Purdue University, Spring 2021   Instructor: Marius Dadarlat

MA265 Linear Algebra — Practice Exam 2

Date: April 7th, 2021   *Duration:* 60 min

Name: ______________________________

PUID: ______________________________

- We will use Gradescope for Exam 2. There is a link to Gradescope in the Content page of Brightspace. When you go to that link and open Gradescope, you should see the Exam 2 assignment on April 7th. Exam 2 will be available on April 7th at 8:00AM and it will be due on April 8th by 9:00AM. Exam 2 will cover the following sections:

  4.1, 4.2, 4.3, 4. 5, 5.1, 5.2, 5.3, 5.4, 5.5, 5.7 and Appendix B

- The exam will consist of 10 questions. Once you open the exam, you will have 80 minutes to complete it and upload 10 files, 1 file for each question. The estimated time to work on the problems is 60 minutes and you are given an extra 20 minutes to upload the files. It is not acceptable to answer multiple questions in a single file.

- In other words, you need to submit each of your solutions separately. You can submit as a pdf or as a picture from your phone (but if you use your phone, the picture must be in a standard format like jpeg or png, NOT heic). For each problem, you should click "select file" and then select a file to upload and then click "submit answer." To check that you did it right, you can click "View your submission." Please make sure that your solution is all there and that it is readable and that it is oriented correctly (vertically). If you want to change anything you submitted, use the "Resubmit" button in the bottom right corner.

- To get credit, you will need to show your work and explain your answers, unless it is specified in the body of the question that no explanations are required.

- The exam is open book. You are not allowed to use computer software. You should work independently. Penalties for cheating include an F in the course

1. Read the instructions on page 1 as they will apply to Exam 2. [0pt]

2. Let $\mathcal{P}_2$ be the vector space of all polynomials p of degree at most 2. [10pt]

Let W be the subspace of $\mathcal{P}_2$ consisting of those polynomials p that satisfy the conditions:

$$\boxed{p(0) + p(-1) = 0.}$$

(a) What is the dimension of W?

(b) Find a basis of W.

$\mathcal{P}_2 = \{ p(t) = a_0 + a_1 t + a_2 t^2 : a_0, a_1, a_2 \text{ in } \mathbb{R} \}$

$\dim(\mathcal{P}_2) = 3$ basis $\{1, t, t^2\}$

$\underbrace{p(0)}_{}+ p(-1) = \underbrace{a_0}_{p(0)} + \underbrace{a_0 - a_1 + a_2}_{p(-1)} = 0$

$2a_0 - a_1 + a_2 = 0$ $a_1 = 2a_0 + a_2$

$\downarrow$

$p(t)$

$p(t) = a_0 + \underbrace{(2a_0 + a_2)}_{a_1} t + a_2 t^2$

$p(t) = a_0(1 + 2t) + a_2(t + t^2)$

$1 + 2t$ $t + t^2$ linearly ind.

they span W

$\dim W = 2$ basis $\{1 + 2t, t + t^2\}$

3. Let A be a 5×7 matrix of rank 3. [10pt]

(a) What is the dimension of the null space of the homogeneous system $A^T\mathbf{x} = \mathbf{0}$? (2)

(b) What is the dimension of the row space of A^T? (3)

$A \quad m \times n$

$$\text{rank}(A) + \underbrace{\dim(\text{Nul}(A))}_{\text{nullity of } A} = n$$

5×7 A: $\text{rank}(A) + \dim(\text{Nul}(A)) = 7$

A^T 7×5: $\underbrace{\text{rank}(A^T)}_{3} + \underbrace{\dim(\text{Nul}(A^T))}_{?} = 5$

$\text{rank}(A) = \text{rank}(A^T)$ (always the case)

$= 3$

in our case

$\dim(\text{Nul}(A^T)) = 2 = 5 - 3$

$\text{Row}(A^T) = \text{Col}(A) \leftarrow$ whose dimension is $\text{rank}(A) = 3$

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 1 & 2 \end{bmatrix}$$ row A is spanned by $[2, 1, 3]$, $[0, 1, 2]$

$$A^T = \begin{bmatrix} 2 & 0 \\ 1 & 1 \\ 3 & 2 \end{bmatrix}$$ row A spanned by $[2, 0]$, $[1, 1]$, $[3, 2]$

4. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, where [10pt]

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} 0 \\ -4 \\ 1 \end{bmatrix}, \ \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}.$$

For each of the following assertions indicate whether it is **true** or **false**.

(i) A basis for span S is $\{\mathbf{v}_1, \mathbf{v}_2\}$. TRUE

(ii) The vector $\mathbf{u} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$ belongs to span S. TRUE

(iii) S is a linearly independent set. FALSE

Span (S) subspace of $\mathbb{R}^3$

(i) $\{v_1, v_2\}$ basis for Span(S) $\iff$ $\underbrace{\text{rank}\,[v_1, v_2]}_{2} = \underbrace{\text{rank}\,[v_1, v_2, v_3]}_{2}$

v_1, v_2 lin. indep.

