbb{R}^3) = 3$, one can have at most 3 linearly independent vectors in $\mathbb{R}^3$.

5. Suppose that A and B are 2×2 matrices satisfying $\det(B) = 8$ and $A^3 = B^2$. Determine the value of $\det(3\,A^T\,BA^{-1}B^{-1}A)$. [10pt]

If X is a 2×2 matrix

$\det(cX) = c^2 \det(X)$ for any c in $\mathbb{R}$

Also $\det(XY) = \det(X)\det(Y)$ $\quad \det(X^T) = \det(X)$

$\det(X^{-1}) = \frac{1}{\det(X)}$ if X is invertible

$\det(3A^T B A^{-1} B^{-1} A) =$

$\det(3A^T)\det(B)\det(A^{-1})\det(B^{-1})\det(A)$

$= 3^2 \det(A^T)\det(B) \frac{1}{\det(A)} \frac{1}{\det(B)} \det(A) \quad \det(A) = 9\det(A^T)$

$= 9\det(A)$

On the other hand $A^3 = B^2$

$\Rightarrow \det(A)^3 = \det(B)^2 = 8^2 = 64$

$\det(A)^3 = 64 \Rightarrow \det(A) = 4$

Thus $9\det(A) = 9 \cdot 4 = 36$

Answer: 36

6. Consider the matrix $A = \begin{pmatrix} 1 & 1 & 4 & 2 \\ 2 & 2 & 10 & 0 \\ 0 & 3 & 1 & 0 \\ 1 & 0 & 0 & 5 \end{pmatrix}$. Compute the $(3, 2)$ entry of the adjugate matrix $\mathrm{adj}(A)$. [10pt]

The (3,2) entry of adj(A) is the cofactor C_{23} of A which we now compute

$$\begin{pmatrix} 1 & 1 & 4 & 2 \\ 2 & 2 & 10 & 0 \\ 0 & 3 & 1 & 0 \\ 1 & 0 & 0 & 5 \end{pmatrix} \qquad \begin{matrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{matrix}$$

$$C_{23} = -\begin{vmatrix} 1 & 1 & 2 \\ 0 & 3 & 0 \\ 1 & 0 & 5 \end{vmatrix} = -3 \begin{vmatrix} 1 & 2 \\ 1 & 5 \end{vmatrix} = -3(5-2) = -9$$

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

Answer: $\boxed{\mathrm{adj}(A)_{3,2} = -9}$

$\mathrm{adj}(A)_{32} = C_{23} = -9$

7. Consider the matrices [10pt]

$$A = \begin{bmatrix} a & b & c & d \\ x & y & z & 0 \\ -3 & 7 & 2 & 11 \\ -1 & 1 & 2 & 10 \end{bmatrix}, \quad B = \begin{bmatrix} x & y & z & 0 \\ -3+bx & 7+by & 2+bz & 11 \\ a & b & c & d \\ -1 & 1 & 2 & 10 \end{bmatrix}.$$

Suppose that $\det(A) = 3$. Find $\det(2B)$.

B is 4×4 $\quad \det(2B) = 2^4 \det(B) = 16 \det(B)$

Row replacement (replace 2^{nd} row) in B with (itself) + $(-b)\cdot$(first row)

$$\det(B) = \begin{vmatrix} x & y & z & 0 \\ -3 & 7 & 2 & 11 \\ a & b & c & d \\ -1 & 1 & 2 & 10 \end{vmatrix} = \begin{vmatrix} a & b & c & d \\ x & y & z & 0 \\ -3 & 7 & 2 & 11 \\ -1 & 1 & 2 & 10 \end{vmatrix} = \det(A) = 3$$

2 row interchanges: $(-1)(-1) = 1$

Thus $\det(2B) = 16 \det(B) = 16 \cdot \det(A) = 16 \cdot 3 = 48$.

Answer: $\boxed{\det(2B) = 48}$

8. Consider a linear system whose augmented matrix is of the form [10pt]

$$[A|\vec{b}] = \left[\begin{array}{ccc|c} 1 & 0 & -2 & a \\ 0 & 1 & a & a-3 \\ 0 & 0 & a-4 & a-3 \end{array}\right]$$

$$\underbrace{}_{A} \qquad \underbrace{}_{\vec{b}}$$

(i) For what values of a will the system have no solution?

