MA265 Linear Algebra — Exam 2

Date: April 7th, 2021 *Duration:* 60 min

Name:

PUID:

- All answers must be justified and you must show all your work in order to receive full credit.
- Academic integrity is strictly observed

Problem	**Points**	**Score**
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total:	100	

1. Find all numbers a for which the following vectors are linearly independent. [10pt]

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ -1 \\ a \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} a^2 \\ 0 \\ -4 \\ 1 \end{bmatrix}, \ \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 1-a \\ -2 \end{bmatrix}, \ \mathbf{v}_4 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \ \mathbf{v}_5 = \begin{bmatrix} 0 \\ 2 \\ a \\ 1 \end{bmatrix}.$$

Since $\dim(\mathbb{R}^4) = 4$ any 5 vectors in $\mathbb{R}^4$ are linearly dependent.

Thus there exists no a such that the given vectors are linearly independent.

2. Let V be the vector space of all polynomials p of degree at most 3. [10pt]

Let W be the subspace of V consisting of those polynomials p that satisfy the conditions:

$$p(0) = p(-1) = 0.$$

Find a basis of W.

$p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$

$p(0) = a_0 = 0$

$p(-1) = a_0 - a_1 + a_2 - a_3 = 0$

$a_0 = 0 \qquad a_2 = a_1 + a_3$

$p(t) = a_1 t + (a_1 + a_3) t^2 + a_3 t^3$

$= a_1 (t + t^2) + a_3 (t^2 + t^3)$

$t + t^2$, $t^2 + t^3$ are linearly independent and they span W

Thus $\{t + t^2, t^2 + t^3\}$ is a basis of W.

3. Let A be a 3×5 matrix with $rank(A) = 2$. For each of the following assertions indicate whether it is **true** or **false**. (*No explanations required*). [10pt]

(1) For some $\mathbf{b}$ in $\mathbb{R}^3$ the system $A\mathbf{x} = \mathbf{b}$ has a unique solution.

(2) If the rank of the augmented matrix $[A\,\mathbf{b}]$ is also 2, then the system $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions.

(3) If the rank of the augmented matrix $[A\,\mathbf{b}]$ is 3, then the system $A\mathbf{x} = \mathbf{b}$ has no solution.

(4) The system $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is in the row space of A.

(5) The null space of A has dimension 1.

$\text{rank}(A) + \text{nullity}(A) = 5 \Rightarrow \text{nullity}(A) = 5 - 2 = 3$

(1) False — because if x_0 is any solution then $x_0 + v$ is also a solution whenever v is in Null (A)

(2) True — $\text{rank}(A) = \text{rank}[A \mid b] \Rightarrow Ax = b$ has solutions.

if x_0 is any solution then $x_0 + v$ is also a solution whenever v is in Null (A)

(3) True — $\text{rank}(A) < \text{rank}[A \mid b]$

(4) False — $Ax = b$ has a solution $\Leftrightarrow$ b is in the column space of A

(5) False — $\dim(\text{Null}(A)) = 3$

4. Let $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be a basis of $\mathbb{R}^3$. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that [10pt]

$T(\mathbf{v}_1) = \mathbf{v}_1 + \mathbf{v}_2$, $\quad T(\mathbf{v}_2) = \mathbf{v}_1 + \mathbf{v}_2$ and $T(\mathbf{v}_3) = \mathbf{v}_1 + 2\mathbf{v}_2 + 3\mathbf{v}_3$.

Find the eigenvalues of T.

If $\mathcal{B} = \{v_1, v_2, v_3\}$

$$A = [T]_{\mathcal{B}} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 & 1 \\ & 1-\lambda & 2 \\ 0 & 0 & 3-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{vmatrix} (3-\lambda)$$

$$= \left((1-\lambda)^2 - 1\right)(3-\lambda) = (\lambda^2 - 2\lambda)(3-\lambda) = 0$$

$$\lambda(\lambda - 2)(3-\lambda) = 0$$

$\lambda_1 = 0$

$\lambda_2 = 2$

$\lambda_3 = 3$

5. Let $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ be three vectors in a vector space. For each of the following assertions indicate whether it is **true** or **false**. (*No explanations required*). [10pt]

(1) If $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are linearly independent, then $Span\{\mathbf{v}_1, \mathbf{v}_2\} \neq Span\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

(2) If $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are linearly dependent, then $Span\{\mathbf{v}_1, \mathbf{v}_2\} = Span\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

(3) If $Span\{\mathbf{v}_1, \mathbf{v}_2\} \neq Span\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, then $\mathbf{v}_3$ is not in $Span\{\mathbf{v}_1, \mathbf{v}_2\}$.