(iii) $\{v_1, v_2, v_3\}$ lin. indep $\iff$ rank $[v_1, v_2, v_3] = 3$

(ii) $\vec{u}$ belongs to span $\{v_1, v_2, v_3\}$

$\iff$ $\underbrace{\text{rank}\,[v_1, v_2, v_3]}_{\dim(\text{span}(S))} = \underbrace{\text{rank}\,[v_1, v_2, v_3, u]}_{2.}$

RREF $\begin{bmatrix} 1 & 0 & 1 & 3 \\ -2 & -4 & 2 & 2 \\ -1 & 1 & -2 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

5. Consider the vector space $\mathcal{P}_4$ consisting of all polynomials of degree ≤ 4. Define a linear transformation $T : \mathcal{P}_4 \to \mathcal{P}_4$ by $T(p) = p''$ where p'' denotes the second derivative of p. Find all the eigenvalues and the corresponding eigenvectors of T. [10pt]

$p = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4$

$p' = a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3$

$p'' = 2a_2 + 6a_3 t + 12 a_4 t^2$

$T(p) = p''$ $\qquad T(p) = \lambda p$ $\qquad \underline{\underline{p''}} = \underline{\underline{\lambda p}}$

$\underline{2a_2 + 6a_3 t + 12 a_4 t^2} = \lambda \left(a_0 + \underline{a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4}\right)$

two polynomials are equal $\iff$ have same coefficients

$$\begin{cases} 2a_2 = \lambda a_0 \\ 6a_3 = \lambda a_1 \\ 12 a_4 = \lambda a_2 \\ 0 = \lambda a_3 \\ 0 = \lambda a_4 \end{cases}$$

case $\lambda \neq 0$

$a_4 = 0$

$a_3 = 0$ $\Rightarrow$ $\underline{\underline{p = 0}}$

$a_1 = 0$

$a_2 = 0$

$a_0 = 0$

Thus no number $\lambda \neq 0$ can be an eigenvalue

See what if $\boxed{\lambda = 0}$

$2a_2 = 0$ $\qquad a_2 = 0$

$6a_3 = 0$ $\qquad a_3 = 0$

$12 a_4 = 0$ $\qquad a_4 = 0$

$\lambda = 0$

$p'' = \lambda p$

$\boxed{p'' = 0}$

$p(t) = a_0 + a_1 t$

$p''(t) = 0 = 0 \cdot p(t)$

Thus $\lambda = 0$ only eigenvalue

eigenspace $\{ \underline{\underline{a_0 + a_1 t}} : a_0, a_1 \text{ in } \mathbb{R} \}$

$p(0) = 1 \cdot 4 \cdot (-1)^2 = 4.$

6. The characteristic polynomial of a 5×5 matrix A is $p(\lambda) = (1-\lambda)(2-\lambda)^2(-1-\lambda)^2$. Which of the following assertions are true? (*No explanations required*). [10pt]

(1) A is diagonalizable FALSE

(2) A is similar to $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$. FALSE

(3) A is similar to $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$. FALSE

(4) A is not diagonalizable. FALSE

(5) A has at least two linearly independent eigenvectors ← TRUE

(6) $det(A) = 4$ True

$\lambda_1 = 1$ $m_1 = 1$

$\lambda_2 = 2$ $m_2 = 2$

$\lambda_3 = -1$ $m_3 = 2$

nullity $(A - 2I) \geq 1$

nullity $(A - (-1)I) \geq 1$

unknown

Review: If a real matrix A has all eigenvalues real numbers $\lambda_1, \lambda_2, \ldots, \lambda_k$
$m_1 \quad m_2 \quad m_k$

then A is diagonalizable

$\iff$ nullity $(A - \lambda_i I) = m_i$ for each $i = 1, 2, \ldots, k$

In general

$1 \leq$ nullity $(A - \lambda_i I) \leq m_i$

If all eigenvalues have multiplicity $= 1$ then A is diagonalizable.

(6) $p(\lambda) = \det(A - \lambda I)$ Take $\lambda = 0$

$\boxed{p(0) = \det(A)}$ $A - 0 \cdot I = A$

7. Let V be the subspace of $\mathbb{R}^4$ consisting of vectors [10pt]

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

that satisfy the relations

$$\begin{aligned} &a + 3b + 2c - d = 0 \\ &b + c + 2d = 0 \\ &a + 2b + c - 3d = 0 \end{aligned}$$

What is the dimension of V?

V is the null space of

$$A = \begin{bmatrix} 1 & 3 & 2 & -1 \\ 0 & 1 & 1 & 2 \\ 1 & 2 & 1 & -3 \end{bmatrix}$$

$$RREF(A) = \begin{bmatrix} 1 & 0 & -1 & -7 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\text{rank}(A) = 2$ $\quad$ $\text{Nullity}(A) = 4 - 2 = 2.$

$\dim(V) = 2.$

8. Let A be an $n \times n$ matrix such that $A\mathbf{v} = 0$ for some nonzero vector $\mathbf{v}$ in $\mathbb{R}^n$. Determine whether or not $\lambda = 0$ must be an eigenvalue of the transpose matrix A^T. [10pt]

A $n \times n$ $\quad$ $\text{rank}(A) \leq n-1$ since $Av = 0$, $v \neq 0$

$\text{nullity}(A) \geq 1$

$\text{rank}(A) = \text{rank}(A^T)$

$\text{rank}(A^T) \leq n-1$

$\text{nullity}(A^T) \geq 1$

$\Rightarrow \exists\ w \neq 0$ $\quad$ $A^T w = 0$

$\lambda = 0$ eigenvalue of A^T.