(ii) For what values of a will the system have a unique solution?

(iii) For what values of a will the system have infinitely many solutions?

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & a \\ 0 & 0 & a-4 \end{bmatrix} \qquad \det(A) = a-4$$

If $\det(A) \neq 0$ then A is invertible and hence the system will have a unique solution.

$\det A = a-4 \neq 0 \iff a \neq 4$.

Thus if $a \neq 4$ we have a unique solution

what happens if $a = 4$

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

the system is inconsistent (have one equation $0 \cdot x_3 = 1$)

In conclusion: Answer:

(i) $a = 4$

(ii) $a \neq 4$

(iii) There is no such a.

9. Consider the system: [10pt]

$$\begin{aligned} x + y + z &= 5 \\ x + 2y + z &= 9 \\ x + y + (a^2 - 5)z &= a \end{aligned}$$

For which value of a does the system have infinitely many solutions?

The matrix of the system is $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & a^2-5 \end{bmatrix}$

$\det(A) = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & a^2-6 \end{vmatrix} = a^2 - 6$

If $\det(A) \neq 0 \Rightarrow$ unique solution.

If $\det(A) = 0 \Rightarrow a^2 = 6 \Rightarrow a = \pm\sqrt{6}$

The system becomes

$$\begin{aligned} x + y + z &= 5 \\ x + 2y + z &= 9 \\ x + y + z &= +\sqrt{6} \end{aligned}$$ (or $-\sqrt{6}$)

inconsistent

Thus the system has either a unique solution or it is inconsistent.

$\boxed{\text{Answer:}}$ There is no a such that the system has infinitely many solutions

10. Find a subset T of the set $S = \left\{ \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}3\\2\\1\end{pmatrix}, \begin{pmatrix}2\\2\\1\end{pmatrix}, \begin{pmatrix}1\\2\\-1\end{pmatrix} \right\}$ such that T is a basis for the subspace of $\mathbf{R}^3$ spanned by S. [10pt]

"$\bar{u}$" "$\bar{v}$" "$\bar{w}$" "$\vec{e}$"

S is a subspace of $\mathbb{R}^3$ and so $\dim(S) \leq 3$

The first two vectors $v = \begin{pmatrix}1\\2\\1\end{pmatrix}$ and $\bar{v} = \begin{pmatrix}3\\2\\1\end{pmatrix}$ are linearly independent

Thus $\dim S = 2$ or $\dim S = 3$

$\begin{vmatrix}1&3&2\\2&2&2\\1&1&1\end{vmatrix} = 2\begin{vmatrix}1&3&2\\1&1&1\\1&1&1\end{vmatrix} = 0$. Thus $\bar{w} = \begin{pmatrix}2\\2\\1\end{pmatrix}$ is a linear combination of $\bar{u}$ and $\bar{v}$ (in fact $\bar{w} = \frac{1}{2}\bar{u} + \frac{1}{2}\bar{v}$)

Consider now $\{\bar{u}, \bar{v}, \vec{e}\}$

$\begin{vmatrix}1&2&1\\2&2&2\\1&1&-1\end{vmatrix} = 2\begin{vmatrix}1&2&1\\1&1&1\\1&1&-1\end{vmatrix} = 2\begin{vmatrix}1&2&1\\0&-1&0\\0&0&-2\end{vmatrix} = 2\begin{vmatrix}-1&0\\0&-2\end{vmatrix} = 4 \neq 0$

Thus $\bar{u}, \bar{v}, \vec{e}$ are linearly independent

$\dim(S) = 3$ so that $S = \mathbb{R}^3$

Answer $T = \left\{ \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}3\\2\\1\end{pmatrix}, \begin{pmatrix}1\\2\\-1\end{pmatrix} \right\}$

<u>Remark:</u> This is not the only choice for T

For example $\left\{ \begin{pmatrix}3\\2\\1\end{pmatrix}, \begin{pmatrix}2\\2\\1\end{pmatrix}, \begin{pmatrix}1\\2\\-1\end{pmatrix} \right\}$

is also a correct answer since

$\begin{vmatrix}3&2&1\\2&2&2\\1&1&-1\end{vmatrix} \neq 0.$