(4) If $\mathbf{v}_3$ is in $Span\{\mathbf{v}_1, \mathbf{v}_2\}$, then $Span\{\mathbf{v}_1, \mathbf{v}_2\} = Span\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

(5) If $\mathbf{v}_3$ is in $Span\{\mathbf{v}_1, \mathbf{v}_2\}$, then $\mathbf{v}_3 - \mathbf{v}_1$ is in $Span\{\mathbf{v}_1, \mathbf{v}_2\}$.

(1) True since $\dim(\text{Span}\{v_1, v_2\}) = 2$
$\dim(\text{Span}\{v_1, v_2, v_3\}) = 3$

(2) False (ex: $v_1 = 0$ $v_2 = 0$ $v_3 \neq 0$)

(3) True

(4) True

(5) True

6. Consider the matrix $A = \begin{bmatrix} 1 & 0 & 1 & 2 & 1 \\ 1 & 0 & 1 & 2 & 2 \\ 2 & 1 & 0 & 1 & 2 \\ 1 & 1 & -1 & -1 & 0 \end{bmatrix}$. [10pt]

It is given that $\text{rref}(A) = \begin{bmatrix} 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & -2 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

Consider the subspace W of $\mathbb{R}^4$ consisting of all vectors $\mathbf{b}$ for which the system $A\mathbf{x} = \mathbf{b}$ is consistent (admits solutions). What is the dimension of W?

Since $W = \text{Col}(A) \Rightarrow \dim(\text{Col}(A)) = \text{rank}(A) = 3$

3 pivots

7. Consider the vector space $\mathcal{P}_5$ consisting of all polynomials of degree ≤ 5. For each of the following assertions indicate whether it is **true** or **false**. (*No explanations required*). [10pt]

(1) The set $\{p \text{ in } \mathcal{P}_5 : p(t) = p(-t)\}$ is a subspace of $\mathcal{P}_5$.

(2) The set $\{p \text{ in } \mathcal{P}_5 :$ the degree of p is equal to $5\}$ is a subspace of $\mathcal{P}_5$.

(3) The set $\{p \text{ in } \mathcal{P}_5 : p(0) = p(1)\}$ is a subspace of $\mathcal{P}_5$.

(4) The set $\{p \text{ in } \mathcal{P}_5 : p(0) = 2p(1)\}$ is a subspace of $\mathcal{P}_5$.

(5) The set $\{p \text{ in } \mathcal{P}_5 : p(0) = 1\}$ is a subspace of $\mathcal{P}_5$.

(1) True $\quad p(t) = p(-t) \quad q(t) = q(-t)$

$\Rightarrow (p+q)(t) = (p+q)(-t)$

$\Rightarrow (\lambda p)(t) = (\lambda p)(-t)$

since $\lambda p(t) = \lambda p(-t)$

(2) False since $t^5 + (-t^5) = 0$

(3) True contains 0 and it is closed under addition and multiplication by scalars

(4) True contains 0 and it is closed under addition and multiplication by scalars

(5) False does not contain 0.

8. Consider the matrix $A = \begin{bmatrix} 1 & 1 & 17 \\ 0 & -1 & 101 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$ and let B be a matrix similar to $A^2 + A^{10}$. [10pt]

Compute the determinant of B.

$$P(\lambda) = \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 & 17 \\ 0 & -1-\lambda & 101 \\ 0 & 0 & \sqrt{2}-\lambda \end{vmatrix} = (1-\lambda)(-1-\lambda)(\sqrt{2}-\lambda)$$

$\lambda_1 = 1$ $\quad \lambda_2 = -1$ $\quad \lambda_3 = \sqrt{2}$ $\quad$ all distinct