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when is λ eigenvalue of A

$\Leftrightarrow$ there is $v \neq 0$

s.t. $Av = \lambda v$

Apply this for $\lambda = 0$

$Av = 0$.

$v \neq 0$

9. For each of the following assertions indicate whether it is **true** or **false**. (*No explanations requred*) [10pt]

(1) A homogeneous linear system with five equations in five unknowns is always consistent. True

(2) A linear system with three equations in ten unknowns is always consistent. FALSE

(3) A homogeneous linear system with fewer equations than unknowns must always have infinitely many solutions. True since $\dim(Null(A)) > 0$ using rank num.

(4) If A is a 5×5 square matrix and the system $A\mathbf{x} = \mathbf{0}$ has at least two different solutions, then the rank of A is at most 3. FALSE $A = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 0 \end{bmatrix}$ rank$(A) = 4$ $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

(5) If A is a 4×4 square matrix and the system $A\mathbf{x} = \mathbf{0}$ has at least two different solutions, then the nullity of A is at least 1. TRUE Null space

(1) $A\bar{x} = \bar{b}$ is consistent $\Leftrightarrow$ it has solutions.

homogeneous $A\bar{x} = \bar{0}$ always consistent $\bar{x} = \bar{0}$ is a sol'n.

True

(2) $\begin{cases} x_1 + x_2 + \cdots + x_{10} = 0 \\ 2x_1 + 2x_2 + \cdots + 2x_{10} = 1 \\ 3x_1 + 3x_2 \quad + 3x_{10} = 0 \end{cases}$

inconsistent

(3) $A\bar{x} = \bar{0}$ $m \times n$

m eqn's n unknowns

$\boxed{m < n}$ by assumption

$\text{rank}(A) \le m \Rightarrow \text{nullity}(A) \ge n - \text{rank}(A) > 0$

$Null(A) \neq 0$

10. If $x_1(t)$, $x_2(t)$ satisfy $x_1(0) = 1$, $x_2(0) = 2$ and [10pt]

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

compute $x_1(1) + x_2(1)$.

$= 3e^3.$

$\lambda_1 = -1 \qquad \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\lambda_2 = 3 \qquad \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$x(t) = c_1 e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$x_1(t) = -c_1 e^{-t} + c_2 e^{3t} \qquad x_1(0) = 1$

$x_2(t) = c_1 e^{-t} + c_2 e^{3t} \qquad x_2(0) = 2$

$-c_1 + c_2 = 1 \qquad c_1 = 1/2$

$c_1 + c_2 = 2 \qquad c_2 = 3/2$

$x_1(1) + x_2(1) = 2c_2 e^{3 \cdot 1} = 2 \cdot \frac{3}{2} e^3$

$= \boxed{3e^3}$

$$\det(P^{-1}) = \frac{1}{\det(P)}$$

11. The characteristic polynomial of a 4×4 matrix A is $p(\lambda) = (1-\lambda)(\lambda^2 - 2)(3-\lambda)$. Compute $\det(A)$ and $\det(A^2 + A)$. [10pt]

Hint: argue that A is diagonalizable and find a diagonal matrix D similar to A. Then find a diagonal matrix similar to $A^2 + A$.

eigenvalues of A are $1 \quad \sqrt{2} \quad -\sqrt{2} \quad 3$

all distinct

$\Rightarrow$ A diagonalizable

$\Rightarrow$ $A = PDP^{-1}$

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & -\sqrt{2} & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

$$\det(A) = \det(PDP^{-1}) = \det(P)\det(D)\cdot\det(P)^{-1}$$
$$= \det(D) = 1(\sqrt{2})(-\sqrt{2})(3)$$
$$= -6$$

A similar to D $\qquad \det(A) = \det(D)$

$A^2 + A$ similar to $D^2 + D$ $\qquad$ why?

$$A^2 = PD\underbrace{P^{-1}P}_{I}DP^{-1} = PD^2P^{-1}$$

$$A^2 + A = PD^2P^{-1} + PDP^{-1}$$
$$= P(D^2 + D)P^{-1}$$

$$\det(A^2 + A) = \det(D^2 + D) \qquad D^2 = \begin{pmatrix} 1 & & & 0 \\ & 2 & & \\ & & 2 & \\ 0 & & & 9 \end{pmatrix}$$

$$D^2 + D = \begin{pmatrix} 2 & & & \\ & 2+\sqrt{2} & & \\ & & 2-\sqrt{2} & \\ & & & 12 \end{pmatrix}$$

$$\det(D^2 + D) = 2(4-2)\cdot 12 = \boxed{48}$$