Thus A is diagonalizable and similar to $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}$

$A^2 + A^{10}$ is similar to $D^2 + D^{10}$

$$D^2 + D^{10} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2^5 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 34 \end{bmatrix}$$

$$\det(A^2 + A^{10}) = \det(D^2 + D^{10}) = (2)(2)(34) = 4 \cdot 34 = 136$$

9. For each fixed λ in $\mathbb{R}$ consider the set V_λ consisting of all vectors of the form $\begin{bmatrix} \lambda^2 b + c \\ b + c \\ \lambda^2 - \lambda \end{bmatrix}$ [10pt]

where b, c are in $\mathbb{R}$. In other words

$$V_\lambda = \left\{ \begin{bmatrix} \lambda^2 b + c \\ b + c \\ \lambda^2 - \lambda \end{bmatrix} \quad : \quad b, c \text{ in } \mathbb{R} \right\}$$

(a) For which values of λ is V_λ a vector subspace of $\mathbb{R}^3$?

(b) If V_λ is a vector subspace of $\mathbb{R}^3$ find its dimension $\dim(V_\lambda)$.

(a) If V_λ is a subspace then $\lambda^2 - \lambda = 0$

since $\begin{bmatrix} 0 \\ 0 \\ \lambda^2-\lambda \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \lambda^2-\lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2(\lambda^2-\lambda^2) \end{bmatrix}$

is in $V_\lambda \Rightarrow 2(\lambda^2-\lambda) = \lambda^2-\lambda$

$\lambda^2 - \lambda = 0 \Rightarrow \lambda = 0$ or $\lambda = 1 \Rightarrow \lambda^2 - \lambda = 0$

(b) $\lambda = 0$ $\quad V_0 = \left\{ \begin{bmatrix} c \\ b+c \\ 0 \end{bmatrix} : b, c \text{ in } \mathbb{R} \right\}$ $\quad \begin{bmatrix} c \\ b+c \\ 0 \end{bmatrix} = c\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

has dimension $= 2$

with basis $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

$\lambda = 1$ $\quad V_1 = \begin{bmatrix} b+c \\ b+c \\ 0 \end{bmatrix}$ has dimension $= 1$

with basis $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}$

$\begin{bmatrix} b+c \\ b+c \\ 0 \end{bmatrix} = (b+c)\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

10. The characteristic polynomial of a 4×4 matrix A is $p(\lambda) = -\lambda(1-\lambda)^2(3-\lambda)$. Let I denote the 4×4 identity matrix. For each of the following assertions indicate whether it is **true** or **false**. (*No explanations required*). [10pt]

(1) $\det(A) = 3$.

(2) $\operatorname{rank}(A) = 3$.

(3) If A is not diagonalizable then $\operatorname{rank}(A - I) = 3$.

(4) If A is not diagonalizable then $\operatorname{rank}(A - I) = 2$.

(5) $\det(A + 3I) = 0$.

$$\begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 1 & \\ & & & 3 \end{pmatrix}$$

Eigenvalues are the roots of $p(\lambda) = 0$ $\quad \lambda_1 = 0 \quad \lambda_2 = \lambda_3 = 1 \quad \lambda_4 = 1$

(1) False since $\det(A) = \det(A - 0 \cdot I) = p(0) = 0$

(2) True since $\operatorname{Nul}(A)$ is the eigenspace corresponding to $\lambda_1 = 0$. This has dimension 1 since 0 has multiplicity $= 1$.

$\operatorname{nullity}(A) = 1 \Rightarrow \operatorname{rank}(A) = 4 - 1 = 3$

(3) True $\quad$ A is not diagonalizable $\Longleftrightarrow$ $\operatorname{nullity}(A - I) = 1 \leq 2 =$ multiplicity of $\lambda = 1$

$1 \leq \operatorname{nullity}(A - I) \leq 2$ $\quad \Longleftrightarrow \operatorname{rank}(A - I) = 3$

Thus (3) is true.

(4) False since $\operatorname{rank}(A - I) = 2 \Rightarrow \operatorname{nullity}(A - I) = 2$
$\Downarrow$
A is diagonalizable

(5) False since -3 is not an eigenvalue